

Decomposable Robust Two-Stage Optimization: An Application to Gas Network Operations Under Uncertainty

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We study gas network problems with compressors and control valves under uncertainty that can be formulated as two-stage robust optimization problems. Uncertain data are present in the physical parameters of the pipes as well as in the overall demand. We show how to exploit the special decomposable structure of the problem in order to reformulate the two-stage robust problem as a standard single-stage optimization problem. Since this structure is present in similar problems on e.g., water or direct current electricity networks, we investigate the consequences of the decomposable structure in an abstract setting: The right-hand side of the single-stage problem can be precomputed by solving a series of optimization problems and multiple elements of the right-hand side can be combined into one optimization task. In order to apply our results to gas network problems, we extend piecewise relaxations and preprocessing techniques to incorporate uncertain input data. The practical feasibility and effectiveness of our approach is demonstrated with benchmarks on realistic gas network instances. We observe large speedups due to the described aggregation method together with the developed preprocessing strategies. Furthermore, we are able to solve even comparably large gas network instances quickly for the price of slightly more conservative solutions.

1 Introduction

In this paper, we study a class of two-stage robust network optimization problems with a special decomposable structure. The aim of robust optimization is to solve optimization tasks that are parameterized by a so-called uncertainty set \mathcal{U} . In the easiest setting, solving a robust problem amounts to specifying a solution for the problem variables

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that is feasible for all possible realizations of the uncertainty within \mathcal{U} . Potentially, this approach can lead to too conservative or even infeasible models, for example for applications where part of the variables can adapt to the realization of the uncertainty. In this setting, a two-stage approach should be used instead. For the two-stage model, the problem variables are classified as *here-and-now* variables that have to be decided before the uncertainty is realized and *wait-and-see* or *adjustable* variables whose values can be chosen after the uncertainty is realized.

A two-stage approach is natural for gas network optimization tasks with remotely controllable elements (“active elements”). Uncertainties are present in the physical parameters or in the demand situation. The network operator has to decide the configuration of the controllable network elements before knowing the realization of the uncertainty, whereas the physical state, i.e., pressures and flows within the network, adjusts itself once the uncertainty is revealed.

This two-stage model has two special properties: first, it is known that the physical state of the network is uniquely determined by the uncertain parameters, and second, there is no coupling between first-stage and second-stage variables. In this paper we show how this structure can be exploited in order to reformulate the two-stage problem as a single-stage problem whose right-hand side comprises the optimal values of a series of subproblems arising from this transformation. By further exploiting the problem structure, we show how the number of subproblems to solve can be reduced. We also show that solving the subproblems to global optimality is not absolutely necessary as relaxations can be used instead to obtain a more conservative—but still robust feasible—solution. Using relaxations instead of the original model allows us to apply the presented methods to problems where obtaining global optimal solutions can be very challenging, e.g., non-convex problems. Furthermore, we show how relaxations can be used to tackle even large instances for the price of more conservative solutions. Our experiments indicate a very modest decrease of optimality due to the employed relaxations.

In order to apply these ideas to the nonlinear, non-convex gas network problem, we extend the well-known piecewise linearization technique for the nonlinear pressure drop constraints to incorporate uncertain parameters. The chosen linearization approach is particularly effective for network problems since the nonlinear constraints of these kind of problems often involve univariate functions of e.g., the flow along an arc or the potential at a node. As our approach requires solving a potentially large number of subproblems where each one is very similar to the nominal problem, it is crucial to reduce the problem size as much as possible for the overall solution time to stay within an acceptable time frame. To this end, we describe a variety of preprocessing techniques which are adapted from the literature and generalized to handle uncertain parameters.

For an introduction to robust optimization we refer the reader to the book [3] or the review articles [6, 15]. Our goal is to solve a two-stage nonlinear robust optimization task.

A typical solution approach for solving multi-stage robust problems is to reformulate the wait-and-see variables with decision rules of a predefined class. This introduces a predefined structure on the adaptability of the wait-and-see variables, for example an affine linear dependency of the variables on the uncertain parameters [4]. Due to the

a priori selected structure of the decision rules, this solution approach often produces more conservative solutions when compared to analytic solutions without the restriction to a given function class. As will be explained later, the wait-and-see variables of the gas network problem are uniquely dependent on the uncertainty and therefore only an “optimal” or *fully-adjustable* decision rule could be used for the problem at hand. While there are some simple problem classes where “optimal” decisions rules are known [8], no such result could be found for problems with the nonlinear and random recourse structure of the gas network problem. Reformulations for nonlinear but convex robust optimization tasks [2] are also not applicable since the gas network problem is modeled using non-convex constraints.

A different solution idea for robust (two-stage) problems is the so-called *adversarial approach*, where the continuous uncertainty set is replaced by a finite scenario set which is gradually expanded until robust feasibility of the whole problem can be proven [7, 15, 29]. Under special conditions, the critical scenarios of a two-stage problem can be identified readily which leads to a reformulation as a single-stage problem [27]. In general, the adversarial approach requires solving subproblems to global optimality, a difficult task for the non-convex gas network problem. Our solution method is similar to an adversarial approach in the sense that we also identify critical scenarios by solving a series of mixed-integer linear (MIP) relaxations.

An overview of the challenges arising in gas network operations is given in [23]. The studied stationary gas network problem is known as the *nomination validation problem*. A comprehensive treatment of this and similar problems can be found in the book [18] and in references therein. In previous works concerning the nominal case, a piecewise-linear approximation approach proved to be very effective for this type of problem. The main idea of this method is to replace the univariate nonlinear pressure drop constraints with piecewise-linear relaxations [13, 14, 21]. By casting these models as MIPs, even very large problems can be solved with a small loss of accuracy that arises due to the piecewise-linear relaxations. We extend this concept to similar constraints found in gas network problems under uncertainty.

This work is structured as follows: In Section 2, we introduce the gas network problem and show how reduction techniques for passive networks can be extended to networks with linear compressor models. In Section 3, uncertain parameters are introduced to the gas network problem and the two-stage robust optimization task is formulated. We first describe two structural properties of the gas network problem. On an abstract level, we show how all two-stage problems with this structure can be transformed to a single-stage problem. We also present further simplifications of the obtained single-stage problem. In Section 4, we extend previously known relaxations of the nonlinear constraints found in gas network operations to the uncertain setting. Section 4.1 presents several generalizations of known preprocessing techniques to incorporate uncertain parameters like demand and pressure drop coefficients. In Section 5, the presented methods are benchmarked on a variety of freely available large scale gas network instances to show their practical feasibility for nonlinear two-stage robust optimization problems. This work closes with a summary in Section 6.

2 Stationary Gas Network Operations with Linear Compressor Models

In this section, we introduce the gas network problem with linear compressor models and show how previously known reduction techniques for passive networks can be extended to this setting.

2.1 The Stationary Gas Network Transport Problem with a Linear Compressor Model

For a comprehensive treatment of stationary gas network problems, we refer to the book [18] and references therein. A gas network comprises a number of different components. Apart from pipes which allow gas transport, there are compressors and control valves which increase and decrease the pressure, valves which can prohibit flow along an arc, and resistors which can hinder the gas flow and can decrease pressure. In this article, we only consider networks with pipes and simple compressor models. Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a digraph with $|\mathcal{V}| = |\{1, \dots, n\}| = n$ nodes and $|\mathcal{A}| = m \geq n$ arcs. Without loss of generality, we assume \mathcal{G} to be weakly connected, i.e., the corresponding undirected graph is connected. If \mathcal{G} is not weakly connected, all presented results can be applied to each connected component separately.

Gas can be inserted or withdrawn at the graph's nodes. Let $d_v \in \mathbb{R}$ denote the demand at node $v \in \mathcal{V}$ where inflowing gas is indicated by a negative sign and outflowing gas by a positive sign. In gas network operations, the vector $d \in \mathbb{R}^{|\mathcal{V}|}$ is also called the network's *nomination*. As for any network flow problem, this vector has to be balanced: $\sum_{v \in \mathcal{V}} d_v = 0$.

Let $q_a \in \mathbb{R}$ denote the gas flow over arc $a \in \mathcal{A}$. Flow in arc direction is indicated by a positive sign of q_a , whereas flow in reverse arc direction is indicated by a negative sign of q_a . Just like linear network flow problems, the flow is conserved at each node:

$$\sum_{a=(v,w) \in \mathcal{A}} q_a - \sum_{a=(w,v) \in \mathcal{A}} q_a = d_v \quad \text{for all } v \in \mathcal{V}. \quad (1)$$

Furthermore, we introduce variables $p_v \in \mathbb{R}_{\geq 0}$ to model the pressure at node $v \in \mathcal{V}$. Since pressure variables always appear in squared form in our setting, we define variables $\pi_v = p_v^2$ modeling the squared pressure.

Let $\mathcal{A} = \mathcal{A}_{\text{pi}} \cup \mathcal{A}_{\text{cs}}$ with $\mathcal{A}_{\text{pi}} \cap \mathcal{A}_{\text{cs}} = \emptyset$ be a partition of the network's arcs into a set \mathcal{A}_{pi} of pipes and a set \mathcal{A}_{cs} of compressors.

Gas traveling along a pipe experiences a pressure drop (see [19] for a derivation):

$$\pi_w - \pi_v = -\phi_a q_a |q_a| \quad \text{for all } (v, w) = a \in \mathcal{A}_{\text{pi}}. \quad (2)$$

The magnitude of this pressure loss depends on the amount q_a of gas and the *pressure drop coefficient* $\phi_a \in \mathbb{R}_{>0}$. The factor ϕ_a is computed from a number of physical properties of the pipe and the quality of the gas mixture, see [18, 21] for an in-depth definition. Since many of these parameters can be affected by uncertainty or can only be measured with great effort, ϕ_a lends itself for a robust treatment.

Compressors can be used to increase the pressure of the gas. This is needed e.g., when gas is inserted into a higher pressure network, or in order to compensate for pressure loss when gas is transported over a long distance. There are several compressor models available, ranging from very simple to highly complex [10, 18, 25, 28]. We use a simple linear compressor model where the increase in pressure is independent of the flow through the compressor. The squared pressure difference may be increased linearly by some nonnegative amount $\Delta_a \in [\underline{\Delta}_a, \overline{\Delta}_a] \subseteq \mathbb{R}_{\geq 0}$:

$$\pi_w - \pi_v = \Delta_a \quad \text{for all } (v, w) = a \in \mathcal{A}_{\text{cs}}. \quad (3)$$

We remark that in general compressors can only increase pressure if the passing gas flows through them in the correct direction. As is shown later on in Section 4.1, tight flow bounds for our setting can be calculated easily with (39). The obtained bounds can be used to verify the direction constraint.

