Uniqueness and Multiplicity of Market Equilibria on DC Power Flow Networks

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Abstract. We consider uniqueness and multiplicity of market equilibria in a short-run setup where traded quantities of electricity are transported through a capacitated network in which power flows have to satisfy the classical lossless DC approximation. The firms face fluctuating demand and decide on their production, which is constrained by given capacities. Today, uniqueness of such market outcomes are especially important in more complicated multilevel models for measuring market (in)efficiency. Thus, our findings are important prerequisites for such studies. We show that market equilibria are unique on tree networks under mild assumptions and we also present a priori conditions under which equilibria are unique on cycle networks. On general networks, uniqueness fails to hold and we present simple examples for which multiple equilibria exist. However, we prove different a posteriori criteria for the uniqueness of a given solution and thereby specify properties of unique solutions.

1. Introduction

We consider a short-run model for a liberalized power market in which producers and consumers trade electricity, which is then transported through a capacitated network. In our model, power flows are modeled by the classical lossless DC approximation of AC power flows. For this setting, we study questions of uniqueness and multiplicity of market equilibria on different types of networks like trees, cycles, and general networks. As usual, the wholesale electricity market is modeled by a mixed nonlinear complementarity system that is made up of the optimality conditions of the players of our market model and additional market clearing constraints; cf., e.g., Hobbs and Helman (2004) or the book by Gabriel et al. (2012). The players are electricity consumers with fluctuating and elastic demand, electricity producers that are constrained by given generation capacities, and the transmission system operator (TSO) who operates the network. While producers and consumers are only constrained by simple bound constraints, the network flows controlled by the TSO have to satisfy the lossless DC power flow model constraints. Thus, the TSO has to cope with loop flows, in particular. The consideration of such loop flows in power market models is of great practical importance. In Europe, the market organization is changed from capacity-based to flow-based market coupling (cf., e.g., Aguado et al. (2012) and Van den Bergh, Boury, et al. (2016)) and thus has to deal with loop flows. Moreover, nodal pricing is current practice in parts of the US and Canada; cf., e.g., Department of Energy (2017) and Ehrenmann and Neuhoff (2009).

Our focus in this paper is on questions regarding uniqueness and multiplicity of market equilibria on DC networks. Besides being a classical topic of mathematical economics, uniqueness of market outcomes is an important question both from a theoretical and practical point of view. In today's liberalized electricity markets,

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different agents make decisions that are based on the market design—e.g., nodal or zonal pricing. For instance, the TSOs make investment decisions depending on the anticipation of future market outcomes or regulators adjust more specific regulations—e.g., network fees—based on the properties of the underlying regime. Uniqueness of market outcomes typically is an important precondition for reasonable analyses of complementary decisions of the mentioned agents. In addition, today's operations research literature on energy markets often integrate models similar to the one discussed in this paper into more complex multilevel frameworks in order to evaluate more complicated market models; cf., e.g., Daxhelet and Smeers (2001), Hobbs, C. B. Metzler, et al. (2000), and Hu and Ralph (2007) and the references therein. These frameworks can, for instance, be of bi- or general multilevel type for evaluating the efficiency of specific market designs in which different players act. Moreover, the mentioned frameworks can also capture multiperiod settings in long-run models that incorporate investment decisions of the different players in the market. In both situations, uniqueness of the outcomes of the short-run model (e.g., used as the lower level in a bilevel model and thus considered as a parameterized optimization problem in dependence of the upper level's decisions) discussed in this paper is of particular importance for the theoretical study of the overall model as well as for the development of effective solution methods for these, typically hard, multilevel or multiperiod problems. For a detailed mathematical discussion of the importance of uniqueness of lower levels in multilevel models see, e.g., Dempe (2002); in particular Chapter 4. Multilevel models with a DC power flow model on a lower level can be found in, e.g., Daxhelet and Smeers (2001), Hobbs, C. B. Metzler, et al. (2000), Hu and Ralph (2007), and C. Ruiz and Conejo (2009) or in Grimm, Kleinert, et al. (2017), Grimm, Martin, et al. (2016), and Kleinert and Schmidt (2018), where the lower-level DC formulation also depends on upper-level network design decisions.

This paper builds on the paper Grimm, Schewe, et al. (2017) in which the authors analyze a comparable setting: On the one hand, they consider models of capacitated networks without DC power flow constraints. On the other hand, their model is a long-run model in which investment decisions of electricity producers in new generation capacity is also taken into account. In this paper, we only consider the short run but integrate a more detailed flow model into our setup. We contribute to the rich literature on liberalized power markets in general and on uniqueness questions in particular. For instance, C. Metzler et al. (2003) also consider power market equilibria that are constrained by a linear DC network model with arbitrage. The authors study bilateral contracts between producers and consumers in a Nash–Cournot setting. They also formulate their market model as a mixed linear complementarity problem (MLCP) and prove uniqueness of the corresponding equilibria. The mentioned paper builds on the article Hobbs (1999), in which an arbitrage-free Nash–Cournot model of bilateral and POOLCO markets constrained by a linear DC model is also formulated as a complementarity problem. In the latter paper, uniqueness aspects are not considered but mentioned for future work. Hobbs and Rijkers (2004) consider an oligopolistic market model with arbitrage and a linear DC network to analyze market power of generators. The resulting mixed complementarity model is further studied in J.-S. Pang et al. (2003), where the classical theory of linear complementarity problems (LCPs) is used to prove uniqueness; see Cottle et al. (2009) for a detailed presentation of this LCP theory. C. Ruiz and Conejo (2009) study a pool-based electricity market to determine the optimal offering strategy of a strategic power producer. The authors use a bilevel programming model in which the lower-level problem represents a welfare-maximizing market clearing with respect to a DC network model. This model is very similar to the one studied in this paper. However, uniqueness of solutions is not considered in C. Ruiz and Conejo (2009).

Another discussion using a model very similar to ours is given in Holmberg and Lazarczyk (2012). The authors compare nodal and zonal pricing schemes and prove uniqueness of a DC power flow based market model. However, they assume strictly convex cost functions so that uniqueness of the resulting strictly convex optimization problem follows from standard theory. Due to the effort of calibrating strictly convex models for computational studies, many authors refrain from this assumption and use linear cost functions; cf., e.g., Chao and Peck (1998), Ehrenmann and Smeers (2011), Gabriel et al. (2012), and Hobbs and J. S. Pang (2007) as well as Grimm, Kleinert, et al. (2017), Grimm, Martin, et al. (2016), and Grimm, Schewe, et al. (2017). Very recently, Bertsch et al. (2016) also consider a long-run model and study congestion management regimes in an inter-temporal equilibrium model. The authors of the latter paper discuss the importance of uniqueness of equilibria in such multiperiod models. However, a detailed analysis of this issue remains open and is only partly addressed by satisfying certain assumptions for which we show that they are not sufficient for uniqueness. This example together with our results on multiplicity of market equilibria on general networks indicates that both models and solution methods have to be chosen very carefully in this context—an issue that is also discussed in Wu et al. (1996). In summary, market model outcomes only seem to be proven unique for the mixed LCP case with arbitrage and for the case of strictly convex cost functions. However, the latter assumption is often not satisfied in computational equilibrium models as discussed above.

