

# UNIQUENESS OF MARKET EQUILIBRIA ON NETWORKS WITH TRANSPORT COSTS

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**ABSTRACT.** We study the existence and uniqueness of equilibria for perfectly competitive markets in capacitated transport networks. The model under consideration is rather general so that it captures basic aspects of related models in, e.g., gas or electricity networks. We formulate the market equilibrium model as a mixed complementarity problem and show the equivalence to a welfare maximization problem. Using the latter we prove uniqueness of the resulting equilibrium for piecewise linear and symmetric transport costs under additional mild assumptions. Moreover, we show the necessity of these assumptions by illustrating examples that possess multiple solutions if our assumptions are violated.

## 1. INTRODUCTION

We consider perfectly competitive markets upon capacitated networks, where transport costs are modeled using piecewise linear and symmetric cost functions. In this setting, we prove uniqueness of market equilibria under mild assumptions. Our motivation is the following. On the one hand, uniqueness of market equilibria is a classical topic of mathematical economics by itself. On the other hand, our model including a networked transport infrastructure has important applications in the areas of, e.g., electricity and gas markets. Furthermore, uniqueness of equilibria of such models is an important prerequisite for studying more complicated, e.g., multilevel, market models; see, e.g., Daxhelet and Smeers (2001), Grimm, Grübel, et al. (2017), Grimm, Kleinert, et al. (2017), Grimm, Martin, et al. (2016), Hobbs, Metzler, et al. (2000), Hu and Ralph (2007), and Kleinert and Schmidt (2018) for multilevel models in electricity markets as well Grimm, Grübel, et al. (2017) and Grimm et al. (2017a) for multilevel models of gas markets. Most of the above mentioned papers abstract from transport costs. However, there also exist equilibrium models including transport costs on networks. These are mainly studied in the context of imperfect competition in gas markets, cf., e.g., Cremer, Gasmi, et al. (2003) and Cremer and Laffont (2002). For an application in the electricity sector, see Paz (2015) or Hobbs and Rijkers (2004) for a more general study. In addition to the fields of gas and electricity, one might also think of other networked transport structures like they appear in water or traffic networks. However, these fields are somehow different. In traffic networks the classical concept of equilibria is the Wardrop equilibrium, cf. Wardrop (1952) and Wardrop and Whitehead (1952), which differs from the equilibria yielding market clearing prices in our context. Finally, in the context of water networks, the techno-economic literature focuses on different issues like market power and institutional constraints due to the complex nature of water rights; cf., e.g., Chakravorty et al. (2009).

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Our contribution is in line with the papers by Grimm et al. (2017b) and Krebs et al. (2017). The former proves uniqueness of long-run market equilibria using a network flow transport model as we do in our paper. The latter considers uniqueness and multiplicity of solutions in the context of short-run market models using DC power flows. However, both analyses do not cover transport costs, which are part of many realistic models for electricity or gas markets that also consider the corresponding network infrastructure—see the literature cited above. Moreover, none of these papers considers uniqueness of equilibria. In contrast, we analyze perfectly competitive markets and prove uniqueness of the resulting equilibria.

## 2. MARKET EQUILIBRIUM MODELING

We consider transport networks that we model by using connected and finite digraphs  $G = (N, A)$  with node set  $N$  and arc set  $A$ . Subsequently, all player models of our overall market model are stated. Since we consider perfectly competitive markets, all players are price takers and their optimization problems are formulated using exogenously given market prices  $\pi_u$  at every node  $u \in N$ .

The first type of players are producers. We assume that there exists exactly one producer at each node  $u \in N$ , which we model by a fixed generation capacity  $\bar{y}_u > 0$  and variable production costs  $w_u > 0$ . Production at node  $u$  is denoted by  $y_u \geq 0$  and is bounded from above by the generation capacity. The objective of the producer is to maximize its profit and, thus, its linear optimization problem reads

$$\max_{y_u} (\pi_u - w_u) y_u \quad \text{s.t.} \quad 0 \leq y_u \leq \bar{y}_u.$$

Its solutions are characterized by the corresponding Karush–Kuhn–Tucker (KKT) conditions

$$\pi_u - w_u + \beta_u^- - \beta_u^+ = 0, \quad 0 \leq y_u \perp \beta_u^- \geq 0, \quad 0 \leq \bar{y}_u - y_u \perp \beta_u^+ \geq 0, \quad (1)$$

where  $\beta_u^\pm$  are the dual variables of the production constraints. Here and in what follows, we use the standard  $\perp$ -notation, which abbreviates

$$0 \leq a \perp b \geq 0 \iff a, b \geq 0, \quad ab = 0.$$

Consumers, as our second players, are also located at the nodes  $u \in N$  and decide on their demand  $d_u \geq 0$ . Their demand elasticity is modeled by inverse demand functions  $p_u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ , for which we make the following assumption.