Let $\Delta \in \mathbb{R}_{\geq 0}^{|\mathcal{A}_{\text{cs}}|}$ be the vector of all compressor “power levels” within the network. The optimization task consists in minimizing the cost of compressor operations (e.g., due to fuel consumption). To this end, we define a linear cost function

$$\Delta \mapsto w^T \Delta \quad (4)$$

where $w \in \mathbb{R}_{\geq 0}^{|\mathcal{A}_{\text{cs}}|}$ is a cost vector associated with the given compressors.

Let $\mathbf{A} \in \mathbb{R}^{n \times m}$ be the node-arc incidence matrix of \mathcal{G} , that is $(\mathbf{A})_{av} = -1$ and $(\mathbf{A})_{aw} = +1$ for $a = (v, w) \in \mathcal{A}$. With this matrix, the gas transport problem can be expressed in a very concise fashion. Flow conservation (1) can be written as

$$\mathbf{A}q = d. \quad (5)$$

Let $F: \mathbb{R}^{|\mathcal{A}|} \times \mathbb{R}^{|\mathcal{A}_{\text{cs}}|} \rightarrow \mathbb{R}$ be an aggregation of pipe (2) and compressor (3) models:

$$F_a(q, \Delta) = \begin{cases} -\phi_a q_a |q_a|, & \text{if } a \in \mathcal{A}_{\text{pi}}, \\ \Delta_a, & \text{if } a \in \mathcal{A}_{\text{cs}}. \end{cases}$$

Hence, the constraint for the network component on arc $a = (v, w) \in \mathcal{A} = \mathcal{A}_{\text{pi}} \cup \mathcal{A}_{\text{cs}}$ is given by

$$\pi_w - \pi_v = F_a(q, \Delta). \quad (6)$$

Using the node-arc incidence matrix \mathbf{A} , constraint (6) for all arcs amounts to

$$\mathbf{A}^T \pi = F(q, \Delta). \quad (7)$$

Combining flow conservation (5), pressure constraints (7), linear compressor model (3),

and objective function (4) yields the following minimum cost gas transport problem:

$$\begin{aligned} \min w^T \Delta \\ \mathbf{A}q = d, \end{aligned} \tag{8a}$$

$$\mathbf{A}^T \pi = F(q, \Delta), \tag{8b}$$

$$\Delta \in [\underline{\Delta}, \overline{\Delta}] \subseteq \mathbb{R}_{\geq 0}^{|\mathcal{A}_{cs}|} \tag{8c}$$

$$\pi \in [\underline{\pi}, \overline{\pi}] \subseteq \mathbb{R}_{\geq 0}^{|\mathcal{V}|}, \tag{8d}$$

$$q \in \mathbb{R}^{|\mathcal{A}|}. \tag{8e}$$

2.2 A Reformulation of the Gas Transport Problem

We next present an equivalent formulation of problem (8) where a number of variables are eliminated. This is an extension of a result in [16] for passive networks to active gas networks with a linear compressor model. For our result, we use the following assumption:

Assumption 1. *Let a gas network problem with compressors over graph \mathcal{G} be given. Then no compressor is part of a cycle in the undirected counterpart of \mathcal{G} .*

In [16], the authors show that all squared pressure variables and $|\mathcal{V}| - 1$ flow variables can be eliminated from a gas network model without compressors. This is done by expressing the flows within the network as a combination of flows on a spanning tree together with flows on the remaining cycles. The pressure at each node is expressed relative to an arbitrary chosen root node $r \in \mathcal{V}$ by defining an aggregated pressure drop function between root node and each node in \mathcal{G} . Similar to Kirchhoff's loop rule in electrical circuits, another set of constraints forces the aggregated pressure drop on every fundamental cycle to be zero. The next paragraph introduces the required concepts from [16] used in the reformulation.

Since we assume \mathcal{G} to be weakly connected, its node-arc incidence matrix \mathbf{A} has rank $n - 1$ and an arbitrary row can be removed. We choose an arbitrary root node $r \in \mathcal{V}$ and discard the row corresponding to r . In the same fashion, a reduced demand vector \tilde{d} is obtained from demand vector d by removing the entry concerning the root node. The remaining matrix $\tilde{\mathbf{A}} \in \mathbb{R}^{(n-1) \times m}$ has full rank. After fixing a basis \mathcal{B} of $\tilde{\mathbf{A}}$, matrix $\tilde{\mathbf{A}}$ is partitioned into a basic submatrix $\tilde{\mathbf{A}}_{\mathcal{B}} \in \mathbb{R}^{(n-1) \times (n-1)}$ and a nonbasic submatrix $\tilde{\mathbf{A}}_{\mathcal{N}} \in \mathbb{R}^{(n-1) \times (m-n+1)}$. Moreover, let $(F_{\mathcal{B}}, F_{\mathcal{N}})$ and $(q_{\mathcal{B}}, q_{\mathcal{N}})$ be the respective partitions of F and q . In the reduced model only nonbasic flows $q_{\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$ remain. Since every nonbasic element corresponds to a fundamental cycle in \mathcal{G} , we call the remaining flow variables $q_{\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$ *cycle flows*.

It is well known from linear algebra that the solution space of the linear equation system $\tilde{\mathbf{A}}q = \tilde{d}$ can be parameterized by the nonbasic variables $q_{\mathcal{N}}$. We define a function $q(\cdot)$ that maps nonbasic flows to flows in the whole network:

$$\begin{aligned} q: \mathbb{R}^{|\mathcal{N}|} &\rightarrow \mathbb{R}^{|\mathcal{A}|}, \\ q_a(q_{\mathcal{N}}) &= \begin{cases} \left(\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} (\tilde{d} - \tilde{\mathbf{A}}_{\mathcal{N}} q_{\mathcal{N}}) \right)_a, & \text{if } a \in \mathcal{B}, \\ (q_{\mathcal{N}})_a, & \text{if } a \in \mathcal{N}. \end{cases} \end{aligned} \tag{9}$$

Next we define an aggregated pressure loss function:

$$\tilde{g}(q, \Delta) = \left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)^{-1} F_{\mathcal{B}}(q, \Delta). \quad (10)$$

We notice that each entry F_a depends either on q_a or on Δ_a . Thus, the arguments of $F_{\mathcal{B}}$ can be restricted to their basis parts $q_{\mathcal{B}}$ and $\Delta_{\mathcal{B}}$. Next, with (9) we rewrite (10) such that \tilde{g} only depends on the nonbasic part $q_{\mathcal{N}}$ of q :

$$\tilde{g}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) = \left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)^{-1} F_{\mathcal{B}}(q_{\mathcal{B}}(q_{\mathcal{N}}), \Delta_{\mathcal{B}}). \quad (11)$$

We extend \tilde{g} to incorporate the root node $r \in \mathcal{V}$ and obtain the final definition of the aggregated pressure drop function g :

$$g: \mathbb{R}^{|\mathcal{N}|} \times \mathbb{R}_{\geq 0}^{|\mathcal{B} \cap \mathcal{A}_{\text{cs}}|} \rightarrow \mathbb{R}^{|\mathcal{V}|},$$

$$g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) = \begin{cases} 0, & \text{if } v = r, \\ \tilde{g}_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}}), & \text{if } v \neq r, \end{cases} \quad \text{for all } v \in \mathcal{V}. \quad (12)$$

By construction, $g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}})$ is the sum of all pressure loss values on the unique path between root node $r \in \mathcal{V}$ and node $v \in \mathcal{V}$ on the spanning tree as induced by \mathcal{B} . This includes the pressure drops along the pipes as well as the pressure changes due to compressors.

Elimination of Variables and Equations The next theorem establishes an equivalent formulation of the gas network problem (8). This result was proven for pipe-only networks without active elements in [16]. We observe next that the result of [16] can easily be generalized to networks with compressors, providing Assumption 1 is satisfied. With $\mathcal{A}_{\text{cs}} \cap \mathcal{N} = \emptyset$, it is $\Delta \equiv \Delta_{\mathcal{B}}$ and thus we can remove any $\Delta_{\mathcal{N}}$ from the model.

Theorem 2.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ be a weakly connected digraph with nodes \mathcal{V} and arcs \mathcal{A} . Let set $\mathcal{A} = \mathcal{A}_{\text{pi}} \cup \mathcal{A}_{\text{cs}}$ of arcs be partitioned into a set \mathcal{A}_{pi} of pipes and a set \mathcal{A}_{cs} of compressors. Let $\tilde{\mathbf{A}}$ be the node-arc incidence matrix of \mathcal{G} after removing the row corresponding to the root node, with an arbitrary partition $(\tilde{\mathbf{A}}_{\mathcal{B}}, \tilde{\mathbf{A}}_{\mathcal{N}})$ into basis and nonbasis as described above. Let $(F_{\mathcal{B}}, F_{\mathcal{N}})$ and $(q_{\mathcal{B}}, q_{\mathcal{N}})$ be the corresponding partitions of F and q , respectively. Let g be the aggregate pressure drop function as defined in (12).*

Then (8a)–(8e) has a feasible solution if and only if the following reduced system in variables $q_{\mathcal{N}}, \Delta$ has a solution:

$$\tilde{\mathbf{A}}_{\mathcal{N}}^T \tilde{g}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) = F_{\mathcal{N}}(q_{\mathcal{N}}, \Delta_{\mathcal{N}}), \quad (13)$$

$$g_w(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) - g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) \leq \bar{\pi}_v - \underline{\pi}_w \quad \text{for all } v, w \in \mathcal{V}, \quad (14)$$

$$q_{\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}, \quad (15)$$

$$\Delta \in \mathbb{R}^{|\mathcal{A}_{\text{cs}}|}.$$

Moreover, any solution of the reduced system can be expanded to a solution of the original system. If $q_{\mathcal{N}}$ is feasible for (13) and (14) exists, the remaining original variables $q_{\mathcal{B}}, \pi$

can be recovered through $q_{\mathcal{B}} = \tilde{\mathbf{A}}_{\mathcal{B}}^{-1} (\tilde{d} - \tilde{\mathbf{A}}_{\mathcal{N}} q_{\mathcal{N}})$ and $\pi_v = \pi_r - g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}})$ for $v \in \mathcal{V}$. The value of π_r is an arbitrary given element of

$$\left[\max_{v \in \mathcal{V}} [\underline{\pi}_v + g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}})], \min_{v \in \mathcal{V}} [\bar{\pi}_v + g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}})] \right].$$

Conversely, a vector $q_{\mathcal{N}}$ that was extracted from a solution q^*, π^* of (8a)–(8e) is feasible for (13)–(15).

