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DARMSTADT

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Master's Thesis

Analysis and Numerical Approximation of Transport Equations on Networks

Nora Marie Philippi

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Supervisor: Prof. Dr. Herbert Egger

Contents

Introduction

In this thesis we investigate scalar, linear transport equations with piecewise constant coefficients on networks. They describe, for instance, the passive transport of a tracer, e.g. a chemical substance, in a fluid flow within a pipe network. In every pipe e the dynamics is modeled by the hyperbolic partial differential equation

$$a^e \partial_t u^e + b^e \partial_x u^e = 0,$$

where a^e $[\mathrm{m}^2]$ denotes the cross-sectional area, b^e $[\mathrm{m}^3/\mathrm{s}]$ the volume flow rate, which is determined by the flow dynamics, and u^e the molar concentration $[\mathrm{mol}/\mathrm{m}^3]$ of the tracer within the fluid flow. Under the assumption that the flow is steady and the fluid is incompressible, we have a constant volume flow rate b^e in every pipe. On networks, additional coupling conditions are necessary that model the mixing process of the tracer at inner junctions. We assume that the inflowing tracer is mixed and flows out with the molar concentration of this mixture. Together with suitable initial and boundary conditions the passive transport process on a network is fully described.

This work deals with the analysis and the numerical approximation of such transport problems on networks. In the first chapter, we present the network topology, the basic notations, and the problem under investigation. The second chapter focuses on the well-posedness of the transport problem on networks. We use semigroup theory to verify existence and uniqueness of solutions. Moreover, we derive a stability estimate for the exact solution. In the third chapter, we investigate the numerical approximation through Galerkin methods, more precisely the Discontinuous Galerkin method. We show a discrete stability estimate and derive a convergence result for the semi-discretization on a single edge and on networks. Additionally, we consider a suitable time discretization and also derive a convergence result for the fully discrete scheme on networks. Finally, the theoretical results are illustrated by some numerical tests in the last chapter.

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Chapter 1

Transport Equations on Networks - Problem and Notation

This work deals with transport equations on networks. In this section, we define the network topology and introduce function spaces on networks. Moreover, we present the problem under investigation. The notations and basic definitions are taken from [3], but they do not coincide completely.

1.1 The network topology

The network is described by a finite, directed, and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a set of vertices $\mathcal{V} = \{v_1, \dots, v_n\}$ and edges $\mathcal{E} = \{e_1, \dots, e_m\} \subset \mathcal{V} \times \mathcal{V}$. Every edge $e \in \mathcal{E}$ has a specific length l^e and we identify e with the interval $(0, l^e)$. The values l^e are stored in a vector $l := (l^e)_{e \in \mathcal{E}}$ of length m. The triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, l)$ is then called a *geometric graph* and represents the basic geometric model for the network. For every edge $e = (v_1, v_2)$ we define two values

$$n^e(v_1) := -1 \quad \text{and} \quad n^e(v_2) := +1, \tag{1.1}$$

which indicate the start and end point of the edge by -1 and $+1$, respectively, and for $v \in \mathcal{V} \setminus \{v_1, v_2\}$ we set $n^e(v) := 0$. For every vertex $v \in \mathcal{V}$ we then define the set of incident edges by $\mathcal{E}(v) := \{e \in \mathcal{E} : n^e(v) \neq 0\}$. We distinguish between inner vertices $\mathcal{V}_0 := \{v \in \mathcal{V} : |\mathcal{E}(v)| \geq 2\}$ and boundary vertices $\mathcal{V}_\partial := \mathcal{V} \setminus \mathcal{V}_0$ and assume that the

set of boundary vertices is non-empty. A geometric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, l)$ satisfying the properties noted above will be called a network in the sequel. An example of a network is given in figure 1.1.

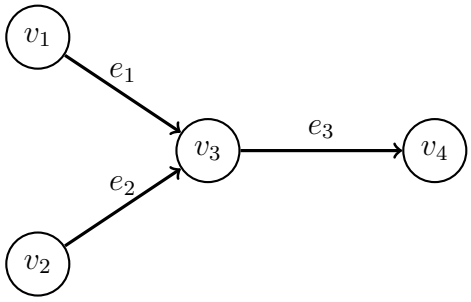


Figure 1.1: A network with vertices $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ and edges $\mathcal{E} = \{e_1, e_2, e_3\}$, inner vertex $\mathcal{V}_0 = \{v_3\}$, and boundary vertices $\mathcal{V}_\partial = \{v_1, v_2, v_4\}$. For the vertex v_3 we have $\mathcal{E}(v_3) = \{e_1, e_2, e_3\}$ and $n^{e_1}(v_3) = n^{e_2}(v_3) = +1,\ n^{e_3}(v_3) = -1$.

1.2 Function spaces on networks

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, l)$ be a network. We denote by $L^2(e) = L^2(0, l^e)$ the space of square integrable functions on the edge $e \in \mathcal{E}$ and by

$$L^2(\mathcal{E}) = L^2(e_1) \times \cdots \times L^2(e_m) = \{u : u^e \in L^2(e) \text{ for all } e \in \mathcal{E}\}$$

the space of square integrable functions on the network. Here $u^e = u|_e$ denotes the restriction of $u \in L^2(\mathcal{E})$ to e. The natural norm and scalar product are given by

$$\|u\|^2_{L^2(\mathcal{E})} = \sum_{e\in\mathcal{E}} \|u^e\|^2_{L^2(e)} \qquad \text{and} \qquad (u, v)_{L^2(\mathcal{E})} = \sum_{e\in\mathcal{E}} (u^e, v^e)_{L^2(e)}.$$

In a similar manner, we define the broken Sobolev spaces

$$H^s_{pw}(\mathcal{E}) = \{u \in L^2(\mathcal{E}) : u^e \in H^s(e) \text{ for all } e \in \mathcal{E}\}.$$

Let us note that for $s \geq 1$ the functions $u \in H^s_{pw}$ are continuous along edges $e \in \mathcal{E}$, but may be discontinuous across junctions $v \in \mathcal{V}_0$. We will frequently use the space H^1_{pw} that can also be written as

$$H^1_{pw}(\mathcal{E}) = \{u \in L^2(\mathcal{E}) : \partial'_x u \in L^2(\mathcal{E})\},$$

where $\partial'_x u|_e = \partial_x u^e$ is the broken weak derivative. This space is equipped with the norm

$$\|u\|^2_{H^1_{pw}(\mathcal{E})} = \|u\|^2_{L^2(\mathcal{E})} + \|\partial'_x u\|^2_{L^2(\mathcal{E})}$$

and the corresponding scalar product

$$(u, v)_{H^1_{pw}(\mathcal{E})} = (u, v)_{L^2(\mathcal{E})} + (\partial'_x u, \partial'_x v)_{L^2(\mathcal{E})}.$$

1.3 Transport equations on a single edge

The transport equation on a single edge $e = (0, l)$ is given by the scalar, linear, hyperbolic partial differential equation

$$a\partial_t u + b\partial_x u = 0, \quad x \in e,\ t > 0, \tag{1.2}$$

which models the transport of a certain quantity, e.g. a substance with a concentration u, through a pipe of length l and cross-sectional area a with a volume flow rate b. We assume that $a, b > 0$ are positive and constant. Equation (1.2) describes the conservation of the total amount of the substance, i.e. the change of the total amount in a control volume is only caused by inflow and outflow across its boundaries.

Lemma 1.1. *Let $T > 0$ and $u \in C^1([0,T]; L^2(e)) \cap C^0([0,T]; H^1(e))$ be a solution of (1.2). Then for any $x_1, x_2 \in e,\ x_1 < x_2$, it holds*

$$\frac{d}{dt}\int_{x_1}^{x_2} au(x,t)\,dx = -bu(x,t)\Big|_{x=x_1}^{x_2}$$

Proof. Since u solves (1.2) and a, b are constant, one has

$$\frac{d}{dt}\int_{x_1}^{x_2} au(x,t)\,dx = \int_{x_1}^{x_2} a\partial_t u(x,t)\,dx = -\int_{x_1}^{x_2} b\partial_x u(x,t)\,dx = -bu(x,t)\Big|_{x=x_1}^{x_2},$$

which proves the result. □

In order to uniquely specify the solution, we need initial conditions and boundary conditions at the inflow boundary $x = 0$ of e given by

$$u(x,0) = u_0(x), \qquad x \in e, \tag{1.3}$$
$$u(0,t) = g(t), \qquad t > 0, \tag{1.4}$$

with $u_0 \in L^2(e)$ and $g \in L^2(0,\infty)$. We can now derive a stability estimate for the solution of (1.2)–(1.4), which directly implies an a-priori bound.

Lemma 1.2 (Stability)**.** *Let $u \in C^1([0,T];L^2(e)) \cap C^0([0,T];H^1(e))$ be a solution of (1.2)–(1.4). Then*

$$\frac{d}{dt}\|a^{1/2}u(t)\|^2_{L^2(e)} + bu(l,t)^2 = bg(t)^2 \qquad \text{for all } t \in (0,T). \tag{1.5}$$

Proof. Since u solves (1.2) with inflow boundary condition (1.4), we have

$$\begin{aligned}\frac{d}{dt}\|a^{1/2}u(t)\|^2_{L^2(e)} &= 2(a\partial_t u(t), u(t))_{L^2(e)} = -2(b\partial_x u(t), u(t))_{L^2(e)} = -\int_e b\frac{d}{dx}u(t)^2\,dx \\ &= bu(0,t)^2 - bu(l,t)^2 = bg(t)^2 - bu(l,t)^2.\end{aligned}$$

□

Lemma 1.2 ensures the uniqueness of solutions of the transport problem (1.2)–(1.4).

Corollary 1.3. *A solution of the transport problem (1.2)–(1.4) is unique.*

Proof. Assume that u_1 and u_2 both solve (1.2)–(1.4). Then $u := u_1 - u_2$ solves (1.2)–(1.4) with $u_0 \equiv 0$ and $g \equiv 0$. The identity (1.5) in lemma 1.2 then yields

$$\frac{d}{dt}\|a^{1/2}u(t)\|^2_{L^2(e)} = 0.$$

Since $a > 0$, we can conclude $\|u(t)\|_{L^2(e)} = 0$ for all $t \geq 0$. Consequently $u \equiv 0$, and the solution is thus unique. □

In the case of constant coefficients a and b, we can explicitly determine the solution of the transport problem (1.2)–(1.4) by using the method of characteristics.

Lemma 1.4. *The unique solution of* (1.2)–(1.4) *is given by*

$$u(x,t) = \begin{cases} u_0(x - \frac{b}{a}t), & ax - bt \geq 0, \\ g(t - \frac{a}{b}x), & ax - bt < 0. \end{cases} \tag{1.6}$$

Proof. Let us recall that a curve $t \mapsto (x(t), t)$, along which a solution of (1.2) is constant, is called characteristic. Characteristics can be determined by solving

$$\frac{d}{dt}x(t) = \frac{b}{a}, \quad x(0) = x_0. \tag{1.7}$$

We obtain $x(t) = x_0 + \frac{b}{a}t$ as solution. Then one has

$$\frac{d}{dt}u(x(t),t) = \partial_t u(x(t),t) + \partial_x u(x(t),t)\partial_t x(t) = \partial_t u(x(t),t) + \frac{b}{a}\partial_x u(x(t),t) = 0.$$

Consequently

$$u(x,t) = \begin{cases} u_0(x - \frac{b}{a}t), & ax - bt \geq 0, \\ g(t - \frac{a}{b}x), & ax - bt < 0, \end{cases}$$

solves the transport problem (1.2)–(1.4). □

Remark 1.5. If the initial and boundary conditions are discontinuous or do not satisfy the compatibility condition $u_0(0) = g(0)$, the function given by (1.6) is discontinuous, but can be considered as a generalized solution of (1.2)–(1.4).

An extensive introduction to conservation laws can be found in [10].

1.4 Transport equations on networks

We now extend the transport problem on a single edge given by (1.2)–(1.4) to a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, l)$. On every edge $e \in \mathcal{E}$ we have the equation

$$a^e \partial_t u^e + b^e \partial_x u^e = 0, \qquad x \in e,\ t > 0 \tag{1.8}$$

with initial conditions

$$u^e(x,0) = u_0^e(x), \qquad x \in e. \tag{1.9}$$

We assume that the volume flow rate $b^e > 0$ and the cross-sectional area $a^e > 0$ are positive and constant on every edge $e \in \mathcal{E}$. Additionally, b^e is assumed to satisfy a conservation condition at inner vertices $v \in \mathcal{V}_0$ given by

$$\sum_{e \in \mathcal{E}(v)} n^e(v) b^e = \sum_{e \in \mathcal{E}^-(v)} b^e - \sum_{e \in \mathcal{E}^+(v)} b^e = 0. \tag{1.10}$$

We define the set of inflow edges $\mathcal{E}^-(v) := \{e \in \mathcal{E} : n^e(v) b^e > 0\}$ and the set of outflow edges $\mathcal{E}^+(v) := \{e \in \mathcal{E} : n^e(v) b^e < 0\}$ for every vertex $v \in \mathcal{V}$. Furthermore, we define the set of inflow boundary vertices $\mathcal{V}_\partial^- := \{v \in \mathcal{V}_\partial : b^e n^e(v) < 0 \text{ for } e \in \mathcal{E}(v)\}$ and outflow boundary vertices $\mathcal{V}_\partial^+ := \{v \in \mathcal{V}_\partial : b^e n^e(v) > 0 \text{ for } e \in \mathcal{E}(v)\}$. We assume that there exists at least one inflow and one outflow boundary vertex. Note that every $v \in \mathcal{V}_\partial^-$ has a unique outflow edge and every $v \in \mathcal{V}_\partial^+$ has a unique inflow edge. For ease of notation we just identify this edge by e, if the context is clear.

Coupling and boundary conditions

In order to fully describe the transport problem, we need boundary conditions at the inflow boundary of every edge $e \in \mathcal{E}$. Since $b^e > 0$, the inflow boundary is at $x = 0$ and the outflow boundary at $x = l^e$. Unlike the transport problem on a single edge (1.2)–(1.4), the inflow boundary condition for edges, whose tails are inner vertices, is determined by the inflowing quantity of adjacent edges. Consequently, we need coupling conditions at inner vertices and boundary conditions at the inflow boundary vertices. In the following we write $u^e(v_1, t) = u^e(0, t)$ and $u^e(v_2, t) = u^e(l^e, t)$ for $e = (v_1, v_2)$.

