Γ-Robust Linear Complementarity Problems

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Abstract. Complementarity problems are often used to compute equilibria made up of specifically coordinated solutions of different optimization problems. Specific examples are game-theoretic settings like the bimatrix game or energy market models like for electricity or natural gas. While optimization under uncertainties is rather well-developed, the field of equilibrium models represented by complementarity problems under uncertainty—especially using the concepts of robust optimization—is still in its infancy. In this paper, we extend the theory of strictly robust linear complementarity problems (LCPs) to Γ-robust settings, where existence of worst-case-hedged equilibria cannot be guaranteed. Thus, we study the minimization of the worst-case gap function of Γ-robust counterparts of LCPs. For box and ℓ_1-norm uncertainty sets we derive tractable convex counterparts for monotone LCPs and study their feasibility as well as the existence and uniqueness of solutions. To this end, we consider uncertainties in the vector and in the matrix defining the LCP. We additionally study so-called ρ-robust solutions, i.e., solutions of relaxed uncertain LCPs. Finally, we illustrate the Γ-robust concept applied to LCPs in the light of the above mentioned classical examples of bimatrix games and market equilibrium modeling.

1. Introduction

Optimization under uncertainty has become a very active field of research in the last decades. While often deterministic assumptions are used regarding the parameters of optimization models, optimization under uncertainty explicitly takes into account that many parameters of practical models are not known. In mathematical optimization, there are two main approaches towards uncertainty—namely stochastic [6, 20] and robust optimization [1, 3, 33]. Both methodologies have been proven to be useful in various fields. However, in the area of equilibrium models much less research has been done although this setting seems to be a rather canonical field of application of optimization under uncertainty.

In this paper, we consider a special type of equilibrium models—namely the linear complementarity problem (LCP), which is defined as follows: Given a matrix $M \in \mathbb{R}^{n\times n}$ and vector $q \in \mathbb{R}^n$, the LCP(q, M) consists in finding a point $x \in \mathbb{R}^n$ satisfying

$$0 \leq x \perp Mx + q \geq 0, \tag{1}$$

or to show that no such point exists. Here and in what follows, we use the standard $\perp$-notation, which abbreviates

$$0 \leq a \perp b \geq 0 \iff 0 \leq a,\ b \geq 0,\ a^\top b = 0$$

for $a, b \in \mathbb{R}^n$. For an overview and many details we refer to the seminal textbook [11].

In most cases, if at all, uncertain LCPs are treated in a stochastic setting. To this end, one uses the gap function formulation of the LCP, which is the quadratic

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optimization problem (QP)

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & g(x) := x^\top (Mx+q) \\ \text{s.t.} \quad & x \in \mathcal{X} := \{x \in \mathbb{R}^n \colon x \geq 0,\, Mx+q \geq 0\}. \end{aligned} \tag{2}$$

Note that the the so-called gap function g, i.e., the objective function of Problem (2), is bounded below by zero on the polyhedral feasible set $\mathcal{X}$, which ensures the existence of a minimizer by the theorem of Frank–Wolfe [14] if $\mathcal{X} \neq \emptyset$. Obviously, a point $x \in \mathbb{R}^n$ is a solution of the LCP (1) if and only if it is global minimizer of (2) with objective function value 0.

In the stochastic setting, one assumes knowledge about the distribution of the uncertain parameters of the LCP and then minimizes the expected value of the gap function, i.e., one considers the problem

$$\min_{x\in\mathcal{X}} \mathbb{E}_{(u_M,u_q)}\left[g(x;u_M,u_q)\right], \quad g(x;u_M,u_q) := x^\top (M(u_M)x + q(u_q)), \tag{3}$$

instead of (2). For more details on stochastic LCPs, we refer to [8–10, 24] and the references therein. One important drawback of stochastic optimization and thus also of the stochastic consideration of LCPs is that one needs knowledge about the distributions of the uncertain parameters. This, however, is often not the case in practice, which is why robust optimization has become important.

In robust optimization, one does not impose distributions but considers given uncertainty sets in which the parameters may vary. Hence, one does not minimize expected values but hedges against the worst-case realization within these uncertainty sets. For LCPs, this means that one considers the situation in which the entries in the problem's data M and q are uncertain, i.e., we have $M(u_M)$ and $q(u_q)$ with $u_M \in \mathcal{U}_M$, $u_q \in \mathcal{U}_q$, and $\mathcal{U}_M, \mathcal{U}_q$ are given uncertainty sets. For example, we define $q(u_q) := \bar{q} + u_q$ in the following, where $\bar{q}$ is the vector containing the so-called nominal values of the LCP vector q. The definition of $M(u_M)$ will be given later. Taking these uncertainty sets into account means that we have an infinite family of complementarity problems

$$\{0 \leq x \perp M(u_M)x + q(u_q) \geq 0\}_{(u_M,u_q)\in\mathcal{U}_M\times\mathcal{U}_q} \tag{4}$$

instead of the single nominal LCP (1). We call Problem (4) an uncertain linear complementarity problem (ULCP). Moreover, we call a point x strictly robust feasible if $x \geq 0$ and $M(u_M)x + q(u_q) \geq 0$ holds for all $(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q$ and the point is called a strictly robust LCP solution if it additionally satisfies

$$x^\top (M(u_M)x + q(u_q)) = 0 \quad \text{for all } (u_M,u_q) \in \mathcal{U}_M \times \mathcal{U}_q. \tag{5}$$

The ULCP (4) can also be stated in terms of the gap function. Instead of minimizing the expected value of the gap function as in (3), in the robust case, we consider worst-case minima, i.e., we study the robust counterpart

$$\min_{x\in\mathcal{X}(u_M,u_q)} \sup_{(u_M,u_q)\in\mathcal{U}_M\times\mathcal{U}_q} g(x;u_M,u_q) \tag{6}$$

of (2), where the robust feasible set is given by

$$\mathcal{X}(u_M,u_q) := \{x \in \mathbb{R}^n \colon x \geq 0,\, M(u_M)x + q(u_q) \geq 0\}.$$

Note that this can be seen as the feasible set of a semi-infinite optimization problem; see, e.g., [30].

Regarding the application of robust techniques to complementarity problems there are only a few publications. The earliest one—at least to the best of our knowledge—is the paper [34]. The authors consider the concept of strict robustness and develop the notion of ρ-robust solutions. These are strictly robust feasible points that satisfy $x^\top (M(u_M)x + q(u_q)) \leq \rho$, $\rho \geq 0$, for all $(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q$,

which is a relaxation of (5). The main contribution is the development of sufficient and necessary conditions for ρ-robust solutions for the case of different uncertainty sets. More recently, tractable strictly robust counterparts of uncertain LCPs have been analyzed in detail in [35, 36]. The authors consider different uncertainty sets and distinguish between monotone and non-monotone LCPs. However, existence and uniqueness of robust solutions are not considered in the mentioned articles. There are also only a few papers that apply robust techniques to complementarity problems. We are only aware of [7, 21, 27] that consider robust equilibrium problems in the context of energy market models.

The major criticism of strictly robust optimization is that it tends to yield highly conservative solutions, which lead to the development of less conservative notions of robustness; see, e.g., [5] for Γ-robustness, [23] for recoverable robustness, [13] for light robustness, or [2] for adjustable robustness. In this paper we adopt the concept of Γ-robustness, which has been developed in [4, 5, 32], to consider uncertain LCPs. This means that the parameters of the LCP may vary in given uncertainty sets as above but that only $\Gamma_M \in \{1, \dots, n^2\}$ many values in $M(u_M)$ and $\Gamma_q \in \{1, \dots, n\}$ many values in $q(u_q)$ are allowed to realize in a worst-case way. For $\Gamma_M = n^2$ and $\Gamma_q = n$ being the total number of parameters, this corresponds to the classical concept of strict robustness. However, by adjusting Γ_M and Γ_q the modeler can control on how conservative the solutions will be. Let us also briefly comment on the min-sup-structure of (6). This corresponds to the classical structure that one faces in robust optimization. However, also the reverted structure could be considered in principle, which would correspond to a setting as it is studied in adjustable robustness; see, e.g., [2].

The contribution of this paper is the following. First, we state Γ-robust LCPs and derive the equivalence to a properly chosen gap minimization problem. This equivalence reveals that existence of robust LCP solutions cannot be expected in the classical sense of robust optimization. This has also been discussed in [35] for the case of strict robustness. As a consequence, we afterward analyze global minimizers of the worst-case gap function and study the settings of uncertain LCP vector q and LCP matrix M for different types of uncertainty sets $\mathcal{U}_M$ and $\mathcal{U}_q$. We are able to derive finite-dimensional robust counterparts, which are typically convex if the LCP matrix is positive semidefinite. Thus, we can explicitly characterize the tractable cases, which is a classical topic in robust optimization; see, e.g., [3] for a survey on tractability of robust counterparts for linear programming, quadratic programming, and even more general but still convex problems. The tractable convex cases then allow us to study the worst-case gap minimization problem again as an LCP for which we then consider the classical questions of existence and uniqueness of solutions. Finally, we also study the concept of ρ-robust solutions as it is introduced in [34]. Due to its relation to the worst-case gap function minimization, we readily obtain results in analogy to those in [34] also for the Γ-robust case. Finally, we briefly study the impact of the degree of uncertainty on the robust outcomes for a market equilibrium model. This case study again reveals that robust LCP solutions cannot be expected—even for rather simple models. Instead, the global minimizers of the LCP's gap function yield points that are "almost equilibria". In this context we also want to mention LCPs with additional integer restrictions where "almost equilibria" are considered as well; see [15–17, 19].

The remainder of this paper is structured as follows. In Section 2, we discuss two very classical applications of LCPs, namely market equilibrium modeling and the bimatrix game, and illustrate the meaning of robustness for these settings. Uncertainty in the LCP vector q is analyzed in Section 3, whereas uncertainty in the LCP matrix M is the topic of Section 4. Afterward, we also discuss the case

of uncertain q and M if the uncertainty sets prescribing $q(u_q)$ and $M(u_M)$ are unrelated; see Section 5. We then return to the example of robust market equilibria in Section 6 and briefly study the dependence of the global LCP gap function minimizers on uncertain LCP data. Finally, the paper closes with some concluding remarks and possible future directions of research in Section 7.

2. Motivating Examples

In this section, we provide two classical examples of linear complementarity problems and illustrate the meaning of Γ-robustifications. We start with an easy model of a market equilibrium and afterward discuss the LCP model of a bimatrix game. The nominal problems are both taken from the seminal textbook [11].

2.1. Market Equilibrium Modeling.

We consider a stylized example of an economy in which production of goods satisfies the respective demands at equilibrium prices. To this end, we model the production side using the linear optimization problem

$$\begin{aligned} \min_{z\in\mathbb{R}^n} \quad & c^\top z && \text{(7a)}\\ \text{s.t.} \quad & Az \geq b, && \text{(7b)}\\ & Bz \geq r^*, && \text{(7c)}\\ & z \geq 0 && \text{(7d)} \end{aligned}$$

with vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $r^* \in \mathbb{R}^k$ and matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times n}$. Here, z represents the vector of production activity levels and the optimization goal is to minimize production costs given by the vector c. Constraint (7b) models technological production constraints and (7c) ensures that production meets the demand r^*. The latter depends on the market prices p^* and is given by the market demand function Q that is chosen to be affine-linear here, i.e.,

$$r^* = Q(p^*) = Dp^* + d \quad \text{with} \quad D \in \mathbb{R}^{k\times k},\ d \in \mathbb{R}^k.$$

Finally, we need the equilibrating condition $p^* = \pi^*$, where $\pi^* \in \mathbb{R}^k$ is the vector of dual variables of the primal demand constraints in (7c). If we now state the Karush–Kuhn–Tucker (KKT) conditions of the production problem (7) and use both the equilibrating condition as well as the market demand function, we obtain the system

$$\begin{aligned} &0 \leq z \perp c - A^\top\lambda - B^\top p \geq 0,\\ &0 \leq \lambda \perp -b + Az \geq 0,\\ &0 \leq p \perp -Dp - d + Bz \geq 0 \end{aligned}$$

by simplifying the complementarity conditions and solving for r^* and π^*. That is, with

$$x = \begin{pmatrix} z\\ \lambda\\ p \end{pmatrix}, \quad M = \begin{bmatrix} 0 & -A^\top & -B^\top\\ A & 0 & 0\\ B & 0 & -D \end{bmatrix}, \quad q = \begin{pmatrix} c\\ -b\\ -d \end{pmatrix}$$

we obtain the LCP

$$x \geq 0, \quad Mx + q \geq 0, \quad x^\top(Mx + q) = 0.$$

What does it now mean to consider a Γ-robustification of this LCP? The modeling reveals that A, B, c, b are the problem's data for the production and D, d are the problem's data for the demand. Consider now for the moment certain production data but uncertain demand (see, e.g., [7, 21, 27], where a similar setting is considered in electricity market models). We assume that D is also certain. Thus, only the demand function's quantity intercept d is uncertain. A Γ-robust model now considers

the situation in which Γ many quantity intercepts may vary in given uncertainty sets, whereas all other demands are certain. In this setting, a robust LCP asks for a market equilibrium for every possible demand within the uncertainty set.