**Assumption 1.** *All inverse demand functions  $p_u$ ,  $u \in N$ , are strictly decreasing and continuously differentiable.*

Under Assumption 1 the concave problem of a surplus maximizing consumer at node  $u$  is given by

$$\max_{d_u} \int_0^{d_u} p_u(x) dx - \pi_u d_u \quad \text{s.t.} \quad 0 \leq d_u$$

and its again necessary and sufficient first-order optimality conditions comprise

$$p_u(d_u) - \pi_u + \alpha_u = 0, \quad 0 \leq d_u \perp \alpha_u \geq 0, \quad (2)$$

where  $\alpha_u$  is the dual variable of the lower demand bound.

The third player in our market model is the transport system operator (TSO). He operates the transport network, in which every arc  $a \in A$  is described by its lower and upper capacities  $f_a^-$  and  $f_a^+$  and by its transport cost function  $c_a : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ . In what follows we assume that all transport cost functions  $c_a$ ,  $a \in A$ , are convex. The goal of the TSO is to control the arc flows  $f_a$ ,  $a \in A$ , such that the transport is realized from low- to high-price regions and the earnings to be maximized result

from the corresponding price differences; cf., e.g., Hobbs and Helman (2004) for the case of electricity networks. Thus, the convex problem of the TSO reads

$$\max_f \sum_{a=(u,v) \in A} (\pi_v - \pi_u) f_a - c_a(f_a) \quad (3a)$$

$$\text{s.t. } f_a^- \leq f_a \leq f_a^+, \quad a \in A. \quad (3b)$$

Here and in what follows, a variable without index denotes the vector containing all corresponding node or arc variables, e.g.,  $f := (f_a)_{a \in A}$ . Constraints (3b) reflect the network's capacities and have the dual variables  $\delta_a^\pm$ . The optimality conditions of (3) are given by

$$\begin{aligned} \pi_v - \pi_u - c'_a(f_a) + \delta_a^- - \delta_a^+ &= 0, \quad a = (u, v) \in A, \\ 0 \leq f_a - f_a^- \perp \delta_a^- &\geq 0, \quad a \in A, \\ 0 \leq f_a^+ - f_a \perp \delta_a^+ &\geq 0, \quad a \in A. \end{aligned} \quad (4)$$

Putting all first-order optimality conditions as well as the flow balance conditions

$$0 = d_u - y_u + \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a, \quad u \in N, \quad (5)$$

together, we obtain the mixed complementarity problem (MCP)

$$\text{Producers: (1), Consumers: (2), TSO: (4), Market Clearing: (5).} \quad (6)$$

This MCP models the considered market for the case of perfect competition. Hence, solutions of (6) are market equilibria. It can be easily seen that this complementarity system is equivalent to the welfare maximization problem

$$\max_{d,y,f} \sum_{u \in N} \int_0^{d_u} p_u(x) dx - \sum_{u \in N} w_u y_u - \sum_{a \in A} c_a(f_a) \quad (7a)$$

$$\text{s.t. } 0 \leq y_u \leq \bar{y}_u, \quad u \in N, \quad (7b)$$

$$0 \leq d_u, \quad u \in N, \quad (7c)$$

$$f_a^- \leq f_a \leq f_a^+, \quad a \in A, \quad (7d)$$

$$0 = d_u - y_u + \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a, \quad u \in N. \quad (7e)$$