The main contribution of this paper is to close this gap in the literature: namely to study uniqueness and multiplicity of market equilibria that are subject to DC power flow networks without arbitrage and not necessarily strictly convex cost functions. We show that uniqueness of market equilibria on general networks typically fails to hold by presenting simple examples with multiple solutions. Furthermore, we characterize the situations in which multiple solutions appear. We can, however, prove uniqueness in special cases: Market equilibria are unique on radial, i.e., tree-like, networks where no loop flows need to be considered. This is a direct consequence of the results shown in Grimm, Schewe, et al. (2017). Moreover, we derive a priori conditions on cycle networks, i.e., conditions that solely rely on properties of the problem's data, under which we can prove uniqueness of equilibria. It turns out that these criteria both depend on production costs and the data of the network's lines. Finally, we prove a posteriori conditions for uniqueness on general networks. That means, the latter conditions can be used ex post to check whether a given solution is unique. Our results cover the case of perfect competition, which is a commonly used economic setting in the context of power market modeling; cf., e.g., Boucher and Smeers (2001) and Daxhelet and Smeers (2007). Since models of strategic behavior typically makes it much harder to establish uniqueness results, cf., e.g., Zöttl (2010), we refrain from discussing the case of imperfect competition—all the more in the light of multiplicity of equilibria that we already obtain under perfect competition in the case of general networks. In comparison to Grimm, Schewe, et al. (2017) the following is noteworthy: Both the market model with a simple network flow model studied in Grimm, Schewe, et al. (2017) yields unique solutions and, as we will show, the physics model studied in this paper without a market model yields unique solutions. However, the combination of both yields multiple solutions.

The rest of the paper is structured as follows. In Section 2 we present our market model both as a mixed nonlinear complementarity problem and as an equivalent optimization problem that we study in the following. Section 3 contains basic known and new results that are used throughout the rest of the paper. Afterward, Section 4 proves uniqueness for tree networks and Section 5 for cycle networks. Then, in Section 6 we show that multiple equilibria arise quite naturally on general networks, derive different a posteriori uniqueness conditions for the general case, and thereby describe properties of unique solutions. The paper closes with some concluding remarks in Section 7.

2. Market Equilibrium Modeling

We consider electricity networks that we model by using a connected digraph $G := (N, A)$ with node set N and arc set A. Subsequently, all player models of our overall market model are stated. Since we consider perfectly competitive markets, all players are price takers and their optimization problems are formulated using exogenously given market prices π_u at every node $u \in N$. The model is based on standard electricity market models as discussed in, e.g., Gabriel et al. (2012) and Hobbs and Helman (2004).

The first type of players are electricity producers. Without loss of generality, we assume that there exists exactly one producer at each node $u \in N$, which we model by a fixed generation capacity $\bar{y}_u > 0$ and variable production costs $w_u > 0$. The assumption of a single producer per node is only used to simplify the presentation. In practice, multiple producers at one node can simply be split by introducing artificial nodes that are connected to the original node by lines with "infinite" capacity.

For later references we state the following assumption on the variable production costs.

Assumption 1. *All variable production costs w_u, $u \in N$, are pairwise distinct.*

This assumption is obviously required for proving uniqueness of market equilibria since, otherwise, producers cannot be sufficiently distinguished from each other. Hence, the assumption is frequently used in the literature on peak-load pricing (cf., e.g., Crew et al. (1995) for a survey) and on power markets. For the latter see, e.g., Grimm, Schewe, et al. (2017) or Bertsch et al. (2016), where this assumption is explicitly discussed in the context of power markets. Moreover, even if the assumption is not discussed explicitly, it is usually satisfied in the numerical experiments; cf. Hobbs (1999) and P. A. Ruiz and Rudkevich (2010) for examples. Despite the necessity of the assumption for uniqueness, we want to briefly highlight two possibilities for tackling situations in which the assumption is not satisfied. The first one a priorily resolves this situation by perturbing those variable costs that violate the assumption. A clear disadvantage of this strategy is that "the winner" of this perturbation will fully benefit since the entire ambiguous production will be realized by this producer due to its lower perturbed variable costs. Second, problem-tailored tie breaking rules can be applied to resolve ambiguous production situations in non-unique market equilibria. By doing so, fairness considerations can typically be taken into account more easily compared to perturbation strategies.

Production at node u is denoted by $y_u \geq 0$ and bounded from above by the generation capacity. The objective of the producer at node u is to maximize its profit and, thus, its linear optimization problem reads

$$\max_{y_u} \quad (\pi_u - w_u)\, y_u \quad \text{s.t.} \quad 0 \leq y_u \leq \bar{y}_u.$$

Its solutions are characterized by the corresponding Karush–Kuhn–Tucker (KKT) conditions

$$\pi_u - w_u + \beta_u^- - \beta_u^+ = 0, \quad 0 \leq y_u \perp \beta_u^- \geq 0, \quad 0 \leq \bar{y}_u - y_u \perp \beta_u^+ \geq 0, \qquad (1)$$

where $\beta_u^{\pm}$ are the dual variables of the production constraints. Here and in what follows, we use the standard $\perp$-notation, which abbreviates

$$0 \le a \perp b \ge 0 \iff 0 \le a,\, b \ge 0,\, ab = 0.$$

Consumers, as our second players, are also located at the nodes $u \in N$ and decide on their demand $d_u \ge 0$. Their demand elasticity is modeled by inverse demand functions $p_u : \mathbb{R}_{\ge 0} \to \mathbb{R}$, for which we make the following assumption.

Assumption 2. *All inverse demand functions p_u, $u \in N$, are strictly decreasing and continuously differentiable.*

Under Assumption 2 the concave problem of a surplus maximizing consumer at node u is given by

$$\max_{d_u} \quad \int_0^{d_u} p_u(x)\,\mathrm{d}x - \pi_u d_u \quad \text{s.t.} \quad 0 \le d_u$$

and its again necessary and sufficient first-order optimality conditions comprise

$$p_u(d_u) - \pi_u + \alpha_u = 0, \quad 0 \le d_u \perp \alpha_u \ge 0, \tag{2}$$

where α_u is the dual variable of the lower demand bound.

The third player in our market model is the transmission system operator (TSO). He operates the transmission network, in which every arc $a \in A$ is described by its susceptance $B_a > 0$ and its transmission capacity $f_a^+ > 0$. The latter bounds the flows f_a by $|f_a| \le f_a^+$. The goal of the TSO is to transport electricity from low- to high-price regions and the earnings to be maximized result from the corresponding price differences; cf., e.g., Hobbs and Helman (2004). Power flow in the network is modeled using the standard linear lossless DC approximation—cf., e.g., Van den Bergh, Delarue, et al. (2014) and Wood et al. (2014)—which is often used for economic analysis, e.g., in Ehrenmann and Neuhoff (2009) and Jing-Yuan and Smeers (1999). Thus, we obtain additional phase angle variables Θ_u for all nodes $u \in N$. With this notation the linear problem of the TSO reads

$$\max_{f,\Theta} \quad \sum_{a=(u,v)\in A} (\pi_v - \pi_u) f_a \tag{3a}$$

$$\text{s.t.} \quad -f_a^+ \le f_a \le f_a^+, \quad a \in A, \tag{3b}$$

$$f_a = B_a(\Theta_u - \Theta_v), \quad a = (u,v) \in A. \tag{3c}$$

Here and in what follows, a variable without index denotes the vector containing all corresponding node or arc variables, e.g., $f := (f_a)_{a\in A}$. Constraints (3c) model the linear lossless DC flow approximation and ε_a are the corresponding dual variables. Constraints (3b) reflect the network's transmission capacities and have the dual variables $\delta_a^{\pm}$. The optimality conditions of (3) are given by

$$\begin{aligned} \pi_v - \pi_u + \delta_a^- - \delta_a^+ + \varepsilon_a = 0, &\quad a = (u,v) \in A,\\ \sum_{a\in\delta^{\text{out}}(u)} B_a\varepsilon_a - \sum_{a\in\delta^{\text{in}}(u)} B_a\varepsilon_a = 0, &\quad u \in N,\\ f_a = B_a(\Theta_u - \Theta_v), &\quad a = (u,v) \in A,\\ 0 \le f_a + f_a^+ \perp \delta_a^- \ge 0, &\quad a \in A,\\ 0 \le f_a^+ - f_a \perp \delta_a^+ \ge 0, &\quad a \in A. \end{aligned} \tag{4}$$

Here we use the standard δ-notation for the in- and outgoing arcs of a node $u \in N$, i.e., $\delta^{\text{in}}(u) := \{(v,u) \in A\}$ and $\delta^{\text{out}}(u) := \{(u,v) \in A\}$.