The same can be done with, e.g., the production matrix A. Thus, a certain number of technological data, for instance production capacities, may vary in an uncertainty set and one wants to hedge against the worst-case of Γ many of such uncertainties. In the former case, the uncertainty appears in the LCP vector q, whereas in the latter case the matrix M is affected. We will later return to this example in the case study in Section 6, where we numerically analyze the dependence of robust solutions on the uncertain data.

2.2. **Bimatrix Games.** Another classical example is the bimatrix game. Consider two players 1 and 2 with m and n pure strategies, respectively. The cost incurred for player 1 if she plays strategy $i \in \{1, \dots, m\}$ and if player 2 plays strategy $j \in \{1, \dots, n\}$ is given as the entry a_{ij} of the non-negative matrix $A \in \mathbb{R}^{m \times n}$. The analogous costs for player 2 are given in the non-negative matrix $B \in \mathbb{R}^{m \times n}$. A mixed strategy for player 1 is a non-negative vector $x \in \mathbb{R}^m$ with $\sum_{i=1}^m x_i = 1$. A mixed strategy for the other player is defined in the same way. The expected costs of the players thus are $x^\top Ay$ and $x^\top By$, respectively, and a pair (x^*, y^*) of mixed strategies is called a Nash equilibrium [28, 29] if

$$(x^*)^\top Ay^* \leq x^\top Ay^* \quad \text{for all } x \geq 0 \text{ with } \sum_{i=1}^m x_i = 1,$$

$$(x^*)^\top By^* \leq (x^*)^\top By \quad \text{for all } y \geq 0 \text{ with } \sum_{j=1}^n y_j = 1.$$

It can be shown that computing a Nash equilibrium of a bimatrix game is equivalent to solving the LCP(q, M) with data

$$q = \begin{pmatrix} -e_m \\ -e_n \end{pmatrix}, \quad M = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix};$$

see [22] for an early study of this relation. This LCP is of rather special type since q does not depend on the problem's data but only contains -1's and also M has a special structure. This example shows that for some LCPs, the consideration of uncertain q is not reasonable. Here, perturbations in q would yield an LCP that has no connection anymore to the original bimatrix game. As a consequence, only M can be reasonably considered uncertain, which corresponds to uncertain payoffs of the players; cf., e.g., [18].

Of course, there are many other important classical LCP models in which the robust treatment of the problem's data is reasonable. Let us finally mention the modeling of traffic equilibria [12]. Here, e.g., traveling times are usually uncertain. See also [35] for a brief discussion of robust LCPs in the context of traffic equilibria.

3. Γ-Uncertainty in q

In this section, we consider uncertainty in the vector q of the ULCP (4). The entries in M are considered to be certain. That is, the uncertain LCP reads

$$\{0 \leq x \perp Mx + q(u) \geq 0\}_{u \in \mathcal{U}}$$

for a given uncertainty set $\mathcal{U} \subset \mathbb{R}^n$. In this setting, the concept of strict robustness is discussed in detail in [35]. We instead focus on Γ-robustifications, i.e., we consider the uncertainty set

$$\mathcal{U}_\Gamma := \{u \in \mathcal{U} \colon |\{i \in [n] \colon u_i \neq 0\}| \leq \Gamma\}.$$

Here and in what follows, $\Gamma \in \{1, \dots, n\}$ describes the number of deviations in q we hedge against and we use the abbreviation $[n] := \{1, \dots, n\}$. We now study the Γ-robust counterpart of the LCP gap minimization problem, i.e.,

$$\min_{x \in \mathcal{X}(u)} \sup_{u \in \mathcal{U}_\Gamma} \quad x^\top (Mx + q(u)), \quad \mathcal{X}(u) := \{x \in \mathbb{R}^n \colon x \geq 0,\, Mx + q(u) \geq 0\},$$

which is equivalent to

$$\min_{x} \sup_{u \in \mathcal{U}_\Gamma} \left\{ x^\top Mx + x^\top q(u) \colon x \geq 0,\ Mx \geq -q(u) \text{ for all } u \in \mathcal{U}_\Gamma \right\}. \tag{8}$$

Note that the notions of a robust feasible point and a robust solution as stated in the last section carry over directly to the Γ-robust case. To be more specific, a point $0 \leq x \in \mathbb{R}^n$ is called Γ-robust feasible if it satisfies $Mx \geq -q(u)$ for all $u \in \mathcal{U}_\Gamma$ and it is called a Γ-robust solution if it is a global minimizer of (8) with objective function value 0.

In analogy to Proposition 3.1 in [35], we directly obtain the following proposition.

Proposition 3.1. *A vector $x \in \mathcal{X}$ solves*

$$0 \leq x \perp Mx + q(u) \geq 0 \quad \textit{for all } u \in \mathcal{U}_\Gamma \tag{9}$$

if and only if x is a solution of (8) with optimal objective function value of zero.

The meaning of this proposition is that a robust feasible point for the LCP is a robust LCP solution if and only if it satisfies complementarity for every realization of the uncertainty. In other words, we face the standard $\exists$-$\forall$ quantifier structure of robust optimization. Unfortunately, it is unlikely that there exists a point that is a Γ-robust LCP solution, i.e., a solution of (9). The same has already been commented on for the case of strict robustness in [35] as well. Thus, in what follows, we consider the global optima of the worst-case gap minimization problem (8) instead of the original problem (9). To this end, we choose different uncertainty sets $\mathcal{U}_\Gamma$ and study whether (8) has a tractable convex counterpart. Moreover, we investigate the feasibility of this counterpart and the existence as well as, if possible, uniqueness of its solution.

3.1. **Box Uncertainty $\mathcal{U}^{\mathbf{box}}_{\Gamma,\bar{u}}$.** In this section, we consider $\mathcal{U}_\Gamma$ to be the box uncertainty set

$$\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}} := \{u \in \mathbb{R}^n \colon\ -\bar{u}_i \leq u_i \leq \bar{u}_i,\, i \in [n],\, |\{i \in [n] \colon u_i \neq 0\}| \leq \Gamma\}$$

with $\bar{u}_i \geq 0$ for all $i \in [n]$. Hence, we write $q(u) := \bar{q} + u$ with $u \in \mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}$ for the uncertain LCP vector. Note that $\bar{u}_i \geq -M_i x - \bar{q}_i$ holds for all $i \in [n]$. The reason is the following: For all $i \in [n]$, we know that $M_i x + \bar{q}_i + u_i \geq 0$ holds for all $u_i \in [-\bar{u}_i, \bar{u}_i]$. This is equivalent to $u_i \geq -M_i x - \bar{q}_i$ and by choosing $u_i = \bar{u}_i$ we obtain $\bar{u}_i \geq -M_i - \bar{q}_i$. Thus, the robust counterpart (8) in this case reads

$$\min_{x \geq 0} \quad x^\top Mx + x^\top \bar{q} + \max_{\{I \subseteq [n] \colon |I| \leq \Gamma\}} \sum_{i \in I} \bar{u}_i x_i \tag{10a}$$

$$\text{s.t.} \quad Mx \geq -\bar{q} + \sum_{i \in I} \bar{u}_i e_i \quad \text{for all } I \subseteq [n],\ |I| \leq \Gamma, \tag{10b}$$

where e_i is the ith unit vector in $\mathbb{R}^n$. Note first that we—in contrast to [32]—do not need absolute values of x in the last sum of the objective function because we restrict x to be non-negative. Second, we can write "max" instead of "sup" because of the boundedness of all u_i. The hardness of Problem (10) stems from the min-max objective function and the fact that it is made up of exponentially many (in n) constraints in (10b). Fortunately, the robust counterpart (10) can be reformulated in a tractable way. First, we derive an equivalent problem of polynomial size and without an inner maximization problem.

Theorem 3.2. *The robust counterpart (10) is equivalent to*

$$\min_{x,\alpha,\beta} \quad x^\top Mx + x^\top \bar{q} + \alpha\Gamma + \sum_{i=1}^{n} \beta_i \tag{11a}$$

$$s.t. \quad M_{i,.}x \geq -\bar{q}_i + \bar{u}_i, \quad i \in [n], \tag{11b}$$

$$\alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n], \tag{11c}$$

$$\alpha \geq 0, \tag{11d}$$

$$x_i \geq 0, \ \beta_i \geq 0, \quad i \in [n], \tag{11e}$$

i.e., if (x, α, β) *solves* (11)*, then* x *solves* (10) *and if* x *solves* (10)*, then there exists* (α, β) *so that* (x, α, β) *solves* (11)*.*

Before we prove the theorem, we notice that a variable without index denotes the vector containing all corresponding variables with indices, e.g., $\beta := (\beta_i)_{i\in[n]}$.

Proof. First, we rewrite the robust counterpart (10) as

$$\min_{x\geq 0,\eta} \quad \eta \tag{12a}$$

$$\text{s.t.} \quad \eta \geq x^\top Mx + x^\top \bar{q} + \max_{\{I\subseteq[n]:|I|\leq\Gamma\}} \sum_{i\in I} \bar{u}_i x_i, \tag{12b}$$

$$Mx \geq -\bar{q} + \sum_{i\in I} \bar{u}_i e_i \quad \text{for all } I \subseteq [n], \ |I| \leq \Gamma. \tag{12c}$$

Thus, all uncertainties are moved to the constraints. The number of constraints in (12c) is exponential in n, so we first reformulate these constraints. For each realization of the uncertainty and each row $i \in [n]$ of M, we need to satisfy $M_{i,.}x + q_i(u) \geq 0$.