Proof. The original result of [16] is established for pipe-only networks, i.e., for $\mathcal{A}_{\text{cs}} = \emptyset$. However, the form of the pressure drop law is never exploited explicitly. Therefore, it is not difficult to see that their result still holds for networks with compressors or more general constraints of the form

$$\pi_w - \pi_v = \alpha_a(q_a) \quad \text{for all } (v, w) = a \in \mathcal{A},$$

where α_a is some scalar-valued function of the flow q_a . □

Existence and Uniqueness of Flow We consider a connected gas network without pressure bounds and without compressors. In this setting a feasible flow always exists, see [9, 24].

Theorem 2.2 ([9]). *Consider a connected gas network without pressure bounds and compressors. Then the set*

$$\mathcal{X} = \{(q, \pi) \mid (q, \pi) \text{ satisfy (8a)–(8c) and (8e)}\}$$

of feasible solutions has the following properties:

1. A flow solution always exists and is unique, i.e., $|\{q \mid \exists \pi: (q, \pi) \in \mathcal{X}\}| = 1$.
2. Given a flow solution q^* , the set \mathcal{X} of feasible solutions has the form

$$\mathcal{X} = \{(q^*, \pi) \mid \pi = \pi^* + \eta(1, \dots, 1)^T, \eta \in \mathbb{R}\}.$$

A feasible π^* can be computed by first fixing the pressure π_r^* at the root node to an arbitrary value. The remaining squared pressures can be computed via $\pi_v = \pi_r - g_v(q_{\mathcal{N}}^*)$ for all $v \in \mathcal{V}$.

Theorem 2.3. *Consider a gas network where the compressor power vector Δ^* is fixed. Let Assumption 1 be fulfilled, i.e., no compressors are part of a cycle. Then Theorem 2.2 still holds.*

Proof. We proof both parts of Theorem 2.2 separately.

1. The solution space of the linear flow problem (8a) can be parameterized by the nonbasic flows $q_{\mathcal{N}}$, see (9). A nonbasic flow $q_{\mathcal{N}}^*$ is feasible for the gas network problem without pressure bounds if it satisfies equation system (13):

$$\tilde{\mathbf{A}}_{\mathcal{N}}^T \tilde{g}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}^*) = F_{\mathcal{N}}(q_{\mathcal{N}}, \Delta_{\mathcal{N}}^*). \quad (16)$$

Due to Assumption 1, all terms in (16) related to Δ^* cancel out since no compressor is part of a cycle. This can be seen as follows:

For ease of explanation, let us assume that \mathcal{G} contains a single fundamental cycle and thus the set of nonbasic arcs \mathcal{N} contains only a single arc $a = (v, w) \in \mathcal{N}$. Our reasoning can be generalized to graphs with multiple fundamental cycles by considering each cycle separately. Plugging the definition (11) of \tilde{g} into the left-hand side of (13), we obtain

$$\tilde{\mathbf{A}}_{\mathcal{N}}^T \tilde{g}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}^*) = \tilde{\mathbf{A}}_{\mathcal{N}}^T \left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)^{-1} F_{\mathcal{B}}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}^*) = \left(\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{\mathbf{A}}_{\mathcal{N}} \right)^T F_{\mathcal{B}}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}^*).$$

The matrix product $\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{\mathbf{A}}_{\mathcal{N}}$ can be interpreted as the solution of linear network flow problem on the tree induced by the basis \mathcal{B} with demands given by the column vector $\tilde{\mathbf{A}}_{\mathcal{N}}$. We recall that $\tilde{\mathbf{A}}_{\mathcal{N}}$ has entries $(\tilde{\mathbf{A}}_{\mathcal{N}})_v = -1$, $(\tilde{\mathbf{A}}_{\mathcal{N}})_w = +1$ and zero entries everywhere else. Thus $\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{\mathbf{A}}_{\mathcal{N}}$ describes a linear network flow solution where one unit is transported from v to w in the spanning tree as induced by \mathcal{B} . Since the spanning tree \mathcal{B} emerges from \mathcal{G} by removing the nonbasic arc $a = (v, w) \in \mathcal{N}$ from the single fundamental cycle in \mathcal{G} , the linear network flow $\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{\mathbf{A}}_{\mathcal{N}}$ must take the unique path between v and w over the remaining arcs of the fundamental cycle. This shows that $(\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{\mathbf{A}}_{\mathcal{N}})^T F_{\mathcal{B}}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}^*)$ is a linear combination of the pressure drops on each arc of the fundamental cycle in \mathcal{B} . Together with the right-hand side of (16), we conclude that this equation system only contains pressure drops on arcs of fundamental cycles. Since no compressor is part of a cycle due to Assumption 1, this system is independent of Δ^* .

We apply Theorem 2.2 to system (16) to obtain a unique solution $q_{\mathcal{N}}^*$. The remaining flows on basis arcs $a \in \mathcal{B}$ can be computed using (9):

$$q_a(q_{\mathcal{N}}^*) = \left(\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \left(\tilde{d} - \tilde{\mathbf{A}}_{\mathcal{N}} q_{\mathcal{N}}^* \right) \right)_a \quad \text{for all } a \in \mathcal{B}.$$

2. After having fixed the squared pressure at the root node to an arbitrary value, the squared pressure at all other nodes $v \in \mathcal{V}$ can be computed with (11):

$$\pi_v = \pi_r - g_v(q_{\mathcal{N}}^*, \Delta_{\mathcal{B}}^*).$$

Since the compressor power Δ^* is fixed and is linked to the adjacent node's squared pressures via

$$\pi_w - \pi_v = \Delta_a^* \quad \text{for all } (v, w) = a \in \mathcal{A}_{\text{cs}},$$

any feasible π^* remains feasible after adding $\eta(1, \dots, 1)^T$, where $\eta \in \mathbb{R}$.

□

3 Robust Treatment of the Gas Transport Problem

A robust optimization task is a family of optimization problems that is parameterized by a so-called uncertainty set $\mathcal{U} \subseteq \mathbb{R}^{n_u}$, $n_u \in \mathbb{N}$:

$$\left\{ \min_{z \in \mathbb{R}^n} \{f(z) : g(z, u) = 0, h(z, u) \leq 0\} \right\}_{u \in \mathcal{U}}. \quad (17)$$

In addition to the problem variables $z \in \mathbb{R}^n$ with $n \in \mathbb{N}$, the constraint functions $g: \mathbb{R}^n \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{m_1}$ and $h: \mathbb{R}^n \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{m_2}$ for $m_1, m_2 \in \mathbb{N}_0$ accept an additional *data* vector $u \in \mathcal{U}$, where \mathcal{U} is a compact and convex set. Without loss of generality, we assume an objective function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ that is certain, see [4]. The goal of robust optimization is to find solutions for (17) which are immunized against all possible realizations $u \in \mathcal{U}$. In the most basic setting, all problem variables have to be fixed before the uncertainty becomes known. This leads to the *strict robust counterpart* (see [5])

$$\min_{z \in \mathbb{R}^n} \{f(z) \mid g(z, u) = 0, h(z, u) \leq 0 \quad \forall u \in \mathcal{U}\}, \quad (\text{RC})$$

where the feasible region is the set of all z which are feasible for all possible realizations of the uncertainty.

For some applications, strict robustness is the wrong modeling choice, e.g., when problem variables can adjust to the revealed uncertainty. In this case, a two-stage approach has to be used. Here, the problem variables are partitioned into *first-stage* and *second-stage* variables: $z = (x, y) \in \mathbb{R}^n = \mathbb{R}^{n_1+n_2}$ with $n_1 \in \mathbb{N}_0$, $n_2 \in \mathbb{N}$. First-stage or *here-and-now* variables $x \in \mathbb{R}^{n_1}$ have to be fixed before the uncertainty becomes known, whereas second-stage or *wait-and-see* variables $y \in \mathbb{R}^{n_2}$ can be decided with knowledge of the revealed uncertainty. Since the second-stage variables thus depend on the uncertain parameter, they are also called *adjustable variables*. The notion of different actions happening at different points in time directly leads to the *adjustable robust counterpart* (see [4])

$$\min_x \{f(x) \mid \exists x \in \mathbb{R}^{n_1} \quad \forall u \in \mathcal{U} \quad \exists y \in \mathbb{R}^{n_2} \text{ with } g(x, y, u) = 0, h(x, y, u) \leq 0\}. \quad (\text{ARC})$$

Again, without loss of generality, the objective only depends on here-and-now variables.

A typical solution approach in this setting consists in modeling the wait-and-see variables with decision rules [4]. The second-stage variable vector y is replaced by an unknown function $y: \mathcal{U} \rightarrow \mathbb{R}^{n_2}$ that maps elements of the uncertainty set to values of the second-stage variables. Finding the function $y(\cdot)$ is now part of the optimization task. In order to solve this problem in practice, the decision rule function $y(\cdot)$ has to be restricted to some function class that can be expressed as part of an optimization problem. Several classes of decision rules have been investigated in the literature, e.g., affine linear functions [4], piecewise linear [29], and polynomials [8]. The predefined function class is often not rich enough to contain the best possible or *fully adjustable* decision rule $y^*(\cdot)$. If this is the case, all obtained solutions are more conservative when compared to the fully adjustable solution as only suboptimal approximations of $y^*(\cdot)$ can be found. As will be shown later, the wait-and-see variables of the gas network problem introduced in Section 2 depend uniquely on the uncertain parameters, i.e., there is exactly one function $y(\cdot)$ that is feasible for the problem. Since the gas network problem is nonlinear with random recourse (see [4] for a definition), we don't expect this function to be contained in any of the studied decision rule classes and therefore, we cannot use a decision rule approach. This motivates the approach outlined in Section 3.3.

3.1 The Gas Network Problem Under Uncertainty

There are two main sources of uncertainty that are studied in this article: fluctuations in the demand vector d as well as uncertainties in the pressure drop coefficient ϕ . We define two polyhedral uncertainty sets, $\mathcal{P} \subseteq \mathbb{R}_{>0}^{|\mathcal{A}_{\text{pi}}|}$ for uncertain pressure drop coefficients, and $\mathcal{D} \subseteq \mathbb{R}^{|\mathcal{V}|}$ for uncertain demands. The efficiency of the presented methods is tied to the choice of uncertainty set as its definition is part of the developed optimization tasks. Therefore, other compact convex sets could be used instead, like e.g., sets defined by convex quadratic constraints.

When not explicitly stated otherwise, the considered problems are affected by both types of uncertainty. Thus, we typically use an uncertainty set of the form $\mathcal{U} = \mathcal{P} \times \mathcal{D}$.