For every inner vertex $v \in \mathcal{V}_0$ we introduce a value u_v^-, which is determined as mixture of the values coming from inflow edges of v, i.e.

$$u_v^-(t) := \frac{\sum_{e \in \mathcal{E}^-(v)} b^e u^e(v, t)}{\sum_{e \in \mathcal{E}^-(v)} b^e}, \qquad v \in \mathcal{V}_0. \tag{1.11}$$

This value then serves as the inflow boundary value for the outflow edges of v, more precisely

$$u^e(v, t) = u_v^-(t) \qquad \text{for } e \in \mathcal{E}^+(v),\ v \in \mathcal{V}_0. \tag{1.12}$$

Note that u_v^- for $v \in \mathcal{V}_0$ has the meaning of g in (1.4) for a single edge. As a consequence of the coupling condition (1.11)–(1.12) we obtain the following result, that yields the

conservation of the total amount of substance at inner vertices.

Lemma 1.6. *Let $u \in C^0([0,T];L^2(\mathcal{E}))$ be a function satisfying the coupling condition (1.11)–(1.12). Then*

$$\sum_{e\in\mathcal{E}} n^e(v)b^e u^e(v,t) = \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t) - \sum_{e\in\mathcal{E}^+(v)} b^e u^e(v,t) = 0.$$

Proof. Inserting the coupling condition (1.11)–(1.12) yields

$$\begin{aligned}
\sum_{e\in\mathcal{E}(v)} n^e(v)b^e u^e(v,t) &= \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t) - \sum_{e\in\mathcal{E}^+(v)} b^e u^e(v,t) \\
&= \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t) - \sum_{e\in\mathcal{E}^+(v)} b^e u_v^-(t) \\
&= \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t) - \sum_{e\in\mathcal{E}^+(v)} b^e \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t)}{\sum_{e\in\mathcal{E}^-(v)} b^e} \\
&= \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t)\big(1 - \frac{\sum_{e\in\mathcal{E}^+(v)} b^e}{\sum_{e\in\mathcal{E}^-(v)} b^e}\big) \\
&= 0.
\end{aligned}$$

Here we used the conservation condition (1.10) for the volume flow rates. □

In addition to the coupling condition (1.11)–(1.12) we need boundary conditions at the inflow boundary vertices $v \in \mathcal{V}_\partial^-$ given by

$$u^e(v,t) = g_v(t) \qquad \text{for } v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v), \tag{1.13}$$

with $g_v \in L^2(0,\infty)$.

The complete transport problem on the network is the given by (1.8) with initial conditions (1.9), inflow boundary conditions (1.13) at the inflow boundary vertices and coupling conditions (1.11)–(1.12) at the inner vertices.

Outlook

The uniqueness of a solution of the transport problem on networks can then again be verified with a stability estimate similar to lemma 1.2 for a single edge. The statement on

existence and uniqueness of solutions together with the proof is given in chapter 2. But on particular networks it is possible to explicitly determine the solution of the transport problem, more precisely on tree-like networks, i.e. networks without loops. One can just extend the exact solution on a single edge given by (1.6) to the network, using the exact solutions on the inflow edges to obtain the coupling conditions for the outflow edges at every inner vertex. Clearly, this procedure is only possible if the network has no loops. Otherwise it is not trivial at all to find the exact solution of the transport problem. In the next section we give a simple example of a tree-like network and determine the exact solution of the transport problem.

1.5 Exact solution of the transport problem for a tree-like network

We consider the tree-like network given in figure 1.1. We distinguish between the inflow boundary vertices $\mathcal{V}_\partial^- = \{v_1, v_2\}$, the outflow boundary vertex $\mathcal{V}_\partial^+ = \{v_4\}$ and the inner vertex $\mathcal{V}_0 = \{v_3\}$. We assume the same cross-sectional area $a^e = 1$ and the same length $l^e = 1$ for all edges. The volume flow rates on the edges are given by $b^{e_1} = b^{e_2} = 1,\ b^{e_2} = 2$, which satisfy the conservation condition (1.10). The initial conditions for the edges are given by $u_0^{e_i} \equiv 0$ for $i = 1, 2, 3$, and the inflow boundary conditions for the inflow boundary vertices in $\mathcal{V}_\partial^-$ are given by $g_{v_1}(t) = \frac{1}{2}$ and $g_{v_2}(t) = 1$ for $t > 0$. According to lemma 1.4, the exact solution on every edge $e_i,\ i = 1, 2, 3$, is given by

$$u^{e_i}(x,t) = \begin{cases} u_0^{e_i}(x - b^{e_i}t), & x - b^{e_i}t \geq 0, \\ g_{v_i}(t - x/b^{e_i}), & x - b^{e_i}t < 0, \end{cases}$$

where $x \in [0, 1],\ t \geq 0$. The inflow boundary condition for the edge e_3 is determined by the transport in the inflow edges e_1, e_2 and the coupling conditions (1.11)–(1.12). We obtain

$$g_{v_3}(t) = \begin{cases} 0, & t \leq 1, \\ \frac{3}{4}, & t > 1. \end{cases}$$

Since the initial and boundary conditions do not satisfy the compatibility condition $u_0^e(0) = g_v(0)$ for $v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v)$, we obtain a discontinuous solution.

Chapter 2

Analysis of Transport Equations on Networks

In this section, we study the well-posedness of the transport problem on networks introduced in section 1.4. For this we will rewrite the problem as an abstract evolution problem in a Hilbert space and use semigroup theory to establish the existence of a unique solution for given initial condition. Additionally, we derive a stability estimate for the solution. Before we state and prove the results on the well-posedness, we give a short introduction to semigroup theory.

2.1 Introduction to semigroup theory

Let us briefly recall some basic results from semigroup theory. We refer to [6, p.433–445] for details and proofs, and to [5, 11] for further reading.

Let X be a real Hilbert space and $A : D(A) \subset X \to X$ be a linear operator with domain $D(A)$. Consider the abstract Cauchy problem

$$u'(t) = Au(t), \qquad t > 0, \tag{2.1}$$

$$u(0) = u_0. \tag{2.2}$$

A function $u \in C^1([0,\infty); X) \cap C^0([0,\infty); D(A))$, which solves (2.1)–(2.2), is called a *classical solution*. Note that (2.1) can be interpreted as an ordinary differential equation

in a Hilbert space.

We now want to investigate the existence and uniqueness of classical solutions of the abstract initial value problem (2.1)–(2.2). For $X = \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ the problem is an ordinary differential equation and the Picard-Lindelöf theorem yields the existence of a unique solution for all $u_0 \in \mathbb{R}^n$. This solution is given by $u(t) = S(t)u_0$ with $S(t) = e^{At}$ defined by the matrix exponential function. One can directly verify that in this case

(i) $S(0)u_0 = u_0 \quad$ for all $u_0 \in X$,

(ii) $S(t+s)u_0 = S(t)S(s)u_0 \quad$ for all $t, s \geq 0,\ u_0 \in X$,

(iii) the mapping $t \mapsto S(t)u_0$ is continuous from $[0, \infty)$ to X.

Now let X be a general real Hilbert space. A family $\{S(t)\}_{t\geq 0}$ of linear and bounded operators $S(t) : X \to X$ satisfying the conditions (i)–(iii) is called a *strongly continuous semigroup*. If, additionally, $\|S(t)\|_X \leq 1$ for all $t \geq 0$, then $\{S(t)\}_{t\geq 0}$ is called a *contraction semigroup*.

To every strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space X, one can associate an operator $A : D(A) \subset X \to X$ defined by

$$Au := \lim_{t\to 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A),$$

which is called the *infinitesimal generator* of the semigroup $\{S(t)\}_{t\geq 0}$. The domain $D(A)$ of A is given by

$$D(A) := \{u \in X : \lim_{t\to 0^+} \frac{S(t)u - u}{t} \quad \text{exists in } X\}$$

and one can show that $D(A)$ is dense in X, see [11].

Semigroup theory allows us to investigate the well-posedness of the abstract Cauchy problem (2.1)–(2.2) and the corresponding inhomogeneous problem. The following result together with the proof can be found in [5, p.435ff.].

Theorem 2.1. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$. Then for any $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; X)$ the*

inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t), \qquad 0 < t < T, \tag{2.3}$$

$$u(0) = u_0 \tag{2.4}$$

has a unique classical solution $u \in C^1([0,T];X) \cap C^0([0,T];D(A))$*, which is given by the abstract variation-of-constants formula*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds. \tag{2.5}$$

Note that for $X = \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ we have $S(t) = e^{At}$ and (2.5) is the well-known solution for the corresponding inhomogeneous ODE.

The objective now is to determine which operators generate strongly continuous semigroups. For contraction semigroups this question is completely answered by the Hille-Yosida theorem, that can be found in [6, p.439]. The Lumer-Phillips theorem gives simpler sufficient conditions for the operator A to generate a contraction semigroup, see [5, p.83] for details. Let us recall that a contraction semigroup is a special case of a strongly continuous semigroup.

Theorem 2.2 (Lumer-Phillips)**.** *Let X be a real Hilbert space and $A : D(A) \subset X \to X$ be a linear, densely defined, and closed operator. If A is dissipative, i.e.*

$$(Au, u)_X \leq 0 \qquad \textit{for all } u \in D(A), \tag{2.6}$$

and $\lambda_0 I - A : D(A) \to X$ has a dense image for some $\lambda_0 > 0$, then A is the generator of a contraction semigroup.

The existence of a unique solution of certain classes of inhomogeneous Cauchy problems of the form (2.3)–(2.4) can thus be verified using theorem 2.1 and the Lumer-Phillips theorem 2.2.

2.2 Well-posedness of transport problems on networks

We now establish the well-posedness of the transport problem on networks introduced in section 1.4 by using the results from the previous section. Let us note that the well-

posedness of evolution equations on networks has also been investigated in [2] for infinite networks using semigroup theory.

Let us recall the transport problem introduced in section 1.4. On every edge $e \in \mathcal{E}$, it reads

$$a^e \partial_t u^e + b^e \partial_x u^e = 0, \qquad x \in e,\ t > 0, \tag{2.7}$$

$$u^e(v,t) = u_v^-(t), \qquad e \in \mathcal{E}^+(v),\ t > 0, \tag{2.8}$$

$$u^e(x,0) = u_0^e(x), \qquad x \in e. \tag{2.9}$$

The inflow values u_v^- are defined by

$$u_v^-(t) := g_v(t), \qquad v \in \mathcal{V}_\partial^-, \tag{2.10}$$

$$u_v^-(t) := \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t)}{\sum_{e\in\mathcal{E}^-(v)} b^e}, \qquad v \in \mathcal{V}_0. \tag{2.11}$$

For our analysis we will assume that

(i) $a \in L^2(\mathcal{E})$ is piecewise constant with $a^e > 0$ for all $e \in \mathcal{E}$,

(ii) $b \in L^2(\mathcal{E})$ is piecewise constant with $b^e > 0$ for all $e \in \mathcal{E}$ and satisfies

$$\sum_{e\in\mathcal{E}(v)} n^e(v) b^e = \sum_{e\in\mathcal{E}^-(v)} b^e - \sum_{e\in\mathcal{E}^+(v)} b^e = 0 \qquad \text{for all } v \in \mathcal{V}_0. \tag{2.12}$$

The main result of this section is the following.

Theorem 2.3 (Well-posedness). *Let $T > 0$, $u_0 \in H^1_{pw}(\mathcal{E})$ satisfying the coupling conditions at inner vertices (2.8) with (2.11), and $g_v \in W^{2,1}(0,T)$ for all $v \in \mathcal{V}_\partial^-$. Further assume that the compatibility condition $u_0^e(v) = g_v(0)$ holds for all $v \in \mathcal{V}_\partial^-$, $e \in \mathcal{E}^+(v)$. Then the transport problem (2.7)–(2.11) has a unique classical solution $u \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$.*

The proof is given in the remainder of this section. In order to show the well-posedness of (2.7)–(2.11), we rewrite the problem as an abstract inhomogeneous Cauchy problem of the form (2.3)–(2.4), and apply theorem 2.1 and the Lumer-Phillips theorem 2.2.

Step 1: Transformation into an abstract Cauchy problem

First we get rid of the inhomogeneous inflow boundary conditions (2.10). We set

$$u^e = u_z^e + u_g^e \quad \text{with} \quad u_z^e(v,t) = 0 \quad \text{and} \quad u_g^e(v,t) = g_v(t) \quad \text{for } v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v).$$

If a suitable function u_g satisfying the coupling conditions at inner vertices (2.8) with (2.11) is given, then we can formulate an inhomogeneous Cauchy problem for u_z on every edge $e \in \mathcal{E}$ by

$$a^e \partial_t u_z^e + b^e \partial_x u_z^e = -(a^e \partial_t u_g^e + b^e \partial_x u_g^e), \qquad x \in e,\ t > 0, \tag{2.13}$$

$$u_z^e(v,t) = u_{z,v}^-(t), \qquad e \in \mathcal{E}^+(v),\ t > 0, \tag{2.14}$$

$$u_z^e(x,0) = u_{z,0}^e(x), \qquad x \in e, \tag{2.15}$$

with homogeneous inflow boundary conditions

$$u_{z,v}^-(t) = 0, \qquad v \in \mathcal{V}_\partial^-, \tag{2.16}$$

and coupling conditions

$$u_{z,v}^-(t) = \frac{\sum_{e \in \mathcal{E}^-(v)} b^e u_z^e(v,t)}{\sum_{e \in \mathcal{E}^-(v)} b^e}, \qquad v \in \mathcal{V}_0. \tag{2.17}$$

Next we define suitable function spaces on the network, see section 1.2, where we include the homogeneous inflow boundary and coupling conditions, by

$$X := L^2(\mathcal{E}) \ \text{ and } \ D(A) := \{v \in H_{pw}^1(\mathcal{E}) :\ v \text{ satisfies (2.14) with (2.16)–(2.17)}\} \subset X$$

with norm and scalar product

$$\|v\|_X := \|a^{1/2} v\|_{L^2(\mathcal{E})} \qquad \text{and} \qquad (u,v)_X := (au, v)_{L^2(\mathcal{E})}.$$

We then formally define

$$A : D(A) \subset X \to X, \quad A|_e := -\frac{b^e}{a^e}\partial_x,$$

$$f \in L^1([0,T]; L^2(\mathcal{E})), \quad f(t)|_e := -(\partial_t u_g^e + \frac{b^e}{a^e}\partial_x u_g^e).$$

Then problem (2.13)–(2.17) can be written as an abstract Cauchy problem on X given by

$$\partial_t' u_z = A u_z + f(t), \qquad t > 0, \tag{2.18}$$

$$u_z(0) = u_{z,0}. \tag{2.19}$$

If a suitable $u_g \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$ satisfying the inflow boundary and coupling conditions (2.8) with (2.10)–(2.11) exists, and if $u_z \in C^1([0,T]; X) \cap C^0([0,T]; D(A))$ is a classical solution of (2.13)–(2.17) resp. (2.18)–(2.19), then $u = u_z + u_g$ is a classical solution of the transport problem (2.7)–(2.11). The uniqueness will be established later by an energy estimate.