Here we use the standard  $\delta$ -notation for the in- and outgoing arcs of a node  $u \in N$ , i.e.,  $\delta^{\text{in}}(u) := \{(v, u) \in A\}$  and  $\delta^{\text{out}}(u) := \{(u, v) \in A\}$ . The mentioned equivalence can be shown by comparing the first-order optimality conditions of Problem (7) with the MCP (6) and by identifying the dual variables  $\gamma_u$  of the flow balance constraints (7e) with equilibrium prices  $\pi_u$  of the complementarity problem. Furthermore, we use the fact that the KKT conditions are again necessary and sufficient optimality conditions of Problem (7) under Assumption 1. The equivalence between the optimization problem (7) and the MCP (6) now allows us to consider Problem (7) in order to obtain results for the MCP (6). This is exactly the road that we follow in order to prove the uniqueness of the equilibrium of (6). By doing so, we remark one structural difference between these two versions of the same problem. In the MCP formulation we consider nodal prices  $\pi_u$  to be given exogenously. In this setting the question then is whether there exists (an unique) equilibrium, i.e., a solution of (6). On the other hand, solving the welfare maximization problem (7) does not require these prices to be given. Instead, equilibrium prices are part of the dual solution of the problem as we noted above.

So far we formulated a short-run market model that does not depend on multiple scenarios. It is, however, straight forward to extend the setting to multiple scenarios,

which then yields a time-separable problem for which all of our results carry over directly.

Since existence of equilibria is trivial because  $(d, y, f) = (0, 0, 0)$  is feasible and the problem is bounded from above, we focus on the study of uniqueness of a solution of Problem (7). We first note that strictly convex cost functions  $c_a$  in (7) yield a unique solution.

**Theorem 2.1.** *Suppose Assumption 1 holds. Consider Problem (7) with strictly convex cost functions  $c_a$ ,  $a \in A$ . Then, the solution of Problem (7) is unique.*

*Proof.* Uniqueness of  $(d, f)$  follows directly from Theorem 1a in Mangasarian (1988). Uniqueness of the productions then follows from the flow conservation constraints (7e).  $\square$

In the rest of the paper, we consider the case without having the assumption of strictly convex cost functions. First, we show uniqueness of the demands.

**Theorem 2.2.** *Suppose Assumption 1 holds. Let  $(d, y, f)$  and  $(d', y', f')$  be two solutions of Problem (7). Then,  $d = d'$  holds.*

*Proof.* The claim follows directly from Theorem 1a in Mangasarian (1988).  $\square$

### 3. UNIQUENESS FOR PIECEWISE LINEAR AND SYMMETRIC TRANSPORT COSTS

In what follows, we consider piecewise linear and symmetric transport cost functions. This is formalized in the following assumption.

**Assumption 2.** *All transport cost functions  $c_a$ ,  $a \in A$ , have the form  $c_a(f_a) := m_a |f_a|$  with  $m_a \in \mathbb{R}_{\geq 0}$ .*

Note that all transport cost functions satisfying this assumption are convex.

In the following we use the concept of flow-induced partitions; cf., e.g., Grimm et al. (2017b). A flow-induced partition of the network  $G = (N, A)$  w.r.t. a solution  $(d, y, f)$  of Problem (7) is the partition  $\{G^i\}_{i \in I}$ ,  $I \subseteq \mathbb{N}$ , where each  $G^i := (N^i, A^i)$  is a connected component of the graph  $(N, A \setminus \bar{A})$  with  $\bar{A} := \{a \in A : f_a = f_a^- \vee f_a = f_a^+\}$ . Each  $G^i$ ,  $i \in I$ , is called a flow-induced component. Additionally, for two arbitrary nodes  $u, v \in N$  we denote by  $P_{uv}$  an undirected path from  $u$  to  $v$  and define

$$r_a := \begin{cases} 1, & \text{if } a \text{ is directed along } P_{uv}, \\ -1, & \text{otherwise,} \end{cases}$$

for each arc  $a \in P_{uv}$ . To show uniqueness of the productions we make the following assumption.

**Assumption 3.** *For each two nodes  $u \neq v \in N$  and all paths  $P_{uv}$  we have*

$$w_u - w_v + \sum_{a \in P_{uv}} m_a < 0 \quad \text{or} \quad w_u - w_v - \sum_{a \in P_{uv}} m_a > 0.$$