For every row i, the worst case for all realizations $I \subseteq [n]$, $|I| \leq \Gamma$, is given by

$$M_{i,.}x + q_i(u) = \begin{cases} M_{i,.}x + \bar{q}_i - \bar{u}_i, & \text{if } i \in I, \\ M_{i,.}x + \bar{q}_i, & \text{if } i \notin I. \end{cases}$$

As $\Gamma \geq 1$, for each index $i \in [n]$ exists at least one realization I with $i \in I$. This implies that $M_{i,.}x + \bar{q}_i - \bar{u}_i \geq 0$ needs to hold for all i in order to ensure robust feasibility. Thus, we replace (12c) with

$$M_{i,.}x + \bar{q}_i - \bar{u}_i \geq 0, \quad i \in [n].$$

Next, we reformulate the term

$$\pi(x, \Gamma) := \max_{\{I\subseteq[n]:\ |I|\leq\Gamma\}} \sum_{i\in I} \bar{u}_i x_i.$$

in Constraint (12b). This maximum can be computed by solving the linear optimization problem

$$\max_{z\in\mathbb{R}^n} \quad \sum_{i=1}^{n} \bar{u}_i x_i z_i \tag{13a}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} z_i \leq \Gamma, \tag{13b}$$

$$0 \leq z_i \leq 1, \quad i \in [n]; \tag{13c}$$

see Proposition 1 in [5] or [32]. This is a bounded and feasible linear optimization problem in z and its dual problem reads

$$\begin{aligned} \min_{\alpha,\beta} \quad & \alpha\Gamma + \sum_{i=1}^{n} \beta_i \\ \text{s.t.} \quad & \alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n], \\ & \alpha \geq 0, \\ & \beta_i \geq 0, \quad i \in [n], \end{aligned}$$

where α is the dual variable of Constraint (13b) and the β_i are the dual variables of the constraints in (13c). We can now apply the strong duality theorem and replace the inner maximization problem in (12b) with the corresponding dual minimization problem. We notice that we do not need the minimum here because if

$$\eta \geq x^\top M x + x^\top \bar{q} + \alpha\Gamma + \sum_{i=1}^{n} \beta_i, \quad \alpha \geq 0, \quad \beta_i \geq 0,\ \alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n],$$

holds, it also holds for the minimum value of $\alpha\Gamma + \sum_{i=1}^{n} \beta_i$ over the dual feasible set. Thus, we obtain

$$\begin{aligned} \min_{x \geq 0, \eta, \alpha, \beta} \quad & \eta \\ \text{s.t.} \quad & \eta \geq x^\top M x + x^\top \bar{q} + \alpha\Gamma + \sum_{i=1}^{n} \beta_i, \\ & M_{i,\cdot} x \geq -\bar{q}_i + \bar{u}_i, \quad i \in [n], \\ & \alpha + \beta_i \geq \bar{u}_i x_i, \quad i \in [n], \\ & \alpha \geq 0, \\ & \beta_i \geq 0, \quad i \in [n], \end{aligned}$$

and the claim follows by eliminating η. □

Remark 3.3. (i) For the nominal model we know that $x = 0$ is a solution of the LCP(q, M) if $q \geq 0$ holds. A generalization for the robust LCP (4) for the box uncertainty set is the following: If $\bar{q} - \bar{u} \geq 0$ holds, then $x = 0$ is a robust solution. The reason is as follows. With the definition of $q(u)$ and $\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}$ one has $\bar{q}_i - \bar{u}_i \leq \bar{q}_i + u_i \leq \bar{q}_i + \bar{u}_i$ for all $i \in [n]$. This, together with the assumption $\bar{q}_i - \bar{u}_i \geq 0$ for all $i \in [n]$, yields $\bar{q}_i + u_i \geq 0$ for all $i \in [n]$. Hence, $x = 0$ is a solution of (4) because $Mx + q(u) = q(u) \geq \bar{q} - \bar{u} \geq 0$ and complementarity directly follows.

(ii) As already mentioned, we briefly explain why we consider the Relaxation (10), respectively (11), instead of the robust LCP formulation (4). By Proposition 3.1 we know that the worst-case gap has to be zero to guarantee that we also have a robust LCP solution. This means that

$$x^\top M x + x^\top \bar{q} + \alpha\Gamma + \sum_{i=1}^{n} \beta_i = 0$$

holds for a solution (x, α, β) of (11). As α, Γ, and β are all non-negative, we have that $x^\top(Mx + \bar{q}) \leq 0$ needs to hold. Since every robust solution also needs to be a nominal solution, it follows $x^\top(Mx + \bar{q}) \geq 0$ and we obtain $x^\top(Mx + \bar{q}) = 0$. Hence, α and β have to be zero as well. By Constraints (11c) we get $x = 0$ as the only robust solution—which is a solution of the nominal LCP. Thus, for x being feasible for the nominal LCP this requires $\bar{q} \geq 0$. To sum up, the only robust solution in the case of box uncertainties in q is $x = 0$, which is only a solution for $\bar{q} \geq 0$.

Given the robust counterpart (11) we can now easily characterize the tractable cases.

Corollary 3.4. *Suppose that $M \in \mathbb{R}^{n\times n}$ is positive semidefinite. Then, Problem (11) is a convex optimization problem.*

In the case of a positive semidefinite LCP matrix M, we can also show that Problem (11) is again equivalent to a suitably chosen LCP.

Theorem 3.5. *Suppose that $M \in \mathbb{R}^{n\times n}$ is positive semidefinite. Then, the Γ-robust counterpart (11) is equivalent to the LCP(q', M') with*

$$M' = \begin{bmatrix} M + M^\top & -M^\top & \operatorname{diag}(\bar{u}) & 0_{n\times n} & 0_n \\ M & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_n \\ -\operatorname{diag}(\bar{u}) & 0_{n\times n} & 0_{n\times n} & I_{n\times n} & \mathbb{1}_n \\ 0_{n\times n} & 0_{n\times n} & -I_{n\times n} & 0_{n\times n} & 0_n \\ 0_n^\top & 0_n^\top & -\mathbb{1}_n^\top & 0_n^\top & 0 \end{bmatrix} \in \mathbb{R}^{(4n+1)\times(4n+1)} \tag{14}$$

and

$$q' = \left(\bar{q}^\top, (\bar{q} - \bar{u})^\top, 0_n^\top, \mathbb{1}_n^\top, \Gamma\right)^\top \in \mathbb{R}^{4n+1}, \tag{15}$$

where $\mathbb{1}_n$ is the vector of all ones in $\mathbb{R}^n$.

Proof. Under the assumption that M is positive semidefinite, we obtain by Corollary 3.4 that Problem (11) is convex. Since all constraints are linear, no further constraint qualifications are required and the KKT conditions are sufficient and necessary optimality conditions for Problem (11). They comprise

$$\begin{aligned} 0 \le Mx + \bar{q} - \bar{u} \perp \lambda \ge 0, \\ 0 \le \alpha + \beta_i - \bar{u}_i x_i \perp \mu_i \ge 0, \quad i \in [n], \\ 0 \le x \perp M^\top x + Mx - M^\top \lambda + (\bar{u}_1\mu_1, \dots, \bar{u}_n\mu_n)^\top + \bar{q} \ge 0, \\ 0 \le \alpha \perp \Gamma - \sum_{i=1}^n \mu_i \ge 0, \quad i \in [n], \\ 0 \le \beta_i \perp 1 - \mu_i \ge 0, \quad i \in [n]. \end{aligned}$$

The solutions $x' := (x^\top, \lambda^\top, \mu^\top, \beta^\top, \alpha)^\top$ of this system are exactly the solutions of the LCP(q', M'). □

As mentioned above, existence of solutions for the original robustified LCP cannot be expected in general. Thus, we consider the existence and uniqueness of solutions of the worst-case gap minimization problem (10). Since its tractable version (11) is equivalent to an LCP again, it is natural to study the existence and uniqueness of solutions for this LCP. As usual for LCPs, results on feasibility of an LCP as well as the existence and uniqueness of solutions of LCPs are stated such that they hold for all LCP vectors q. Moreover, both kinds of results depend on the LCP matrix being a P or Stiemke matrix (S-matrix), respectively. We refer to the seminal book [11] for the definitions of these and other matrix classes. Unfortunately, the LCP matrix M' is neither a P-matrix nor an S-matrix.

Lemma 3.6. *The matrix M' defined in (14) is not an S-matrix and, in particular, not a P-matrix.*

Proof. We show that M' is not an S-matrix by proving that there exists no vector $x' := (x^\top, \lambda^\top, \mu^\top, \beta^\top, \alpha)^\top \ge 0_{4n+1}$ with $M'x' > 0_{4n+1}$. To this end, we use the same notation as in the proof of Theorem 3.5. From $x' \ge 0$, we especially obtain

$\mu_i \geq 0$ for all $i \in [n]$. However, as

$$M'x' = \begin{bmatrix} (M + M^\top)x - M^\top\lambda + \operatorname{diag}(\bar{u})\mu \\ Mx \\ -\operatorname{diag}(\bar{u})x + I_{n\times n}\beta + \mathbb{1}_n\alpha \\ -I_{n\times n}\mu \\ -\sum_{i=1}^n \mu_i \end{bmatrix}$$

holds, we need $\mu_i < 0$ for all $i \in [n]$ to obtain $M'x' > 0$.

Since every P-matrix is an S-matrix (see Corollary 3.3.5 in [11]), M' cannot be a P-matrix. □

Using Proposition 3.1.5 and Theorem 3.3.7 in [11], the last result shows that the LCP with matrix M' is neither uniquely solvable nor feasible for all possible LCP vectors q'. However, taking a closer look on the specific vector in (15), we now ask whether a feasible point of (11)—and hence of (10)—exists for all such specific q'.

First, for positive definite matrices M we are able to prove that Problem (11) is always feasible.

Theorem 3.7. *Suppose that the matrix M in Problem (11) is positive definite. Then, Problem (11) is feasible.*

Proof. From Section 3.1 in [11] it follows that every positive definite matrix is an S-matrix. Hence, as M is positive definite, there exists a vector $x > 0$ with $Mx > 0$. Thus, we can choose a scalar $\lambda > 0$ sufficiently large such that $\lambda Mx \geq -\bar{q} + \bar{u}$ and we define $\hat{x} := \lambda x$. Then, $\bar{u}_i\hat{x}_i \geq 0$ holds for all $i \in [n]$ and the right-hand sides of the constraints in (11c) are non-negative and fixed. Hence, $\beta_i := \bar{u}_i\hat{x}_i$ for all $i \in [n]$ and $\alpha = 0$ is a feasible solution of Problem (11). □

The next question is whether Problem (11), and hence (10), is solvable if it is feasible. This is equivalent to the question if M' is a $\mathcal{Q}_0$-matrix. In our setting, this is guaranteed if the original LCP matrix M is positive semidefinite.

Theorem 3.8. *Let M be a positive semidefinite matrix. Then, the matrix M' defined in (14) is positive semidefinite and, thus, a $\mathcal{Q}_0$-matrix. Together with Theorem 3.7, this guarantees the existence of a solution of Problem (10) if M is positive definite.*

Proof. Let $x' := (x^\top, \lambda^\top, \mu^\top, \beta^\top, \alpha)^\top$, then we have

$$\begin{aligned}(x')^\top M'x' &= (x^\top, \lambda^\top, \mu^\top, \beta^\top, \alpha)\begin{pmatrix} (M + M^\top)x - M^\top\lambda + \operatorname{diag}(\bar{u})\mu \\ Mx \\ -\operatorname{diag}(\bar{u})x + I_{n\times n}\beta + \mathbb{1}_n\alpha \\ -I_{n\times n}\mu \\ -\sum_{i=1}^n \mu_i \end{pmatrix} \\ &= x^\top(M + M^\top)x - x^\top M^\top\lambda + x^\top\operatorname{diag}(\bar{u})\mu + \lambda^\top Mx - \mu^\top\operatorname{diag}(\bar{u})x \\ &\quad + \mu^\top I_{n\times n}\beta + \mu^\top\mathbb{1}_n\alpha - \beta^\top I_{n\times n}\mu - \sum_{i=1}^n \alpha\mu_i \\ &= x^\top(M + M^\top)x \geq 0.\end{aligned}$$

Thus, M' is positive semidefinite. □

Finally, we can show x-uniqueness of Problem (11) for positive definite matrices M. Since feasibility and existence is already shown, the next result is a direct consequence of Theorem 1a in [26].

Proposition 3.9. *Suppose that the matrix M in Problem (11) is positive definite. Then, the solution of Problem (11) is unique in x.*

However, the entire primal solution of Problem (11) is not unique because it is not unique in α and β in general—even for the case of M being positive definite. This is shown in the following example.

Example 3.10. Let $n = 3$, $\Gamma = 1$, $M = I_{3\times 3}$, $\bar{q} = (-4, 2, 0)^\top$, and $\bar{u} = (3, 2, 10)^\top$ be the input data of Problem (11). One solution with value 221 is $x = (7, 0, 10)^\top$, $\alpha = 76$, and $\beta = (0, 0, 24)^\top$. A second solution reads $x = (7, 0, 10)^\top$, $\alpha = 80$, and $\beta = (0, 0, 20)^\top$. These are two solutions with the same values for x but different values for α and β.

Next, we briefly discuss the concept of ρ-relaxations of robust LCPs, which has been introduced in [34] and which is strongly related to the classical regularization technique for mathematical programs with complementarity constraints as proposed in [31].