Putting all first-order optimality conditions as well as the flow balance conditions

$$0 = d_u - y_u + \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a, \quad u \in N, \tag{5}$$

together, we obtain the mixed complementarity problem

$$\text{Producers: (1),} \quad \text{Consumers: (2),} \quad \text{TSO: (4),} \quad \text{Market Clearing: (5),} \tag{6}$$

which models the wholesale electricity market under consideration for the case of perfect competition. Hence, solutions of (6) are market equilibria. It can be easily seen that this complementarity system is equivalent to the welfare maximization problem

$$\max_{d,y,f,\Theta} \quad \sum_{u \in N} \int_0^{d_u} p_u(x) \, \mathrm{d}x - \sum_{u \in N} w_u y_u \tag{7a}$$

$$\text{s.t.} \quad 0 \le y_u \le \bar{y}_u, \quad u \in N, \tag{7b}$$

$$0 \le d_u, \quad u \in N, \tag{7c}$$

$$-f_a^+ \le f_a \le f_a^+, \quad a \in A, \tag{7d}$$

$$0 = d_u - y_u + \sum_{a \in \delta^{\text{out}}(u)} f_a - \sum_{a \in \delta^{\text{in}}(u)} f_a, \quad u \in N, \tag{7e}$$

$$f_a = B_a(\Theta_u - \Theta_v), \quad a = (u, v) \in A. \tag{7f}$$

The equivalence can be shown by comparing the first-order optimality conditions of Problem (7) with the mixed complementarity system (6) and by identifying the dual variables γ_u of the flow balance constraints (7e) with the equilibrium prices π_u of the complementarity problem. Further, we need that the KKT conditions are again necessary and sufficient optimality conditions of Problem (7) under Assumption 2.

So far we formulated a short-run market model that does not depend on multiple scenarios or time periods. This is a simplification that we make for the ease of presentation—but no restriction of generality. Nothing changes when more than one scenario is considered. It leads to a time-separable optimization problem that consists of problems of type (7) for every time period because there are no constraints coupling different time periods.

From now on we study the uniqueness of a solution of Problem (7). Existence is trivial because $(d, y, f, \Theta) = (0, 0, 0, 0)$ is feasible and the problem is bounded from above. Before we start with our uniqueness considerations, we note that replacing the linear cost functions $w_u y_u$ in (7a) by strictly convex cost functions $w_u(y_u)$ yields a strictly concave maximization problem in the space of demand and production variables. The next theorem is then a direct consequence of Mangasarian (1988).

Theorem 2.1. *Suppose Assumption 2 holds. Consider Problem (7) with strictly convex cost functions. Then, the solution of Problem (7) is unique in (d, y).*

In the rest of the paper, we therefore consider the case without the assumption of strictly convex cost functions.

3. Basic Results

In this section we collect and prove auxiliary results concerning the uniqueness of the flows, phase angles, and demands of Problem (7). We use the theory of an earlier work on gas networks of Ríos-Mercado et al. (2002) to obtain uniqueness of flows and phase angles for given supply and demand decisions. For the following, we need the next assumption.

Assumption 3. *The demands d and productions y satisfy (7b), (7c), and $\sum_{u \in N}(y_u - d_u) = 0$. Additionally, the phase angle Θ_r at an arbitrary node $r \in N$ is fixed.*

The node r in Assumption 3 is called the reference node of the network. Using Theorem 2 of Ríos-Mercado et al. (2002) we directly get the following result.

Theorem 3.1. *Suppose Assumption 3 holds. If a solution of System (7d)–(7f) exists, then it is unique.*

Actually, we are interested in the uniqueness of a solution of the entire Problem (7) and not only of Subsystem (7d)–(7f). However, to answer the question of uniqueness of Problem (7), we can now restrict ourselves to study under which conditions the solution of Problem (7) is unique in the demands and productions. This is the assertion of the following corollary.

Corollary 3.2. *Let $r \in N$ be a reference node with fixed phase angle Θ_r. Further, let the demands and productions in the solutions of Problem (7) be unique. Then, Problem (7) has a unique solution.*

Proof. Let d and y be the unique demands and productions in a solution of Problem (7). Thus, d and y satisfy Assumption 3. The solutions (f, Θ) of System (7d)–(7f) are exactly the flows and phase angles corresponding to the demands d and productions y such that (d, y, f, Θ) is a solution of Problem (7). The existence of a solution of System (7d)–(7f) is a consequence of the existence of a solution of Model (7). Hence, applying Theorem 3.1 yields a unique flow and unique phase angles w.r.t. the fixed phase angle at node r. Thus, the solution of Problem (7) is unique. □

Next, we state uniqueness of the demands, which follows directly from Corollary 3 in Mangasarian (1988).

Theorem 3.3. *Suppose Assumption 2 holds. Let (d, y, f, Θ) and (d', y', f', Θ') be two solutions of Problem (7). Then, $d = d'$ holds.*

Both previous theorems together state that we only have to consider the productions in a solution of Problem (7) to obtain uniqueness of the overall solution. With the following lemma, which is mainly taken from Grimm, Schewe, et al. (2017), it is sufficient to show uniqueness of productions for fixed binding production and flow bounds (7b), (7d).

Lemma 3.4. *Suppose Assumption 2 holds. Then, exactly one of the two following cases occurs:*

(a) There exist demands d^* and productions y^* such that every solution of Problem (7) is of the form (d^*, y^*, f, Θ) for some flow f and phase angles Θ.

(b) There exist two solutions $z := (d, y, f, \Theta)$ and $z' := (d, y', f', \Theta')$ of Problem (7) with $y \neq y'$ and

$$\begin{aligned}
\left\{a \in A : f_a = -f_a^+\right\} &= \left\{a \in A : f'_a = -f_a^+\right\}, \\
\left\{a \in A : f_a = f_a^+\right\} &= \left\{a \in A : f'_a = f_a^+\right\}, \\
\{u \in N : y_u = 0\} &= \{u \in N : y'_u = 0\}, \\
\{u \in N : y_u = \bar{y}_u\} &= \{u \in N : y'_u = \bar{y}_u\}.
\end{aligned}$$

The next lemma contains a helpful fact about the production differences of two distinct solutions.

Lemma 3.5. *Suppose Assumption 1 and $|N| \geq 3$ hold. Let $z := (d, y, f, \Theta)$ and $z' := (d, y', f', \Theta')$ be two solutions of Problem (7) with different productions $y \neq y'$. Then, y and y' differ at least in three entries.*

Proof. Summing up the flow balance constraints (7e) for all nodes yields

$$\sum_{u\in N} y_u = \sum_{u\in N} d_u = \sum_{u\in N} y'_u \tag{8}$$

because demands are the same in both solutions. Assume y and y' differ at exactly one node, i.e., $y_v \neq y'_v$ for a node $v \in N$ and $y_u = y'_u$ for all $u \in N \setminus \{v\}$. This directly contradicts (8). Assume now that the productions y and y' differ at exactly two nodes $u \neq v \in N$. Then, (8) implies

$$y_u + y_v = y'_u + y'_v \iff y_u - y'_u = y'_v - y_v. \tag{9}$$

As z and z' are both solutions of Problem (7), they have the same objective function value, i.e.,

$$\sum_{i\in N}\int_0^{d_i} p_i(x)\,\mathrm{d}x - \sum_{i\in N} w_i y_i = \sum_{i\in N}\int_0^{d_i} p_i(x)\,\mathrm{d}x - \sum_{i\in N} w_i y'_i.$$

Uniqueness of demands at all nodes and of productions at nodes $i \in N\setminus\{u,v\}$ yields

$$w_u y_u + w_v y_v = w_u y'_u + w_v y'_v \iff w_u(y_u - y'_u) = w_v(y'_v - y_v). \tag{10}$$

Substituting (9) in (10) results in $w_u(y_u - y'_u) = w_v(y_u - y'_u)$, which implies $w_u = w_v$. This is a contradiction to Assumption 1. Hence, the productions y and y' differ at least at three nodes. □

We directly obtain the following corollary.