Uncertain pressure loss coefficients This is an *arc-wise* uncertainty and influences the pressure drop between two incident nodes. For all arcs $a \in \mathcal{A}_{\text{pi}}$, let $\underline{\phi}_a, \bar{\phi}_a \in \mathbb{R}_{>0}$ with $0 < \underline{\phi}_a \leq \bar{\phi}_a$ be given. This leads to the uncertainty set

$$\mathcal{P} = \left\{ \phi \in \mathbb{R}_{>0}^{|\mathcal{A}_{\text{pi}}|} \mid \underline{\phi}_a \leq \phi_a \leq \bar{\phi}_a \quad \text{for all } a \in \mathcal{A}_{\text{pi}} \right\}, \quad (18)$$

which is also known as a *box uncertainty*.

Uncertain demand This *node-wise* uncertainty has an impact on the solution space of the linear network flow problem. For all nodes $v \in \mathcal{V}$, let $\underline{d}_v, \bar{d}_v \in \mathbb{R}$ with $\underline{d}_v \leq \bar{d}_v$ be given: As the overall demand always has to be balanced, the uncertainty set includes a balancing constraint.

$$\mathcal{D} = \left\{ d \in \mathbb{R}^{|\mathcal{V}|} \mid \begin{array}{l} \underline{d}_v \leq d_v \leq \bar{d}_v \quad \text{for all } v \in \mathcal{V} \\ \sum_{v \in \mathcal{V}} d_v = 0 \end{array} \right\}. \quad (19)$$

A polyhedral set of the form \mathcal{D} is also known as *hose polytope*, see [11].

3.2 The Reduced Gas Network Problem as a Two-Stage Problem

Robust treatment of the gas network problem (8) requires a two-stage model. We assume for a moment that single-stage (also known as *strict*) robustness is applied to the problem at hand. In the single-stage setting, a solution for the flow vector q and the squared pressure vector π has to be found which is valid for all possible realizations of data contained in the uncertainty set. When considering flowing gas as a physical system, such a solution is very unlikely to exist as flows and pressures within the network are highly dependent on the given parameters, i.e., demands and pressure loss coefficients. Therefore, a two-stage robust approach is appropriate.

For the gas network problem, the compressor power vector Δ is a first-stage decision, whereas the flow vector q and the squared pressure vector π are adjustable second-stage variables. Thus, the two-stage problem can be formulated as

$$\min \left\{ w^T \Delta \mid \exists \Delta \text{ such that } \forall u = (\phi, d) \in \mathcal{U} \exists q, \pi \text{ that satisfy (8a)–(8e)} \right\}.$$

This is also known as the *(fully) adjustable robust counterpart* of problem (8). With Theorem 2.1, the squared pressure variables and some equations can be eliminated from the model to obtain an equivalent formulation:

$$\min \left\{ w^T \Delta \mid \exists \Delta \text{ such that } \forall u = (\phi, d) \in \mathcal{U} \exists q_{\mathcal{N}} \text{ that satisfies (13) and (14)} \right\}. \quad (20)$$

3.3 Transforming the Adjustable Robust Model to a Single-Stage Problem

By exploiting structural properties of the gas network problem, we show how the two-stage problem (20) can be transformed to a single-stage problem. In particular, we exploit separability of the constraints (see (21)) and the weak connection between first-stage and second-stage variables. As this transformation can be applied to all two-stage problems with this structure, we abstract our presentation from gas networks and state our results for the general two-stage problem (ARC). Problems with a similar structure include e.g., direct current electricity networks, water networks, and other potential driven networks.

The necessary conditions for our approach are summarized in the following assumption:

Assumption 2. (a) *The equation system $g(x, y, u) = 0$ does not depend on x and admits a unique solution $y^*(u)$ for all $u \in \mathcal{U}$.*

(b) *The inequality constraints $h(x, y, u) \leq 0$ are separable in the following way:*

$$h(x, y, u) = s(x) + t(y, u). \quad (21)$$

Before proceeding to the abstract presentation, we first show that the gas network problem satisfies Assumption 2.

Theorem 3.1. *Suppose that Assumption 1 holds. Then the constraints of the two-stage gas network problem (20) satisfy Assumption 2.*

Proof. First we restate equation system (13):

$$\tilde{\mathbf{A}}_{\mathcal{N}}^T \tilde{g}(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) = F_{\mathcal{N}}(q_{\mathcal{N}}).$$

Assumption 1 implies that this system is independent of the first-stage variables Δ . Moreover, uniqueness and existence of a solution $q_{\mathcal{N}}^*$ holds due to Theorem 2.3. Thus, the constraint system satisfies the first part of Assumption 2.

With the assumption's second part in mind, we take a closer look at inequality system (14):

$$g_w(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) - g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) \leq \bar{\pi}_v - \underline{\pi}_w \quad \text{for all } v, w \in \mathcal{V}.$$

We have

$$\begin{aligned} g_v(q_{\mathcal{N}}, \Delta_{\mathcal{B}}) &= \left[\left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)^{-1} F_{\mathcal{B}}(q_{\mathcal{B}}(q_{\mathcal{N}}), \Delta_{\mathcal{B}}) \right]_v, \\ &= \left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)_v^{-1} F_{\mathcal{B}}(q_{\mathcal{B}}(q_{\mathcal{N}}), \Delta_{\mathcal{B}}), \end{aligned}$$

where $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)_v^{-1}$ denotes the v -th row of $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}$. Since there is no row in $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}$ corresponding to the root node r , we let $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)_r^{-1} = (0, \dots, 0)$ for ease of notation.

From this definition it is evident that g_v is a linear combination of the entries of $F_{\mathcal{B}}$. Every entry in $F_{\mathcal{B}}$ corresponds to either a pipe and depends on $q_{\mathcal{N}}$ or a compressor and depends on Δ . Therefore the inequalities are separable:

$$\begin{aligned} (\tilde{\mathbf{A}}_{\mathcal{B}}^T)_w^{-1} F_{\mathcal{B}}(q_{\mathcal{B}}(q_{\mathcal{N}}), \Delta_{\mathcal{B}}) - (\tilde{\mathbf{A}}_{\mathcal{B}}^T)_v^{-1} F_{\mathcal{B}}(q_{\mathcal{B}}(q_{\mathcal{N}}), \Delta_{\mathcal{B}}) - \bar{\pi}_v + \underline{\pi}_w \\ = s_{vw}(\Delta) + t_{vw}(\phi_{\mathcal{B}}, q_{\mathcal{B}}(q_{\mathcal{N}})) \leq 0 \quad \text{for all } v, w \in \mathcal{V} \end{aligned} \quad (22)$$

for suitable functions s_{vw} and t_{vw} . \square

In the remainder of this chapter, we present the transformation of the abstract two-stage problem (ARC) to single-stage problem along with further simplifications.

Lemma 3.2. *Under Assumption 2, the set of feasible first-stage decisions x of the adjustable robust counterpart (ARC) is given by*

$$\mathcal{X} = \{x \in \mathbb{R}^{n_1} \mid s(x) \leq b\}, \quad (23)$$

where $b = (b_i)_{i=1, \dots, m_1} \in \mathbb{R}^{m_1}$ with

$$b_i = -\max_{u \in \mathcal{U}} \{t_i(y, u) \mid g(y, u) = 0, y \in \mathbb{R}^{n_2}\}.$$

Proof. Due to Assumption 2, the equality constraints of (ARC) only depend on y and u , i.e., $g(x, y, u) \equiv g(y, u)$. Furthermore the inequality constraints are separable: $h(x, y, u) = s(x) + t(y, u)$. Therefore, the set of feasible first-stage decisions x of (ARC) can be written as

$$\{x \mid \exists x \in \mathbb{R}^{n_1} \forall u \in \mathcal{U} \exists y \in \mathbb{R}^{n_2} \text{ with } g(y, u) = 0, s(x) + t(y, u) \leq 0\}. \quad (24)$$

From Assumption 2 it follows that there is a function $y^*(u)$ which maps values u of the uncertainty set \mathcal{U} to solutions of $g(y, u) = 0$. This function exists and is well defined since solutions of the equality system exist for all $u \in \mathcal{U}$ and are unique. With this in mind, the exists quantor for variable y and the equality constraints can be eliminated from (24) and thus we obtain the feasible region of a single-stage robust optimization problem of form (RC):

$$\{x \in \mathbb{R}^{n_1} \mid s(x) + t(y^*(u), u) \leq 0 \quad \forall u \in \mathcal{U}\}.$$

This semi-infinite problem can be reformulated by maximizing the left-hand side of the inequality:

$$\begin{aligned} & s(x) + t(y^*(u), u) \leq 0 && \text{for all } u \in \mathcal{U}, \\ \iff & \max_{u \in \mathcal{U}} (s_i(x) + t_i(y^*(u), u)) \leq 0 && \text{for all } i = 1, \dots, m_1, \\ \iff & s_i(x) + \max_{u \in \mathcal{U}} (t_i(y^*(u), u)) \leq 0 && \text{for all } i = 1, \dots, m_1. \end{aligned}$$

After rewriting the solution function y^* in terms of g and letting

$$b_i := -\max_{u \in \mathcal{U}} \{t_i(y, u) \mid g(y, u) = 0, y \in \mathbb{R}^{n_2}\} \quad \text{for all } i = 1, \dots, m_1, \quad (25)$$

we obtain

$$s_i(x) + \max_{u \in \mathcal{U}} (t_i(y^*(u), u)) = s_i(x) - b_i \leq 0$$

and thus set of feasible first-stage decisions x of (ARC) is equivalent to

$$\{x \in \mathbb{R}^{n_1} \mid s(x) \leq b\}. \quad (26)$$

□

We want to highlight two properties of (23) which may be beneficial for solving such problems in practice.

Remark 3.3. Calculating b_i involves solving an optimization task to global optimality. In situations where this is not possible or where it can only be done with great effort, e.g., due to non-convex constraints, relaxations of the problem can be used instead. We propose to use convex or mixed-integer linear (MIP) relaxations as they lead to optimization tasks which can be solved to global optimality using available software. We consider the definition of b_i :

$$b_i = -\underbrace{\max_{u \in \mathcal{U}} \{t_i(y, u) \mid g(y, u) = 0, y \in \mathbb{R}^{n_2}\}}_{(\star)}. \quad (27)$$

Replacing (\star) by a relaxed optimization problem leads to an optimal value $b'_i \leq b_i$. Plugging b' into (26) yields a smaller feasible region for the first-stage decision variables x :

$$\{x \in \mathbb{R}^{n_1} \mid s(x) \leq b'\} \subseteq \{x \in \mathbb{R}^{n_1} \mid s(x) \leq b\}.$$

Since the feasible region of the relaxed problem's first-stage variables is a subset of the original feasible region, solutions obtained in this fashion are still robust feasible. Depending on the quality of used relaxations, solutions obtained in this way can be more conservative, i.e., have a worse objective function value. On the other hand, using relaxations of non-convex problems typically allows us to solve much larger instances compared to using the original problem formulation.