Definition 3.11. Let $\mathcal{U}$ be the uncertainty set of the ULCP (4) and let $\rho \geq 0$ be given. Then, the system

$$x \geq 0, \tag{16a}$$
$$M(u)x + q(u) \geq 0, \quad u \in \mathcal{U}, \tag{16b}$$
$$x^\top (M(u)x + q(u)) \leq \rho, \quad u \in \mathcal{U}, \tag{16c}$$

is called the *ρ-relaxation* of the ULCP (4). Solutions of System (16) are called *ρ-robust solutions*.

As before, we only consider uncertainty in the vector q. Thus, System (16) for the box-uncertainty set $\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}$ reads

$$\begin{aligned} x &\geq 0, \\ Mx + \bar{q} + u &\geq 0, \quad u \in \mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}, \\ x^\top (Mx + \bar{q} + u) &\leq \rho, \quad u \in \mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}. \end{aligned}$$

As in [34], our goal now is to derive a finite system of equations and inequalities that characterizes ρ-robust solutions. This is achieved by the following theorem, which is closely related to Theorem 3.2.

Theorem 3.12. *Let $\mathcal{U} = \mathcal{U}^{box}_{\Gamma,\bar{u}}$ be the given uncertainty set. Then, x is a ρ-robust LCP solution if and only if there exist $\alpha \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$, $i \in [n]$, that satisfy*

$$x^\top (Mx + \bar{q}) + \alpha\Gamma + \sum_{i=1}^{n} \beta_i \leq \rho, \tag{17a}$$
$$Mx + \bar{q} - \bar{u} \geq 0, \tag{17b}$$
$$\alpha + \beta_i - \bar{u}_i x_i \geq 0, \quad i \in [n], \tag{17c}$$
$$\alpha \geq 0, \tag{17d}$$
$$x_i, \beta_i \geq 0, \quad i \in [n]. \tag{17e}$$

In particular, this implies that x is a ρ-robust LCP solution if and only if the quadratic program (11) has an optimal solution with objective function value not larger than ρ.

Proof. First, assume that x, α, β are given that satisfy System (17). This means that x, α, β are feasible for Problem (11) and that the corresponding objective function value is not larger than ρ. By Theorem 3.2, this is equivalent to x being a ρ-robust solution. On the other hand, let x be a ρ-robust solution. Again by Theorem 3.2 this means that Problem (11) has a feasible point with objective function value less than ρ, which directly implies the existence of α and β so that (17) is satisfied. □

We close this section with a remark about the connection of the uncertainty set $\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}$ and the ℓ_∞-norm uncertainty set

$$\mathcal{U}^{\infty}_{\Gamma,\delta} := \{u \in \mathbb{R}^n \colon \|u\|_\infty \leq \delta,\ |\{i \in [n] \colon u_i \neq 0\}| \leq \Gamma\}.$$

It is easy to see that $\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}} = \mathcal{U}^{\infty}_{\Gamma,\delta}$ if $\bar{u}_i = \bar{u} = \delta$ holds. That is, all results in this section also hold for $\mathcal{U}^{\infty}_{\Gamma,\delta}$.

3.2. **Uncertainty set $\mathcal{U}^1_{\Gamma,\delta}$.** In this section, we consider the ℓ_1-norm uncertainty set

$$\mathcal{U}^1_{\Gamma,\delta} := \{u \in \mathbb{R}^n \colon \|u\|_1 \leq \delta,\ |\{i \in [n] \colon u_i \neq 0\}| \leq \Gamma\}$$

for a given $\delta > 0$. The robust counterpart (8) in this case reads

$$\min_{x \geq 0} \quad x^\top M x + x^\top \bar{q} + \max_{u \in \mathcal{U}^1_{\Gamma,\delta}} \sum_{i \in [n]} u_i x_i \tag{18a}$$

$$\text{s.t.} \quad M_{i.} x + \bar{q}_i + \min_{u \in \mathcal{U}^1_{\Gamma,\delta}} u_i \geq 0, \quad i \in [n]. \tag{18b}$$

This optimization model can be reformulated in a tractable way. To prove this, we apply the strategy used in [35].

Theorem 3.13. *Let $\delta > 0$ be given. Then, the robust counterpart (18) is equivalent to*

$$\min_{x \geq 0, t \geq 0} \quad x^\top M x + x^\top \bar{q} + \delta t \tag{19a}$$

$$s.t. \quad x_i \leq t, \quad i \in [n], \tag{19b}$$

$$M_{i.} x + \bar{q}_i - \delta \geq 0, \quad i \in [n]. \tag{19c}$$

Proof. We can rewrite the inner maximization problem in the objective function of (18) as $\delta||x||_\infty$. Thus, the robust counterpart (18) is equivalent to

$$\begin{aligned} \min_{x \geq 0, t \geq 0} \quad & x^\top M x + x^\top \bar{q} + \delta t \\ \text{s.t.} \quad & x_i \leq t, \quad i \in [n], \\ & M_{i.} x + \bar{q}_i + \min_{u \in \mathcal{U}^1_{\Gamma,\delta}} u_i \geq 0, \quad i \in [n]. \end{aligned}$$

With

$$\min_{u \in \mathcal{U}^1_{\Gamma,\delta}} u_i = -\delta,$$

the claim follows. □

Remark 3.14. Here, something interesting is happening: The Γ-robust counterpart (19) is independent of Γ, which is not the case for the uncertainty set $\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}$ discussed in the last section. The reason is that the condition $\|u\|_1 = \sum_{i=1}^n |u_i| \leq \delta$ aggregates all uncertain components.

Moreover, the ℓ_1-norm counterpart yields some kind of ℓ_∞-norm regularization of the original LCP since Problem (19) can be rewritten as

$$\min_{x \geq 0} \quad x^\top M x + x^\top \bar{q} + \delta \|x\|_\infty \quad \text{s.t.} \quad M x + \bar{q} \geq \mathbb{1}\delta.$$

That is, we have the original gap function extended by an ℓ_∞-regularizer as well as a δ-tightened constraint set.

Given the robust counterpart (19) we can now easily characterize the tractable case.

Corollary 3.15. *Suppose that $M \in \mathbb{R}^{n \times n}$ is positive semidefinite. Then, the Γ-robust counterpart (19) is a convex optimization problem.*

Under the assumption of the last corollary, we can also show that Problem (19) is again equivalent to a suitably chosen LCP.

Theorem 3.16. *Suppose that $M \in \mathbb{R}^{n\times n}$ is positive semidefinite. Then, the Γ-robust counterpart (19) is equivalent to the LCP(q', M') with*

$$M' = \begin{bmatrix} M + M^\top & 0_{n\times 1} & I_{n\times n} & -M^\top \\ 0_{1\times n} & 0_{1\times 1} & -\mathbb{1}_n^\top & 0_{1\times n} \\ -I_{n\times n} & \mathbb{1}_n & 0_{n\times n} & 0_{n\times n} \\ M & 0_{n\times 1} & 0_{n\times n} & 0_{n\times n} \end{bmatrix} \in \mathbb{R}^{(3n+1)\times(3n+1)} \tag{20}$$

and

$$q' = \left(\bar{q}^\top, \delta, 0_n^\top, (\bar{q} - \delta\mathbb{1}_n)^\top\right)^\top \in \mathbb{R}^{3n+1}.$$

Proof. Using the assumption that M is positive semidefinite, we obtain by Corollary 3.15 that Problem (19) is convex. Since all constraints are linear, no further constraint qualifications are required and the KKT conditions are sufficient and necessary optimality conditions of Problem (19). They comprise

$$\begin{aligned} 0 \le x \perp M^\top x + Mx - M^\top\gamma + \beta + \bar{q} \ge 0, \\ 0 \le \delta - \sum_{i\in[n]} \beta_i \perp t \ge 0, \\ 0 \le t - x_i \perp \beta_i \ge 0, \quad i \in [n], \\ 0 \le Mx + \bar{q} - \delta\mathbb{1}_n \perp \gamma \ge 0. \end{aligned}$$

Here, β_i, $i \in [n]$, are the duals of the Constraints (19b) and the vector γ contains the dual variables of the constraints in (19c). The solutions $x' := (x^\top, t, \beta^\top, \gamma^\top)^\top$ of this system are solutions of the LCP(q', M'). □

As in Section 3.1 we now investigate existence and uniqueness of solutions of the robust counterpart (19). To this end, we again try to make use of classical LCP theory. However, the LCP matrix M' is neither a P- nor an S-matrix.

Theorem 3.17. *The matrix M' defined in (20) is neither an S-matrix nor a P-matrix.*

Proof. We proceed as in the proof of Theorem 3.6. We show that M' is not an S-matrix by proving that there exists no vector $x' := (x^\top, t, \beta^\top, \gamma^\top)^\top \ge 0_{3n+1}$ with $M'x' > 0_{3n+1}$. From $x' \ge 0$ we especially obtain $\beta_i \ge 0$ for all $i \in [n]$. However, as

$$M'x' = \begin{bmatrix} (M + M^\top)x - M^\top\gamma + \beta \\ -\sum_{i\in[n]} \beta_i \\ -x + \mathbb{1}_n t \\ Mx \end{bmatrix}$$

holds, we see that $-\sum_{i\in[n]} \beta_i \le 0$ follows and, thus, $-\sum_{i\in[n]} \beta_i > 0$ is impossible and, thus, M' is not an S-matrix and, consequently, also not a P-matrix; see also the proof of Lemma 3.6. □

As in the last section, we can also derive conditions under which Problem (19) is feasible.

Theorem 3.18. *Suppose that the matrix M in Problem (19) is positive definite. Then, Problem (19) is feasible.*

Proof. As M is positive definite, there exists a vector $x > 0$ with $Mx > 0$. Hence, we can choose $\lambda > 0$ sufficiently large such that $\lambda Mx \ge \delta\mathbb{1}_n - \bar{q}$ and we define $\hat{x} := \lambda x$. Then, choosing $\hat{t} := \max\{\hat{x}_i : i \in [n]\}$ yields a feasible solution $(\hat{x}, \hat{t})$. □

In addition, we also obtain existence and uniqueness if the original LCP matrix M is positive definite.

Theorem 3.19. *Suppose that the matrix M in Problem (19) is positive definite. Then, a solution of the Γ-robust counterpart (19) exists and is unique.*

Proof. First, existence of a solution follows from Theorem 3.18 and the theorem of Frank–Wolfe [14] since the objective function is bounded below on the polyhedral feasible set. The uniqueness of the solution in x again follows from Theorem 1a in [26]. Then, the solution is also unique in t because $\delta > 0$ holds and we obtain the unique optimal value $t = \max\{x_i \colon i \in [n]\}$ by Conditions (19b). □

Again, we close this section with a brief discussion of ρ-relaxations of the uncertain LCPs. Thus, we consider the System (16) for $\mathcal{U}^1_{\Gamma,\delta}$, which reads

$$\begin{aligned} x &\geq 0, \\ Mx + \bar{q} + u &\geq 0, \quad u \in \mathcal{U}^1_{\Gamma,\delta}, \\ x^\top (Mx + \bar{q} + u) &\leq \rho, \quad u \in \mathcal{U}^1_{\Gamma,\delta}. \end{aligned}$$

Theorem 3.20. *Let $\mathcal{U} = \mathcal{U}^1_{\Gamma,\delta}$ be the given uncertainty set. Then, x is a ρ-robust LCP solution if and only if there exists a scalar $t \in \mathbb{R}$ that satisfies*

$$\begin{aligned} x^\top (Mx + \bar{q}) + \delta t &\leq \rho, \\ x &\geq 0, \\ x_i &\leq t, \quad i \in [n], \\ Mx + \bar{q} - \delta \mathbb{1}_n &\geq 0. \end{aligned}$$

In particular, this implies that x is a ρ-robust LCP solution if and only if the quadratic program (19) has an optimal solution with objective function value not larger than ρ.