Since the latter assumption is the key precondition for uniqueness, we discuss it in more detail. The assumption mainly states that the aggregated transport costs between two nodes need to be less than the difference of production costs at the nodes. This needs to hold for all possible transport paths connecting the two nodes and is a direct generalization of the required assumptions in the cases in which no network or a network without transport costs are considered. In these cases one needs to assume that the variable production costs are pairwise different; cf., e.g., Grimm et al. (2017b). Thus, not surprisingly, this precondition is exactly covered by Assumption 2 if transport costs are zero, i.e., if  $m_a = 0$  holds for all  $a \in A$ . Moreover, we think that Assumption 3 is rather mild in practice. The variable production costs  $w_u$  and  $w_v$  belong to production facilities of different producers

located at the nodes  $u$  and  $v$  and the transport costs depend on the installed network infrastructure that is operated by the TSO. Usually, this network infrastructure has high investment costs but comparably small operating costs. This is, for instance, the case in gas transport networks, where building new infrastructure (like new pipelines or compressor stations) is costly compared to the operation of the network, for which the costs are mainly given by the compressor costs; cf., e.g., Fügenschuh et al. (2014) and Pfetsch et al. (2015) and the references therein. Thus, we think that the assumption is reasonable in the considered setup.

**Theorem 3.1.** *Suppose Assumptions 1, 2, and 3 hold. Let  $z = (d, y, f)$  be a solution of (7) and let  $\{G^i = (N^i, A^i)\}_{i \in I}$ ,  $I \subseteq \mathbb{N}$ , be its flow-induced partition. Then, there exists at most one node  $k \in N^i$  in each flow-induced component  $G^i$  with  $0 < y_k < \bar{y}_k$ .*

*Proof.* Let  $u \neq v \in N^i$  be two nodes in a flow-induced component satisfying  $0 < y_u < \bar{y}_u$  and  $0 < y_v < \bar{y}_v$ . Furthermore, let  $P_{uv}$  be a path from  $u$  to  $v$  that is completely contained in  $G^i$ , i.e.,  $f_a \in (f_a^-, f_a^+)$  holds for all arcs  $a \in P_{uv}$ . We now construct a feasible point  $z' = (d, y', f')$  of Problem (7) with a larger objective function value than  $z$ . This yields a contradiction to the optimality of  $z$ . For a suitable  $\varepsilon \in \mathbb{R}$  we define for each node  $k \in N$  and each arc  $a \in A$

$$y'_k := \begin{cases} y_u + \varepsilon, & \text{if } k = u, \\ y_v - \varepsilon, & \text{if } k = v, \\ y_k, & \text{otherwise,} \end{cases} \quad f'_a := \begin{cases} f_a + r_a \varepsilon, & \text{if } a \in P_{uv}, \\ f_a, & \text{otherwise.} \end{cases}$$

It is easy to see that these productions and flows satisfy the flow-balance conditions (7e). Now we determine an interval for  $\varepsilon$  such that  $z'$  also satisfies the production and flow bounds (7b), (7d). We have

$$0 \leq y_u + \varepsilon \leq \bar{y}_u, \quad 0 \leq y_v - \varepsilon \leq \bar{y}_v, \quad f_a^- \leq f_a + r_a \varepsilon \leq f_a^+ \quad \text{for all } a \in P_{uv},$$

which implies

$$\begin{aligned} \varepsilon &\geq \max\{-y_u, y_v - \bar{y}_v, \max_{a \in P_{uv}, r_a=1} \{f_a^- - f_a\}, \max_{a \in P_{uv}, r_a=-1} \{f_a - f_a^+\}\}, \\ \varepsilon &\leq \min\{\bar{y}_u - y_u, y_v, \min_{a \in P_{uv}, r_a=1} \{f_a^+ - f_a\}, \min_{a \in P_{uv}, r_a=-1} \{f_a - f_a^-\}\}. \end{aligned}$$

The lower bound is negative and the upper bound is positive. In what follows, we denote with  $\Phi(\tilde{z})$  the objective function value of a feasible point  $\tilde{z}$  of Problem (7). The objective function difference of  $z$  and  $z'$  reads

$$\begin{aligned} \Phi(z) - \Phi(z') &= -w_u y_u - w_v y_v + w_u (y_u + \varepsilon) + w_v (y_v - \varepsilon) \\ &\quad - \sum_{a \in P_{uv}} c_a(f_a) + \sum_{a \in P_{uv}} c_a(f_a + r_a \varepsilon) \\ &= (w_u - w_v) \varepsilon - \sum_{a \in P_{uv}} m_a |f_a| + \sum_{a \in P_{uv}} m_a |f_a + r_a \varepsilon|. \end{aligned}$$