Corollary 3.6. *Suppose Assumptions 1 and 2 hold and let G be a network with $|N| \leq 2$. Then, the solution of Problem (7) is unique w.r.t. a fixed phase angle.*

With the preliminaries we can now prove uniqueness on trees.

4. Trees

We now establish a uniqueness result for Problem (7) for radial transmission networks, i.e., for the case of tree-like networks. For this purpose, we use existing uniqueness results given in Grimm, Schewe, et al. (2017) for a similar problem on capacitated networks without DC constraints, i.e., Problem (7a)–(7e). To this end, the next two lemmas consider the relation between the solutions of Problem (7) and its relaxation (7a)–(7e). Note that a solution of (7a)–(7e) only involves the variables d, y, and f; the phase angles Θ are not taken into account.

Lemma 4.1. *Suppose the network $G = (N, A)$ is a tree. Let (d, y, f) be a solution of (7a)–(7e) and let $r \in N$ be an arbitrary reference node with fixed phase angle Θ_r. Then, there exist unique phase angles Θ_u for all nodes $u \in N \setminus \{r\}$ such that (d, y, f, Θ) is a solution of Problem (7).*

Proof. Let $u \in N \setminus \{r\}$ be an arbitrary node. Since G is a tree, there is a unique path $P_{ru} := (r = v_1, \ldots, v_{k+1} = u)$ from r to u and we have

$$\Theta_u = \Theta_r - \sum_{i=1}^{k} (\Theta_{v_i} - \Theta_{v_{i+1}}). \tag{11}$$

Using the DC flow conditions (7f), the difference $\Theta_{v_i} - \Theta_{v_{i+1}}$ is given by

$$\Theta_{v_i} - \Theta_{v_{i+1}} = \begin{cases} f_{v_i v_{i+1}}/B_{v_i v_{i+1}}, & \text{if } (v_i, v_{i+1}) \in A, \\ -f_{v_{i+1} v_i}/B_{v_{i+1} v_i}, & \text{if } (v_{i+1}, v_i) \in A, \end{cases} \tag{12}$$

where the right-hand side is uniquely determined by the flow f. Hence, all Θ_u with $u \in N \setminus \{r\}$ satisfying (7f) are determined by (11) and (12) because the phase angle Θ_r is fixed. The uniqueness of each Θ_u w.r.t. Θ_r is a consequence of the uniqueness of the path P_{ru} from r to u in G. This shows that a solution (d, y, f) of Problem (7a)–(7e) can be uniquely extended to a feasible point (d, y, f, Θ) of Problem (7). Optimality of (d, y, f, Θ) follows directly because the objective is independent of Θ. □

Lemma 4.2. *Suppose the network is a tree. Let (d, y, f, Θ) be a solution of Problem (7). Then, (d, y, f) is a solution of the relaxation (7a)–(7e).*

Proof. As (d, y, f, Θ) is feasible for (7b)–(7f), (d, y, f) is feasible for (7b)–(7e). Assume that (d, y, f) is not optimal and let (d', y', f') be a solution of (7a)–(7e). With Lemma 4.1 we obtain unique phase angles Θ' w.r.t. a fixed phase angle such that (d', y', f', Θ') is a solution of Problem (7). Since (d, y, f, Θ) is also a solution of this problem, the objective function values are the same and independent of the phase angles. Hence, (d, y, f) and (d', y', f') have the same objective function value. Consequently, (d, y, f) is a solution of (7a)–(7e) as well. □

Lemma 4.1 and Lemma 4.2 state that (d, y, f, Θ) is a solution of (7) if and only if (d, y, f) is a solution of (7a)–(7e). This means that the DC flow conditions do not matter if the network is a tree. Finally, using the results of Grimm, Schewe, et al. (2017) we achieve the following uniqueness theorem.

Theorem 4.3. *Let the network be a tree. Suppose Assumptions 1 and 2 hold. Then, Problem (7) has a unique solution w.r.t. a reference node with fixed phase angle.*

Proof. Let (d, y, f, Θ) be a solution of Problem (7). By use of Lemma 4.2, (d, y, f) is a solution of (7a)–(7e). As shown in Grimm, Schewe, et al. (2017), Problem (7a)–(7e) has unique demand and production solutions. Thus, every solution of (7a)–(7e) has demands d and productions y. With this it also follows that every solution of (7) has demands d and productions y. Finally, Theorem 3.2 yields the claim. □

5. Cycles

We now consider the question of uniqueness of Problem (7) if the network is a cycle. We limit ourselves to cycles $G = (N, A)$ with node set $N := \{1, \dots, n\}$ and $n = |N| \geq 3$, as Corollary 3.6 already yields uniqueness for cycles with two nodes. Moreover, we use the arc set $A := \{(1, 2), (2, 3), \dots, (n, 1)\}$. This is possible because it is easy to verify that a change of an arc direction only results in an inverted sign of the flow on this arc in a solution of Problem (7). Finally, whenever we write $(n, n + 1)$ we mean the arc $(n, 1)$ in G.

In Section 4 we applied existing uniqueness results for Problem (7) without the DC flow conditions (7f) shown in Grimm, Schewe, et al. (2017) to prove uniqueness if the network is cycle-free. However, the techniques in Grimm, Schewe, et al. (2017) are not applicable in the DC case with cycles. Consequently, we need another strategy to establish a uniqueness result if the network is a cycle or contains cycles. The key is to exclude special relations between the variable production costs and the susceptances. To this end, we introduce the following definition.

Definition 5.1. Let the network $G = (N, A)$ be a cycle. For the directed path $P_{ij} := (i, (i, i + 1), i + 1, \dots, (j - 1, j), j)$ from node $i \in N$ to node $j \in N$ we define

$$S_{ij} := \sum_{a \in P_{ij}} \frac{1}{B_a} \quad \text{and} \quad S_{ii} := 0.$$

Note that in a cycle we have two paths from one node to another node. In what follows, we always use the path that traverses the nodes in order.

The next assumption summarizes the conditions on susceptances and variable production costs, which we need to prove uniqueness of Problem (7).

Assumption 4. *Let $G = (N, A)$ be a cycle with $|N| \geq 3$. Further, let $P_{i\ell}$ be a path with $i \neq \ell \in N$ and let j, k be internal nodes of $P_{i\ell}$ such that either $j = k$ or j appears before k in $P_{i\ell}$. Then*

$$(w_i - w_j)S_{k\ell} \neq (w_k - w_\ell)S_{ij} \tag{13}$$

holds.

Before we formally prove uniqueness of equilibria on cycles under Assumption 4, let us briefly discuss the necessity of the assumption in an informal way. From the point of view of the TSO, sending flow through the network is driven by two aspects—an economic and a physical one. First, the TSO tries to send flow from low- to high-price nodes; cf. the objective function (3a). Since the nodal prices are determined by the nodal variable production costs, this aspect is covered in the Conditions (13). Second, flows on cycles are driven by the DC power flow law (3c) that depends on the line's susceptances. Thus, this aspect is also contained in the Conditions (13). In specific situations, these two aspects may be related to each other such that variable cost differences and cycle flow laws allow for multiple equilibria. To exclude these situations is the aim of the Conditions (13). Lastly, we remark that the necessity of Assumption 4 is also highlighted by Example 5.5, which possesses multiple solutions due to the violation of Assumption 4.