Proof. The proof is completely analogous to the proof of Theorem 3.12. □

4. Γ-Uncertainty in M

In this section, we consider the ULCP (4) with uncertainties in the matrix M. The entries in q are considered to be certain. That is, the problem now reads

$$\{0 \leq x \perp M(u)x + q \geq 0\}_{u \in \mathcal{U}}.$$

4.1. Box Uncertainty $\mathcal{U}^{\mathbf{box}}_{\Gamma,\bar{u}}$.

We start with a definition of $M(u)$ in analogy to $q(u)$ in the last section. That is, let $\bar{M} := [\bar{m}_{ij}]_{i,j\in[n]}$ be the matrix containing all nominal values and let $M(u) := [\bar{m}_{ij} + u_{ij}]_{i,j\in[n]}$ with $[u_{ij}]_{i,j\in[n]} \in \mathcal{U}$. In this section, we consider box uncertainties for the entries in M. To this end, for every row $i \in [n]$ we define

$$\mathcal{U}^{\text{box}}_{\Gamma,\bar{u},i} := \{u_i \in \mathbb{R}^n \colon -\bar{u}_{ij} \leq u_{ij} \leq \bar{u}_{ij},\ j \in [n],\ |\{j \in [n] \colon u_{ij} \neq 0\}| \leq \Gamma_i\}$$

as the uncertainty set of row i of M and $\Gamma_i \in \{1, \dots, n\}$. In this case, the robust counterpart (8) for uncertainty in M reads

$$\min_{x \geq 0} \quad x^\top \bar{M} x + x^\top q + \sum_{i \in [n]} \max_{\{I_i \subseteq [n] \colon |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_i x_j \tag{21a}$$

$$\text{s.t.} \quad \sum_{j \in [n]} \bar{m}_{ij} x_j - \max_{\{I_i \subseteq [n] \colon |I_i| \leq \Gamma_i\}} \sum_{j \in I_i} \bar{u}_{ij} x_j \geq -q_i, \quad i \in [n]. \tag{21b}$$

With this counterpart at hand, we say that a point $x \in \mathbb{R}^n$ is a Γ-robust feasible point if it satisfies (21b) and we call it a Γ-robust solution if it is a global minimizer of (21) with objective function value 0.

As it was the case for uncertainty in q, our goal is to derive a tractable robust counterpart. However, the following theorem reveals that this is not possible in general for the case of uncertain M.

Theorem 4.1. *Let $\mathcal{U}^{box}_{\Gamma,\bar{u},i}$ be the uncertainty set of row $i \in [n]$ in $M(u)x + q \geq 0$. Then, the robust counterpart (21) is equivalent to*

$$\min_{x,\alpha,\beta,\gamma,\delta,\varepsilon,\xi} \quad x^\top \bar{M}x + x^\top q + \sum_{i\in[n]} \left(\gamma_i \Gamma_i + \sum_{j\in[n]} \delta_{ij} \right) \tag{22a}$$

$$\text{s.t.} \quad \sum_{j\in[n]} \bar{m}_{ij} x_j - \varepsilon_i \Gamma_i - \sum_{j\in[n]} \xi_{ij} \geq -q_i, \quad i \in [n], \tag{22b}$$

$$\varepsilon_i + \xi_{ij} \geq \bar{u}_{ij} x_j, \quad j \in [n], \tag{22c}$$

$$\varepsilon_i \geq 0, \quad i \in [n], \tag{22d}$$

$$\xi_{ij} \geq 0, \quad i,j \in [n], \tag{22e}$$

$$\gamma_i + \delta_{ij} \geq \bar{u}_{ij} x_i x_j, \quad i,j \in [n], \tag{22f}$$

$$\gamma_i \geq 0, \quad i \in [n], \tag{22g}$$

$$\delta_{ij} \geq 0, \quad i,j \in [n]. \tag{22h}$$

Proof. To prove this theorem, we proceed as in the proof of Theorem 3.2. First, we rewrite Problem (21) as

$$\min_{x\geq 0,\eta} \quad \eta \tag{23a}$$

$$\text{s.t.} \quad \eta \geq x^\top \bar{M}x + x^\top q + \sum_{i\in[n]} \max_{\{I_i \subseteq [n]\colon |I_i| \leq \Gamma_i\}} \sum_{j\in I_i} \bar{u}_{ij} x_i x_j, \tag{23b}$$

$$\sum_{j\in[n]} \bar{m}_{ij} x_j - \max_{\{I_i \subseteq [n]\colon |I_i| \leq \Gamma_i\}} \sum_{j\in I_i} \bar{u}_{ij} x_j \geq -q_i \quad i \in [n]. \tag{23c}$$

Now, we reformulate the inner maximization problem of Constraint (23b). To this end, we use the equivalent formulation

$$\begin{aligned} \max_{z_i} \quad & \sum_{j\in[n]} \bar{u}_{ij} x_i x_j z_{ij} \\ \text{s.t.} \quad & \sum_{j\in[n]} z_{ij} \leq \Gamma_i, \\ & 0 \leq z_{ij} \leq 1, \quad j \in [n], \end{aligned}$$

of

$$\max_{\{I_i \subseteq [n]\colon |I_i| \leq \Gamma_i\}} \sum_{j\in I_i} \bar{u}_{ij} x_i x_j.$$

Its dual problem is given by

$$\begin{aligned} \min_{\gamma_i,\delta_i} \quad & \gamma_i \Gamma_i + \sum_{j\in[n]} \delta_{ij} \\ \text{s.t.} \quad & \gamma_i + \delta_{ij} \geq \bar{u}_{ij} x_i x_j, \quad j \in [n], \\ & \gamma_i \geq 0, \\ & \delta_{ij} \geq 0, \quad j \in [n]. \end{aligned}$$

In a second step, we use the equivalent formulation

$$\begin{aligned} \max_{z_i} \quad & \sum_{j\in[n]} \bar{u}_{ij} x_j z_{ij} \\ \text{s.t.} \quad & \sum_{j\in[n]} z_{ij} \leq \Gamma_i, \\ & 0 \leq z_{ij} \leq 1, \quad j \in [n], \end{aligned}$$

for each $i \in [n]$ of the inner maximization problem in (23c). We can again replace this maximization problem by its dual, which reads

$$\begin{aligned} \min_{\varepsilon_i,\xi_i} \quad & \varepsilon_i \Gamma_i + \sum_{j\in[n]} \xi_{ij} \\ \text{s.t.} \quad & \varepsilon_i + \xi_{ij} \geq \bar{u}_{ij} x_j, \quad j \in [n], \\ & \varepsilon_i \geq 0, \\ & \xi_{ij} \geq 0, \quad j \in [n]. \end{aligned}$$

Using these dual reformulations and the arguments used in the previous proofs, the claim follows. □

Unfortunately, Problem (22) is a nonconvex and, thus, intractable optimization problem due to the bilinear terms on the right-hand side of the constraints in (22f). To be more specific, Problem (22) is a (nonconvex) quadratically constrained quadratic program. For such problems, even in the case of a convex objective, i.e., for positive semidefinite $\bar{M}$, the existence of solutions cannot be guaranteed in general; see, e.g., [25]. To avoid the bilinear terms in (22f), we now consider another definition of $M(u)$, which is the same as in [35]. Let now

$$M(u) := \bar{M} + \sum_{\ell\in[L]} u_\ell M^\ell$$

with $L \in \mathbb{N}$ and $M^\ell := [m_{ij}^\ell]_{i,j\in[n]} \in \mathbb{R}^{n\times n}$. This can be interpreted as a linear combination of a uncertainties with given matrices M^ℓ, $\ell \in [L]$. With this, the uncertainty set is defined as

$$\mathcal{U}_{\Gamma,\bar{u}}^{\text{box}} := \{u \in \mathbb{R}^L \colon 0 \leq u_\ell \leq \bar{u}_\ell,\ \ell \in [L],\ |\{\ell \in [L] \colon u_\ell \neq 0\}| \leq \Gamma\}.$$

In this case, the robust counterpart reads

$$\min_{x\geq 0} \quad x^\top \bar{M} x + x^\top q + \max_{u\in\mathcal{U}_{\Gamma,\bar{u}}^{\text{box}}} \sum_{\ell\in[L]} u_\ell x^\top M^\ell x \tag{24a}$$

$$\text{s.t.} \quad \bar{M} x + q + \min_{u\in\mathcal{U}_{\Gamma,\bar{u}}^{\text{box}}} \sum_{\ell\in[L]} u_\ell M^\ell x \geq 0. \tag{24b}$$

The following theorem states that we can rewrite this robust counterpart as a tractable one. To prove this theorem, we proceed as in the proof of Theorem 3.4 in [35].

Theorem 4.2. *Consider the uncertainty set*

$$\mathcal{U}_{\Gamma,\bar{u}}^{box} := \{u \in \mathbb{R}^L \colon 0 \leq u_\ell \leq \bar{u}_\ell,\ \ell \in [L],\ |\{\ell \in [L] \colon u_\ell \neq 0\}| \leq \Gamma\}$$

and $L > \Gamma$. Furthermore, suppose that $\bar{M}$ and M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, Problem (24) is equivalent to the convex, and thus tractable, problem

$$\min_{x,\alpha,\beta,\gamma,\delta} \quad x^\top \bar{M} x + x^\top q + \Gamma\alpha + \sum_{\ell\in[L]} \beta_\ell \tag{25a}$$

$$s.t. \quad \alpha + \beta_\ell \geq \bar{u}_\ell x^\top M^\ell x, \quad \ell \in [L], \tag{25b}$$

$$\alpha \geq 0, \tag{25c}$$
$$\beta_\ell \geq 0, \quad \ell \in [L], \tag{25d}$$
$$\gamma_i \geq 0, \quad i \in [n], \tag{25e}$$
$$\delta_{i\ell} \geq 0, \quad i \in [n],\ \ell \in [L], \tag{25f}$$
$$x \geq 0, \tag{25g}$$
$$\bar{M}_{i,\cdot}x + q_i - \gamma_i\Gamma - \sum_{\ell\in[L]} \delta_{i\ell} \geq 0, \quad i \in [n], \tag{25h}$$
$$\gamma_i + \delta_{i\ell} \geq -\bar{u}_\ell M^\ell_{i,\cdot}x, \quad i \in [n],\ \ell \in [L]. \tag{25i}$$

Proof. First, we rewrite Problem (24) as

$$\min_{x\geq 0,\eta} \quad \eta \tag{26a}$$
$$\text{s.t.} \quad x^\top \bar{M}x + x^\top q + \max_{u\in\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}} \sum_{\ell\in[L]} u_\ell x^\top M^\ell x \leq \eta, \tag{26b}$$
$$\bar{M}x + q + \min_{u\in\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}} \sum_{\ell\in[L]} u_\ell M^\ell x \geq 0 \tag{26c}$$

and reformulate the inner maximization problem in Constraint (26b). Since all M^ℓ, $\ell \in [L]$, are positive semidefinite one has

$$\max_{u\in\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}} \sum_{\ell\in[L]} u_\ell x^\top M^\ell x = \max_{\{I\subseteq[L]\colon |I|\leq\Gamma\}} \sum_{\ell\in I} \bar{u}_\ell x^\top M^\ell x,$$

where we can write the right-hand side as

$$\begin{aligned} \max_{z} \quad & \sum_{\ell\in I} (\bar{u}_\ell x^\top M^\ell x) z_\ell \\ \text{s.t.} \quad & \sum_{\ell\in[L]} z_\ell \leq \Gamma, \\ & 0 \leq z_\ell \leq 1, \quad \ell \in [L]. \end{aligned}$$

Its dual problem reads

$$\begin{aligned} \min_{\alpha,\beta} \quad & \Gamma\alpha + \sum_{\ell\in[L]} \beta_\ell \\ \text{s.t.} \quad & \alpha + \beta_\ell \geq \bar{u}_\ell x^\top M^\ell x, \quad \ell \in [L], \\ & \alpha \geq 0, \\ & \beta_\ell \geq 0, \quad \ell \in [L], \end{aligned}$$

and again we can replace the inner maximization problem in Constraint (26b) with this dual. In a second step, we have a closer look at Constraint (26c). We now consider this constraint componentwise,

$$\bar{M}_{i,\cdot}x + q_i + \min_{u\in\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}} \sum_{\ell\in[L]} u_\ell M^\ell_{i,\cdot}x \geq 0$$

and fix $i \in [n]$ for what follows. Assume now, w.l.o.g., that we have an ascending order of the values $\bar{u}_\ell M^\ell_{i,\cdot}x$ for all $i \in [n]$, i.e.,

$$\bar{u}_1 M^1_{i,\cdot}x \leq \bar{u}_2 M^2_{i,\cdot}x \leq \cdots \leq \bar{u}_L M^L_{i,\cdot}x.$$