Step 2: Definition of the function u_g

Let us start with defining a suitable function $u_g \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$ which satisfies (2.8) with (2.10)–(2.11), so that $u_{z,0}$ and f fulfil the conditions in theorem 2.1, i.e. $u_{z,0} \in D(A)$ and $f \in W^{1,1}([0,T]; X)$. We set

$$u_g^e(x,t) := \begin{cases} g_v(t), & v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v), \\ G_v^-(t) := \frac{\sum_{e \in \mathcal{E}^-(v)} b^e u_g^e(v,t)}{\sum_{e \in \mathcal{E}^-(v)} b^e}, & v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v), \end{cases}$$

and

$$u_{z,0}^e(x) := \begin{cases} u_0^e(x) - g_v(0), & v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v), \\ u_0^e(x) - G_v^-(0), & v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v). \end{cases}$$

Then $u_g \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$, because $g_v \in W^{2,1}(0,T)$ and the space $W^{2,1}(0,T)$ is continuously embedded in $C^1(0,T)$, and since $x \mapsto u_g^e(x,t)$ is a constant function and thus in $H^1(e)$. Moreover, u_g satisfies (2.8) with (2.10)–(2.11) by construction. Since the initial and boundary conditions also fulfil the compatibility condition $g_v(0) = u_0^e(v)$ for $v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v)$, and u_0 itself satisfies the coupling condition at inner vertices (2.8) with (2.11), one can directly see that $u_{z,0} \in D(A)$. For $g_v \in W^{2,1}(0,T)$ it is further clear that

$$f(t)|_e = -(\partial_t u_g^e + \frac{b^e}{a^e} \partial_x u_g^e) = \begin{cases} -\partial_t g_v(t), & v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v), \\ -\partial_t G_v^-(t), & v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v), \end{cases}$$

defines a function $f \in W^{1,1}([0,T]; L^2(\mathcal{E}))$.

Step 3: Existence of a unique solution of (2.18)–(2.19)

According to theorem 2.1 the inhomogeneous Cauchy Problem (2.18)–(2.19) with A, f and $u_{z,0}$ defined in step 1 and step 2 has a unique solution, if the operator A is the infinitesimal generator of a strongly continuous semigroup. We will therefore verify the conditions of theorem 2.2, and show that A is even the infinitesimal generator of a contraction semigroup.

We first check that the operator $A : D(A) \subset X \to X$ is linear, densely defined, and closed. By definition, A is linear. It is also densely defined, since the space $C_0^\infty(\mathcal{E}) = \{u \in L^2(\mathcal{E}) : u^e \in C_0^\infty(e)\}$ is dense in $X = L^2(\mathcal{E})$ and $C_0^\infty(\mathcal{E}) \subset D(A)$. Furthermore, the operator A is closed, i.e. for all sequences $(u^k)_{k\in\mathbb{N}} \subset D(A)$ with $u^k \to u$ and $Au^k \to y$ w.r.t. $\|\cdot\|_X$, it holds that $u \in D(A)$ and $Au = y$. This can be seen by observing that the space $D(A)$ is closed as a proper subspace of the closed space $H^1_{pw}(\mathcal{E})$. Since $(u^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $D(A)$ w.r.t. $\|\cdot\|_X$, the limit u is then also in $D(A)$. From the definition of A it is then immediately clear that $Au = y$.

As a next step, we show that $\lambda_0 I - A : D(A) \subset X \to X$ has a dense image for some $\lambda_0 > 0$. As indicated above, the space $D(A)$ is dense in $X = L^2(\mathcal{E})$ and $A(D(A))$ is also dense in $L^2(\mathcal{E})$ due to the definition of A. Hence the image of $\lambda_0 I - A$ is dense in X for any $\lambda_0 > 0$.

It remains to verify that A is dissipative, i.e. $(Au, u)_X \leq 0$ for all $u \in D(A)$. By definition of A and the scalar product $(\cdot,\cdot)_X$, we have

$$(Au, u)_X = -(\frac{b}{a}\partial'_x u, u)_X = -\sum_{e\in\mathcal{E}} (b^e \partial_x u^e, u^e)_{L^2(e)}.$$

Observe that

$$\begin{aligned} -2\sum_{e\in\mathcal{E}} (b^e \partial_x u^e, u^e)_{L^2(e)} &= -\sum_{e\in\mathcal{E}} \int_e b^e \partial_x (u^e)^2 \, dx = -\sum_{e\in\mathcal{E}} b^e (u^e)^2 \Big|_0^{l^e} \\ &= \sum_{v\in\mathcal{V}_\partial^-} b^e \underbrace{u^e(v)^2}_{=0} - \sum_{v\in\mathcal{V}_\partial^+} b^e u^e(v)^2 \\ &\quad + \sum_{v\in\mathcal{V}_0} \Big(\sum_{e\in\mathcal{E}^+(v)} b^e u^e(v)^2 - \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)^2 \Big), \end{aligned} \tag{2.20}$$

since $u \in D(A)$ implies $u^e(v) = 0$ for $v \in \mathcal{V}_\partial^-$. At the inner vertices $v \in \mathcal{V}_0$, we can insert the coupling condition (2.17) and estimate (2.20) by using Jensen's inequality.

We obtain

$$u^e(v)^2 = (u_v^-)^2 = \left(\frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)}{\sum_{e\in\mathcal{E}^-(v)} b^e} \right)^2 \leq \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)^2}{\sum_{e\in\mathcal{E}^-(v)} b^e} \tag{2.21}$$

for $v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v)$. Inserting (2.21) in (2.20), the sum over the inner vertices can thus be estimated by

$$\begin{aligned}
&\sum_{v\in\mathcal{V}_0} \Big(\sum_{e\in\mathcal{E}^+(v)} b^e u^e(v)^2 - \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)^2 \Big) \\
&\qquad \leq \sum_{v\in\mathcal{V}_0} \Big(\sum_{e\in\mathcal{E}^+(v)} b^e \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)^2}{\sum_{e\in\mathcal{E}^-(v)} b^e} - \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)^2 \Big) \\
&\qquad = \sum_{v\in\mathcal{V}_0} \Big(\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v)^2 \big(\frac{\sum_{e\in\mathcal{E}^+(v)} b^e}{\sum_{e\in\mathcal{E}^-(v)} b^e} - 1 \big) \Big) \\
&\qquad = 0.
\end{aligned} \tag{2.22}$$

The last identity follows by the conservation condition (2.12) for the volume flow rates. In summary, we thus have shown that

$$(Au, u)_X \leq -\frac{1}{2} \sum_{v\in\mathcal{V}_\partial^+} b^e u^e(v)^2 \leq 0 \qquad \text{for all } u \in D(A).$$

This proves that the operator A is dissipative. By theorem 2.2, A is the infinitesimal generator of a contraction semigroup.

Proof of theorem 2.3

We verified all conditions of theorem 2.1. Hence the inhomogeneous Cauchy problem (2.18)–(2.19) has a unique classical solution $u_z \in C^1([0,T]; X) \cap C^0([0,T]; D(A))$ and consequently $u = u_z + u_g \in C^1([0,T]; L^2(\mathcal{E})) \cap C^0([0,T]; H^1_{pw}(\mathcal{E}))$ is a classical solution of (2.7)–(2.11). The uniqueness will be shown in the next section.

2.3 A stability estimate

We now derive a stability estimate for solutions of (2.7)–(2.11), which yields an a-priori bound and allows us to prove uniqueness of solutions. Define

$$E(u) := \frac{1}{2}\|a^{1/2}u\|_{L^2}^2, \tag{2.23}$$

which can be interpreted as mathematical entropy, i.e. a measure for the information content in u. On a network consisting of only one edge $e = (v_1, v_2)$, we already know from lemma 1.2, that $\frac{d}{dt}E(u(t))$ only depends on the inflow and outflow across the boundary, more precisely

$$\frac{d}{dt}E(u^e(t)) + \frac{b^e}{2}u^e(v_2,t)^2 = \frac{b^e}{2}g_{v_1}(t)^2.$$

We now show that a similar property holds on networks.

Lemma 2.4. *Let u be a classical solution of (2.7)–(2.11). Then*

$$\frac{d}{dt}E(u(t)) + \frac{1}{2}\sum_{v\in\mathcal{V}_\partial^+} b^e u^e(v,t)^2 \le \frac{1}{2}\sum_{v\in\mathcal{V}_\partial^-} b^e g_v(t)^2. \tag{2.24}$$

Proof. By elementary computations, one obtains

$$\begin{aligned}
\frac{d}{dt}E(u(t)) &= \frac{1}{2}\frac{d}{dt}\|a^{1/2}u(t)\|_{L^2(\mathcal{E})}^2 = (a\partial_t u(t), u(t))_{L^2(\mathcal{E})} \\
&= -(b\partial_x u(t), u(t))_{L^2(\mathcal{E})} = -\frac{1}{2}\sum_{e\in\mathcal{E}} b^e u^e(t)^2\big|_{x=0}^{l^e} \\
&= \frac{1}{2}\sum_{v\in\mathcal{V}_\partial^-} b^e u^e(v,t)^2 - \frac{1}{2}\sum_{v\in\mathcal{V}_\partial^+} b^e u^e(v,t)^2 \\
&\quad + \frac{1}{2}\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^+(v)} b^e u^e(v,t)^2 - \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t)^2\Big).
\end{aligned}$$

The sum over the inner vertices in the last line can again be estimated by inserting the coupling conditions (2.11) and using Jensen' inequality together with conservation condition (2.12) for the volume flow rates in the same way as we have done in equation (2.22). This yields

$$\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^+(v)} b^e u^e(v,t)^2 - \sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t)^2\Big) \le 0.$$

Inserting the inflow boundary conditions (2.10) then already yields the final stability estimate. □

Hence we verified that no information is produced at junctions. Instead information might get lost due to the mixing process. Moreover, we can use the stability estimate to verify the uniqueness of solutions of (2.7)–(2.11).

Corollary 2.5 (Uniqueness)**.** *A solution of the transport problem on networks* (2.7)–(2.11) *is unique.*

Proof. Assume that u_1 and u_2 both solve (2.7)–(2.11). Then $u := u_1 - u_2$ satisfies (2.7)–(2.11) with $u_0 \equiv 0$ and $g \equiv 0$. Consequently $E(u(0)) = 0$ and lemma 2.4 yields

$$\frac{d}{dt} E(u(t)) \leq 0.$$

It directly follows that $E(u(t)) = 0$, because by definition we have $E(u) \geq 0$. Since $a > 0$, we then obtain $u \equiv 0$. Hence a solution of (2.7)–(2.11) is unique. □

Chapter 3

Numerical Approximation

In this chapter, we study numerical methods for approximating the solution of the transport problem on networks introduced in section 1.4. For the space discretization we use the Discontinuous Galerkin method, which has shown to be especially suitable for hyperbolic problems. First we present this method applied to the transport equation on a single edge and then transfer it to networks. Moreover, we give a stability and a convergence result for the Discontinuous Galerkin semi-discretization. Finally, we discuss a suitable time discretization and state a convergence result for the fully discrete scheme.

3.1 The Discontinuous Galerkin method on a single edge

The Discontinuous Galerkin method is a nonconforming finite element method using discontinuous functions for approximation. Before we focus on the transport problem on networks, we consider the scalar linear transport equation on a single edge $e = [0, l]$ given by

$$a\partial_t u + b\partial_x u = 0, \qquad x \in e,\ t > 0, \tag{3.1}$$

with a constant cross-sectional area $a > 0$ and a constant volume flow rate $b > 0$. Boundary conditions at the inflow boundary $x = 0$ of e and initial conditions are given by

$$u(0, t) = g(t), \qquad t > 0, \tag{3.2}$$

$$u(x, 0) = u_0(x), \qquad x \in e. \tag{3.3}$$

In the following we give an introduction to the basic ideas and present the main results about the Discontinuous Galerkin method based on the expositions in [1, 7, 9, 12].

Notation

We define the spatial grid

$$\mathcal{T}_h := \{T_i := [x_{i-1}, x_i] : i = 1, \dots, M,\ x_0 = 0,\ x_M = l\}$$

with $h_i := x_i - x_{i-1}$ and $h := \max h_i$, and the grid dependent scalar products

$$(u, v)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} (u, v)_{L^2(T)}, \quad \langle nu, v\rangle_{\partial\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} \int_{\partial T} nuv\, ds,$$

where here n denotes the outward unit normal vector on the boundaries of each element $T_i = [x_{i-1}, x_i]$ given by $n|_{T_i} = n_i$ with

$$n_i(x_{i-1}) = -1, \qquad n_i(x_i) = +1.$$

The corresponding grid norm is given by $\|v\|^2_{\mathcal{T}_h} := (v, v)_{\mathcal{T}_h}$. Note that for $v \in L^2(e)$ the grid norm is just the regular L^2-norm. Moreover, we denote by

$$H^s_{pw}(\mathcal{T}_h) := \{v \in L^2(e) : v|_T \in H^s(T) \text{ for all } T \in \mathcal{T}_h\}$$

the grid-dependent space of piecewise H^s-functions.

For the approximation in space we choose piecewise polynomials of degree k, that can be discontinuous at the spatial grid points x_i, from the finite dimensional space

$$V_h := \{v_h \in L^2(e) : v_h|_T \in P_k(T) \text{ for all } T \in \mathcal{T}_h\},$$

where P_k denotes the space of polynomials of degree k. Note that functions in V_h formally take two values at inner grid points. We thus define the lower and upper limit value of $v_h \in V_h$ at $x \in e = [0, l]$ as

$$v_h^-(x) := \lim_{z \to 0^-} v_h(x + z), \quad v_h^+(x) := \lim_{z \to 0^+} v_h(x + z).$$

Note that for $x = 0$ only v_h^+ and for $x = l$ only v_h^- are defined in this way. Additionally,

we define the jump at the grid point $x_i,\ i = 1, \dots, M-1$, as

$$[v_h]_i := v_h^-(x_i) - v_h^+(x_i).$$

For convenience we set $[v_h]_0 := -v_h^+(x_0)$ and $[v_h]_M := v_h^-(x_M)$.