For the following we need the concept of flow-induced components, which has already been used in Grimm, Schewe, et al. (2017). A flow-induced partition of the network $G = (N, A)$ w.r.t. a solution (d, y, f, Θ) of Problem (7) is the partition $\{G^i\}_{i\in I}$, $I \subseteq \mathbb{N}$, where each $G^i := (N^i, A^i)$ is a connected component of the graph $(N, A \setminus A^{\text{sat}})$ with $A^{\text{sat}} := \{a \in A : |f_a| = f_a^+\}$. Each G^i, $i \in I$, is called a flow-induced component.

In the two following lemmas we state properties of a flow-induced partition of a cycle. These are the key ingredients of the proof of our main uniqueness theorem for cycles. The first lemma ensures that in each flow-induced component of a solution exist at most two nodes that do not produce on the lower or upper production bounds (7b).

Lemma 5.2. *Suppose Assumption 4 holds. Let $z := (d, y, f, \Theta)$ be a solution of Problem (7) and let $\{G^i := (N^i, A^i)\}_{i\in I}$, $I \subseteq \mathbb{N}$, be its flow-induced partition. Then, $|\{u \in N^i : 0 < y_u < \bar{y}_u\}| \leq 2$ for all $i \in I$ holds.*

Proof. Assume that there exists a flow-induced component in which at least at three nodes both production bounds (7b) are strictly satisfied. Let $i, j, k \in N$ be three such distinct nodes with $j \in P_{ik}$. Thus,

$$\begin{aligned} &0 < y_i < \bar{y}_i, \quad 0 < y_j < \bar{y}_j, \quad 0 < y_k < \bar{y}_k, \\ &-f_a^+ < f_a < f_a^+ \quad \text{for all } a \in P_{ik} \end{aligned} \tag{14}$$

holds. Now we construct a feasible point $z' := (d, y', f', \Theta')$ with a larger objective function value than the one of z. This will contradict the optimality of z and the assertion of Lemma 5.2 follows. To this end, we define for $u \in N$

$$y'_u := \begin{cases} y_u + \Delta y_u, & \text{if } u \in \{i, j, k\}, \\ y_u, & \text{otherwise,} \end{cases}$$

where $\Delta y_u \in \mathbb{R}$, $u \in \{i, j, k\}$, denote the production differences at nodes i, j, and k between z and z' with

$$\Delta y_i + \Delta y_j + \Delta y_k = 0. \tag{15}$$

The latter holds because we do not change the demands and thus the total amount of production must remain the same. Figure 1 (left) illustrates the considered scenario.

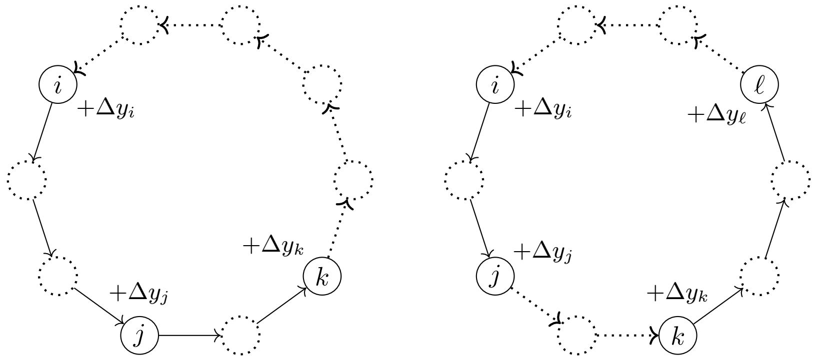

FIGURE 1. Illustration of the scenario in proof of Lemma 5.2 (left) and of Lemma 5.3 (right). Solid lines mark nodes and arcs with inactive production and flow bounds.

The flow f'_a for $a \in A$ is given by

$$f'_a := \begin{cases} f_a + \Delta y_i, & \text{if } a \in P_{ij}, \\ f_a + \Delta y_i + \Delta y_j, & \text{if } a \in P_{jk}, \\ f_a, & \text{otherwise,} \end{cases}$$

and the phase angles Θ'_u, $u \in N$, depending on the phase angle Θ'_i, read

$$\Theta'_u := \begin{cases} \Theta_u - \Theta_i + \Theta'_i - \Delta y_i S_{iu}, & \text{if } u \in \{i+1, \dots, j\}, \\ \Theta_u - \Theta_i + \Theta'_i - \Delta y_i S_{iu} - \Delta y_j S_{ju}, & \text{if } u \in \{j+1, \dots, k\}, \\ \Theta_u - \Theta_i + \Theta'_i, & \text{otherwise.} \end{cases}$$

The following calculation shows that f' and Θ' fulfill the DC flow conditions (7f) if the production differences Δy_i and Δy_j satisfy

$$\Delta y_j = -\frac{S_{ik}}{S_{jk}} \Delta y_i. \tag{16}$$

For $a := (u, v) \in A$ we have

$$B_a(\Theta'_u - \Theta'_v) = B_a(\Theta_u - \Theta_v) + \begin{cases} \Delta y_i, & \text{if } a \in P_{ij}, \\ \Delta y_i + \Delta y_j, & \text{if } a \in P_{jk}, \\ B_a(-\Delta y_i S_{ik} - \Delta y_j S_{jk}), & \text{if } a = (k, k+1), \\ 0, & \text{if } a \in P_{k+1,i}. \end{cases}$$

Since z is feasible for Problem (7), it is $f_a = B_a(\Theta_u - \Theta_v)$ and, as a consequence, the DC flow conditions (7f) are satisfied for z' if (16) holds.

For z' being feasible for Problem (7) the production and flow bounds (7b), (7d) have to be satisfied. To this end, we first substitute Δy_j and Δy_k in f' and y' by Δy_i using the Equations (15) and (16). This yields

$$y'_u := \begin{cases} y_i + \Delta y_i, & \text{if } u = i, \\ y_j - \frac{S_{ik}}{S_{jk}} \Delta y_i, & \text{if } u = j, \\ y_k + \frac{S_{ij}}{S_{jk}} \Delta y_i, & \text{if } u = k, \\ y_u, & \text{otherwise,} \end{cases} \qquad f'_a := \begin{cases} f_a + \Delta y_i, & \text{if } a \in P_{ij}, \\ f_a - \frac{S_{ij}}{S_{jk}} \Delta y_i, & \text{if } a \in P_{jk}, \\ f_a, & \text{otherwise.} \end{cases}$$

Then, we deduce the bounds on Δy_i such that all production and flow bounds for y' and f' are satisfied, i.e.,

$$0 \le y_i + \Delta y_i \le \bar{y}_i, \quad 0 \le y_j - \frac{S_{ik}}{S_{jk}} \Delta y_i \le \bar{y}_j, \quad 0 \le y_k + \frac{S_{ij}}{S_{jk}} \Delta y_i \le \bar{y}_k,$$

and

$$-f_a^+ \le f_a + \Delta y_i \le f_a^+, \quad a \in P_{ij}, \quad -f_a^+ \le f_a - \frac{S_{ij}}{S_{jk}} \Delta y_i \le f_a^+, \quad a \in P_{jk}.$$

Altogether, we obtain

$$\begin{aligned} \Delta y_i \le \min \Bigg\{ &\bar{y}_i - y_i, \min_{a \in P_{ij}} (f_a^+ - f_a), \, y_j \frac{S_{jk}}{S_{ik}}, \\ &(\bar{y}_k - y_k) \frac{S_{jk}}{S_{ij}}, \min_{a \in P_{jk}} (f_a^+ + f_a) \frac{S_{jk}}{S_{ij}} \Bigg\} > 0, \end{aligned} \tag{17a}$$

$$\begin{aligned} \Delta y_i \ge \max \Bigg\{ &-y_i, \max_{a \in P_{ij}} (-f_a^+ - f_a), \, (y_j - \bar{y}_j) \frac{S_{jk}}{S_{ik}}, \\ &-y_k \frac{S_{jk}}{S_{ij}}, \max_{a \in P_{jk}} (-f_a^+ + f_a) \frac{S_{jk}}{S_{ij}} \Bigg\} < 0. \end{aligned} \tag{17b}$$