The second observation explains how the b_i can be calculated efficiently in practice. Suppose there are two constraints $h_i(x, y, u) = s_i(x) + t_i(y, u) \leq 0$ for $i = 1, 2$ which are reformulated as $s_1(x) \leq b_1$ and $s_2(x) \leq b_2$ due to Lemma 3.2. If the functions s_1 and s_2 are identical, both constraints can be aggregated into one constraint by setting

$$s_1(x) = s_2(x) \leq \min(b_1, b_2). \quad (28)$$

In order to implement reduction (28), one can solve a single optimization problem rather than two separate ones:

$$\begin{aligned}
\min(b_1, b_2) &\stackrel{(25)}{=} \min_{i=1,2} \left(-\max_{u \in \mathcal{U}} \{t_i(y, u) \mid g(y, u) = 0, y \in \mathbb{R}^{n_2}\} \right) \\
&= \min_{i=1,2} \left(\min_{u \in \mathcal{U}} \{-t_i(y, u) \mid g(y, u) = 0, y \in \mathbb{R}^{n_2}\} \right) \\
&= \min_{u \in \mathcal{U}} \left\{ \min_{i=1,2} (-t_i(y, u)) \mid g(y, u) = 0, y \in \mathbb{R}^{n_2} \right\}. \tag{29}
\end{aligned}$$

Compared to (25), the aggregated optimization problem (29) has the same feasible set but a different objective function. For mixed-integer models, this minimum-of-functions objective structure can be modeled with binary variables.

This can be generalized to any finite number of constraints h_i ($i = 1, \dots, m_1$). We want to emphasize that this situation may not be that uncommon in practice. For example, all variable bounds $y_i \leq \bar{y}_i$ are independent of x and can therefore be reduced to one constraint of the form $s(x) = 0 \leq c$ for some $c \in \mathbb{R}$.

The Robust Gas Transport Problem as a Single-Stage Problem After applying Lemma 3.2 to the two-stage robust gas transport problem (20), we obtain the single-stage problem

$$\begin{aligned}
&\min_{\Delta \in [\underline{\Delta}, \bar{\Delta}]} w^T \Delta, \\
&s_{vw}(\Delta) \leq b_{vw} \quad \text{for all } v, w \in \mathcal{V}, \tag{30}
\end{aligned}$$

where b_{vw} is precomputed as

$$b_{vw} = -\max \left\{ t_{vw}(\phi_B, q_B(q_N)) \mid \begin{array}{l} \tilde{A}_N^T \tilde{g}(q_N) = F_N(q_N) \\ (\phi, d) \in \mathcal{U} \\ q_N \in \mathbb{R}^{|\mathcal{N}|} \end{array} \right\}. \tag{31}$$

for $v, w \in \mathcal{V}$. Assuming (31) is precomputed, (30) reduces to a linear program (LP). As (31) is a non-convex optimization task, we next explain how relaxations can be computed effectively in this context.

4 Precomputation of the Right-Hand Side of Problem (30) using Piecewise Linear Relaxations

In order to solve the robust gas network problem via the linear problem (30), one first needs to compute b_{vw} by solving a series of nonlinear and non-convex optimization problems to global optimality. Since this is a difficult task in general, we first replace all nonlinear terms by piecewise-linear relaxations and use this surrogate model to compute the right-hand side b . As is discussed in Section 3.3, using relaxations for computing b can lead to a somewhat smaller feasible region but preserves robust feasibility of the obtained solution.

For ease of explanation, we transform problem (31) back into its non-reduced form and show how b can be computed using the original model formulation. We also express the pressure drop in each pipe equivalently by defining sets of feasible pressure drops $\mathcal{L}_{\phi_a}^a$ for each pipe $a \in \mathcal{A}$:

$$(q_a, \lambda_a) \in \mathcal{L}_{\phi_a}^a = \{(q_a, \lambda_a) \mid \lambda_a = -\phi_a |q_a| q_a, q_a \in \mathbb{R}\}, \quad (32)$$

which is equivalent to

$$\pi_w - \pi_v = \lambda_a = -\phi_a q_a |q_a|.$$

For the following Lemma, we remember that the row of $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}$ corresponding to the root node r is a vector of zeros, i.e., $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}_r = (0, \dots, 0)$.

Lemma 4.1. *Let $v, w \in \mathcal{V}$ be two given nodes. Let $c_{vw} = (\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}_w - (\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}_v$, where $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}_v$ denotes the row of $(\tilde{\mathbf{A}}_{\mathcal{B}}^T)^{-1}$ corresponding to v . Furthermore, let $\mathcal{L}_{\phi_a}^a$ be defined for all $a \in \mathcal{A}_{\text{pi}}$ as in (32).*

Then, b_{vw} of (31) is the optimal value of the following optimization problem:

$$b_{vw} = \min - (c_{vw}^T \lambda - \bar{\pi}_v + \underline{\pi}_w) \quad \mathbf{A}q = d, \quad (33a)$$

$$\mathbf{A}_{\mathcal{A}_{\text{cs}}}^T \pi = 0, \quad (33b)$$

$$\mathbf{A}_{\mathcal{A}_{\text{pi}}}^T \pi = \lambda, \quad (33c)$$

$$(q_a, \lambda_a) \in \mathcal{L}_{\phi_a}^a \quad \text{for all } a \in \mathcal{A}_{\text{pi}}, \quad (33d)$$

$$q \in \mathbb{R}^{|\mathcal{A}|}, \quad (33e)$$

$$\lambda \in \mathbb{R}^{|\mathcal{A}_{\text{pi}}|}.$$

Proof. The reduced and non-reduced model of the gas network problem are equivalent due to Theorem 2.1. Therefore, we replace the constraints of the reduced model (31) by their non-reduced counterparts (33a)–(33e). As the compressor power Δ has no influence on (31) due to Assumption 1, the compressor power in the non-reduced model (33b) is set to zero in (33b).

In order to transform the objective to fit the non-reduced model, we observe that the objective of (31) is a linear combination of entries in F , i.e., of the pressure drops. We introduce auxiliary variable $\lambda \in \mathbb{R}^{|\mathcal{A}_{\text{pi}}|}$ to model the pressure drop in equations (33c) and (33d). A closer look at (22) shows that the linear objective function can be written as $c^T \lambda_{vw} - \bar{\pi}_v + \underline{\pi}_w$ using the coefficient vector

$$c_{vw} = \left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)^{-1}_w - \left(\tilde{\mathbf{A}}_{\mathcal{B}}^T \right)^{-1}_v \quad \text{for all } v, w \in \mathcal{V}.$$

□

Our aim is to solve problem (33) or a relaxation thereof to global optimality. To this end, we present several relaxations of the nonlinear and non-convex set $\mathcal{L}_{\phi_a}^a$ that can

be used for that purpose. Since global optimal solutions are required to ensure robust feasibility of the obtained results, we develop piecewise-linear relaxations of $\mathcal{L}_{\phi_a}^a$ which can then be used in a mixed-integer linear program (MIP). Of course other relaxations like linear (see Section 4.1) or semidefinite relaxations arising from polynomial programming (see [1]) are also conceivable, however, we restrict ourselves to piecewise-linear relaxations as a priori error bounds can be computed. In the remainder of this section, we drop the arc-specific indices of $\mathcal{L}_{\phi_a}^a$. Furthermore, we assume the flow variables to be in a finite interval, i.e., $q_a \in [\underline{q}_a, \bar{q}_a]$. This is no restriction as the pressure and flow variables of problem (33) are always bounded, see Section 4.1 for more details regarding their computation.

Relaxations with a priori error bounds are of particular interest as they allow us to compute solutions with arbitrary precision by reducing the error ε .

Definition 4.1 (ε -exact relaxation). Let $\mathcal{L} = \{(x, y) \in [\underline{x}, \bar{x}] \times \mathbb{R} \mid y = f(x)\} \subseteq \mathbb{R}^2$ be the function graph of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ over a finite interval $[\underline{x}, \bar{x}]$ and let $\varepsilon > 0$ be a given error. We call $\tilde{\mathcal{L}} \subseteq [\underline{x}, \bar{x}] \times \mathbb{R}$ an ε -exact relaxation of \mathcal{L} , if

1. $\mathcal{L} \subseteq \tilde{\mathcal{L}}$ and
2. $|y - \tilde{y}| \leq \varepsilon$ for all $x \in [\underline{x}, \bar{x}]$ with $(x, y) \in \mathcal{L}$ and $(x, \tilde{y}) \in \tilde{\mathcal{L}}$

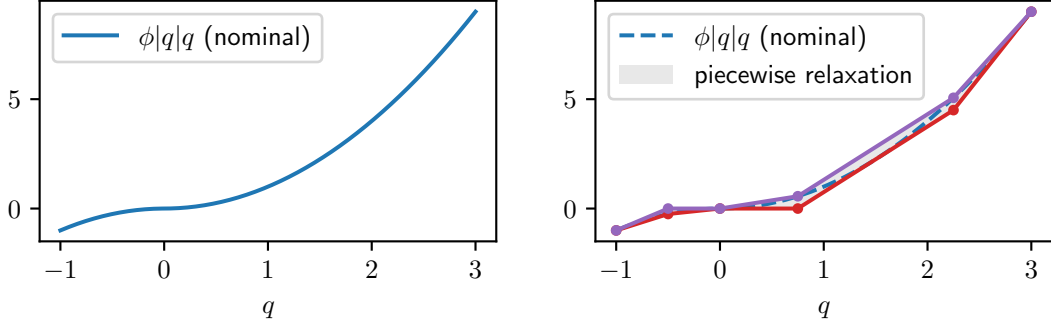
are satisfied.

As all presented relaxations are constructed using piecewise-linear functions, we briefly restate how to express piecewise-linear functions in MIPs using the delta method [20]. Let $(x_i, y_i)_{i=1, \dots, k}$ a series of points in \mathbb{R}^2 with $x_1 < x_2 < \dots < x_k$. Then the graph of the piecewise-linear function with sampling points (x_i, y_i) is described given by the following mixed-integer constraints:

$$\begin{aligned} x &= x_1 + \sum_{i=1, \dots, k-1} (x_{i+1} - x_i) \delta_i, \\ y &= y_1 + \sum_{i=1, \dots, k-1} (y_{i+1} - y_i) \delta_i, \\ \delta_1 &\geq z_1 \geq \delta_2 \geq z_2 \geq \dots \geq z_{k-2} \geq \delta_{k-1}, \\ \delta_i &\in [0, 1] && \text{for all } i = 1, \dots, k-1, \\ z_i &\in \{0, 1\} && \text{for all } i = 1, \dots, k-2. \end{aligned}$$

In our implementation, the delta method was used exclusively since it is known for its good performance for gas network problems in practice (see [12]).