The triangle inequality yields

$$\Phi(z) - \Phi(z') \leq (w_u - w_v) \varepsilon + \sum_{a \in P_{uv}} m_a |r_a \varepsilon|$$

and together with the definition of  $r_a$  we obtain

$$\begin{aligned} \Phi(z') &\geq \Phi(z) - \left( (w_u - w_v) \varepsilon + \sum_{a \in P_{uv}} m_a |\varepsilon| \right) \\ &= \Phi(z) - \left( w_u - w_v + \operatorname{sgn}(\varepsilon) \sum_{a \in P_{uv}} m_a \right) \varepsilon. \end{aligned}$$

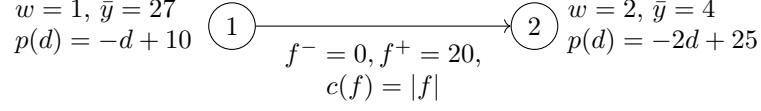


FIGURE 1. Two-node network with multiple solutions.

As we can choose  $\varepsilon$  being positive or negative and since Assumption 3 holds, we can choose an  $\varepsilon \neq 0$  such that  $\Phi(z') > \Phi(z)$  holds. This is a contradiction to the optimality of  $z$ .  $\square$

Next we prove uniqueness of the productions in a solution of Problem (7). To this end, we obtain from the following lemma—which is mainly taken from Grimm et al. (2017b)—that it is sufficient to show uniqueness of productions for fixed binding production and flow bounds (7b), (7d).

**Lemma 3.2.** *Suppose Assumption 1 holds. Then, exactly one of the two following cases occurs:*

- (a) *There exist a demand vector  $d^*$  and a production vector  $y^*$  such that every solution of Problem (7) is of the form  $(d^*, y^*, f)$  for some flow  $f$ .*
- (b) *There exist two solutions  $z = (d, y, f)$  and  $z' = (d, y', f')$  of Problem (7) with  $y \neq y'$  and*

$$\begin{aligned} \{a \in A: f_a = f_a^-\} &= \{a \in A: f'_a = f_a^-\}, \\ \{a \in A: f_a = f_a^+\} &= \{a \in A: f'_a = f_a^+\}, \\ \{u \in N: y_u = 0\} &= \{u \in N: y'_u = 0\}, \\ \{u \in N: y_u = \bar{y}_u\} &= \{u \in N: y'_u = \bar{y}_u\}. \end{aligned}$$

**Theorem 3.3.** *Suppose Assumptions 1, 2, and 3 hold. Then, the productions  $y_u$ ,  $u \in N$ , in a solution of Problem (7) are unique.*

*Proof.* Assume that the productions  $y_u$ ,  $u \in N$ , in a solution of Problem (7) are not unique. Due to Theorem 2.2 the demands are unique. Let  $z = (d, y, f)$  and  $z' = (d, y', f')$  be two solutions satisfying part (b) of Lemma 3.2. This means that the flow-induced partitions  $\{G^i := (N^i, A^i)\}_{i \in I}$  of  $z$  and  $z'$  are the same. Summing up the flow-balance conditions for all nodes  $u \in N^i$  yields

$$\sum_{u \in N^i} y_u = \sum_{u \in N^i} y'_u \quad (8)$$

for all  $i \in I$ . By Theorem 3.1, there exists at most one node  $u \in N^i$  where both production bounds are strict, i.e.,  $0 < y_u < \bar{y}_u$ . As  $z$  and  $z'$  have the same binding production structure,  $z$  and  $z'$  have the same productions in each flow-induced component due to (8). Thus,  $y = y'$  holds. This is a contradiction to the choice of  $y$  and  $y'$ .  $\square$

We now briefly illustrate the necessity of Assumption 3 in Theorem 3.3. To this end, we consider a two-node network with  $N = \{1, 2\}$  and  $A = \{(1, 2)\}$  as depicted in Figure 1, where all parameters of production, demand, and transport are given. Note that  $w_1 - w_2 + m_{12} = 0$  holds and, thus, Assumption 3 is violated. Two solutions of Problem (7) are given by  $(d_1, d_2, y_1, y_2, f_{12}) = (9, 11.5, 20.32, 0.18, 11.32)$  and  $(d_1, d_2, y_1, y_2, f_{12}) = (9, 11.5, 19, 1.5, 10)$  and we obtain the equilibrium prices  $\pi_1 = 1$  and  $\pi_2 = 2$ .