The upper bound (17a) for Δy_i is positive and the lower bound (17b) is negative because (14) holds. In the next step, we determine how we can choose Δy_i such that the objective function value of z' is larger than the one of z. To this end, the difference of the objective function values of z and z' yields the condition $w_i \Delta y_i + w_j \Delta y_j + w_k \Delta y_k < 0$. Using (15) and (16), this is equivalent to

$$\left[(w_i - w_k) - (w_j - w_k) \frac{S_{ik}}{S_{jk}} \right] \Delta y_i < 0.$$

Due to $S_{ik} = S_{ij} + S_{jk}$ it is $0 \neq (w_i - w_k) - (w_j - w_k) S_{ik}/S_{jk}$ equivalent to $0 \neq (w_i - w_j) S_{jk} - (w_j - w_k) S_{ij}$, which holds under Assumption 4. Thus, we find a suitable $\Delta y_i \neq 0$ such that z' is feasible for Problem (7) and has a larger objective function value than z. This contradicts that z is optimal and, thus, yields the claim. □

The next lemma shows that there exists at most one flow-induced component of G with respect to a solution of (7) in which at least two nodes do not produce on the lower or upper production bound.

Lemma 5.3. *Suppose Assumption 4 holds. Let $z := (d, y, f, \Theta)$ be a solution of Problem (7) and let $\{G^i := (N^i, A^i)\}_{i \in I}$, $I \subseteq \mathbb{N}$, be its flow-induced partition. Then, $|\{i \in I : \exists u \neq v \in N^i, \ 0 < y_u < \bar{y}_u, \ 0 < y_v < \bar{y}_v\}| \le 1$ holds.*

Proof. Assume there exist two distinct flow-induced components $P_{u_1 v_1}$ and $P_{u_2 v_2}$, each containing at least two nodes at which the production bounds in z are not binding. We again show that this yields a contradiction by constructing a feasible point $z' := (d, y', f', \Theta')$ of Problem (7) with larger objective function value than z. Let

$$\begin{aligned} i \neq j \in P_{u_1 v_1} \quad &\text{with} \quad 0 < y_i < \bar{y}_i, \quad 0 < y_j < \bar{y}_j, \\ k \neq \ell \in P_{u_2 v_2} \quad &\text{with} \quad 0 < y_k < \bar{y}_k, \quad 0 < y_\ell < \bar{y}_\ell, \end{aligned} \tag{18}$$

and assume that node i is located before node j in $P_{u_1 v_1}$ and that node k is located before node ℓ in $P_{u_2 v_2}$. As $P_{u_1 v_1}$ and $P_{u_2 v_2}$ are flow-induced components, we further have $-f_a^+ < f_a < f_a^+$ for all $a \in P_{u_1 v_1} \cup P_{u_2 v_2}$. We modify the productions at the four nodes i, j, k, ℓ by

$$y'_u := \begin{cases} y_u + \Delta y_u, & \text{if } u \in \{i, j, k, \ell\}, \\ y_u, & \text{otherwise,} \end{cases}$$

and we define the flow f'_a for each arc $a \in A$ as

$$f'_a := \begin{cases} f_a + \Delta y_i, & \text{if } a \in P_{ij}, \\ f_a + \Delta y_k, & \text{if } a \in P_{k\ell}, \\ f_a, & \text{otherwise.} \end{cases}$$

Figure 1 (right) illustrates the considered scenario. The flow balance conditions (7e) for z' are satisfied if

$$\Delta y_j = -\Delta y_i \quad \text{and} \quad \Delta y_\ell = -\Delta y_k \tag{19}$$

holds. The phase angles Θ'_u, $u \in N$, depending on the phase angle Θ'_i, read

$$\Theta'_u := \begin{cases} \Theta_u - \Theta_i + \Theta'_i - \Delta y_i S_{iu}, & \text{if } u \in \{i+1, \dots, j\}, \\ \Theta_u - \Theta_i + \Theta'_i - \Delta y_i S_{ij}, & \text{if } u \in \{j+1, \dots, k\}, \\ \Theta_u - \Theta_i + \Theta'_i - \Delta y_i S_{ij} - \Delta y_k S_{ku}, & \text{if } u \in \{k+1, \dots, \ell\}, \\ \Theta_u - \Theta_i + \Theta'_i, & \text{otherwise.} \end{cases}$$

Similarly to the proof of Lemma 5.2 we can show that z' fulfills the DC flow conditions (7f) if the relation

$$\Delta y_k = -\frac{S_{ij}}{S_{k\ell}} \Delta y_i \tag{20}$$

holds. In order to be feasible, z' also has to satisfy the production and flow bounds (7b), (7d). Replacing in y' and f' all occurring Δy_j, Δy_k, and Δy_ℓ by Δy_i using (19) and (20) we obtain—as in the proof of Lemma 5.2—the following bounds for Δy_i:

$$\begin{aligned} \Delta y_i \leq \min \Bigg\{ & \bar{y}_i - y_i, y_j, \min_{a \in P_{ij}} (f_a^+ - f_a), \frac{S_{k\ell}}{S_{ij}} y_k, \\ & \frac{S_{k\ell}}{S_{ij}} (\bar{y}_\ell - y_\ell), \min_{a \in P_{k\ell}} (f_a^+ + f_a) \frac{S_{k\ell}}{S_{ij}} \Bigg\} > 0, \end{aligned} \tag{21a}$$

$$\begin{aligned} \Delta y_i \geq \max \Bigg\{ & -y_i, y_j - \bar{y}_j, \max_{a \in P_{ij}} (-f_a^+ - f_a), \frac{S_{k\ell}}{S_{ij}} (y_k - \bar{y}_k), \\ & -\frac{S_{k\ell}}{S_{ij}} y_\ell, \max_{a \in P_{k\ell}} (f_a - f_a^+) \frac{S_{k\ell}}{S_{ij}} \Bigg\} < 0. \end{aligned} \tag{21b}$$

As Conditions (18), (19) are satisfied, the upper bound (21a) for Δy_i is positive and the lower bound (21b) is negative. We now choose Δy_i to obtain a larger objective function value for z' than for z. To this end, the objective function difference must be

$$0 > w_i \Delta y_i + w_j \Delta y_j + w_k \Delta y_k + w_\ell \Delta y_\ell = (w_i - w_j) \Delta y_i + (w_k - w_\ell) \Delta y_k,$$

which is, due to (20), equivalent to $[(w_i - w_j) - (w_k - w_\ell) S_{ij} / S_{k\ell}] \Delta y_i < 0$. Since $(w_i - w_j) S_{k\ell} - (w_k - w_\ell) S_{ij} \neq 0$ by Assumption 4, we always find a suitable $\Delta y_i \neq 0$ such that z' has a larger objective function value than z. This contradicts the optimality of z. □

We are now ready to prove our main uniqueness result for cycles.