Then, we obtain

$$\min_{u\in\mathcal{U}^{\text{box}}_{\Gamma,\bar{u}}} \sum_{\ell\in[L]} u_\ell M^\ell_{i,\cdot}x = \sum_{\ell=1}^{\Gamma} \min\left\{0, \bar{u}_\ell M^\ell_{i,\cdot}x\right\}.$$

So, we can rewrite for each $i \in [n]$ the Constraint (26c) as

$$\bar{M}_{i,\cdot}x + q_i + \min_{\{I \subseteq [L]\colon |I| \leq \Gamma\}} \sum_{\ell \in I} \bar{u}_\ell M^\ell_{i,\cdot} x \geq 0.$$

Now, we reformulate the inner minimization problem in this constraint as

$$\begin{aligned} \min_{z_i} \quad & \sum_{\ell \in [L]} (\bar{u}_\ell M^\ell_{i,\cdot} x) z_{i\ell} \\ \text{s.t.} \quad & \sum_{\ell \in [L]} z_{i\ell} \leq \Gamma, \\ & 0 \leq z_{i\ell} \leq 1, \quad \ell \in [L]. \end{aligned}$$

Its dual reads

$$\begin{aligned} \max_{\gamma_i, \delta_i} \quad & -\gamma_i \Gamma - \sum_{\ell \in [L]} \delta_{i\ell} \\ \text{s.t.} \quad & \gamma_i \geq 0, \\ & \delta_{i\ell} \geq 0, \quad \ell \in [L], \\ & \gamma_i + \delta_{i\ell} \geq -\bar{u}_\ell M^\ell_{i,\cdot} x, \quad \ell \in [L]. \end{aligned}$$

Using this dual problem and the arguments used in the previous proofs, the claim follows. □

Remark 4.3. Let us make two remarks regarding the last theorem. First, note that we only consider the case $L > \Gamma$ because otherwise we are in the strictly robust case. Second, the latter theorem is qualitatively different to the results that we obtained for uncertain q in the last section. For uncertain q, we are able to state a finite-dimensional counterpart without inner minimization or maximization problems independent on whether the original LCP matrix M is positive semidefinite or not. Only the convexity of the counterpart depends on whether M is positive semidefinite or not. Here, we are only able to state a finite-dimensional counterpart without inner minimization or maximization problems in the case of positive semidefinite M.

In Section 3.1 on uncertain q we have been able to reformulate the robust counterpart as an equivalent LCP again. This is not possible anymore in the case of uncertain M. The reason is the quadratic term on the right-hand side of the constraints in (25b).

Next, we derive conditions for the existence and uniqueness of solutions to (25).

Theorem 4.4. *Assume that Problem (25) is feasible and that $\bar{M}$ and M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, there exists a solution of Problem (25).*

Proof. From Constraint (25h) it follows that $\bar{M}x + q \geq 0$ holds, which implies $x^\top \bar{M} x + x^\top q \geq 0$ due to $x \geq 0$. Thus, the objective function of Problem (25) is bounded below on the feasible set of the problem. We can thus apply Theorem 3 of [25] that ensures the existence of a solution. □

Note that the feasibility of Problem (25) is assumed in the theorem. Unfortunately, we have not been able to prove the feasibility of the problem in general like we did in, e.g., Theorem 3.7.

If the matrix $\bar{M}$ in Problem (25) is positive definite, we obtain x-uniqueness of the solution as a consequence of [26].

Proposition 4.5. *Suppose that the matrix $\bar{M}$ in (25) is positive definite. Then, the solution of Problem (25) is unique in x.*

In the light of Example 3.10, we think that uniqueness of the other variables α, β, γ, and δ cannot be achieved.

Finally, we again consider ρ-robust solutions. The result has the same flavor as the corresponding one in Section 3 and can be proven in the same way.

Theorem 4.6. *Let $\mathcal{U} = \mathcal{U}^{box}_{\Gamma,\bar{u}}$ be the given uncertainty set and let $\bar{M}$, M_ℓ, $\ell \in [L]$, be positive semidefinite. Then, x is a ρ-robust LCP solution if and only if there exist $\alpha, \beta, \gamma, \delta$ such that the system*

$$\begin{aligned}
x^\top \bar{M} x + x^\top q + \Gamma\alpha + \sum_{\ell\in[L]} \beta_\ell &\le \rho, && \\
\alpha + \beta_\ell &\ge \bar{u}_\ell x^\top M^\ell x, && \ell \in [L], \\
\alpha &\ge 0, && \\
\beta_\ell &\ge 0, && \ell \in [L], \\
\gamma_i &\ge 0, && i \in [n], \\
\delta_{i\ell} &\ge 0, && i \in [n],\ \ell \in [L], \\
x &\ge 0, && \\
\bar{M}_{i,.} x + q_i - \gamma_i\Gamma - \sum_{\ell\in[L]} \delta_{i\ell} &\ge 0, && i \in [n], \\
\gamma_i + \delta_{i\ell} &\ge -\bar{u}_\ell M^\ell_{i,.} x, && i \in [n],\ \ell \in [L],
\end{aligned}$$

is satisfied. Moreover, this is equivalent to the case that Problem (25) has a solution with objective function value not larger than ρ.

4.2. **Uncertainty set $\mathcal{U}^1_{\Gamma,\delta}$.** As for the case of uncertain q, we now also consider ℓ_1-norm uncertainty in M and we again define

$$M(u) := \bar{M} + \sum_{\ell\in[L]} u_\ell M^\ell$$

with $L \in \mathbb{N}$, $M^\ell := [m^\ell_{ij}]_{i,j\in[n]} \in \mathbb{R}^{n\times n}$, and

$$u \in \mathcal{U}^1_{\Gamma,\delta} := \{u \in \mathbb{R}^L \colon \sum_{\ell\in[L]} u_\ell \le \delta,\ 0 \le u_\ell,\ \ell \in [L],\ |\{\ell \in [L] \colon u_\ell \ne 0\}| \le \Gamma\}.$$

The robust counterpart reads

$$\min_{x\ge 0} \quad x^\top \bar{M} x + x^\top q + \max_{u\in\mathcal{U}^1_{\Gamma,\delta}} \sum_{\ell\in[L]} u_\ell x^\top M^\ell x \tag{27a}$$

$$\text{s.t.} \quad \bar{M}_{i,.} x + q_i - \max_{u\in\mathcal{U}^1_{\Gamma,\delta}} \sum_{\ell\in[L]} u_\ell M^\ell_{i,.} x \ge 0, \quad i \in [n]. \tag{27b}$$

The definitions of Γ-robust feasible points and Γ-robust solutions directly carry over from Section 4.1 to the robust counterpart (27) that we study in this section.

Now, we proceed as in the proof of Theorem 3.4 in [35] to obtain for the latter optimization problem an equivalent reformulation.

Theorem 4.7. *Let $\mathcal{U}^1_{\Gamma,\delta}$ be the uncertainty set. Furthermore, let M^ℓ, $\ell \in [L]$, be positive semidefinite. Then, Problem (27) is equivalent to the problem*

$$\min_{x\ge 0, t\ge 0, s\ge 0} \quad x^\top \bar{M} x + x^\top q + \delta t \tag{28a}$$

$$s.t. \quad x^\top M^\ell x \le t, \quad \ell \in [L], \tag{28b}$$

$$\bar{M} x + q \ge \delta s, \tag{28c}$$

$$s \ge M^\ell x, \quad \ell \in [L]. \tag{28d}$$

Proof. First, we rewrite the inner maximization problem in the objective function of Problem (27). Since

$$\max_{u\in\mathcal{U}^1_{\Gamma,\delta}} \sum_{\ell\in[L]} u_\ell x^\top M^\ell x = \delta \max_{\ell\in[L]} \left\{x^\top M^\ell x\right\},$$

holds, the problem can be reformulated as

$$\min_{x\geq 0, t\geq 0} \quad x^\top \bar{M} x + x^\top q + \delta t \tag{29a}$$

$$\text{s.t.} \quad x^\top M^\ell x \leq t, \quad \ell \in [L], \tag{29b}$$

$$\bar{M}_{i,\cdot} x + q_i - \max_{u\in\mathcal{U}^1_{\Gamma,\delta}} \sum_{\ell\in[L]} u_\ell M^\ell_{i,\cdot} x \geq 0, \quad i \in [n]. \tag{29c}$$

In a second step, we eliminate the minimization problem in Constraints (29c). As all u_ℓ, $\ell \in [L]$, are non-negative, we have

$$\max_{u\in\mathcal{U}^1_{\Gamma,\delta}} \sum_{\ell\in[L]} u_\ell M^\ell_{i,\cdot} x = \delta \max_{\ell\in[L]} \left\{M^\ell_{i,\cdot} x, 0\right\}$$

for all $i \in [n]$ and, thus, Conditions (29c) can be replaced by

$$\bar{M}_{i,\cdot} x + q_i \geq \delta \max_{\ell\in[L]} \left\{M^\ell_{i,\cdot} x, 0\right\}, \quad i \in [n].$$

This last reformulation is equivalent to

$$\begin{aligned} \bar{M}_{i,\cdot} x + q_i &\geq \delta s_i, \quad i \in [n], \\ s_i &\geq M^\ell_{i,\cdot} x, \quad i \in [n],\ \ell \in [L], \\ s_i &\geq 0, \quad i \in [n]. \end{aligned}$$

So, the claim of the theorem follows. □

We notice that again the equivalent reformulation (28) of the robust counterpart (27) is independent of the parameter Γ, as it is the case for the uncertainty set $\mathcal{U}^1_{\Gamma,\delta}$ for uncertainty in q; cf. Section 3.2.

The tractable case is again easy to determine.

Corollary 4.8. *Let $\bar{M}$, M^ℓ, $\ell \in [L]$, be positive semidefinite. Then, the Γ-robust counterpart (28) is a convex optimization problem.*

Theorem 4.9. *Assume that Problem (28) is feasible and that $\bar{M}$ and M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, there exists a solution of Problem (28). If $\bar{M}$ is, in addition, positive definite, then the solution is unique in x and t.*

Proof. From Constraint (28c) follows $\bar{M}x + q \geq 0$, which implies $x^\top \bar{M} x + x^\top q \geq 0$ due to $x \geq 0$. Thus, the objective function is bounded below on the feasible set of the problem. We can thus apply Theorem 3 of [25] that ensures the existence of a solution. If $\bar{M}$ is positive definite, the uniqueness of x again follows from Theorem 1a in [26]. Using this unique part of the solution, it is easy to see that $t = \max\{x^\top M^\ell x \colon \ell \in [L]\}$ is unique as well. □

We close this section with the straightforward result about ρ-robust solutions. The proof can be done by following the lines of the corresponding proofs in Section 3.

Theorem 4.10. *Let $\mathcal{U} = \mathcal{U}^1_{\Gamma,\delta}$ be the given uncertainty set and assume that all M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, x is a ρ-robust LCP solution if and only if*

there exist s, t such that the system

$$\begin{aligned} x^\top \bar{M} x + x^\top q + \delta t &\le \rho, \\ x^\top M^\ell x &\le t, && \ell \in [L], \\ \bar{M} x + q &\ge \delta s, \\ s &\ge M^\ell x, && \ell \in [L], \\ t, s &\ge 0 \end{aligned}$$

is satisfied. Moreover, this implies that x is a ρ-robust solution if and only if the quadratic program (28) has an optimal solution with objective function value not larger than ρ.

5. Γ-Uncertainty in q and M

In this section we consider the ULCP (4) with uncertainties in the vector q and uncertainties in the matrix M. To this end, we assume that the uncertainty sets prescribing $q(u)$ and $M(u)$ are independent, i.e., the problem now reads

$$\{0 \le x \perp M(u_M)x + q(u_q) \ge 0\}_{(u_M, u_q) \in \mathcal{U}_M \times \mathcal{U}_q}.$$

5.1. Box Uncertainty $\mathcal{U}^{\mathbf{box}}_{\Gamma_M, \bar{u}_M} \times \mathcal{U}^{\mathbf{box}}_{\Gamma_q, \bar{u}_q}$.