Up to now, we derived conditions under which the solution of Problem (7) is unique in the demands and productions if the transport costs are piecewise linear

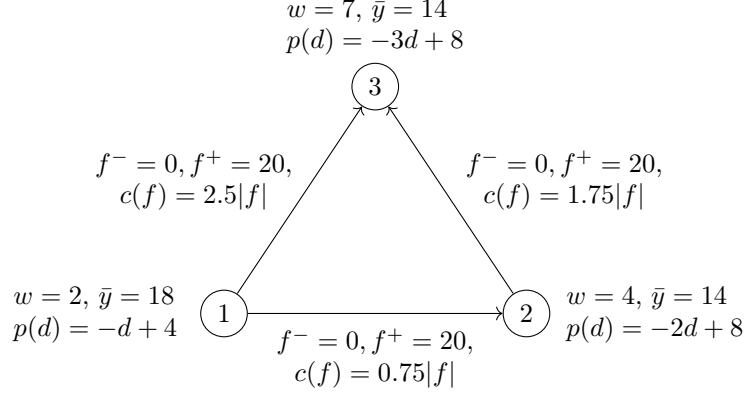


FIGURE 2. Three-node cycle with multiple solutions.

and symmetric. It remains to prove the uniqueness of the flows. In the following we first consider uniqueness of flows for tree networks.

**Theorem 3.4.** *Suppose Assumptions 1, 2, and 3 hold and let the network  $G$  be a tree. Then, the solution of Problem (7) is unique.*

*Proof.* Due to Theorem 2.2 and 3.3 the solution of Problem (7) has unique demands  $d_u$ ,  $u \in N$ , and productions  $y_u$ ,  $u \in N$ . Let  $N(u) := \{v \in N : (u, v) \in A \vee (v, u) \in A\}$  be the neighborhood of a node  $u \in N$  and let  $L := \{u \in N : |N(u)| = 1\}$  be the set of leaf nodes. Without loss of generality, we assume that  $G$  is an out-tree. Then, we can compute the unique flow as follows. As long as we have arcs in our graph we do the following: For every leaf node  $u \in L$  and unique arc  $a \in \delta^{\text{in}}(u)$  we set  $f_a \leftarrow d_u - y_u$  and then update  $y_v \leftarrow y_v + f_a$  for the unique neighbor  $v \in N(u)$ . We then set  $N \leftarrow N \setminus L$ ,  $A \leftarrow A \setminus \bigcup_{u \in L} \delta^{\text{in}}(u)$ , and iterate. Obviously, this procedure computes a unique flow and terminates after  $|A|$  iterations.  $\square$

We now illustrate that the conditions that are sufficient for uniqueness in the tree case are no longer sufficient if the network contains cycles. To this end, we consider a cycle with three nodes as depicted in Figure 2. In this situation all assumptions of Theorem 3.4 are satisfied but we obtain multiple solutions of Problem (7). Two solutions are given by common demands and productions  $(d_1, d_2, d_3, y_1, y_2, y_3) = (2, 2.63, 1.17, 5.79, 0, 0)$  and different flows  $(f_{12}, f_{13}, f_{23}) = (3.21, 0.58, 0.58)$  as well as  $(f_{12}, f_{13}, f_{23}) = (2.96, 0.83, 0.34)$ . Here we obtain the equilibrium prices  $\pi_1 = 2$ ,  $\pi_2 = 2.74$ , and  $\pi_3 = 4.49$ . This example indicates that we need additional assumptions to obtain uniqueness of Problem (7) for non-tree networks.

In the rest of this section we consider general networks. The previous Theorems 2.2 and 3.3 together state that we only have to consider the flows of a solution of Problem (7) to establish uniqueness of the overall solution. Thus, we can write Problem (7) as the following min-cost flow problem for given net supplies  $y_u - d_u$  for all  $u \in N$ :

$$\min_f \sum_{a \in A} m_a |f_a| \quad (9a)$$

$$\text{s.t. } f_a^- \leq f_a \leq f_a^+, \quad a \in A, \quad (9b)$$

$$y_u - d_u = \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a, \quad u \in N. \quad (9c)$$

Solving this problem then yields optimal flows for given productions and demands. We now transform the arc set of the network such that all arc capacities are non-negative and all transportation costs are linear. To this end, we make the following case analysis for each arc  $a = (u, v) \in A$  with transport costs  $c_a(f_a) = m_a|f_a|$  and use the notation  $a^{-1} := (v, u)$  for the corresponding backward arc.