Furthermore, we introduce the L^2-projection of a function $w \in L^2(e)$ onto the space V_h, which is given by the mapping $\Pi_h : L^2(e) \to V_h$, such that

$$(w - \Pi_h w, v_h)_{\mathcal{T}_h} = 0 \qquad \text{for all } v_h \in V_h. \tag{3.4}$$

Let us recall the standard interpolation error estimates.

Lemma 3.1 (Interpolation error estimates)**.** *Let $u \in H^{k+1}_{pw}(\mathcal{T}_h)$ and $T_i \in \mathcal{T}_h$, and denote by $u_i = u|_{T_i}$. Then*

(i) $\|u - \Pi_h u\|_{L^2(T_i)} \le C h^{k+1} |u|_{H^{k+1}(T_i)}$,

(ii) $|(u_i - \Pi_h u_i)(x)| \le C h_i^{k+1/2} |u|_{H^{k+1}(T_i)} \qquad$ *for $x \in \partial T_i$.*

Proof. The first estimate (i) is a standard result, see e.g. [8, Appendix C]. The second estimate (ii) is a direct consequence from the standard trace inequality, the Bramble-Hilbert lemma and (i), see also [8, Appendix C] for the proof. □

The Discontinuous Galerkin method

Denoting by $\Gamma^- = \{0\}$ the inflow boundary of $e = [0, l]$, the Discontinuous Galerkin approximation in space of problem (3.1)–(3.3) is then given by the following problem.

Problem 3.2 (Discontinuous Galerkin method)**.** Find $u_h \in H^1([0,T]; V_h)$, such that

$$u_h(0) = \Pi_h u_0, \tag{3.5}$$

$$(a \partial_t u_h(t), v_h)_{\mathcal{T}_h} + B_h(u_h(t), v_h) = l_h(t, v_h) \tag{3.6}$$

for all $v_h \in V_h,\ t \in (0, T)$, where

$$B_h(u, v) := -(bu, \partial_x v)_{\mathcal{T}_h} + \langle nbu^-, v_h \rangle_{\partial \mathcal{T}_h \setminus \Gamma^-}, \tag{3.7}$$

$$l_h(t, v) := -\langle nbg(t), v \rangle_{\Gamma^-}. \tag{3.8}$$

Auxiliary results

In the following we collect some important properties of the Discontinuous Galerkin method.

Lemma 3.3 (Consistency)**.** *Let u be the classical solution of* (3.1)–(3.3)*. Then*

$$(a\partial_t u(t), v_h)_{\mathcal{T}_h} + B_h(u(t), v_h) = l_h(t, v_h) \qquad \textit{for all } v_h \in V_h,\ t \in (0, T).$$

Proof. Inserting u in (3.6) and applying integration-by-parts yields

$$\begin{aligned}
&(a\partial_t u(t), v_h)_{\mathcal{T}_h} + B_h(u(t), v_h) - l_h(t, v_h) \\
&\quad = (a\partial_t u(t), v_h)_{\mathcal{T}_h} - (bu(t), \partial_x v_h)_{\mathcal{T}_h} + \langle nbu^-(t), v_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-} + \langle nbg(t), v_h\rangle_{\Gamma^-} \\
&\quad = (a\partial_t u(t), v_h)_{\mathcal{T}_h} + (b\partial_x u(t), v_h)_{\mathcal{T}_h} + \langle nb(u^-(t) - u(t)), v_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-} \\
&\qquad + \langle nb(g(t) - u(t)), v_h\rangle_{\Gamma^-} \\
&\quad = 0.
\end{aligned}$$

Here we used that u solves (3.1), is continuous on e and $u(0, t) = g(t)$. □

As a direct consequence of consistency we obtain Galerkin orthogonality.

Corollary 3.4 (Galerkin orthogonality)**.** *Let u be the classical solution of* (3.1)–(3.3) *and u_h be the solution of problem 3.2. Then*

$$(a\partial_t u(t) - a\partial_t u_h(t), v_h)_{\mathcal{T}_h} + B_h(u(t) - u_h(t), v_h) = 0$$

for all $v_h \in V_h,\ t \in (0, T)$.

The well-posedness of problem 3.2 can be verified by the Picard-Lindelöf theorem.

Lemma 3.5 (Well-posedness)**.** *Let $u_0 \in L^2(e)$ and $g \in C^0(0, T)$. Then problem 3.2 has a unique solution $u \in C^1([0, T]; V_h)$.*

Proof. Consider a basis of V_h given by $(\psi_l)_{l=1}^d$, where d is the dimension of V_h, and define

$$\begin{aligned}
\underline{M}_{jl} &= (a\psi_l, \psi_j)_{\mathcal{T}_h}, \\
\underline{B}_{jl} &= -(b\psi_l, \partial_x\psi_j)_{\mathcal{T}_h} + \langle nb\psi_l^-, \psi_j\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-}, \\
\underline{l}_j(t) &= -\langle nbg(t), \psi_j\rangle_{\Gamma^-},
\end{aligned}$$

$j, l = 1, \dots, d$. Then (3.5)–(3.6) can be transformed into

$$\begin{aligned} \underline{M}\partial_t \underline{u}(t) + \underline{B}\underline{u}(t) &= \underline{l}(t), \\ \underline{u}(0) &= \underline{u}_0, \end{aligned}$$

with coefficient vector $\underline{u}(t) = (\underline{u}_l(t))_{l=1}^d$. The initial condition is given by $\Pi_h u_0 \in V_h$, which can thus be written as $\sum_{l=1}^d \underline{u}_{0,l}\psi_l(x)$. The well-posedness of problem 3.2 then directly follows from the Picard-Lindelöf theorem, since the mass matrix $\underline{M}$ is symmetric and positive definite and therefore invertible, and $\underline{l}(t) \in C^0(0,T)$ for $g \in C^0(0,T)$. The corresponding solution of problem 3.2 is then given by $u(x,t) = \sum_{l=1}^d \underline{u}_l(t)\psi_l(x)$. □

Discrete stability

We now study the discrete stability of the Discontinuous Galerkin semi-discretization on a single edge. For this we need to investigate more closely the bilinearform B_h. Let us recall that $a, b > 0$ are assumed to be positive and constant.

Lemma 3.6. *For B_h defined in (3.7) it holds that*

$$B_h(v_h, v_h) = \sum_{i=0}^{M} \frac{b}{2}[v_h]_i^2 \qquad \textit{for all } v_h \in V_h.$$

Proof. From the definition of B_h, one can see that

$$\begin{aligned} B_h(v_h, v_h) &= -(bv_h, \partial_x v_h)_{\mathcal{T}_h} + \langle nbv_h^-, v_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-} \\ &= -\frac{1}{2}\langle nbv_h, v_h\rangle_{\partial\mathcal{T}_h} + \langle nbv_h^-, v_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-} \\ &= -\sum_{i=0}^{M}\frac{b}{2}[v_h^2]_i + \sum_{i=1}^{M} bv_h^-(x_i)[v_h]_i \\ &= \sum_{i=0}^{M}\frac{b}{2}[v_h]_i^2. \end{aligned}$$

□

Lemma 3.6 directly yields a discrete stability estimate for the solution of problem 3.2.

Corollary 3.7 (Discrete stability)**.** *For the solution u_h of problem 3.2, we have*

$$\frac{d}{dt}\|a^{1/2}u_h(t)\|^2_{\mathcal{T}_h} + \sum_{i=1}^{M} b[u_h(t)]_i^2 \leq bg(t)^2.$$

Proof. Since u_h is the solution of problem 3.2 and thus satisfies (3.6), we obtain

$$\frac{1}{2}\frac{d}{dt}\|a^{1/2}u_h(t)\|^2_{\mathcal{T}_h} = (a\partial_t u_h(t), u_h(t))_{\mathcal{T}_h} = -B_h(u_h(t), u_h(t)) + l_h(u_h(t)).$$

Lemma 3.6 yields an identity for B_h given by

$$B_h(u_h(t), u_h(t)) = \sum_{i=0}^{M} \frac{b}{2}[u_h(t)]_i^2.$$

Using Young's inequality, we can estimate l_h by

$$l_h(t, u_h(t)) = -\langle nbg(t), u_h(t)\rangle_{\Gamma^-} = bg(t)u_h^+(0,t) \leq \frac{b}{2}g(t)^2 + \frac{b}{2}u_h^+(0,t)^2.$$

Inserting this into the first identity then leads to the desired result. □

Remark 3.8. Let us recall that the analytical solution of the transport problem on a single edge (3.1)-(3.3) satisfies the stability identity of lemma 1.2, which was given by

$$\frac{d}{dt}\|a^{1/2}u(t)\|^2_{L^2(e)} + bu(l,t)^2 = bg(t)^2.$$

Corollary 3.7 yields a similar result for the discrete solution u_h of problem 3.2. The jumps introduced by the Discontinuous Galerkin methods can be interpreted as additional numerical dissipation, which promotes stability.

Convergence analysis

It remains to investigate the convergence of the Discontinuous Galerkin method applied to the transport problem on a single edge (3.1)–(3.3). The main ingredients of the corresponding proof are the consistency of the method in lemma 3.3, the resulting Galerkin orthogonality in corollary 3.4, the discrete stability in corollary 3.7, and the interpolation error estimates in lemma 3.1.

Theorem 3.9. *Let $T > 0$ and let u be the exact classical solution of (3.1)–(3.3) and u_h be the solution of problem 3.2. If additionally $u(t) \in H^{k+1}(\mathcal{T}_h)$ for all $0 \leq t \leq T$, then*

$$\|u - u_h\|_{L^\infty([0,T];L^2(e))} \leq Ch^{k+1/2}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h))}.$$

We will not go into details here, but refer to the proof of the convergence result on networks given in the next section. This proof directly transfers to a single edge, which can be considered as a network that consists of only one edge. Basically, the procedure in both proofs is the same, except that on networks we have to deal with inner junctions. The convergence result on a single edge and the corresponding proof can also be found in the literature, see e.g. [1, Theorem 3.1].

Remark 3.10. Assuming more regularity of the exact solution, namely $u(t) \in H^{k+2}_{pw}(\mathcal{T}_h)$ for $0 \leq t \leq T$, it is possible to derive an improved error estimate of order $\mathcal{O}(h^{k+1})$. The proof exploits the superconvergence property of the orthogonal projection with the Gauss-Radau points, see [1, Theorem 3.2] for details.

3.2 The Discontinuous Galerkin method on networks

We now extend the Discontinuous Galerkin method stated in problem 3.2 to networks. Let us start with recalling the transport problem on networks introduced in section 1.4. On every edge $e \in \mathcal{E}$ we have

$$a^e \partial_t u^e + b^e \partial_x u^e = 0, \qquad x \in e,\ t > 0, \tag{3.9}$$

$$u^e(v,t) = u_v^-(t), \qquad e \in \mathcal{E}^+(v),\ t > 0, \tag{3.10}$$

$$u^e(x,0) = u_0^e(x), \qquad x \in e, \tag{3.11}$$

with inflow values u_v^- defined by

$$u_v^-(t) := g_v(t), \qquad v \in \mathcal{V}_\partial^-, \tag{3.12}$$

$$u_v^-(t) := \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^e(v,t)}{\sum_{e\in\mathcal{E}^-(v)} b^e}, \qquad v \in \mathcal{V}_0. \tag{3.13}$$

Let us also recall the basic assumptions on the coefficients, i.e.

(i) $a \in L^2(\mathcal{E})$ is piecewise constant with $a^e > 0$ for all $e \in \mathcal{E}$. We denote by $a_{\min} = \min_{e\in\mathcal{E}} a^e$ and by $a_{\max} = \max_{e\in\mathcal{E}} a^e$.

(ii) $b \in L^2(\mathcal{E})$ is piecewise constant with $b^e > 0$ for all $e \in \mathcal{E}$ and satisfies

$$\sum_{e\in\mathcal{E}(v)} n^e(v)b^e = \sum_{e\in\mathcal{E}^-(v)} b^e - \sum_{e\in\mathcal{E}^+(v)} b^e = 0 \qquad \text{for all } v \in \mathcal{V}_0. \tag{3.14}$$

Notation

Now we extend the Discontinuous Galerkin method to the transport problem on networks (3.9)–(3.13). For this we define the spatial grid by

$$\mathcal{T}_h(\mathcal{E}) := \{T_i^e := [x_{i-1}^e, x_i^e] : i = 1, \dots, M^e,\ x_0^e = 0,\ x_{M^e}^e = l^e,\ e \in \mathcal{E}\},$$

and the approximation space by

$$V_h(\mathcal{E}) := \{v_h \in L^2(\mathcal{E}) : v_h|_T \in P_k(T) \text{ for all } T \in \mathcal{T}_h(\mathcal{E})\}.$$

We denote by $h_i^e := x_i^e - x_{i-1}^e$ and by $h := \max h_i^e$. For ease of notation we drop the index e if the meaning is clear from the context. We further introduce the grid dependent scalar products

$$(u, v)_{\mathcal{T}_h(\mathcal{E})} := \sum_{T\in\mathcal{T}_h(\mathcal{E})} (u, v)_{L^2(T)}, \quad \langle nu, v\rangle_{\partial\mathcal{T}_h(\mathcal{E})} := \sum_{T\in\mathcal{T}_h(\mathcal{E})} \int_{\partial T} nuv\, ds,$$

and the corresponding grid norm $\|v\|^2_{\mathcal{T}_h(\mathcal{E})} := (v, v)_{\mathcal{T}_h(\mathcal{E})}$, where n denotes the outward unit normal vector on the boundary of each element $T_i^e = [x_{i-1}^e, x_i^e] \in \mathcal{T}_h(\mathcal{E})$ given by $n|_{T_i^e} = n_i^e$ with

$$n_i^e(x_{i-1}^e) = -1, \qquad n_i^e(x_i^e) = +1.$$

For $e = (v_1, v_2)$ we set $n_1^e(x_0^e) = n^e(v_1)$ and $n_{M_e}^e(x_{M_e}^e) = n^e(v_2)$, where $n^e(v)$ is defined in (1.1). Furthermore, we define the grid-dependent space of piecewise H^s-functions by

$$H^s_{pw}(\mathcal{T}_h(\mathcal{E})) := \{v \in L^2(\mathcal{E}) : v|_T \in H^s(T) \text{ for all } T \in \mathcal{T}_h(\mathcal{E})\}.$$

The L^2-projection of $w \in L^2(\mathcal{E})$ onto the space $V_h(\mathcal{E})$ is given by the mapping $\Pi_h : L^2(\mathcal{E}) \to V_h(\mathcal{E})$, such that

$$(w - \Pi_h w, v_h)_{L^2(\mathcal{E})} = 0 \qquad \text{for all } v_h \in V_h(\mathcal{E}). \tag{3.15}$$

Due to the discontinuity of the functions in $V_h(\mathcal{E})$, the L^2-projection can be defined locally on every element, and therefore the interpolation error estimates given in lemma 3.1 can be used to obtain corresponding estimates for Π_h on networks by summation.