Finding approximations for \mathcal{L} , i.e., the function graph of $cx|x|$ for a fixed $c > 0$ is straightforward as follows. After eliminating the absolute value by splitting the function graph into negative and positive parts, only square functions cx^2 need to be treated. We show next that the error of approximating cx^2 by a (piecewise) linear function only depends on the parameter c and on the distance between two adjacent sampling points but not on the position of the chosen sampling points. Indeed pick any two points (x_1, y_1) , (x_2, y_2)



(a) Nominal pressure drop

(b) Nominal piecewise-linear relaxation

Figure 1: Nominal pressure drop and piecewise-linear relaxation for constant pressure drop coefficient $\phi = 1$ and approximation quality $\varepsilon = 1.0$.

on the graph of $f^{\text{sq}}(x) = cx^2$ with $x_1 < x_2$ and let $f^{\text{lin}}(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$ be the line connecting both points. To calculate the maximum deviation $\varepsilon = \max_x f^{\text{lin}}(x) - f^{\text{sq}}(x)$, we observe that $f^{\text{lin}}(x) - f^{\text{sq}}(x)$ is a degree two polynomial and thus attains its extreme value between its two roots x_1 and x_2 at $x^* = \frac{1}{2}(x_1 + x_2)$. A short calculation shows that the maximum error is given by $\varepsilon = f^{\text{lin}}(x^*) - f^{\text{sq}}(x^*) = \frac{c}{2}(x_2 - x_1)^2$. Since the approximation error only depends on c and on the distance $x_2 - x_1$ between two sampling points, we conclude that cx^2 can be approximated by a piecewise-linear function with a given error ε by equidistant sampling points. See Fig. 1 for an example.

Two different cases need to be distinguished when building relaxations for $\mathcal{L}_{\phi_a}^a$, depending on whether ϕ_a is constant or affected by uncertainty. If the coefficient ϕ_a is constant, the standard relaxation from the literature with equidistant sampling points can be applied to obtain an ε -exact relaxation $\tilde{\mathcal{L}}$, see [14]. For the second case, we assume the pressure drop coefficient ϕ_a to be uncertain. In general, this requires a relaxation \mathcal{L}_{ϕ_a} which is parameterized by ϕ_a so that it can adjust to the different realizations of ϕ_a to preserve the ε -approximation quality. However, due to uncorrelated pressure drop coefficients (18), a simplification can be applied. Since the realization of pressure drop coefficient ϕ_a at arc a is independent of all other uncertainties, it is sufficient to construct an ε -exact relaxation of the union

$$\bigcup_{\phi_a \in [\underline{\phi}_a, \bar{\phi}_a]} \mathcal{L}_{\phi_a}^a.$$

Due to continuity and monotonicity of the function value $\phi_a q_a |q_a|$ in ϕ_a , the union $\bigcup_{\phi_a \in [\underline{\phi}_a, \bar{\phi}_a]} \mathcal{L}_{\phi_a}^a$ has no holes and its boundary can be described by piecewise functions of the form $\phi_a q_a |q_a|$:

$$\bigcup_{\phi_a \in [\underline{\phi}_a, \bar{\phi}_a]} \mathcal{L}_{\phi_a}^a = \left\{ (q_a, \lambda_a) \left| \begin{array}{l} -\bar{\phi}_a q_a |q_a| \leq \lambda_a \leq -\underline{\phi}_a q_a |q_a|, \quad \text{if } q_a \geq 0 \\ -\underline{\phi}_a q_a |q_a| \leq \lambda_a \leq -\bar{\phi}_a q_a |q_a|, \quad \text{if } q_a \leq 0 \end{array} \right. \right\}. \quad (35)$$

Finally, the relaxations for functions with constant ϕ_a can be applied to the boundary

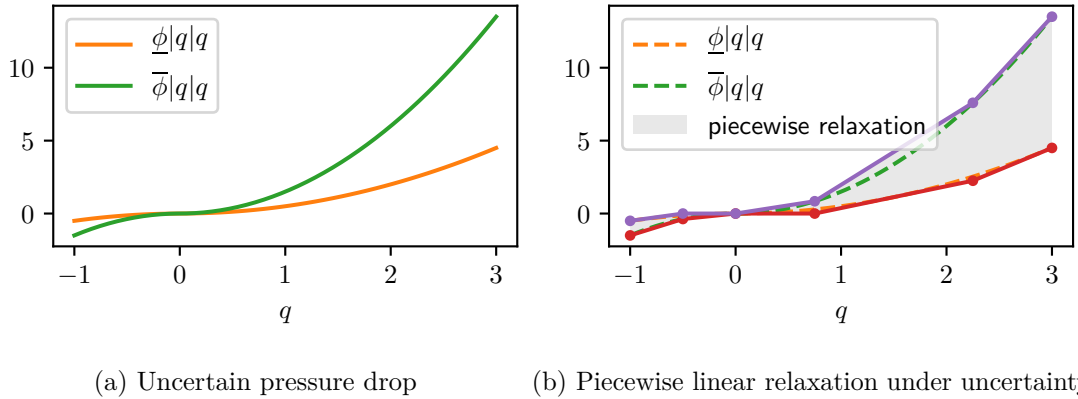


Figure 2: Pressure drop and piecewise-linear relaxation for uncertain pressure drop coefficient $\phi \in [0.5, 1.5]$ and approximation quality $\varepsilon = 1.0$.

functions of (35) in order to obtain an ε -exact relaxation of the union of all possible pressure drops for the given uncertain parameters, see Fig. 2 for an example.

We want to emphasize that the shown relaxation ideas for nonlinear constraints under uncertainty can be applied to other nonlinear problems as well. Although the presented model for piecewise-linear functions (34) can be generalized to higher dimensions [13], the relaxation approach is most effective for univariate functions. This is often the case for problems on networks where some quantities of interest solely depend on e.g., the flow along an arc or the potential at a specific node. As was mentioned earlier, first-stage solutions obtained through relaxations of (27) are always robust feasible but can be more conservative compared to exact solutions.

4.1 Reducing Model Size by Preprocessing

Binary variables are used for the construction of the presented piecewise-linear relaxations. The overall complexity of solving a MIP typically depends heavily on the number of discrete variables. Our settings requires solving not a single but a series of MIPs to determine the right-hand sides b_i . It is therefore very desirable to speed up the solution process as much as possible.

The number of binary variables in our setting depends on the flow bounds $[q_a, \bar{q}_a]$ and on the approximation error ε . Since the approximation error is given, we can decrease the number of required binary variables by providing strong bounds for q_a . To this end, our methods comprise two trivial bounds and an optimization method using linear relaxations. Preprocessing ideas for gas network problems can be found in [12], including more complex procedures like pressure and flow propagation heuristics. For a broader overview of different preprocessing ideas for mixed-integer nonlinear programs, see the review article [22]. However, we cannot use most of the mentioned ideas as is since they are tailored towards nominal problems without uncertainty. Due to the nature of our problem setting where we optimize over the uncertainty set, our preprocessing methods

must preserve the full range of states in the network depending on the uncertainty. All methods are presented with the full range of uncertainty in mind, i.e., uncertain demand and uncertain pressure drop coefficients. For problems where only one or no uncertainty is given, the presented methods can often be simplified considerably.

Trivial bounds We present two trivial flow bounds, one resulting from the maximum overall total demand and one resulting from the decomposition of the linear flow solution space into tree and cycle flows. The problem is assumed to be affected by uncertain demands and uncertain pressure drop coefficients.

A trivial flow bound can be derived by calculating the maximum possible positive demand:

$$d^{\text{total}} = \frac{1}{2} \max \{ \|d\|_1 \mid d \in \mathcal{D} \}$$

In general this problem is NP-hard and has to be reformulated as a MIP. Typically, the preprocessing has to be very fast (compared to solving the actual problem MIPs), so we calculate a simple upper bound on d^{total} instead. We consider uncertainty set (19). After omitting the balancing hyperplane, every demand parameter d_v is only affected by upper and lower bounds:

$$\underline{d}_v \leq d_v \leq \bar{d}_v \quad \text{for all } v \in \mathcal{V}.$$

With this in mind, we first bound the maximum total gas injection and withdrawal. A simple bound can be obtained from these quantities by taking the minimum of their absolute values:

$$\begin{aligned} d^+ &= \sum_{v \in \mathcal{V}} \max(0, \bar{d}_v), & d^- &= \sum_{v \in \mathcal{V}} \min(0, \underline{d}_v), \\ d^{\text{relax}} &= \min(d^+, -d^-). \end{aligned} \tag{36}$$

The gas flow over each arc can never exceed the total injection, thus

$$q_a \in [-d^{\text{relax}}, d^{\text{relax}}] \tag{37}$$

is a feasible bound for all arcs $a \in \mathcal{A}$.

The previous bound can be improved considerably for certain arcs if the structure of the linear network flow solution space is exploited. Recall from (9) that any feasible flow q can be written as

$$q = \tilde{\mathbf{A}}_{\mathcal{B}}^{-1} (\tilde{d} - \tilde{\mathbf{A}}_{\mathcal{N}} q_{\mathcal{N}}), \tag{38}$$

where $q_{\mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$ is a free parameter. It is known from linear flow theory that due to (38), the flow over all arcs which are not part of a cycle is independent of $q_{\mathcal{N}}$, i.e., can be written as $q_a = (\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{d})_a$. This allows us to find tight bounds for q_a over non-cycle arcs a by optimizing over the demand uncertainty set:

$$q_a \in \left[\min_{\tilde{d} \in \mathcal{D}} (\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{d})_a, \max_{\tilde{d} \in \mathcal{D}} (\tilde{\mathbf{A}}_{\mathcal{B}}^{-1} \tilde{d})_a \right]. \tag{39}$$

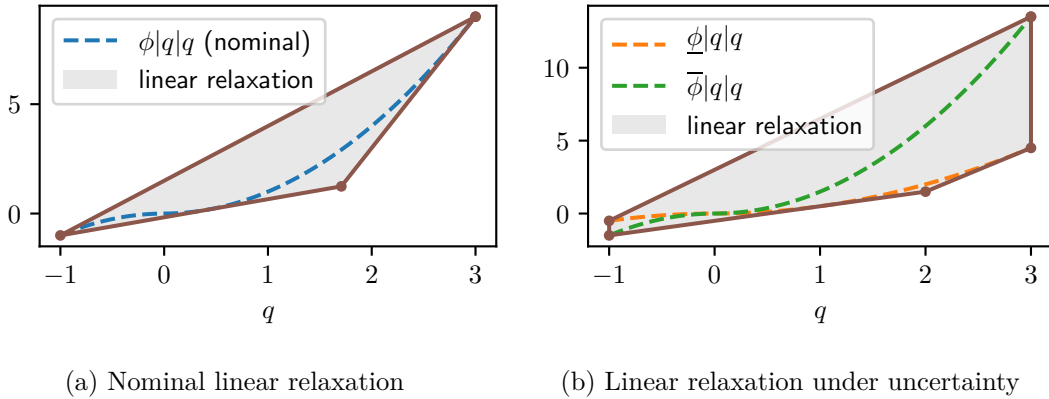


Figure 3: Linear convex hull of nominal ($\phi = 1.0$) and uncertain pressure drop coefficient ($\phi \in [0.5, 1.5]$).