- (i) If  $0 \leq f_a^- \leq f_a^+$  holds, the transport costs are given by  $c_a(f_a) = m_a f_a$ . We denote the corresponding arc set with  $A'_1 := \{a \in A : 0 \leq f_a^- \leq f_a^+\}$ .
- (ii) If  $f_a^- \leq f_a^+ \leq 0$ , we replace  $a$  by  $a^{-1}$  with transport costs  $c_{a^{-1}}(f_{a^{-1}}) = m_a f_{a^{-1}}$  and capacities  $f_{a^{-1}} \in [ |f_a^+|, |f_a^-| ]$ . We denote the corresponding arc set with  $A'_2 := \{a \in A : f_a^- \leq f_a^+ \leq 0\}$ .
- (iii) If  $f_a^- \leq 0 \leq f_a^+$ , we use transport costs  $c_a(f_a) = m_a f_a$ , capacities  $f_a \in [0, f_a^+]$ , and we add the backward arc  $a^{-1}$  with transport costs  $c_{a^{-1}}(f_{a^{-1}}) = m_a f_{a^{-1}}$  and capacities  $f_{a^{-1}} \in [0, |f_a^-|]$ . We denote the corresponding arc set with  $A'_3 := \{a \in A : f_a^- \leq 0 \leq f_a^+\}$ .

Thus, Problem (9) is equivalent to the min-cost flow problem

$$\min_f \sum_{a \in A'} m_a f_a \quad (10a)$$

$$\text{s.t. } f_a^- \leq f_a \leq f_a^+, \quad a \in A'_1, \quad (10b)$$

$$|f_a^+| \leq f_{a^{-1}} \leq |f_a^-|, \quad a \in A'_2, \quad (10c)$$

$$0 \leq f_a \leq f_a^+, \quad 0 \leq f_{a^{-1}} \leq |f_a^-|, \quad a \in A'_3, \quad (10d)$$

$$y_u - d_u = \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a, \quad u \in N \quad (10e)$$

with non-negative flows on the graph  $G' := (N, A')$  with arc set  $A'$ , which we obtain by the network transformation (i)–(iii). Hence, we can restrict ourselves to consider uniqueness of solutions of Problem (10). To obtain unique flows in this setting we need the following assumption.

**Assumption 4.** *Let  $G = (N, A)$  be an arbitrary network. Then*

$$\sum_{a \in A_C} r_a m_a \neq 0$$

*holds for all cycles  $C = (N_C, A_C)$  of  $G$ .*

**Theorem 3.5.** *Suppose Assumption 4 holds. Then, the solution of Problem (10) is unique.*

*Proof.* Assume  $f \neq f'$  are two solutions of Problem (10). Then, the flow difference  $f - f'$  is a circulation because at each node  $u \in A'$  the flow conservation reads

$$\begin{aligned} & \sum_{a \in \delta^{\text{out}}(u)} (f_a - f'_a) - \sum_{a \in \delta^{\text{in}}(u)} (f_a - f'_a) \\ &= \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a - \left( \sum_{a \in \delta^{\text{out}}(u)} f'_a - \sum_{a \in \delta^{\text{in}}(u)} f'_a \right) = 0. \end{aligned}$$

Thus,  $f - f'$  decomposes into flows on a set  $Z$  of cycles and there exists a cycle  $C \in Z$  with

$$\Delta_C := r_a (f_a - f'_a) \neq 0 \quad (11)$$

for all arcs  $a \in C$ . We next show that  $Z = \emptyset$ , which yields a contradiction to  $f \neq f'$ . As  $f$  and  $f'$  are both solutions of Problem (10),

$$0 = \sum_{a \in A'} m_a (f_a - f'_a) = \sum_{C \in Z} \Delta_C \sum_{a \in C} r_a m_a$$



follows from the objective function difference. Assume that there exists a cycle  $\tilde{C} \in Z$  with  $\Delta_{\tilde{C}} \sum_{a \in \tilde{C}} r_a m_a \neq 0$  and, w.l.o.g.,

$$\Delta_{\tilde{C}} \sum_{a \in \tilde{C}} r_a m_a > 0. \quad (12)$$

Then, we define a new flow  $\tilde{f}$  by

$$\tilde{f}_a := \begin{cases} f_a, & \text{if } a \notin \tilde{C}, \\ f_a - r_a \Delta_{\tilde{C}}, & \text{if } a \in \tilde{C}, \end{cases}$$