Theorem 5.4. *Suppose Assumptions 1, 2, and 4 hold. Then, the solution of Problem (7) is unique w.r.t. a reference node with fixed phase angle.*

Proof. Using Theorem 3.3, the demands are unique. Assume that the productions are not unique. Then, Lemma 3.4 states that there exist two solutions $z := (d, y, f, \Theta)$ and $z' := (d, y', f', \Theta')$ with $y \neq y'$ such that z and z' have the same binding flow and production bounds. Thus, z and z' have the same flow-induced partition

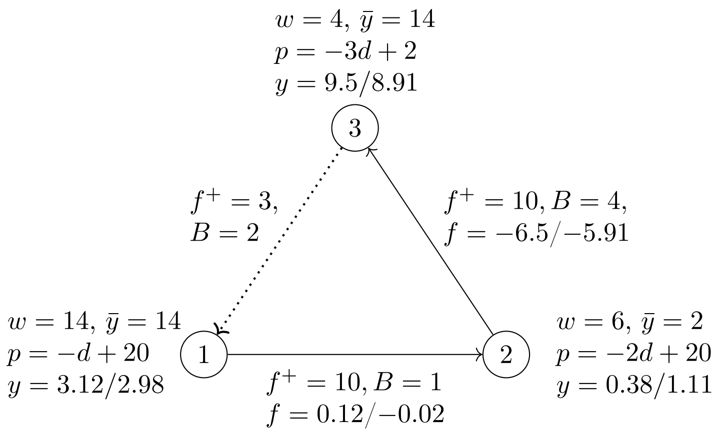

FIGURE 2. Three node cycle with multiple solutions. Deleting the dotted arc yields the flow-induced partition of the solutions listed in Table 1. Variable costs are denoted by w, capacities by $\bar{y}$ and f^+, inverse demand functions by p, and flow and production values of the two solutions listed in Table 1 by f and y.

$\{G^i := (N^i, A^i)\}_{i\in I}$ and summing up the flow balance conditions (7e) over all nodes in a component yields $\sum_{u\in N^i} y_u = \sum_{u\in N^i} y'_u$ for all $i \in I$. As a consequence, the production differences between y and y' only occur in components G^i, $i \in I$, in which at least at two distinct nodes both production bounds are strictly fulfilled. Using Lemma 3.5, the productions y and y' differ at least at three distinct nodes. Due to Lemma 5.2 the number of nodes $u \in N^i$ at which $0 < y_u, y'_u < \bar{y}_u$ holds is at most two for each component $i \in I$. This means that there must be at least two flow-induced components in which at two nodes the production bounds are not binding in z and z'. This is not possible because we stated in Lemma 5.3 that there is at most one component G^i in which the production at two nodes is not on the lower or upper capacity bound. Consequently, there cannot exist two solutions z, z' of Problem (7) with $y \neq y'$ that have the same pattern of binding inequalities as required in part (2) of Lemma 3.4. Thus, the productions are the same in all solutions and, finally, by Theorem 3.2 we obtain uniqueness of the solution w.r.t. a fixed phase angle. □

If the relations for the variable production costs and susceptances given in Assumption 4 do not hold, the solution is not always unique. This is illustrated by the following example.

Example 5.5. We consider a cycle with three nodes as depicted in Figure 2. Here $(w_2 - w_1)B_{12} = (w_3 - w_2)B_{23}$ holds, i.e., Assumption 4 is violated. Let y denote the productions in the first solution given in Table 1 w.r.t. the fixed phase angle at node 1. Its flow-induced partition consists of one component and $0 < y_u < \bar{y}_u$ holds at all nodes $u \in N$; see Figure 2. By Lemma 5.2, this situation cannot appear if Assumption 4 holds. Applying the techniques of the proof of Lemma 5.2 one can construct the second solution given in Table 1.

Finally, let us point out that Assumption 4 is sufficient (due to Theorem 5.4) but not necessary as the following example shows.

Example 5.6. We modify the data of Example 5.5 by changing the intercept of the inverse demand function of the consumer located at node 1. Consequently, all parameters are as depicted in Figure 2 with $p_1(d) = -d + 14$ instead of $p_1(d) =$

TABLE 1. Two different solutions of Problem (7) for the scenario in Example 5.5.

Solution	d_1; p_1	d_2; p_2	d_3; p_3	y_1	y_2	y_3
1	6; 14	7; 6	0; 2	3.12	0.38	9.5
2				2.98	1.11	8.91
	f_{12}	f_{23}	f_{31}	Θ_1	Θ_2	Θ_3
1	0.12	-6.5	3	0	-0.12	1.5
2	-0.02	-5.91			0.02	

TABLE 2. Unique solutions of Problem (7) for the scenario in Example 5.6.

d_1; p_1	d_2; p_2	d_3; p_3	y_1	y_2	y_3	f_{12}	f_{23}	f_{31}	Θ_1	Θ_2	Θ_3
2.75; 11.25	7.27; 5.46	0; 2	0	0	10.02	0.25	-7.02	3	0	-0.25	1.5

$-d + 20$. In this case, the unique solution (with $\Theta_1 = 0$) is listed in Table 2. Why is this the unique solution? The demands are unique by Theorem 3.3. At node 1 we have $d_1 > 0$ and the market price $p_1(d_1) = 14 - 2.75 = 11.25$ is less than the variable production costs $w_1 = 14$. We later show in Lemma 6.7 that then $y_1 = 0$ has to hold in each solution. As the productions of two different solutions differ at all three nodes by Lemma 3.5 the solution is unique.

6. General Networks

The previous Section 5 shows that the solution of Problem (7) is not always unique if the network is a cycle but that a priori conditions for uniqueness can be derived. In Section 6.1 we state an example with multiple solutions of a more complex network. This shows that the considered market model does not possess unique equilibria on general networks. Subsequently, in Section 6.2 we provide different a posteriori uniqueness criteria for a solution of (7) and thereby specify properties of unique solutions.

6.1. **Multiplicity of Solutions.** We state an example with multiple solutions of Problem (7). To this end, we consider the network $G = (N, A)$ shown in Figure 3. The arc parameters read

$$f_a^+ := \begin{cases} 5, & \text{if } a = (1,2), \\ 10, & \text{if } a = (4,5), \\ 15, & \text{otherwise,} \end{cases} \qquad B_a := \begin{cases} 3, & \text{if } a = (1,2), \\ 2.6, & \text{if } a = (4,5), \\ 1, & \text{otherwise.} \end{cases} \tag{22}$$

At all nodes $u \in N$ the production capacity is $\bar{y}_u = 13$ and the remaining generation and demand parameters are given in Figure 3. In this situation $z := (d, y, f, \Theta)$ denotes the first solution given in Table 3. Its flow-induced partition consists of the two components that we obtain by deleting the dotted arcs $(1,2)$ and $(4,5)$. In each component exist two nodes without active production bounds. These are the nodes 4 and 7 as well as 5 and 6. To construct a second solution $z' := (d, y', f', \Theta')$ we use production differences $\Delta y_4, \Delta y_5 \in \mathbb{R}$ at node 4 and 5 and define productions and

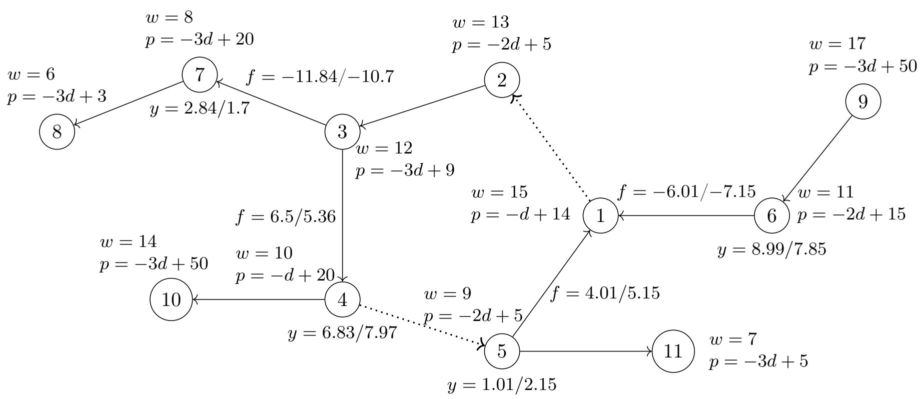

FIGURE 3. Network with multiple solutions. Variable costs are denoted by w, inverse demand functions by p, and flow and production values of the two solutions listed in Table 3 by f and y. Deleting the dotted arcs yields the flow-induced components of the first solution given in Table 3.