The Discontinuous Galerkin method

A direct combination of the Discontinuous Galerkin approximation for the problem (3.1)–(3.3) on a single edge given in problem 3.2 together with the boundary and coupling conditions (3.12)–(3.13) on networks now leads to the following scheme.

Problem 3.11 (Discontinuous Galerkin method)**.** Find $u_h \in H^1([0,T]; V_h(\mathcal{E}))$, such that

$$u_h(0) = \Pi_h u_0, \tag{3.16}$$

$$(a\partial_t u_h(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u_h(t), v_h) = l_h(t, v_h) \tag{3.17}$$

for all $v_h \in V_h(\mathcal{E}),\ 0 \le t \le T$, where

$$B_h(u, v) := -(bu, \partial_x v)_{\mathcal{T}_h(\mathcal{E})} + \langle nbu^-, v\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-}, \tag{3.18}$$

$$l_h(t, v) := -\langle nbg(t), v\rangle_{\mathcal{V}_\partial^-}, \tag{3.19}$$

with

$$u^{e-}(v) := \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^{e-}(v)}{\sum_{e\in\mathcal{E}^-(v)} b^e}, \qquad v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v). \tag{3.20}$$

Auxiliary results

In the very same manner as on a single edge, we obtain the following properties of the Discontinuous Galerkin method on networks given by problem 3.11.

Lemma 3.12 (Consistency)**.** *Let u be the classical solution of the transport problem on networks (3.9)–(3.13). Then*

$$(a\partial_t u(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u(t), v_h) = l_h(t, v_h) \qquad \text{for all } v_h \in V_h(\mathcal{E}),\ t \in (0,T).$$

Proof. Applying integration-by-parts yields

$$\begin{aligned}
&(a\partial_t u(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u(t), v_h) - l_h(t, v_h) \\
&\quad = (a\partial_t u(t), v_h)_{\mathcal{T}_h(\mathcal{E})} - (bu(t), \partial_x v_h)_{\mathcal{T}_h(\mathcal{E})} + \langle nbu^-(t), v_h\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-} + \langle nbg(t), v_h\rangle_{\mathcal{V}_\partial^-} \\
&\quad = (a\partial_t u(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + (b\partial_x u(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + \langle nb(u^-(t) - u(t)), v_h\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-} \\
&\qquad + \langle nb(g(t) - u(t)), v_h\rangle_{\mathcal{V}_\partial^-} \\
&\quad = \langle nb(u^-(t) - u(t)), v_h\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-}.
\end{aligned}$$

The other terms vanish, because u solves (3.9) and satisfies the inflow boundary condition (3.10) with (3.12), so $u(v,t) = g_v(t)$ for $v \in \mathcal{V}_\partial^-$. For the remaining term we obtain

$$\langle nb(u^-(t) - u(t)), v_h\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-} = \sum_{v\in\mathcal{V}_0} \sum_{e\in\mathcal{E}^+(v)} -b^e(u^{e-}(v,t) - u^e(v,t))v_h^e(v),$$

since u is continuous on every edge $e \in \mathcal{E}$ and thus $u^e(t) = u^{e-}(t)$ on $(0, l^e]$. For any $v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v)$, it holds $u^e(v,t) = u_v^-(t)$, where $u_v^-(t)$ is defined in (3.13). We observe that this definition is identical to the definition of $u^{e-}(v,t)$ for $v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v)$ in (3.20), also since u is continuous on e. Hence, this term vanishes as well and we obtain the desired consistency result. □

As before, consistency again directly implies Galerkin orthogonality.

Corollary 3.13 (Galerkin orthogonality)**.** *Let u be the classical solution of the transport problem on networks (3.9)–(3.13) and u_h be the solution of problem 3.11. Then*

$$(a\partial_t u(t) - a\partial_t u_h(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u(t) - u_h(t), v_h) = 0$$

for all $v_h \in V_h(\mathcal{E}),\ t \in (0,T)$.

As a next step we investigate the well-posedness of the Discontinuous Galerkin semi-discretization on networks. It can be shown with exactly the same argumets as on a single edge, see lemma 3.5.

Lemma 3.14 (Well-posedness)**.** *Let $u_0 \in L^2(\mathcal{E})$ and $g_v \in C^0(0,T)$ for all $v \in \mathcal{V}_\partial^-$. Then problem 3.11 has a unique solution $u \in C^1([0,T]; V_h(\mathcal{E}))$.*

Discrete stability

The analytical solution of the transport problem on networks (3.9)–(3.13) satisfies the stability estimate of lemma 2.4, which was given by

$$\frac{d}{dt}\|a^{1/2}u(t)\|^2_{L^2(\mathcal{E})} + \sum_{v\in\mathcal{V}_\partial^+} b^e u^e(v,t)^2 \leq \sum_{v\in\mathcal{V}_\partial^-} b^e g_v(t)^2.$$

We now show that the solution of the Discontinuous Galerkin semi-discretization defined in problem 3.11 satisfies a similar stability result. For this we first present an estimate for the Discontinuous Galerkin bilinearform B_h defined in (3.18) with (3.20). Note that we assume that $a^e, b^e > 0$ are positive and constant for all $e \in \mathcal{E}$.

Lemma 3.15. *For the bilinearform B_h defined in (3.18) with (3.20) it holds*

$$B_h(v_h, v_h) \geq \sum_{e\in\mathcal{E}} \sum_{i=1}^{M^e-1} \frac{b^e}{2}[v_h]_i^2 + \sum_{v\in\mathcal{V}_\partial^-} \frac{b^e}{2} v_h^+(v)^2 + \sum_{v\in\mathcal{V}_\partial^+} \frac{b^e}{2} v_h^-(v)^2$$

for all $v_h \in V_h(\mathcal{E})$.

Proof. On every edge $e \in \mathcal{E}$ we can use lemma 3.6 to obtain

$$\begin{aligned} B_h(v_h, v_h) &= -(bv_h, \partial_x v_h)_{\mathcal{T}_h(\mathcal{E})} + \langle nbv_h, v_h\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-} \\ &= \sum_{e\in\mathcal{E}} \sum_{i=0}^{M^e} \frac{b^e}{2}[v_h]_i^2 - \sum_{v\in\mathcal{V}_0} \sum_{e\in\mathcal{E}^+(v)} b^e v_h^-(v) v_h^+(v) \\ &= \sum_{e\in\mathcal{E}} \sum_{i=1}^{M^e-1} \frac{b^e}{2}[v_h]_i^2 + \sum_{v\in\mathcal{V}_\partial^-} \frac{b^e}{2} v_h^+(v)^2 + \sum_{v\in\mathcal{V}_\partial^+} \frac{b^e}{2} v_h^-(v)^2 + \\ &\quad \sum_{v\in\mathcal{V}_0} \Big(\sum_{e\in\mathcal{E}^-(v)} \frac{b^e}{2} v_h^-(v)^2 + \sum_{e\in\mathcal{E}^+(v)} \frac{b^e}{2} v_h^+(v)^2 - b^e v_h^-(v) v_h^+(v) \Big). \end{aligned} \tag{3.21}$$

The sum over the inner vertices in the last line can be further estimated by

$$\begin{aligned}
&\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2+\sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}v_h^+(v)^2-u_h^-(v,t)v_h^+(v)\Big)\\
&\qquad=\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2+\sum_{e\in\mathcal{E}^+(v)}\underbrace{\frac{b^e}{2}(v_h^+(v)-v_h^-(v))^2}_{\geq 0}-\frac{b^e}{2}v_h^-(v)^2\Big)\\
&\qquad\geq\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2-\sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}v_h^-(v)^2\Big).
\end{aligned}$$

By inserting the definition of $v_h^{e-}(v)$ for $v\in\mathcal{V}_0,\ e\in\mathcal{E}^+(v)$ given in (3.20), we obtain

$$\begin{aligned}
&\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2-\sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}v_h^-(v)^2\Big)\\
&\qquad=\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2-\sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}\Big(\frac{\sum_{e\in\mathcal{E}^-(v)}b^e v_h^-(v)}{\sum_{e\in\mathcal{E}^-(v)}b^e}\Big)^2\Big)\\
&\qquad\geq\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2-\sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}\frac{\sum_{e\in\mathcal{E}^-(v)}b^e v_h^-(v)^2}{\sum_{e\in\mathcal{E}^-(v)}b^e}\Big)\\
&\qquad=\sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}v_h^-(v)^2\Big(1-\frac{\sum_{e\in\mathcal{E}^+(v)}b^e}{\sum_{e\in\mathcal{E}^-(v)}b^e}\Big)\Big)\\
&\qquad=0.
\end{aligned}$$

Here we used Jensen's inequality and the conservation condition (3.14) for the volume flow rates. Inserting this result in (3.21) thus yields the final estimate

$$B_h(v_h,v_h)\geq\sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e-1}\frac{b^e}{2}[v_h]_i^2+\sum_{v\in\mathcal{V}_\partial^-}\frac{b^e}{2}v_h^+(v)^2+\sum_{v\in\mathcal{V}_\partial^+}\frac{b^e}{2}v_h^-(v)^2.$$

Note that this shows that the bilinearform B_h is positive semi-definite. □

As a direct consequence, we obtain a discrete stability estimate.

Corollary 3.16 (Discrete stability)**.** *For the solution u_h of problem 3.11, one has*

$$\frac{d}{dt}\|a^{1/2}u_h(t)\|_{\mathcal{T}_h(\mathcal{E})}^2+\sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e-1}b^e[u_h(t)]_i^2+\sum_{v\in\mathcal{V}_\partial^+}b^e u_h^-(v,t)^2\leq\sum_{v\in\mathcal{V}_\partial^-}b^e g_v(t)^2.$$

Proof. Testing (3.17) with $v_h = u_h(t)$ leads to

$$\frac{1}{2}\frac{d}{dt}\|a^{1/2}u_h(t)\|^2_{\mathcal{T}_h(\mathcal{E})} = (a\partial_t u_h(t), u_h(t))_{\mathcal{T}_h(\mathcal{E})} = -B_h(u_h(t), u_h(t)) + l_h(t, u_h(t)).$$

Lemma 3.15 yields an estimate for B_h given by

$$B_h(u_h(t), u_h(t)) \geq \sum_{e\in\mathcal{E}} \sum_{i=1}^{M^e-1} \frac{b^e}{2}[u_h(t)]_i^2 + \sum_{v\in\mathcal{V}_\partial^-} \frac{b^e}{2}u_h^+(v,t)^2 + \sum_{v\in\mathcal{V}_\partial^+} \frac{b^e}{2}u_h^-(v,t)^2,$$

and using Young's inequality we can estimate $l_h(t, u_h(t))$ by

$$l_h(t, u_h(t)) = -\langle nbg(t), u_h(t)\rangle_{\mathcal{V}_\partial^-} \leq \sum_{v\in\mathcal{V}_\partial^-} \frac{b^e}{2}g_v(t)^2 + \frac{b^e}{2}u_h^+(v,t)^2.$$

Inserting both estimations into the first identity leads to the desired result. □

Convergence analysis

As a final step of our analysis, we now investigate the convergence of the Discontinuous Galerkin method applied to the transport problem on networks (3.9)–(3.13). The main result is the following.

Theorem 3.17 (Convergence)**.** *Let $T > 0$, and let u be the exact classical solution of (3.9)–(3.13) and u_h be the solution of problem 3.11. If $u(t) \in H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))$ for all $0 \leq t \leq T$, then*

$$\|u - u_h\|_{L^\infty([0,T];L^2(\mathcal{E}))} \leq C(T)h^{k+1/2}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))},$$

where $C(T)$ is a constant only depending on T.

The proof is given in the remainder of this section. The basic procedure is as follows: First we split the error $u-u_h$ into a projection and a discrete error component and consider both separately. The projection error can be estimated by the interpolation error estimates given in lemma 3.1. For the estimation of the discrete error we exploit consistency and discrete stability of the Discontinuous Galerkin scheme. Further computations to handle the inner junctions then yield the stated convergence result.

Let us define $\eta_h := u - \Pi_h u$ and $e_h := u_h - \Pi_h u_h$. Then we can split the error via

$$u(t) - u_h(t) = u(t) - \Pi_h u(t) + \Pi_h u(t) - u_h(t) = \eta_h(t) - e_h(t)$$

into a projection and a discrete error component. By the triangle inequality we obtain

$$\|u(t) - u_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \leq \|\eta_h(t)\|_{\mathcal{T}_h(\mathcal{E})} + \|e_h(t)\|_{\mathcal{T}_h(\mathcal{E})}. \tag{3.22}$$

It thus remains to estimate the two contributions separately.

Lemma 3.18. *The projection error $\eta_h(t)$ can be estimated by*

$$\|\eta_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \leq Ch^{k+1}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))} \tag{3.23}$$

for all $0 \leq t \leq T$. Note that the constant C is independent of T.

Proof. Since $u(t) \in H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))$ for all $0 \leq t \leq T$, the standard interpolation error estimates given in lemma 3.1 yield

$$\|\eta_h(t)\|_{T_i^e} \leq C{h_i^e}^{k+1}|u(t)|_{H^{k+1}_{pw}(T_i^e)}$$

for all $T_i^e \in \mathcal{T}_h(\mathcal{E})$. Summing over all elements then gives

$$\|\eta_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \leq Ch^{k+1} \max_{s\in[0,T]} |u(s)|_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))} = Ch^{k+1}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))}.$$

The last identity holds, because u is continuous in time. □

Lemma 3.19. *The discrete error $e_h(t)$ can be estimated by*

$$\|e_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \leq C(T)h^{k+1/2}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))} \tag{3.24}$$

for all $0 \leq t \leq T$ with a constant $C(T)$ that only depends on T.