for all  $a \in A'$ . Using (11) and the definition of  $r_a$  we have  $\tilde{f}_a = f'_a$  for all arcs  $a \in \tilde{C}$ . Hence,  $\tilde{f}$  satisfies the flow bounds (10b)–(10d) because both  $f$  and  $f'$  are feasible. We next show that  $\tilde{f}$  satisfies the flow conservation constraints (10e). For all nodes  $u \notin \tilde{C}$  we have

$$\sum_{a \in \delta^{\text{out}}(u)} \tilde{f}_a - \sum_{a \in \delta^{\text{in}}(u)} \tilde{f}_a = \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a = y_u - d_u.$$

At all nodes  $u \in \tilde{C}$  there exists either (i) exactly one arc  $a^{\text{in}} \in \delta^{\text{in}}(u) \cap \tilde{C}$ , exactly one arc  $a^{\text{out}} \in \delta^{\text{out}}(u) \cap \tilde{C}$ , and it is  $r_{a^{\text{in}}} = r_{a^{\text{out}}}$ , (ii)  $a_1^{\text{in}} \neq a_2^{\text{in}} \in \delta^{\text{in}}(u) \cap \tilde{C}$ ,  $\delta^{\text{out}}(u) \cap \tilde{C} = \emptyset$ , and  $r_{a_1^{\text{in}}} = -r_{a_2^{\text{in}}}$ , or (iii)  $a_1^{\text{out}} \neq a_2^{\text{out}} \in \delta^{\text{out}}(u) \cap \tilde{C}$ ,  $\delta^{\text{in}}(u) \cap \tilde{C} = \emptyset$ , and  $r_{a_1^{\text{out}}} = -r_{a_2^{\text{out}}}$ . Hence, the flow  $\tilde{f}$  satisfies the flow conservation at each node  $u \in \tilde{C}$  due to

$$\begin{aligned} & \sum_{a \in \delta^{\text{out}}(u)} \tilde{f}_a - \sum_{a \in \delta^{\text{in}}(u)} \tilde{f}_a \\ &= \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a \\ &+ \begin{cases} -(r_{a^{\text{in}}} - r_{a^{\text{out}}}) \Delta_{\tilde{C}} & \text{if } \delta^{\text{in}}(u) \cap \tilde{C} \neq \emptyset, \delta^{\text{out}}(u) \cap \tilde{C} \neq \emptyset, \\ -(r_{a_1^{\text{in}}} + r_{a_2^{\text{in}}}) \Delta_{\tilde{C}} & \text{if } \delta^{\text{in}}(u) \cap \tilde{C} \neq \emptyset, \delta^{\text{out}}(u) \cap \tilde{C} = \emptyset, \\ (r_{a_1^{\text{out}}} + r_{a_2^{\text{out}}}) \Delta_{\tilde{C}} & \text{if } \delta^{\text{in}}(u) \cap \tilde{C} = \emptyset, \delta^{\text{out}}(u) \cap \tilde{C} \neq \emptyset \end{cases} \\ &= y_u - d_u. \end{aligned}$$

Thus,  $\tilde{f}$  is feasible for Problem (10) and its objective function value is given by

$$\sum_{a \in A'} m_a \tilde{f}_a = \sum_{a \in A'} m_a f_a - \Delta_{\tilde{C}} \sum_{a \in \tilde{C}} r_a m_a.$$

Using (12), we obtain an upper bound

$$\sum_{a \in A'} m_a f_a - \Delta_{\tilde{C}} \sum_{a \in \tilde{C}} r_a m_a < \sum_{a \in A'} m_a f_a.$$

The latter is a contradiction to the optimality of  $f$ . So, it implies  $\Delta_C \sum_{a \in C} r_a m_a = 0$  for all cycles  $C \in Z$  and as Assumption 4 holds, we have  $\Delta_C = 0$  for all  $C \in Z$  and thus  $Z = \emptyset$ .  $\square$

Combining Theorem 2.2, 3.3, and 3.5, we obtain the final uniqueness theorem.

**Theorem 3.6.** *Suppose Assumptions 1, 2, 3, and 4 hold. Then, the solution of Problem (7) is unique.*

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