If it is not desirable to solve an optimization task, lower and upper bounds for (39) can be found with a similar approach as (36). We remark that for problems without demand uncertainty, there is nothing to optimize and the exact, constant flow q_a can be evaluated by calculating $(\tilde{\mathbf{A}}_{\mathcal{B}}^{-1}\tilde{\mathbf{d}})_a$. In this case, the nonlinear pressure drop equation can be removed by evaluating the signed square function at q_a .

Bounds due to linear relaxations Every fundamental cycle introduces a free parameter $q_{\mathcal{N}}$ into the description of the flow (38). The flow on every arc that is part of a cycle depends on the unbounded parameter $q_{\mathcal{N}}$. Consequently, equation (38) cannot be used to derive finite bounds for the flow on arcs that are part of cycle. In order to derive bounds in this setting, we supplement (38) with a very rough approximation of the pressure drop constraints. This was done previously for the nominal case in [12] by defining a convex hull of the pressure drop constraint’s graph through linear inequalities. When compared to the piecewise linearization approach in Section 4, the defined set is still a relaxation of the original constraint but does not guarantee an ε -approximation.

Next, we generalize the linear model to incorporate uncertain pressure drop coefficients $\phi \in \mathcal{U}$. To this end, we construct a convex hull $\hat{\mathcal{L}}^a$ of the set of all possible pressure drops when given an uncertain coefficient $\phi_a \in [\underline{\phi}_a, \bar{\phi}_a]$:

$$\hat{\mathcal{L}}^a \supset \bigcup_{\phi_a \in [\underline{\phi}_a, \bar{\phi}_a]} \mathcal{L}_{\phi_a}^a.$$

See Fig. 3 for a linear convex hull for nominal pressure drop as well as for pressure drop under uncertainty.

Given relaxations $\hat{\mathcal{L}}^a$ for each pipe $a \in \mathcal{A}_{\text{pi}}$, a very similar problem as (33) is defined by replacing the nonlinear sets (33d) with their corresponding relaxations. Let $\hat{\mathcal{X}}$ be the feasible region of the resulting problem. Then bounds $q_a \in [q_{a,\text{lb}}^*, q_{a,\text{ub}}^*]$ for the flow

along each arc $a \in \mathcal{A}$ can be derived by minimizing and maximizing q_a over $\hat{\mathcal{X}}$:

$$q_{a,\text{lb}}^* = \min_{q,\pi \in \hat{\mathcal{X}}} q_a, \quad q_{a,\text{ub}}^* = \max_{q,\pi \in \hat{\mathcal{X}}} q_a. \quad (40)$$

Since the problems in (40) are linear programs (LPs), we also call this the LP-based relaxation preprocessing approach. In the literature, this type of preprocessing strategy is also known as *feasibility-based bounds tightening*, see [22].

We remark that in case of demand uncertainty, the problems (40) contain description of the uncertainty set. For pressure drop uncertainty, the uncertainty set is incorporated into the relaxations of the pressure drop equations and therefore is not present in the linear programs.

Preprocessing strategy In our implementation, all three bounds—trivial total demand, linear flows on non-cycle arcs and LP relaxation based bounds—are combined into an iterative bound tightening procedure. Initially, trivial flow bounds (37) are derived for each arc to obtain finite bounds. Next, the flow bounds of all non-cycle arcs are tightened with (39). The procedure then enters a loop where the linear programs (40) are solved repeatedly for all remaining arcs. In one iteration step, model (40) is build only once and then reused with different objectives for each flow variable. At the end of each iteration step, all flow bounds are updated with the newly calculated bound information. The algorithm terminates if either a maximum number of iterations is reached or if the Euclidean norm of the difference between the bounds of two subsequent iterations is smaller than a specified cutoff value. We use a maximum number of fifteen iterations and a cutoff value of 1.0 in our computations.

5 Numerical Experiments

In this section, the performance of the developed methods is evaluated on a family of gas network instances. We first examine the influence of the preprocessing routines and of the aggregation idea (29) to determine the best possible combination of both. Using this as a basis for further study, problem running times are studied in more detail under different aspects such as relaxation quality and magnitude of the uncertainty.

Instances and setup The studied problems are taken from GASLIB [26], a freely available collection of realistic gas network instances comprising of topology and nomination data. We used networks GASLIB-11, GASLIB-24, and GASLIB-40 with their supplemented demand and pressure nominations. The employed instances were slightly modified to fit the context of this article: Any element which is no compressor and no pipe is replaced by a so-called *short pipe*—an special type of pipe with zero pressure drop coefficient that can be traversed freely by gas without experiencing a pressure drop. In order to satisfy Assumption 1, the compressor of GASLIB-40 which is part of a cycle was replaced by a short pipe as well. Furthermore, the demands of GASLIB-24 and GASLIB-40 were scaled to obtain nominations whose corresponding robust problems are feasible for all studied

uncertainty sets and have nonzero optimal solutions, i.e., compressors have to be used to reach feasibility. Table 1 gives an overview over the features of the used instances. We abbreviate *compressors* with “comprs.” and *control valves* with “ctrl. valves”.

Table 1: Instances for numerical experiments.

	scaling	#nodes	#pipes	#comprs.	#ctrl. valves	#short pipes
GASLIB-11	1.00	11	8	2	0	1
GASLIB-24	2.05	24	19	3	1	2
GASLIB-40	0.67	40	39	5	0	1

Each instance can be affected by uncertainty. We use relative perturbations around the nominal demand values or pressure drop coefficients as uncertainty sets. A unified naming scheme of the defined uncertainty sets is utilized for both demand and pressure drop uncertainty. The network’s demand or pressure drop coefficients may be affected independently by four levels of uncertainty: *nominal* (i.e., no uncertainty), *small*, *medium*, and *large* uncertainty, see Table 2. Any combination of the provided levels defines an uncertainty set for the numerical experiments, ranging from no uncertainty (“nominal demand and nominal pressure drop uncertainty”) to the combination of large demand uncertainty with large pressure drop uncertainty. The chosen uncertainty level is then applied to all affected elements, e.g., demands or pressure drop coefficients. Thus, there are 16 uncertainty sets in total. The concrete definitions can be found in Table 2.

Table 2: Every combination of demand and pressure drop uncertainty level defines an uncertainty set used in the numerical study.

	demand	pressure drop coefficient
nominal	$\{d\}$	$\{\phi\}$
small	$[0.95 \cdot d, 1.05 \cdot d]$	$[\phi, 1.10 \cdot \phi]$
medium	$[0.90 \cdot d, 1.10 \cdot d]$	$[\phi, 1.50 \cdot \phi]$
large	$[0.80 \cdot d, 1.20 \cdot d]$	$[\phi, 2.00 \cdot \phi]$

The linearization error for the piecewise-linear relaxation of the pressure drop equations was chosen as $\varepsilon \in \{0.01, 0.1, 1.0\}$.

All experiments were carried out on a machine with a four core Xeon E3-1240 v5 CPU running at 3.5GHz each and 16 GB of RAM. The linear and mixed-integer linear problems were solved using Gurobi 7.5 [17] using 4 threads.

Running time improvements due to preprocessing and aggregation We compare preprocessing strategies on GASLIB-11 and GASLIB-24 since GASLIB-40 is already too large to be solved in an acceptable timespan without preprocessing. In order to cover a wide range of problems, we derive average running times of instance groups where each

group contains all possible combinations of the remaining parameters like approximation quality ε and uncertainty set.

First we take a look at the average running times depending on the employed preprocessing method, see Table 3 and Fig. 4a. The columns denote the different preprocessing choices: *trivial* from equation (37), *treeflows* (39), and LP-based bound tightening (40) (“*opt*”). Each entry of Table 3 is the averaged running time of 96 instances arising from all combinations of uncertainty set, $\varepsilon \in \{0.01, 0.1, 1.0\}$, and choice of individual or aggregated model. The number in brackets displays the relative speedup, when compared to the trivial preprocessing bounds. Speedups from treeflows preprocessing is negligible, possibly due to the fact that the studied GASLIB instances contain only few arcs which are not part of cycle. We observe a dramatic speedup of a factor of about 30–50 when the LP-based bound tightening is used.

Table 3: Mean running times when using different preprocessing strategies. The number in brackets denotes the speedup compared to “trivial” preprocessing bounds.

	trivial	treeflows	opt
GASLIB-11	14.7 s (1.0×)	11.5 s (1.3×)	0.5 s (31.6×)
GASLIB-24	327.1 s (1.0×)	327.1 s (1.0×)	6.2 s (52.4×)

Next, the influence of aggregation is benchmarked. We only consider instances after applying LP-based bound tightening. Recall that in order to calculate the right-hand side of (30) with problem (33), we can either solve $\mathcal{O}(|\mathcal{V}|^2)$ problems individually to obtain each b_{vw} or a smaller number of problems after applying an aggregation step (28). In Table 4 we compare the mean running times on instances GASLIB-11, GASLIB-24, and GASLIB-40 when choosing to solve all problems *individually* or in an *aggregated* fashion. As with the preprocessing strategies, the numbers in brackets denote the relative speedup compared to the slowest method, see also Fig. 4b. Each cell of Table 4 is an average over all instances with varying approximation quality and varying uncertainty set. We observe a mean speedup factor of about 8.5 for the larger instances and a smaller speedup of a factor of about 2.6 for the smallest instance when using the aggregated model. In total,

Table 4: Mean running times of individual and aggregated models after LP-based preprocessing. The number in brackets denotes the speedup compared to the individual model.

	individual	aggregated
GASLIB-11	0.7 s (1.0×)	0.3 s (2.6×)
GASLIB-24	11.1 s (1.0×)	1.4 s (7.8×)
GASLIB-40	193.8 s (1.0×)	21.4 s (9.1×)

a combination of LP-based bound tightening and an aggregation of subproblems yields a mean speedup factor of up to 400 for GASLIB-24 when compared to no preprocessing

and to solving all problems individually (see Fig. 4).