Proof. By corollary 3.13 we know that

$$(a\partial_t u(t) - a\partial_t u_h(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u(t) - u_h(t), v_h) = 0$$

for all $v_h \in V_h(\mathcal{E}),\ t \in (0,T)$. As a consequence we have

$$(a\partial_t \eta_h(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(\eta_h(t), v_h) = (a\partial_t e_h(t), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(e_h(t), v_h) \tag{3.25}$$

for all $v_h \in V_h(\mathcal{E})$. Inserting $v_h = e_h(t)$ into (3.25) leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|a^{1/2}e_h(t)\|_{\mathcal{T}_h(\mathcal{E})} &= -B_h(e_h(t),e_h(t)) + (a\partial_t\eta_h(t),e_h(t))_{\mathcal{T}_h(\mathcal{E})} + B_h(\eta_h(t),e_h(t)) \\ &=: \text{ (I)} + \text{(II)} + \text{(III)}. \end{aligned} \tag{3.26}$$

Now we estimate the terms in this identity separately.

Estimation of (I): The identity (3.21) in the proof of lemma 3.15 yields

$$\begin{aligned}B_h(e_h(t),e_h(t)) = &\sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e-1}\frac{b^e}{2}[e_h(t)]_i^2 + \sum_{v\in\mathcal{V}_\partial^-}\frac{b^e}{2}e_h^+(v,t)^2 + \sum_{v\in\mathcal{V}_\partial^+}\frac{b^e}{2}e_h^-(v,t)^2 \\ &+ \sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\frac{b^e}{2}e_h^-(v,t)^2 + \sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}e_h^+(v,t)^2 - b^e e_h^-(v,t)e_h^+(v,t)\Big).\end{aligned}$$

Estimation of (II): The term $(a\partial_t\eta_h(t),e_h(t))_{\mathcal{T}_h(\mathcal{E})}$ vanishes due to the definition of η_h and the property of the L^2-projection (3.15), since $e_h \in V_h(\mathcal{E})$, a is constant on every edge $e \in \mathcal{E}$, and $\partial_t\Pi_h u = \Pi_h\partial_t u$.

Estimation of (III): We obtain

$$\begin{aligned}B_h(\eta_h(t),e_h(t)) &= -\underbrace{(b\eta_h(t),\partial_x e_h(t))_{\mathcal{T}_h(\mathcal{E})}}_{=0} + \langle nb\eta_h(t),e_h(t)\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-} \\ &= \sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e} b^e\eta_h^-(x_i,t)[e_h(t)]_i - \sum_{v\in\mathcal{V}_0}\sum_{e\in\mathcal{E}^+(v)} b^e\eta_h^-(v,t)e_h^+(v,t) \\ &\le \sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e-1}\Big(\frac{b^e}{2}\eta_h^-(x_i,t)^2 + \frac{b^e}{2}[e_h(t)]_i^2\Big) + \sum_{v\in\mathcal{V}_\partial^+}\Big(\frac{b^e}{2}\eta_h^-(v,t)^2 + \frac{b^e}{2}e_h^-(v,t)^2\Big) \\ &\quad + \sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)} b^e\eta_h^-(v,t)e_h^-(v,t) - \sum_{e\in\mathcal{E}^+(v)} b^e\eta_h^-(v,t)e_h^+(v,t)\Big).\end{aligned}$$

The term $(b\eta_h(t),\partial_x e_h(t))_{\mathcal{T}_h(\mathcal{E})}$ also vanishes due to the definition of η_h, the property of the L^2-projection (3.15), the assumption that b is constant on every edge $e \in \mathcal{E}$, and the fact that $\partial_x e_h \in V_h(\mathcal{E})$. Additionally, we used Young's inequality here to estimate the sum over the outflow boundary vertices.

Inserting the results for (I)–(III) into (3.26), where some of the terms are already cancelling out, then gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|a^{1/2}e_h(t)\|_{\mathcal{T}_h(\mathcal{E})} &= -B_h(e_h(t),e_h(t)) + B_h(\eta_h(t),e_h(t))\\ &\leq \sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e-1}\frac{b}{2}\eta_h^-(x_i,t)^2 - \sum_{v\in\mathcal{V}_\partial^-}\frac{b^e}{2}e_h^+(v,t)^2 + \sum_{v\in\mathcal{V}_\partial^+}\frac{b^e}{2}\eta_h^-(v,t)^2\\ &\quad + \sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)}\big(-\frac{b^e}{2}e_h^-(v,t)^2 + b^e\eta_h^-(v,t)e_h^-(v,t)\big)\\ &\quad + \sum_{e\in\mathcal{E}^+(v)}\big(-\frac{b^e}{2}e_h^+(v,t)^2 + b^e e_h^-(v,t)e_h^+(v,t) - b^e\eta_h^-(v,t)e_h^+(v,t)\big)\Big). \quad (3.27)\end{aligned}$$

As a next step, we estimate the sum over the inner vertices. For $v\in\mathcal{V}_0,\ e\in\mathcal{E}^-(v)$ it holds that

$$-\frac{b^e}{2}e_h^-(v,t)^2 + b^e\eta_h^-(v,t)e_h^-(v,t) = -\frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2 + \frac{b^e}{2}\eta_h^-(v,t)^2$$

and for $v\in\mathcal{V}_0,\ e\in\mathcal{E}^+(v)$, using Young's inequality again, we obtain

$$\begin{aligned}&-\frac{b^e}{2}e_h^+(v,t)^2 + b^e e_h^-(v,t)e_h^+(v,t) - b^e\eta_h^-(v,t)e_h^+(v,t)\\ &\quad = -\frac{b^e}{2}(e_h^+(v,t)-e_h^-(v,t))^2 + \frac{b^e}{2}e_h^-(v,t)^2 - b^e\eta_h^-(v,t)\big(e_h^+(v,t)-e_h^-(v,t)\big)\\ &\qquad - b^e\eta_h^-(v,t)e_h^-(v,t)\\ &\quad \leq \frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2.\end{aligned}$$

Inserting the estimates in (3.27) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|a^{1/2}e_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \leq \sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e}\frac{b^e}{2}\eta_h^-(x_i,t)^2 + \sum_{v\in\mathcal{V}_0}\Big(\sum_{e\in\mathcal{E}^-(v)} -\frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2\\ + \sum_{e\in\mathcal{E}^+(v)}\frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2\Big),\end{aligned}$$

where we dropped the term $-\sum_{v\in\mathcal{V}_\partial^-}\frac{b^{ev}}{2}e_h^+(v,t)^2 \leq 0$. By inserting the coupling conditions (3.20) in the same way as we have done in the proof of lemma 3.15, we obtain for

every inner vertex $v \in \mathcal{V}_0$ that

$$\begin{aligned}
&\sum_{e\in\mathcal{E}^-(v)} -\frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2 + \sum_{e\in\mathcal{E}^+(v)} \frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2\big) \\
&\quad\le \sum_{e\in\mathcal{E}^-(v)} -\frac{b_e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2 + \sum_{e\in\mathcal{E}^+(v)} \frac{b_e}{2}\frac{\sum_{e\in\mathcal{E}^-(v)} b^e(\eta_h^-(v,t)-e^-(v,t))^2}{\sum_{e\in\mathcal{E}^-(v)} b^e} \\
&\quad= \sum_{e\in\mathcal{E}^-(v)} \frac{b^e}{2}(\eta_h^-(v,t)-e_h^-(v,t))^2\big(\frac{\sum_{e\in\mathcal{E}^+(v)} b^e}{\sum_{e\in\mathcal{E}^-(v)} b^e}-1\big) \\
&\quad= 0.
\end{aligned}$$

Here we again used Jensen's inequality for the estimate, and the conservation condition (3.14) for the volume flow rates in the last identity.

In summary we can thus estimate the discrete error by the projection error via

$$\frac{1}{2}\frac{d}{dt}\|a^{\frac{1}{2}}e_h(t)\|^2_{\mathcal{T}_h(\mathcal{E})} \le \sum_{e\in\mathcal{E}}\sum_{i=1}^{M^e}\frac{b^e}{2}\eta_h^-(x_i,t)^2.$$

Using the interpolation error estimates given in lemma 3.1, one can see that

$$|(w-\Pi_h w)^-(x_i^e)| \le Ch^{k+1/2}|w|_{H^{k+1}_{pw}(T_i^e)}$$

for $w \in H^{k+1}(T_i^e)$ and all $i = 1,\dots,M^e,\ e \in \mathcal{E}$. Hence

$$\frac{d}{dt}\|a^{1/2}e_h(t)\|^2_{\mathcal{T}_h(\mathcal{E})} \le Ch^{2k+1}|u(t)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}.$$

Integrating over $[0,t]$ then leads to

$$\begin{aligned}
\|a^{1/2}e_h(t)\|^2_{\mathcal{T}_h(\mathcal{E})} \le& \|a^{1/2}e_h(0)\|^2 + Ch^{2k+1}\int_0^t |u(s)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\,ds \\
\le& C(T)h^{2k+1}|u|^2_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))}.
\end{aligned}$$

Here we used that $\|a^{1/2}e_h(0)\|_{\mathcal{T}_h(\mathcal{E})} = 0$, because $u_h(0) = \Pi_h u_0$. Hence we can further estimate

$$\|e_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \le \frac{1}{a_{\min}}\|a^{1/2}e_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \le C(T)h^{k+1/2}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))},$$

which is the final estimate for the discrete error. □

Proof of theorem 3.17

We sucessfully estimated the projection error by (3.23) in lemma 3.18 and the discrete error by (3.24) in lemma 3.19. Inserting both results in (3.22) then yields

$$\|u(t) - u_h(t)\|_{\mathcal{T}_h(\mathcal{E})} \leq C(T)h^{k+1/2}|u|_{L^\infty([0,T];H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E})))}$$

for all $t \in [0,T]$, where $C(T)$ is a constant only depending on T. This gives the desired convergence result.

Remark 3.20. Using the arguments mentioned in remark 3.10 it should be possible to derive an improved error estimate for the Discontinuous Galerkin method of order $\mathcal{O}(h^{k+1})$.

3.3 Time discretization

To obtain a fully discrete scheme, we simply use the implicit Euler method for the time discretization of problem 3.11. We consider N equidistant time steps of length τ up to the time point $T > 0$ and set $t^n = n\tau,\ n = 0,\dots,N$. We denote by u_h^n the numerical solution of the fully discrete scheme at time t^n and by

$$\overline{\partial}_\tau u_h^n := \frac{u_h^n - u_h^{n-1}}{\tau}$$

the corresponding backward difference quotient.

Problem 3.21 (Fully discrete scheme)**.** Set $u_h^0 = \Pi_h u_0$. For $n = 1,\dots,N$ find $u_h^n \in V_h(\mathcal{E})$, such that

$$(a\overline{\partial}_\tau u_h^n, v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u_h^n, v_h) = l_h(t^n, v_h) \tag{3.28}$$

for all $v_h \in V_h(\mathcal{E})$ with $B_h,\ l_h$ and $u^-(v)$ defined as in (3.18), (3.19) and (3.20), respectively. Let us recall the definitions that are given by

$$B_h(u,v) := -(bu, \partial_x v)_{\mathcal{T}_h(\mathcal{E})} + \langle nbu^-, v\rangle_{\partial\mathcal{T}_h(\mathcal{E})\setminus\mathcal{V}_\partial^-}, \tag{3.29}$$

$$l_h(t,v) := -\langle nbg(t), v\rangle_{\mathcal{V}_\partial^-} \tag{3.30}$$

with

$$u^{e-}(v) := \frac{\sum_{e\in\mathcal{E}^-(v)} b^e u^{e-}(v)}{\sum_{e\in\mathcal{E}^-(v)} b^e} \qquad \text{for } v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v). \tag{3.31}$$

Auxiliary results

Now we investigate the properties of the fully discrete scheme. We observe that consistency and Galerkin orthogonality are not satisfied, since the solution $u(t^n)$ of the transport problem on networks (3.9)–(3.13) does not solve (3.28). But we obtain the following similar results including an additional error term introduced by the time discretization.

Lemma 3.22. *Let u be the classical solution of problem (3.9)–(3.13). Then*

$$(a\overline{\partial}_\tau u(t^n), v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u(t^n), v_h) = (a\overline{\partial}_\tau u(t^n) - a\partial_t u(t^n), v_h)_{\mathcal{T}_h(\mathcal{E})} + l_h(t^n, v_h)$$

for all $v_h \in V_h(\mathcal{E}),\ n = 1, \dots, N$.

Proof. This result is a direct consequence of the consistency of the Discontinuous Galerkin semi-discretization, see lemma 3.12. □

We can immediately conclude the following result.

Corollary 3.23. *Let u be the classical solution of (3.9)–(3.13) and $\{u_h^n\}_{0\le n\le N}$ be the solution of problem 3.21. Then*

$$(a\overline{\partial}_\tau u(t^n) - a\overline{\partial}_\tau u_h^n, v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u(t^n) - u_h^n, v_h) = (a\overline{\partial}_\tau u(t^n) - a\partial_t u(t^n), v_h)_{\mathcal{T}_h(\mathcal{E})}$$

for all $v_h \in V_h(\mathcal{E}),\ n = 1, \dots, N$.

It remains to investigate the well-posedness of the fully discrete scheme. It can be verified by using the Lax-Milgram theorem, that gives the existence and uniqueness of variational problems under some requirements on the corresponding bilinear- and linearform.