Here, we consider the box uncertainty set $\mathcal{U}^{\text{box}}_{\Gamma_M, \bar{u}_M} \times \mathcal{U}^{\text{box}}_{\Gamma_q, \bar{u}_q}$ with

$$\mathcal{U}^{\text{box}}_{\Gamma_M, \bar{u}_M} := \{u_M \in \mathbb{R}^L \colon 0 \le u_{M,\ell} \le \bar{u}_{M,\ell},\ \ell \in [L],\ |\{\ell \in [L] \colon u_{M,\ell} \ne 0\}| \le \Gamma_M\}$$

and $\bar{u}_{M,\ell} \ge 0$ for all $\ell \in [L]$, as well as

$$\mathcal{U}^{\text{box}}_{\Gamma_q, \bar{u}_q} := \{u_q \in \mathbb{R}^n \colon -\bar{u}_{q,i} \le u_{q,i} \le \bar{u}_{q,i},\ i \in [n],\ |\{i \in [n] \colon u_{q,i} \ne 0\}| \le \Gamma_q\}$$

with $\bar{u}_{q,i} \ge 0$ for all $i \in [n]$. As in Section 3.1 and 4.1, we have $q(u_q) = \bar{q} + u_q$ and $M(u_M) = \bar{M} + \sum_{\ell \in [L]} u_{M,\ell} M^\ell$ with $L \in \mathbb{N}$ and $M^\ell := [m^\ell_{ij}]_{i,j \in [n]} \in \mathbb{R}^{n \times n}$.

As the uncertainties in q and M are independent, the robust counterpart reads

$$\min_{x \ge 0} \quad x^\top \bar{M} x + x^\top \bar{q} + \max_{\{I \subseteq [n] \colon |I| \le \Gamma_q\}} \sum_{i \in I} \bar{u}_{q,i} x_i + \max_{u_M \in \mathcal{U}^{\text{box}}_{\Gamma_M, \bar{u}_M}} \sum_{\ell \in [L]} u_{M,\ell} x^\top M^\ell x \tag{30a}$$

$$\text{s.t.} \quad \bar{M} x + \bar{q} + \min_{u_M \in \mathcal{U}^{\text{box}}_{\Gamma_M, \bar{u}_M}} \sum_{\ell \in [L]} u_{M,\ell} M^\ell x \ge \sum_{i \in I} \bar{u}_{q,i} e_i \quad \text{for all } I \subseteq [n],\ |I| \le \Gamma_q. \tag{30b}$$

This robust counterpart with exponential (in the LCP's dimension n) many constraints can be reformulated in a tractable way.

Theorem 5.1. *Let the uncertainty set $\mathcal{U}^{box}_{\Gamma_M, \bar{u}_M} \times \mathcal{U}^{box}_{\Gamma_q, \bar{u}_q}$ be given and $L > \Gamma_M$. Furthermore, suppose that $\bar{M}$ and $M^\ell, \ell \in [L]$, are positive semidefinite. Then, the*

robust counterpart (30) *is equivalent to the convex, and thus tractable, problem*

$$\min_{z} \quad x^\top \bar{M} x + x^\top \bar{q} + \alpha_M \Gamma_M + \alpha_q \Gamma_q + \sum_{i=1}^{n} \beta_{q,i} + \sum_{\ell \in [L]} \beta_{M,\ell} \tag{31a}$$

$$s.t. \quad \bar{M}_{i,.} x - \gamma_i \Gamma_M - \sum_{\ell \in [L]} \delta_{i,\ell} \geq -\bar{q}_i + \bar{u}_{q,i}, \quad i \in [n], \tag{31b}$$

$$\alpha_q + \beta_{q,i} \geq \bar{u}_{q,i} x_i, \quad i \in [n], \tag{31c}$$

$$\alpha_M + \beta_{M,\ell} \geq \bar{u}_{M,\ell} x^\top M^\ell x, \quad \ell \in [L], \tag{31d}$$

$$\alpha_M, \alpha_q \geq 0, \tag{31e}$$

$$x_i, \gamma_i, \beta_{q,i} \geq 0, \quad i \in [n], \tag{31f}$$

$$\beta_{M,\ell} \geq 0, \quad \ell \in [L], \tag{31g}$$

$$\delta_{i,\ell} \geq 0, \quad i \in [n], \ \ell \in [L], \tag{31h}$$

where $z := (x, \alpha_M, \alpha_q, \beta_M, \beta_q, \gamma, \delta)$.

Proof. As the uncertainty sets for q and M are independent, we can combine the techniques and arguments used in the proofs of Theorem 3.2 and 4.2. □

As in Section 3 and 4 we derive conditions for the existence and uniqueness of solutions to (31). In analogy to the proof of Theorem 4.4, one can show the following existence result.

Theorem 5.2. *Assume that Problem* (31) *is feasible and that* $\bar{M}$ *and* $M^\ell, \ell \in [L]$, *are positive semidefinite. Then, there exists a solution of Problem* (31).

Using [26], we can again obtain uniqueness of x in the solution of Problem (31) if $\bar{M}$ is positive definite. As before, we do not think that we can obtain uniqueness of the entire solution of Problem (31).

For completeness, we also state the following result on ρ-robust solutions, which also carries over directly from Section 3 and 4.

Theorem 5.3. *Let* $\mathcal{U}^{box}_{\Gamma_M, \bar{u}_M} \times \mathcal{U}^{box}_{\Gamma_q, \bar{u}_q}$ *be the given uncertainty set and let* $\bar{M}$, M_ℓ, $\ell \in [L]$, *be positive semidefinite. Then,* x *is a* ρ*-robust LCP solution if and only if there exist* $\alpha_M, \alpha_q, \beta_M, \beta_q, \gamma, \delta$ *satisfying* (31b)–(31h) *and*

$$x^\top \bar{M} x + x^\top \bar{q} + \alpha \Gamma_2 + \bar{\alpha} \Gamma_1 + \sum_{i=1}^{n} \bar{\beta}_i + \sum_{\ell \in [L]} \beta_\ell \leq \rho.$$

Moreover, this implies that x *is a* ρ*-robust LCP solution if and only if the quadratic program* (31) *has an objective function value not larger than* ρ.

5.2. **ℓ_1-Norm Uncertainty $\mathcal{U}^1_{\Gamma_M, \delta_M} \times \mathcal{U}^1_{\Gamma_q, \delta_q}$.** Next, we consider the ℓ_1-norm uncertainty set $\mathcal{U}^1_{\Gamma_M, \delta_M} \times \mathcal{U}^1_{\Gamma_q, \delta_q}$ with

$$\mathcal{U}^1_{\Gamma_M, \delta_M} := \Big\{ u_M \in \mathbb{R}^L : \sum_{\ell \in [L]} u_{M,\ell} \leq \delta_M, \, 0 \leq u_{M,\ell}, \, \ell \in [L], \\ |\{\ell \in [L] \colon u_{M,\ell} \neq 0\}| \leq \Gamma_M \Big\}.$$

and

$$\mathcal{U}^1_{\Gamma_q, \delta_q} := \left\{ u_q \in \mathbb{R}^n \colon \ \|u_q\|_1 \leq \delta_1, \, |\{i \in [n] \colon u_{q,i} \neq 0\}| \leq \Gamma_q \right\}.$$

If the uncertainties in q and M are independent, the robust counterpart reads

$$\min_{x\geq 0} \quad x^\top \bar{M}x + x^\top \bar{q} + \max_{u_q \in \mathcal{U}^1_{\Gamma_q,\delta_q}} \sum_{i\in[n]} u_{q,i}x_i + \max_{u_M \in \mathcal{U}^1_{\Gamma_M,\delta_M}} \sum_{\ell\in[L]} u_{M,\ell} x^\top M^\ell x \tag{32a}$$

$$\text{s.t.} \quad \bar{M}_{i\cdot}x + \bar{q}_i + \min_{u_q \in \mathcal{U}^1_{\Gamma_q,\delta_q}} u_{q,i} - \max_{u_M \in \mathcal{U}_{\Gamma_M,\delta_M}} \sum_{\ell\in[L]} u_{M,\ell} M^\ell_{i\cdot} x \geq 0, \quad i \in [n]. \tag{32b}$$

As in Section 5.1, all results directly carry over to the setting considered since the uncertainty sets for $q(u)$ and $M(u)$ are unrelated. Thus, for reasons of brevity, we again omit all proofs.

Theorem 5.4. *Let $\mathcal{U}^1_{\Gamma_M,\delta_M} \times \mathcal{U}^1_{\Gamma_q,\delta_q}$ be the uncertainty set. Furthermore, let M^ℓ, $\ell \in [L]$, be positive semidefinite. Then, Problem (32) is equivalent to the problem*

$$\min_{x\geq 0, t_q \geq 0, t_M \geq 0, s\geq 0} \quad x^\top \bar{M}x + x^\top \bar{q} + \delta_1 t_q + \delta_2 t_M \tag{33a}$$

$$s.t. \quad x_i \leq t_q, \quad i \in [n], \tag{33b}$$

$$x^\top M^\ell x \leq t_M, \quad \ell \in [L], \tag{33c}$$

$$\bar{M}_{i\cdot}x + \bar{q}_i - \delta_q - \delta_M s \geq 0, \quad i \in [n], \tag{33d}$$

$$s \geq M^\ell x, \quad \ell \in [L]. \tag{33e}$$

Furthermore, (33) is a convex and thus tractable optimization problem if $\bar{M}$ and $M^\ell, \ell \in [L]$, are positive semidefinite.

Again note that (33) is independent of the parameters Γ_M and Γ_q, as it is the case for the uncertainty set $\mathcal{U}^1_{\Gamma,\delta}$ for uncertainty in q, cf. Section 3.2, and for uncertainty in M, cf. Section 4.2.

Now, we consider the existence and uniqueness of solutions to Problem (33).

Theorem 5.5. *Assume that Problem (33) is feasible and that $\bar{M}$ and M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, there exists a solution of Problem (33). If $\bar{M}$ is, in addition, positive definite, then the solution is unique in x, t_M, and t_q.*

We close this section with the straightforward result about ρ-robust solutions. The proof can be done by following the lines of the corresponding proofs in Section 3.

Theorem 5.6. *Let $\mathcal{U}^1_{\Gamma_M,\delta_M} \times \mathcal{U}^1_{\Gamma_q,\delta_q}$ be the given uncertainty set and assume that all M^ℓ, $\ell \in [L]$, are positive semidefinite. Then, x is a ρ-robust LCP solution if and only if there exist $s, t_M, t_q \geq 0$ satisfying (33b)–(33e) and*

$$x^\top \bar{M}x + x^\top \bar{q} + \delta_q t_q + \delta_M t_M \leq \rho.$$

Moreover, this implies that x is a ρ-robust LCP solution if and only if the quadratic program (33) has an objective function value not larger than ρ.

Let us close this section with some words on the case of uncertainty in q and M, where these uncertainties are not unrelated but parameterized by the same vector. For the case of strict robustness, this has been done in [35]. However, also in this easier case it is already required to parameterize the matrix uncertainty using the Cholesky factorization of the matrix M and to study semidefinite programming problems (SDPs). By doing so, tractability of the counterpart can be shown for ℓ_2-norm uncertainties in the strict case, whereas this is not possible for ℓ_1- and ℓ_∞-norm uncertainty sets. Thus, it cannot be expected that a tractability result for related uncertainties hold for the cases discussed in this paper.