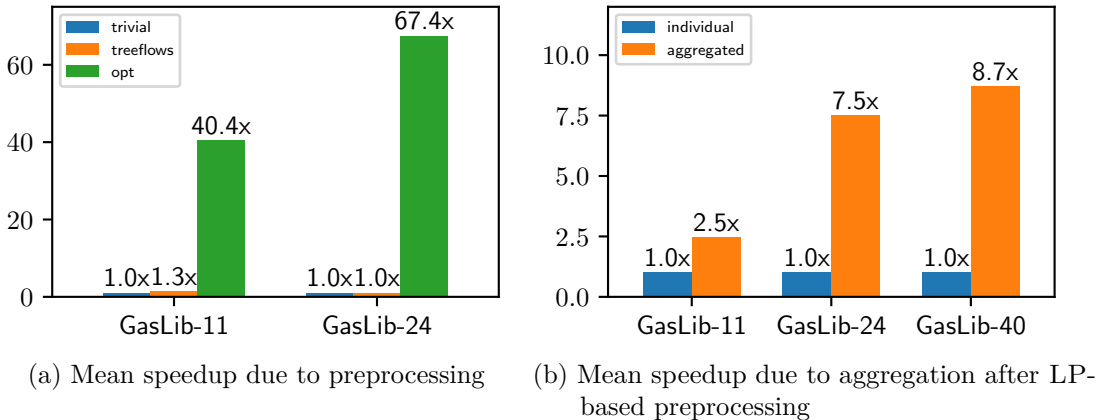


Figure 4: Different speedups due to preprocessing and aggregation. With both techniques, mean speedup reaches a factor of up to 400.

Running times and number of subproblems The overall running time of the gas network problem mainly consists of running times of the preprocessing LPs, running times of the MIP subproblems, and running time of last LP for deciding a solution of the compressors. In our setting, solving all occurring LP problems is trivial and can be done within fractions of a second. Therefore, we focus on the performance of the MIP subproblems when LP-based preprocessing has been applied.

Table 5: Number of MIP subproblems for individual and aggregated models, together with their mean and total running times.

	individual			aggregated		
	#probs	rt mean	rt total	#probs	rt mean	rt total
GASLIB-11	110	0.0 s	0.7 s	7	0.0 s	0.2 s
GASLIB-24	552	0.0 s	11.0 s	21	0.1 s	1.3 s
GASLIB-40	1560	0.1 s	193.3 s	31	0.7 s	20.8 s

Table 5 gives a more detailed summary over the mean running times for the individual and aggregated method. As before, every cell is the average over all possible combinations of approximation quality and uncertainty set. We use LP-based preprocessing for all instances. The columns are partitioned into one group related to solving all problems individually and one group where the aggregated method is applied. In each column group, we list the number of required subproblems together with their mean and total running times. It can be observed that applying the aggregation method drastically reduces the number of subproblems required to solve. For the studied instances, the

running times of the aggregated models increases at a smaller rate compared to the reduction of problems. Thus, the increase in complexity of the aggregated models is more than compensated by the smaller number of instances that need to be solved.

Influence of relaxation parameter: Price of piecewise-linear relaxations Finally, we investigate the influence of the relaxation parameter on the optimal value of the single-stage robust problem. Larger values of ε lead to coarser relaxations of the nonlinear constraints and thus should lead to more conservative solutions. For each GASLIB instance, we solve the robust problem for the “large \times large” combination of uncertainty sets (see Table 2 for a definition) and for varying relaxation parameters $\varepsilon \in \{0.001, 0.01, 0.1, 1.0, 10.0\}$. All comparisons in this paragraph are relative to the solution obtained with the smallest relaxation parameter $\varepsilon = 0.001$, which we assume to be “close enough” to the exact solution of the mixed-integer nonlinear problem. We use LP-based preprocessing and the aggregation method for all problems. The results are summarized in Table 6. Each column corresponds to a different choice of ε . We associate three rows with each gas network instance, where the first row denotes the absolute objective function value, the second row denotes the relative increase when compared to the finest relaxation $\varepsilon = 0.001$, and the third row denotes the running time.

Table 6: Absolute and relative comparison of optimal total compressor cost for different relaxation parameters ε together with their respective running times.

	relaxation parameter ε				
	0.001	0.01	0.1	1.0	10.0
GASLIB-11	763.78	763.79	763.90	764.76	768.15
		+0.002 %	+0.016 %	+0.128 %	+0.572 %
	30.74 s	3.21 s	0.44 s	0.14 s	0.08 s
GASLIB-24	647.83	647.86	648.08	649.93	665.37
		+0.004 %	+0.039 %	+0.325 %	+2.708 %
	130.27 s	16.25 s	2.59 s	0.66 s	0.32 s
GASLIB-40	45.12	45.16	45.69	49.50	79.21
		+0.102 %	+1.281 %	+9.711 %	+75.575 %
	2948.58 s	374.17 s	42.30 s	10.05 s	3.67 s

As a general trend, the objective value increases with the relaxation parameter ε as well as with the instance size, i.e., the number of nodes in the network. Furthermore, we observe an increase in running times for decreasing relaxation parameters. This is to be expected since smaller ε lead to finer approximations and thus to larger MIP models with more binary variables and constraints.

Concerning the optimal value, we observe a very small relative increase for GASLIB-11 and GASLIB-24 of at most 3% for all choices of ε . Even for the largest instance GASLIB-40, the additional cost due the chosen relaxation only amounts to about 10% for

a comparatively small $\varepsilon = 1.0$. Only for the largest choice of $\varepsilon = 10.0$ does the objective increase drastically by 76 %.

Next, we want to highlight the quality of solutions that can be found within the fixed timespan of 1 minute. As is shown in Table 6, we can solve GASLIB-11 for $\varepsilon = 0.001$, GASLIB-24 for $\varepsilon = 0.01$, and GASLIB40 for $\varepsilon = 0.1$ within this timespan. Moreover, the additional cost due to relaxation is very small: +0.004 % for GASLIB24 and +1.281 % for GASLIB40. This demonstrates that our method can be used to find high quality robust solutions for—in the context of robust optimization—large real-world networks within a short timespan.

Finally, we want to investigate the influence of ε on the absolute magnitude of the optimal value and the total running times of the resulting problems. To this end, the left part of Fig. 5 shows the absolute increase of objective relative to the smallest choice of $\varepsilon = 0.001$ for all three networks. The right part displays the corresponding total running times on a logarithmic scale. We observe a seemingly linear dependence of the optimal value on the choice of ε and note that the displayed curves are almost, but not completely, monotonically increasing. The lack of monotonicity is due to the fact that the feasible region of a piecewise-linear relaxation with parameter ε_1 does not necessarily have to be a subset of the feasible region of another relaxation with $\varepsilon_2 > \varepsilon_1$. In fact, two piecewise-linear relaxations that are constructed with different ε can have very different sampling points and thus one is not necessarily contained within the other.

When comparing both figures, we observe the objectives to scale almost linearly with ε whereas the corresponding running times scale exponentially for roughly $\varepsilon \leq 1$, but are influenced only very little by the relaxation parameter for about $\varepsilon \geq 2$. Since the running times for $\varepsilon \in [2, 10]$ are roughly the same, there is no reason not to solve the problem with smaller values of ε and thus profit from higher quality solutions. On the other hand, the improvement of objective for small relaxation parameters, e.g., $\varepsilon \in (0, 1]$ is so small that taking values from that range seems to be unjustified when taking the potentially large increase of running times into account. In total, there seems to be sweet spot at $1 \leq \varepsilon \leq 2$ where high quality solutions can be found for a comparatively small computational effort.

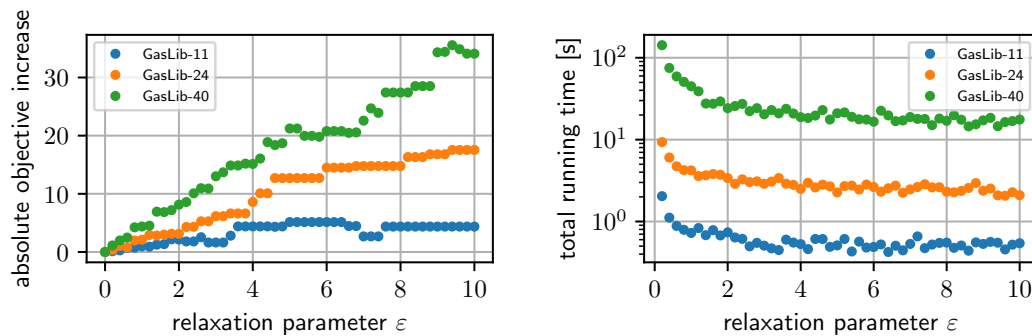


Figure 5: Influence of the relaxation parameter ε on objective and running times. The absolute increase of objective in the left picture is shown relative to $\varepsilon = 0.001$.

6 Concluding Remarks

In this work, gas network problems with a linear compressor model under uncertainties in demand or pressure drop coefficients were studied. The two-stage robust network optimization problem at hand has a special structure: first and second-stage variables are not coupled, and the second-stage variables are uniquely determined by the uncertain data. We showed how these properties can be exploited to reformulate a general, possibly nonlinear, two-stage robust optimization task with this structure as a single-stage problem. The right-hand side of the resulting problem is made up of the optimal objective values of a series of optimization tasks. On closer examination of the newly created single-stage problem, we obtained an aggregated model where certain subsets of subproblems can be treated as a single optimization task.

Even if these subproblems cannot be solved to global optimality, e.g., because the original problem contains very challenging nonlinear and non-convex constraints, it is nevertheless possible to obtain conservative but robust feasible solutions for the overall problem by solving relaxations instead. As the gas network problem is nonlinear and non-convex, we generalized previously known piecewise linearization techniques for the nominal case to incorporate uncertain parameters. This allowed us to construct strong relaxations for the robust gas network problem. To further increase the efficiency of our methods, we developed uncertainty-aware preprocessing techniques by adapting techniques for the nominal setting previously described in the literature.

This work is concluded with a series of benchmarks using realistic gas network instances from the freely available GASLIB in order to demonstrate the practical feasibility of our approach. By combining optimization-based preprocessing techniques with the aggregated model, we observed speedups of a factor of up to 400 when compared to models without preprocessing and aggregation. Further investigations regarding the influence of the strength of the relaxation on the objective function value suggested a relatively small decline in solution quality even for coarser relaxations. A mix of all developed techniques allowed us to obtain high quality robust feasible solutions for a large 40-node instance with both types of uncertainties in under a minute.

As an outlook, we remark that the reformulation of the abstract two-stage problem and the developed aggregation technique can be applied to any robust network problem that admits the studied structure. Moreover, the presented uncertainty-aware preprocessing and linearization techniques could be used on other potential driven network problems like e.g., water networks.

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