Lemma 3.24 (Well-posedness)**.** *Problem 3.21 has a unique solution $u_h^n \in V_h(\mathcal{E})$, $n = 0, \dots, N$, for all $u_0 \in L^2(\mathcal{E})$ and $g_v \in L^2(0,T),\ v \in \mathcal{V}_\partial^-$.*

Proof. We can write (3.28) equivalently as

$$\tilde{B}_h(u_h^n, v_h) := (\frac{a}{\tau}u_h^n, v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(u_h^n, v_h) = (\frac{a}{\tau}u_h^{n-1}, v_h)_{\mathcal{T}_h(\mathcal{E})} + l(t^n, v_h) =: \tilde{l}_h^n(v_h).$$

According to the Lax-Milgram theorem, the problem of finding $u_h^n \in V_h(\mathcal{E})$, such that $\tilde{B}_h(u_h^n, v_h) = \tilde{l}_h^n(v_h)$ for all $v_h \in V_h(\mathcal{E})$ and given $u_h^{n-1} \in V_h(\mathcal{E})$, $1 \leq n \leq N$, has a unique solution, if the bilinearform $\tilde{B}_h$ is continuous and elliptic and $\tilde{l}_h^n$ is continuous. The continuity is obviously satisfied. It remains to show that $\tilde{B}_h$ is elliptic, that means

$$\tilde{B}_h(v_h, v_h) \geq C\|v_h\|^2_{\mathcal{T}_h(\mathcal{E})} \quad \text{for all } v_h \in V_h(\mathcal{E}).$$

Lemma 3.15 yields an estimation for B_h given by

$$B_h(v_h, v_h) \geq \sum_{e\in\mathcal{E}} \sum_{i=1}^{M^e-1} \frac{b^e}{2}[v_h]_i^2 + \sum_{v\in\mathcal{V}_\partial^-} \frac{b^e}{2}v_h^+(v)^2 + \sum_{v\in\mathcal{V}_\partial^+} \frac{b^e}{2}v_h^-(v)^2 \geq 0.$$

Since $a^e > 0$ is assumed to be constant on every edge $e \in \mathcal{E}$, we can estimate

$$(\frac{a}{\tau}v_h, v_h)_{\mathcal{T}_h(\mathcal{E})} \geq \frac{a_{\min}}{\tau}\|v_h\|^2_{\mathcal{T}_h(\mathcal{E})}.$$

In summary, we obtain ellipticity of $\tilde{B}_h$. □

Discrete stability

The discrete stability estimate for the Discontinuous Galerkin semi-discretization given in corollary 3.16 directly transfers to the fully discrete scheme. We define

$$\frac{d}{d\tau}\|u_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} := (\overline{\partial}_\tau u_h^n, u_h^n)_{\mathcal{T}_h(\mathcal{E})},$$

and obtain the following result.

Corollary 3.25 (Discrete stability)**.** *Let $\{u_h^n\}_{0\leq n\leq N}$ be the solution of problem 3.21. Then*

$$\frac{d}{d\tau}\|a^{1/2}u_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} + \sum_{e\in\mathcal{E}} \sum_{i=1}^{M^e-1} b^e[u_h^n]_i^2 + \sum_{v\in\mathcal{V}_\partial^+} b^e u_h^{n-}(v)^2 \leq \sum_{v\in\mathcal{V}_\partial^-} b^e g_v(t^n)^2.$$

Convergence analysis

Finally we investigate the convergence of the fully discrete scheme on networks. We will give an error estimate together with the proof.

Theorem 3.26 (Convergence)**.** *Let u be the sufficiently smooth exact solution of (3.9)–(3.13) and $\{u_h^n\}_{0\le n\le N}$ be the solution of problem 3.21 with time step size $0 < \tau \le \tau_0 < 1$. Then the following error estimate holds*

$$\max_{n=0,\dots,N} \|u(t^n) - u_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} \le C\big(\tau^2 \|\partial_{tt} u\|^2_{L^\infty([0,T];L^2(\mathcal{E}))} + h^{2k+1} \max_{n=0,..,N} |u(t^n)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big).$$

The constant C only depends on T and τ_0.

In the remainder of this section we give the proof of this theorem. The procedure is similar as in the proof of theorem 3.17. We again split the error into a projection part and a discrete part and estimate both separately. The projection error can be estimated using the interpolation error estimates given in lemma 3.1. For the discrete error we use corollary 3.23 and the results derived in the proof of theorem 3.17. Moreover, we need a discrete Gronwall lemma to derive the final convergence result.

Denote by $u^n = u(t^n)$ the exact solution at the time point t^n, $n = 0, \dots, N$. Further denote by $\eta_h^n = u^n - \Pi_h u^n$ and $e_h^n = u_h^n - \Pi_h u^n$, $n = 0, \dots, N$. We can now split the error via

$$u^n - u_h^n = u^n - \Pi_h u^n + \Pi_h u^n - u_h^n = \eta_h^n - e_h^n$$

into a projection and a discrete error component, and obtain

$$\|u^n - u_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} \le 2\big(\|\eta_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} + \|e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})}\big). \tag{3.32}$$

Now we estimate the projection error η_h^n and the discrete error e_h^n separately.

Lemma 3.27. *The projection error η_h^n can be estimated by*

$$\|\eta_h^n\|_{\mathcal{T}_h(\mathcal{E})} \le C h^{k+1} \max_{l=0,..,N} |u(t^l)|_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))} \tag{3.33}$$

for all $n = 0, \dots, N$ with a constant C independent of T.

Proof. Assuming $u(t) \in H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))$ for all $0 \leq t \leq T$, the interpolation error estimates in lemma 3.1 yield

$$\|\eta_h^n\|_{T_i^e} \leq C {h_i^e}^{k+1} |u(t^n)|_{H^{k+1}_{pw}(T_i^e)}$$

for all $T_i^e \in \mathcal{T}_h(\mathcal{E}),\ n = 0, \dots, N$. Summing over all elements then leads to

$$\|\eta_h^n\|_{\mathcal{T}_h(\mathcal{E})} \leq C h^{k+1} |u(t^n)|_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))} \leq C h^{k+1} \max_{l=0,..,N} |u(t^l)|_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}.$$

Note that the constant C is independent of T. □

Lemma 3.28. *The discrete error can be estimated by*

$$\|e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} \leq C\big(\tau^2 \|\partial_{tt} u\|^2_{L^\infty([0,T];L^2(\mathcal{E}))} + h^{2k+1} \max_{l=0,..,N} |u(t^l)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big) \tag{3.34}$$

for all $n = 0, \dots, N$ with a constant $C = C(T, \tau_0)$ that only depends on T and τ_0.

Proof. By corollary 3.23 we can deduce that

$$\begin{aligned}
&(a\overline{\partial}_\tau e_h^n, v_h)_{\mathcal{T}_h(\mathcal{E})} + B_h(e_h^n, v_h) \\
&\quad = \underbrace{(a\overline{\partial}_\tau \eta_h^n, v_h)_{\mathcal{T}_h(\mathcal{E})}}_{=0} + B_h(\eta_h^n, v_h) - (a\overline{\partial}_\tau u^n - a\partial_t u(t^n), v_h)_{\mathcal{T}_h(\mathcal{E})} \\
&\quad \leq B_h(\eta_h^n, v_h) + \frac{1}{2}\|a^{1/2}(\overline{\partial}_\tau u^n - \partial_t u(t^n))\|^2_{\mathcal{T}_h(\mathcal{E})} + \frac{1}{2}\|a^{1/2} v_h\|^2_{\mathcal{T}_h(\mathcal{E})}
\end{aligned} \tag{3.35}$$

for all $v_h \in V_h(\mathcal{E})$. The term vanishes, because of the definition of η_h^n, the property of the L^2-projection (3.15), and the fact that a is constant on every edge. Moreover, we used Young's inequality for the last estimate.

Now we insert $v_h = e_h^n$ in (3.35). From estimates we made in the proof of the convergence theorem 3.17, we know that

$$-B_h(e_h^n, e_h^n) + B_h(\eta_h^n, e_h^n) \leq \sum_{e \in \mathcal{E}} \sum_{i=1}^{M^e} \frac{b^e}{2} \eta_h^{n-}(x_i)^2 \leq \frac{C}{2} h^{2k+1} |u(t^n)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}.$$

The error introduced by the implicit Euler method can be estimated by

$$\|a^{1/2}(\overline{\partial}_\tau u^n - \partial_t u(t^n))\|_{\mathcal{T}_h(\mathcal{E})} \leq C a_{\max}^{1/2} \tau \|\partial_{tt} u\|_{L^\infty([t^{n-1}, t^n];L^2(\mathcal{E}))}$$

using the Taylor series and the fact that $a > 0$ is constant on every edge. Inserting the

estimates in (3.35) yields

$$(a\overline{\partial}_\tau e_h^n, e_h^n)_{\mathcal{T}_h(\mathcal{E})} \leq \frac{1}{2}\|a^{1/2}e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} + \frac{C}{2}\big(\tau^2\|\partial_{tt}u\|^2_{L^\infty([t^{n-1},t^n];L^2(\mathcal{E}))} + h^{2k+1}|u(t^n)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big).$$

We can further estimate

$$\begin{aligned}(a\overline{\partial}_\tau e_h^n, e_h^n)_{\mathcal{T}_h(\mathcal{E})} &= \frac{1}{2\tau}\big(\|a^{1/2}(e_h^n - e_h^{n-1})\|^2_{\mathcal{T}_h(\mathcal{E})} + \|a^{1/2}e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} - \|a^{1/2}e_h^{n-1}\|^2_{\mathcal{T}_h(\mathcal{E})}\big)\\ &\geq \frac{1}{2\tau}\big(\|a^{1/2}e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} - \|a^{1/2}e_h^{n-1}\|^2_{\mathcal{T}_h(\mathcal{E})}\big)\end{aligned}$$

and obtain the following estimate of e_h^n given by

$$\frac{1}{\tau}\big(\|a^{1/2}e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} - \|a^{1/2}e_h^{n-1}\|^2_{\mathcal{T}_h(\mathcal{E})}\big) \leq \|a^{1/2}e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} + C\big(\tau^2\|\partial_{tt}u\|^2_{L^\infty([t^{n-1},t^n];L^2(\mathcal{E}))} + h^{2k+1}|u(t^n)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big). \tag{3.36}$$

Now we need a discrete version of the Gronwall lemma for differential equations that are discretized with the implicit Euler method, see [4].

Lemma 3.29 (Discrete Gronwall lemma: backward difference form)**.**
Let $(a^n)_{n\in\mathbb{N}}$ *and* $(b^n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $1-\tau > 0$ *be given and assume for* $n \geq 1$ *that*

$$\frac{1}{\tau}(a^n - a^{n-1}) \leq a^n + b^n.$$

Then

$$a^n \leq (1-\tau)^{-n}\big(a^0 + \tau\sum_{j=0}^{n-1}(1-\tau)^j b^{j+1}\big).$$

We can now apply lemma 3.29 to the estimate (3.36) and obtain

$$\begin{aligned}\|a^{1/2}e_h^n\|^2_{\mathcal{T}_h(\mathcal{E})} \leq\ & (1-\tau)^{-n}\Big(\|a^{1/2}e_h^0\|^2_{\mathcal{T}_h(\mathcal{E})}\\ &+ C\tau\sum_{j=0}^{n-1}(1-\tau)^j\big(\tau^2\|\partial_{tt}u\|^2_{L^\infty([t^j,t^{j+1}];L^2(\mathcal{E}))} + h^{2k+1}|u(t^{j+1})|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big)\Big)\\ \leq\ & \frac{C\tau\sum_{j=0}^{n-1}(1-\tau)^j}{(1-\tau)^n}\big(\tau^2\|\partial_{tt}u\|^2_{L^\infty([0,T];L^2(\mathcal{E}))} + h^{2k+1}\max_{l=0,..,N}|u(t^l)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big).\end{aligned}$$

Here we used that $e_h^0 = u_h^0 - \Pi_h u(0) = 0$ by our choice of u_h^0, so $\|a^{1/2} e_h^0\| = 0$. Furthermore, it holds

$$\sum_{j=0}^{n-1} (1-\tau)^j = \frac{(1-(1-\tau)^n)}{\tau}.$$

We can thus estimate

$$\frac{\tau \sum_{j=0}^{n-1} (1-\tau)^j}{(1-\tau)^n} = \frac{1}{(1-\tau)^n} - 1 \leq \frac{1}{(1-\tau)^N} \leq e^{\frac{T}{1-\tau_0}}.$$

Here we used that $0 < \tau \leq \tau_0 < 1$ and

$$\frac{1}{(1-\tau)^N} = \left(1 + \frac{\tau}{1-\tau}\right)^N \leq \left(1 + \frac{\tau}{1-\tau_0}\right)^N \leq e^{\frac{N\tau}{1-\tau_0}} = e^{\frac{T}{1-\tau_0}}.$$

Including this in the constant $C = C(T, \tau_0)$, we obtain

$$\begin{aligned} \|e_h^n\|^2 &\leq \frac{1}{a_{\min}} \|a^{1/2} e_h^n\|^2 \\ &\leq C\big(\tau^2 \|\partial_{tt} u\|^2_{L^\infty([0,T];L^2(\mathcal{E}))} + h^{2k+1} \max_{l=0,..,N} |u(t^l)|^2_{H^{k+1}_{pw}(\mathcal{T}_h(\mathcal{E}))}\big). \end{aligned}$$

□

Proof of theorem 3.26

We were able to estimate the projection error by (3.33) in lemma 3.27 and the discrete error by (3.34) in lemma 3.28. Inserting in (3.32) then yields the desired result, since both estimations hold for all $n = 0, \ldots, N$.

Remark 3.30. As stated in remark 3.20 and remark 3.10 it should be possible to derive an improved spatial error estimate of order $\mathcal{O}(h^{k+1})$ for the Discontinuous Galerkin semi-discretization. This spatial superconvergence should directly transfer to the fully discrete scheme, which then yields an improved total error of order $\mathcal{O}(\tau + h^{k+1})$.

Chapter 4

Numerical Tests

In the last chapter, we give some details on the implementation of the fully discrete scheme presented in section 3.3, and illustrate our theoretical results by some numerical experiments. We want to verify the stability and error estimates for the exact solution of the transport problem on networks and the numerical solution of the fully discrete scheme.

Details on the implementation

The fully discrete scheme given in problem 3.21 uses the Discontinuous Galerkin method for space and the implicit Euler method for time discretization. Since the implicit Euler is only first order accurate, it seems reasonable to use the Discontinuous Galerkin method with piecewise constant polynomials. We denote by $u_i^{e,n}$ the constant solution value on T_i^e at time point t^n. For $n = 1, \dots, N$ we then solve

$$\frac{h_i^e}{\tau} a^e \left(u_i^{e,n} - u_i^{e,n-1}\right) + b^e \left(u_i^{e,n} - u_{i-1}^{e,n}\right) = 0 \tag{4.1}$$

for $e \in \mathcal{E},\ i = 1, \dots, M^e$, with inflow boundary and initial values

$$u_0^{e,n} = \begin{cases} g_v(t^n), & v \in \mathcal{V}_\partial^-,\ e \in \mathcal{E}^+(v), \\ \frac{\sum_{e \in \mathcal{E}^-(v)} b^e u_{M^e}^{e,n}}{\sum_{e \in \mathcal{E}^-(v)} b^e}, & v \in \mathcal{V}_0,\ e \in \mathcal{E}^+(v), \end{cases} \tag{4.2}$$

$$u_i^{e,0} = \frac{1}{h_i^e} \int_{T_i^e} u_0(x)\, dx. \tag{4.3}$$

For simplicity we introduced a new variable $u_0^{e,n}$ that replaces $u^{e-}(v)$ in the scheme (3.28)–(3.31) at the inflow boundary of every edge $e \in \mathcal{E}$. It can be understood as additional algebraic condition and (4.1)–(4.3) as an algebraic differential equation, which is discretized with the implicit Euler method. We can write it as a linear system of equations that we solve in every time step. For illustration we give a simple example, that will also be considered in the numerical tests.