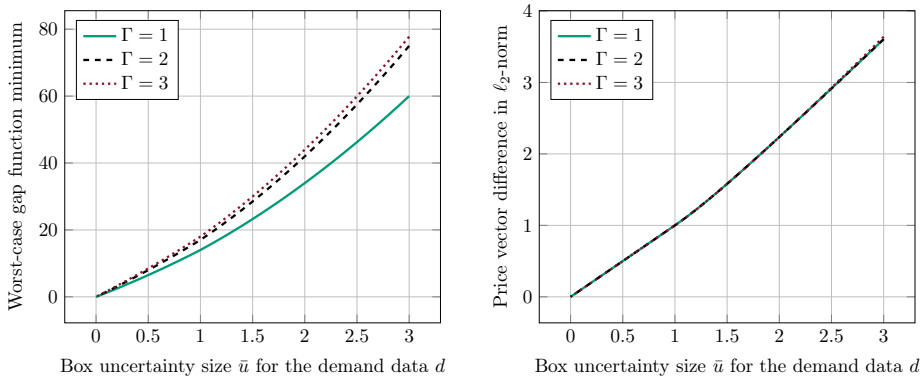

FIGURE 1. Left: Worst-case gap function minimum for box uncertainties in q, for $\Gamma \in \{1, 2, 3\}$, and for varying uncertainty set parameter $\bar{u} \in [0, 3]$. Right: ℓ_2-norm of the nominal price vector and the robust price vector for box uncertainties in q, for $\Gamma \in \{1, 2, 3\}$, and for varying uncertainty set parameter $\bar{u} \in [0, 3]$.

6. Case Study

In this section, we briefly discuss the dependence of the worst-case gap function minima on the considered uncertain data. We try to keep things as simple as possible and, thus, discuss uncertainty in the LCP vector q and matrix M of the market equilibrium model in Section 2.1 separately. Here, we consider three goods and production activity levels as well as three technology constraints. The cost vector is $c = (3, 2, 1)^\top$ and $A = -I_{3\times 3}$, $b = (-4, -5, -10)^\top$ is used. This means that the three productions have capacities 4, 5, and 10. Moreover, we set $B = I_{3\times 3}$, i.e., every production separately yields a certain good. The demand is calibrated by $D = -I_{3\times 3}$ and $d = (6, 9, 3)^\top$, i.e., price sensitivity is the same for all demands but consumers have a different maximum willingness to pay. The resulting LCP matrix M is positive semidefinite and considered certain at a first glance. The uncertain data is the vector $d \in \mathbb{R}^3$ that we parameterize as $d_i + u_i$ with $-\bar{u} \leq u_i \leq \bar{u}$ for all $i \in \{1, 2, 3\}$. Thus, we consider the same box uncertainty for every entry of the vector d. In Figure 1 (left) we show the optimal values of the worst-case gap minimization problem for $\Gamma \in \{1, 2, 3\}$ and different values of $\bar{u}$ ranging from 0 (which yields a certain LCP) and 3. We choose 3 here to ensure that also the smallest possible entry in d is still non-negative. First, the dependence of the minimum worst-case gap, i.e., the objective function value of Problem (11), on the value of Γ is as expected: Larger values of Γ, i.e., more data that is allowed to realize in a worst-case way, lead to larger minimum gaps. Interestingly, a change of Γ from 1 to 2 uncertain demands leads to significantly larger gaps, whereas changing Γ from 2 to 3 does not increase the worst-case gaps that significantly anymore. Moreover, for fixed Γ, the gaps seem to quadratically depend on the size of the uncertainty boxes.

As it can be seen in Problem (11), feasibility does not depend on the actual value of Γ since it only appears in the objective function—which is also the case in classical Γ-robust optimization; cf., e.g., Theorem 1 in [32]. This can also be seen in Figure 1 (right), where we plotted the ℓ_2-norm difference of the optimal prices p^* for different box uncertainty sizes $\bar{u}$ and the optimal (equilibrium) prices for the nominal case. It can be seen that all curves are the same for all Γ. Moreover, the norm of the price differences is linearly increasing in the box uncertainty sizes. This is to be expected since the relation between the variables x and the box uncertainty

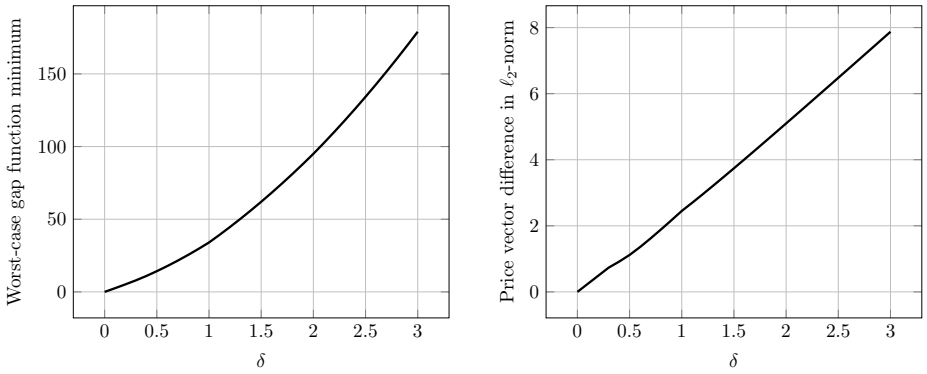

FIGURE 2. Left: Optimal value of the tractable robust counterpart for ℓ_1-norm uncertainties in q and varying uncertainty set parameter $\delta \in [0, 3]$. Right: ℓ_2-norm of the nominal price vector and the robust price vector for ℓ_1-norm uncertainties in q and varying uncertainty set parameter $\delta \in [0, 3]$.

size $\bar{u}$ is linear in the constraints. Consequently, this linear relation translates into a quadratic behavior in the objective value due to the quadratic term $x^\top Mx$.

The same observations can be made for the case of uncertain q and ℓ_1-norm uncertainty sets as discussed in Section 3.2. In Figure 2 (left) we again plot the worst-case gap minima, i.e., the optimal objective function values of Problem (19). For the ℓ_1-norm uncertainty sets we have shown that the robust counterparts are independent of Γ. Thus, we have only one curve that shows the optimal gaps for varying values of δ (from 0 to 3). The quadratic nature is the same as for the other box uncertainties and also the ℓ_2-norm of the difference of the nominal and robust price vectors is again linear in δ; see Figure 2 (right).

Next, we turn to the case in which q is certain but uncertainties arise in the LCP matrix M. Since we considered uncertain demand in case of uncertain q we now study the robustification regarding uncertainties in the production matrix A. The nominal value still is $A = \bar{A} = -I_{3\times 3}$ and the corresponding right-hand side still is given by $b = (-4, -5, -10)^\top$. Uncertainty in A is then modeled via

$$A(u) = \bar{A} + \sum_{\ell \in [L]} u_\ell M^\ell$$

with

$$M^\ell = \omega_\ell I_{3\times 3} \quad \text{with} \quad \omega_\ell \in \{-0.5, -0.15, 0, 0.15, 0.5\} \quad \text{for} \quad \ell = 1, \ldots, 5.$$

Finally, $u_\ell \in [0, 1]$ for all ℓ, and the other matrices B and D are considered to be certain. Using this parameterization, all assumptions regarding positive semidefiniteness of matrices as stated in Section 4 are satisfied.

The results are shown in Figure 3 and 4 as before. Interestingly, the worst-case gaps (left plot in Figure 3) behave similarly as in the case of uncertain q whereas price vector differences now depend sub-linearly on the box uncertainty set size of u_ℓ. Both can also be seen for the case of ℓ_1-norm uncertainties in Figure 4.

7. Conclusion

In this paper, we considered Γ-robustifications of uncertain linear complementarity problems. After defining the problem class we studied the case of uncertainty in the LCP vector q and in the LCP matrix M separately. Moreover, for both cases we considered box as well as ℓ_1-norm uncertainty sets and discussed that one cannot

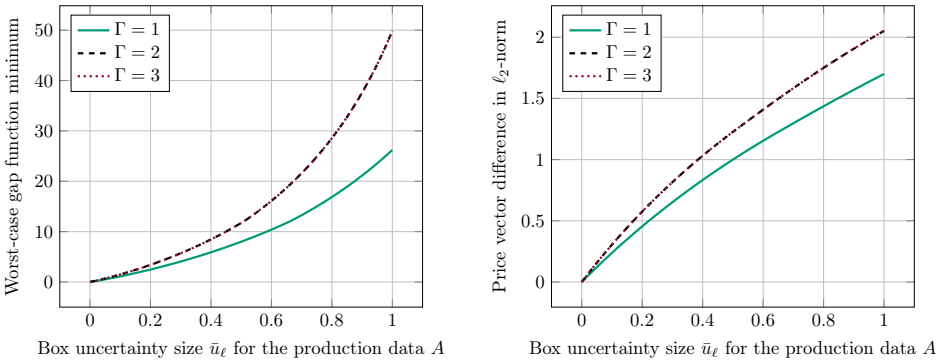

FIGURE 3. Left: Worst-case gap function minimum for box uncertainties in M, for $\Gamma \in \{1, 2, 3\}$, and for varying uncertainty set parameter $\bar{u}_\ell \in [0, 1]$. Right: ℓ_2-norm of the nominal price vector and the robust price vector for box uncertainties in M, for $\Gamma \in \{1, 2, 3\}$, and for varying uncertainty set parameter $\bar{u}_\ell \in [0, 1]$.

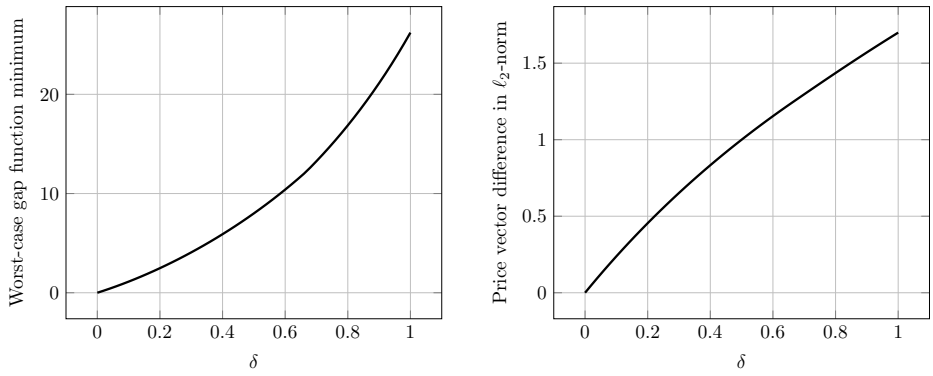

FIGURE 4. Left: Optimal value of the tractable robust counterpart for ℓ_1-norm uncertainties in M and varying uncertainty set parameter $\delta \in [0, 1]$. Right: ℓ_2-norm of the nominal price vector and the robust price vector for ℓ_1-norm uncertainties in M and varying uncertainty set parameter $\delta \in [0, 1]$.

expect the existence of robust equilibria in a pure sense. Thus, we investigated the global minimizers of the worst-case gap function. For both types of uncertainty sets, we derived conditions (typically monotonicity of the original LCP) for the tractability of the robust counterpart. In the case of uncertain q, this then allows to consider the tractable robust counterpart again as an LCP. For this LCP, desired matrix properties like the membership in the matrix classes S and P are shown to be violated in every case. Nevertheless, we derived specific conditions for the feasibility of the robust counterpart as well as for the existence and, if possible, uniqueness of its solution. In the case of uncertain M, the tractable counterparts are convex QCQPs, for which we also study the existence and uniqueness of solutions. Finally, we characterized ρ-robust solutions, i.e., solutions to certain relaxations of the uncertain LCP.

Despite the theoretical results obtained in this paper, there still are some open questions in the context of Γ-robust LCPs. Let us briefly discuss four of them. First, other uncertainty sets instead of box and ℓ_1-norm uncertainties can be discussed. In our opinion, the canonical next step is the study of ellipsoidal, i.e., ℓ_2-norm

uncertainty sets. However, also general polyhedral uncertainty sets might be of interest. Second, the question remains open whether there exist a-priori conditions on q, M, and the uncertainty set that ensure the existence of ρ-robust solutions for a given $\rho > 0$. Third, the tractability of Γ-robust counterparts is still open for non-monotone LCPs. Fourth and finally, the deeper study of relevant applications is of interest. We sketched some possible applications in this paper but many other, e.g., traffic equilibrium problems, might give interesting application-specific insights in this rather new field of robust optimization.

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