Example 4.1. We again consider the network in figure 1.1 that has been investigated in section 1.5. The initial condition is given by $u_0(x)$ and the inflow boundary condition by $g_{v_i}(t)$ for $i = 1, 2$. On every edge we have a constant cross-sectional area $a^e > 0$ and a constant volume flux rate $b^e > 0$, and assume an equidistant grid separating each edge into two intervals. Denoting by $\lambda^e := \frac{\tau b^e}{h^e a^e}$, we then solve the following linear system of equations in every time step:

$$\begin{bmatrix}
1+\lambda^{e_1} & 0 & 0 & 0 & 0 & 0 & -\lambda^{e_1} & 0 & 0 \\
-\lambda^{e_1} & 1+\lambda^{e_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1+\lambda^{e_2} & 0 & 0 & 0 & 0 & -\lambda^{e_2} & 0 \\
0 & 0 & -\lambda^{e_2} & 1+\lambda^{e_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1+\lambda^{e_3} & 0 & 0 & 0 & -\lambda^{e_3} \\
0 & 0 & 0 & 0 & -\lambda^{e_3} & 1+\lambda^{e_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -b^{e_1} & 0 & -b^{e_2} & 0 & 0 & 0 & 0 & b^{e_1}+b^{e_2}
\end{bmatrix}
\begin{bmatrix}
u_1^{e_1,n} \\ u_2^{e_1,n} \\ u_1^{e_2,n} \\ u_2^{e_2,n} \\ u_1^{e_3,n} \\ u_2^{e_3,n} \\ u_0^{e_1,n} \\ u_0^{e_2,n} \\ u_0^{e_3,n}
\end{bmatrix}
=
\begin{bmatrix}
u_1^{e_1,n-1} \\ u_2^{e_1,n-1} \\ u_1^{e_2,n-1} \\ u_2^{e_2,n-1} \\ u_1^{e_3,n-1} \\ u_2^{e_3,n-1} \\ g_{v_1}(t^n) \\ g_{v_2}(t^n) \\ 0
\end{bmatrix}$$

with $u_i^{e,0} = \frac{1}{h^e} \int_{T_i^e} u_0(x)\, dx$.

Numerical tests

Now we present some numerical tests to verify the stability and convergence results for the solution of the transport problem (3.9)–(3.13) and the solution of problem 3.21. We consider two different network topologies and use the fully discrete scheme given in problem 3.21 with piecewise constant polynomials in space for the numerical approximation.

First, we consider the network from example 4.1 given in figure 1.1. We assume the same length $l^e = 1$ and cross-sectional area $a^e = 1$ for all edges $e \in \mathcal{E}$ and volume flow rates $b^{e_1} = 1,\ b^{e_2} = 1$ and $b^{e_3} = 2$, which satisfy the conservation condition (3.14). The exact solution can be explicitly determined, see section 1.5. Details on the implementation

of the fully discrete scheme using piecewise constant polynomials in space are given in example 4.1.

For an illustration of the stability estimates we choose $u_0^e \equiv 1$ as initial condition for all $e \in \mathcal{E}$ and

$$g_v(t) = \begin{cases} 1 - \frac{t}{2}, & 0 \leq t \leq 2, \\ 0, & t > 2, \end{cases}$$

as inflow boundary condition for the inflow boundary vertices v_1 and v_2. As measure we use the mathematical entropy defined in (2.23), which is given by

$$E(u) := \frac{1}{2}\|u\|^2_{L^2(\mathcal{E})}, \tag{4.4}$$

since $a^e = 1$ for all $e \in \mathcal{E}$. The stability estimate for the exact solution in lemma 2.4 and the corresponding estimate for the numerical solution given by corollary 3.25 suggest that for the chosen inflow boundary condition E should quickly tend to zero for both exact and numerical solution. In our test we compute E at time points $T = 0, 1, \ldots, 5$, and expect the same behaviour for the exact and the numerical solution, more precisely E should dissipate quickly and take approximately the same values for both solutions. Table 4.1 shows the results. And indeed we can observe the predicted behaviour.

T	0	1	2	3	4	5
exact	1.5000	1.0833	0.2813	0.0104	0.0000	0.0000
num	1.5000	1.0797	0.2833	0.0106	$3.0600 \cdot 10^{-12}$	$4.8209 \cdot 10^{-41}$

Table 4.1: Dissipation of the exact solution and the numerical solution for $h = 0.01$ and $\tau = 0.005$.

Now we focus on the verification of the error estimate given in theorem 3.26. As initial and inflow boundary conditions we choose $u_0^e \equiv 0$ for all edges e and $g_{v_1}(t) = \frac{1}{4}t^2$ and $g_{v_2}(t) = \frac{1}{8}t^2$. They satisfy the compatibility condition $u_0^{e_i}(0) = g^{v_i}(0)$ for $i = 1, 2$. We measure the error with the standard L^2-norm on every edge in the network at the time point T given by

$$\|u^e(T) - u_h^{e,N}\|_{L^2(e)}. \tag{4.5}$$

We choose $h = 0.5$ and $\tau = 0.2$ as start grid step sizes, and refine the grid by holding the ratio τ/h constant and uniformly reduce h and thus τ by a factor of 2. Theorem 3.26

states a first order convergence rate in time and a convergence rate of order $k + 1/2$ in space. It should even be possible to derive an improved spatial rate of order $k + 1$, if the solution u of the transport problem satisfies $u(t) \in H_{pw}^{k+2}(\mathcal{T}_h(\mathcal{E}))$ for all $0 \leq t \leq T$, see remark 3.30. Since we are using the fully discrete scheme with piecewise constant polynomials and the solution u of the transport problem (3.9)–(3.13) with the chosen initial and inflow boundary conditions satisfies $u(t) \in H^2(\mathcal{T}_h(\mathcal{E}))$ for $t \geq 0$, we expect a convergence rate of 1. The results of our tests are illustrated in the tables 4.2 and 4.3 together with the corresponding convergence rates at time points $T = 1, 2$. They match our theoretical results.

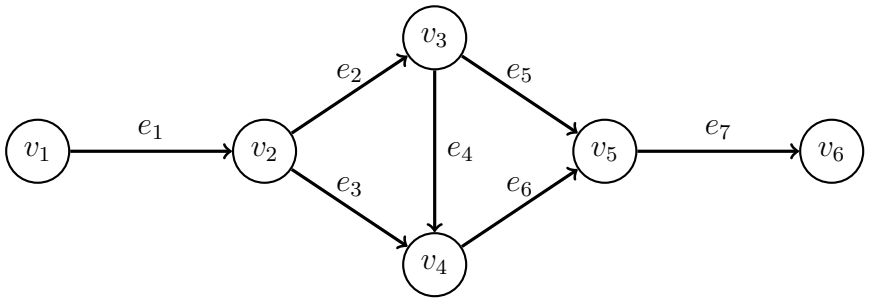
h	0.5	0.25	0.125	0.0625	$0.5 \cdot 2^{-4}$	$0.5 \cdot 2^{-5}$	$0.5 \cdot 2^{-6}$	$0.5 \cdot 2^{-7}$
e_1	0.05086	0.03125	0.01804	0.00995	0.00533	0.00279	0.00144	0.00074
rate	/	0.70270	0.79283	0.85869	0.90323	0.93319	0.95355	0.96754
e_2	0.02543	0.01562	0.00902	0.00497	0.00266	0.00139	0.00072	0.00037
rate	/	0.70270	0.79283	0.85869	0.90323	0.93319	0.95355	0.96754
e_3	0.02134	0.01146	0.00567	0.00265	0.00120	0.00053	0.00023	0.00010
rate	/	0.89641	1.01568	1.09638	1.14581	1.17755	1.19934	1.21450

Table 4.2: Error estimates and corresponding convergence rates for $T = 1$.

h	0.5	0.25	0.125	0.0625	$0.5 \cdot 2^{-4}$	$0.5 \cdot 2^{-5}$	$0.5 \cdot 2^{-6}$	$0.5 \cdot 2^{-7}$
e_1	0.14156	0.07812	0.04107	0.02103	0.01063	0.00535	0.00268	0.00134
rate	/	0.85760	0.92773	0.96535	0.98321	0.99169	0.99586	0.99794
e_2	0.07078	0.03906	0.02053	0.01052	0.00532	0.00268	0.00134	0.00067
rate	/	0.85760	0.92773	0.96535	0.98321	0.99169	0.99586	0.99794
e_3	0.08490	0.05101	0.02843	0.01492	0.00756	0.00378	0.00189	0.00094
rate	/	0.73515	0.84309	0.92995	0.98211	1.00031	1.00205	1.00113

Table 4.3: Error estimates and corresponding convergence rates for $T = 2$.

Now we focus on a more complex network given by



with vertices $\mathcal{V}_0 = \{v_1, \dots, v_6\}$ and edges $\mathcal{E} = \{e_1, \dots, e_7\}$. We distinguish between the inflow boundary vertex $\mathcal{V}_\partial^- = \{v_1\}$, the outflow boundary vertex $\mathcal{V}_\partial^+ = \{v_6\}$ and the inner vertices $\mathcal{V}_0 = \{v_2, v_3, v_4, v_5\}$. Again we assume the same length $l^e = 1$ and cross-sectional area $a^e = 1$ for all edges $e \in \mathcal{E}$, and volume flow rates $b^{e_1} = 2$, $b^{e_2} = b^{e_3} = 1$, $b^{e_4} = b^{e_5} = 0.5$, $b^{e_6} = 1.5$ and $b^{e_7} = 2$, that satisfy the conservation condition (3.14). Given initial and inflow boundary conditions, the exact solution can be determined analogously as in section 1.5.

For a verification of the stability estimates of the exact and the numerical solution given in 2.4 and corollary 3.25, we choose $u_0^e \equiv 1$ as initial condition for all $e \in \mathcal{E}$, and

$$g_{v_1}(t) = \begin{cases} 1 - t, & 0 \leq t \leq 1, \\ 0, & t > 1, \end{cases}$$

as inflow boundary condition at the inflow boundary vertex v_1. As measure we again use (4.4) and compute E for the exact and the numerical solution at time points $T = 0, \dots, 10$. We again expect that E should quickly dissipate and that the values should be approximately the same for both solutions. The test results given in table 4.4 again match our theoretical prediction.

T	0	1	2	3	4	5
exact	3.5000	2.8333	1.8264	0.7308	0.1554	0.0062
num	3.5000	2.8257	1.8057	0.7168	0.1496	$7.98 \cdot 10^{-3}$
T	6	7	8	9	10	
exact	0.0000	0.0000	0.0000	0.0000	0.0000	
num	$1.59 \cdot 10^{-6}$	$1.16 \cdot 10^{-15}$	$3.61 \cdot 10^{-30}$	$8.31 \cdot 10^{-49}$	$1.48 \cdot 10^{-70}$	

Table 4.4: Dissipation of the exact solution and the numerical solution for $h = 0.01$ and $\tau = 0.005$.

Finally, we investigate the convergence of the fully discrete scheme. We want to verify the theoretical result given in theorem 3.26. As initial and inflow boundary condition we choose $u_0^e \equiv 0$ for all $e \in \mathcal{E}$ and $g_{v_1}(t) = \frac{1}{8}t^2$, which fulfil the compatibility condition $u_0^{e_1}(0) = g_{v_1}(0)$. Moreover, the corresponding solution of (3.9)–(3.13) satisfies $u(t) \in H^2(\mathcal{E})$. According to theorem 3.26 and remark 3.30 we expect a first order convergence. For the test we again use (4.5) as measure for the error, choose $h = 0.5$ and $\tau = 0.2$ as start grid step sizes, and refine the grid by holding the ratio τ/h constant and uniformly reduce h and thus τ by a factor of 2. Table 4.5 illustrates the results at the time point $T = 4$ together with the corresponding convergence rates. Again we observe that the convergence rates match our theoretical result derived in theorem 3.26 and remark 3.30.

h	0.5	0.25	0.125	0.0625	$0.5 \cdot 2^{-4}$	$0.5 \cdot 2^{-5}$
e_1	0.11699	0.06026	0.03057	0.01539	0.00772	0.00387
rate	/	0.95716	0.97923	0.98977	0.99492	0.99747
e_2	0.14570	0.07748	0.03994	0.02027	0.01021	0.00512
rate	/	0.91114	0.95596	0.97853	0.98942	0.99475
e_3	0.14570	0.07748	0.03994	0.02027	0.01021	0.00512
rate	/	0.91114	0.95596	0.97853	0.98942	0.99475
e_4	0.14544	0.08464	0.04714	0.02521	0.01305	0.00661
rate	/	0.78103	0.84437	0.90314	0.94974	0.98112
e_5	0.14544	0.08464	0.04714	0.02521	0.01305	0.00661
rate	/	0.78103	0.84437	0.90314	0.94974	0.98112
e_6	0.09731	0.05324	0.02843	0.01512	0.00801	0.00421
rate	/	0.87020	0.90498	0.91147	0.91639	0.92749
e_7	0.12751	0.07011	0.03676	0.01881	0.00955	0.00484
rate	/	0.86283	0.93167	0.96660	0.97792	0.97982

Table 4.5: Error estimates and corresponding convergence rates for $T = 4$.

Conclusion

In this work we investigated scalar, linear transport equations with piecewise constant coefficients on networks. We were able to prove well-posedness by semigroup theory under some restrictions on the regularity of the initial and boundary conditions. Moreover, we derived a stability estimate for the solution also on networks. The semi-discretization using the Discontinuous Galerkin method was shown to preserve this stability estimate, and a convergence result together with the proof was given. We could show that the order of convergence for the transport problem on a single edge also holds on networks. For the time discretization we considered the implicit Euler method and derived an error estimate for the fully discrete scheme. The well-known results on a single edge again transfer to networks. Finally, we illustrated the theoretical results by some numerical experiments. Overall, we were able to verify well-known stability and convergence results for transport equations on a single edge also on networks.

We only considered transport equations with constant volume flow rates on every edge of the network. An extension of the results to space or time dependent volume flow rates is missing, but under some restrictions it should be possible to transfer them. Further investigations are needed. Moreover, other numerical methods than the Discontinuous Galerkin method could be considered, as well as a higher order time discretization. The results derived in this work could also be used to study more complex models that include transport equations.

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