TABLE 3. Two different solutions of Problem (7) for the scenario of Section 6.1.

Solution	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}
1/2	3	0	0.33	10	0	2	4	0	13	13.33	0
	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}
1/2	11	5	8	10	5	11	8	3	11	10	5
	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
1 2	0	0	0	6.83 7.97	1.01 2.15	8.99 7.85	2.84 1.7	13	0	0	13
	$f_{1,2}$	$f_{2,3}$	$f_{3,4}$	$f_{3,7}$	$f_{4,5}$	$f_{4,10}$	$f_{5,1}$	$f_{5,11}$	$f_{6,1}$	$f_{7,8}$	$f_{9,6}$
1 2	-5	-5	6.5 5.36	-11.84 -10.7	-10	13.33	4.01 5.15	-13	-6.01 -7.15	-13	-13
	Θ_1	Θ_2	Θ_3	Θ_4	Θ_5	Θ_6	Θ_7	Θ_8	Θ_9	Θ_{10}	Θ_{11}
1 2	0	1.67	6.67	0.16 1.3	4.01 5.15	-6.01 -7.15	18.5 17.36	31.5 30.36	-19.01 -20.15	-13.17 -12.03	17.01 18.15

flows by

$$y'_u := \begin{cases} y_4 + \Delta y_4, & \text{if } u = 4, \\ y_5 + \Delta y_5, & \text{if } u = 5, \\ y_6 - \Delta y_5, & \text{if } u = 6, \\ y_7 - \Delta y_4, & \text{if } u = 7, \\ y_u, & \text{otherwise,} \end{cases} \qquad f'_a := \begin{cases} f_{3,4} - \Delta y_4, & \text{if } a = (3,4), \\ f_{3,7} + \Delta y_4, & \text{if } a = (3,7), \\ f_{5,1} + \Delta y_5, & \text{if } a = (5,1), \\ f_{6,1} - \Delta y_5, & \text{if } a = (6,1), \\ f_a, & \text{otherwise.} \end{cases}$$

With these definitions z' satisfies the flow balance conditions (7e). In addition, we need phase angles Θ' such that Θ' and f' satisfy the DC flow conditions (7f). Using $\Theta_1 = \Theta'_1 = 0$ and the definition of y' and f' yield feasible phase angles if $\Delta y_4 = B_{34}\Delta y_5 / B_{51}$ holds. The objective function values of z and z' should be the same and since demands are unique, Δy_5 has to fulfill

$$\left[(w_5 - w_6) + (w_4 - w_7)\frac{B_{34}}{B_{51}}\right] \Delta y_5 = 0. \tag{23}$$

With the susceptances and variable production costs given in (22) and Figure 3 we have $(w_5-w_6)+(w_4-w_7)B_{34}/B_{51} = 0$. Hence, z' is an optimal solution if we choose Δy_5 such that all flow and production bounds are satisfied. Using $B_{34} = 1 = B_{51}$ we obtain the following conditions for Δy_5:

$$0 \le y_u + \Delta y_5 \le \bar{y}_u, \quad u \in \{4,5\}, \quad -f_a^+ \le f_a + \Delta y_5 \le f_a^+, \quad a \in \{(3,7),(5,1)\},$$
$$0 \le y_u - \Delta y_5 \le \bar{y}_u, \quad u \in \{6,7\}, \quad -f_a^+ \le f_a - \Delta y_5 \le f_a^+, \quad a \in \{(3,4),(6,1)\}.$$

Thus, Δy_5 has to be in the interval

$$-1.01 \le \Delta y_5 \le 2.84. \tag{24}$$

This shows that there exists an optimal solution of (7) for every generation capacity utilization between 0 % to 29.61 % at node 5, which shows that the present multiplicities are not only of minor relevance but can indeed lead to clearly different solutions. Choosing $\Delta y_5 = 1.14$ yields the second solution given in Table 3.

The reason for multiple solutions is that the coefficient of Δy_5 in the objective function difference (23) is zero. Note that there is a strong similarity compared to our uniqueness criteria for the case having only a single cycle: There exist two flow-induced components each containing two nodes not producing on a capacity bound. We exactly excluded such cases by (13) in the cycle case; cf. Lemma 5.3. Note further that the choice of a nonzero Δy_5 in (23) only depends on the susceptances of the arcs that are located in the cycle as well as on the undirected path P_{74} or P_{65}, and not, e.g., on the susceptances of the arcs $(3,7)$ and $(6,1)$ although the flow solutions differ on these arcs as well.

6.2. **A Posteriori Uniqueness Criteria.** The example with multiple solutions in the previous section indicates that it seems unrealistic to obtain a priori uniqueness criteria for Problem (7) on general networks. However, we state a posteriori uniqueness criteria in this section. This is in line with the general literature since it is usually hard to obtain a priori uniqueness criteria. For instance, for the case of general linear optimization problems, a priori uniqueness criteria are not known but a posteriori criteria are well-studied; cf., e.g., Mangasarian (1979).

The main results in this section are twofold: First, we show that a solution is unique if all flow bounds are strictly fulfilled. Second, we derive an a posteriori result that depends on the relation between variable production costs and resulting nodal prices of the solution.

For our first a posteriori criterion we reformulate Problem (7). It is folklore knowledge the used Power Transfer Distribution Factor (PTDF) formulation is equivalent to the one used in (7). Thus, all results obtained with the reformulated variant of (7) also apply to the original model formulation. We only use the alternative formulation for simplifying the proofs. Hence, in what follows we consider the PTDF matrix $P \in \mathbb{R}^{|A|\times|N|}$ with entries p_{au} for each arc $a \in A$ and each node $u \in N$ that describe the change of the power flow on arc a by injection of one unit of power at node u and withdrawal at a chosen reference node; see, e.g., Van den Bergh, Delarue, et al. (2014). Throughout this section, we set node 1 as reference node with fixed phase angle $\Theta_1 = 0$ and we introduce the node-arc incidence matrix $M \in \mathbb{R}^{|N|\times|A|}$ with entries

$$m_{ua} := \begin{cases} -1, & \text{if } a \in \delta^{\text{in}}(u), \\ 1, & \text{if } a \in \delta^{\text{out}}(u), \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

Then, we can rewrite (7f) as

$$f_a = \sum_{u \in N} p_{au}(y_u - d_u), \quad a \in A, \tag{26}$$

and, thus, an equivalent formulation of Problem (7) is given by

$$\max_{d,y,f} \quad \text{(7a)} \quad \text{s.t.} \quad \text{(7b)–(7e)}, \text{(26)}. \tag{27}$$

Its dual feasibility conditions comprise

$$-p_u(d_u) - \alpha_u + \gamma_u + (P^\top \varepsilon)_u = 0, \quad u \in N, \tag{28a}$$

$$w_u - \beta_u^- + \beta_u^+ - \gamma_u - (P^\top \varepsilon)_u = 0, \quad u \in N, \tag{28b}$$

$$-\delta_a^- + \delta_a^+ + (M^\top \gamma)_a + \varepsilon_a = 0, \quad a \in A, \tag{28c}$$

and its KKT complementarity conditions read

$$\beta_u^- y_u = 0, \quad u \in N, \tag{29a}$$

$$\beta_u^+ (\bar{y}_u - y_u) = 0, \quad u \in N, \tag{29b}$$

$$\alpha_u d_u = 0, \quad u \in N, \tag{29c}$$

$$\delta^-(-f_a - f_a^+) = 0, \quad a \in A, \tag{29d}$$

$$\delta^+(f_a - f_a^+) = 0, \quad a \in A, \tag{29e}$$

where α_u denotes the dual variable of Constraint (7c), $\beta_u^\pm$ the dual variables of Constraints (7b), $\delta_a^\pm$ the dual variables of Constraints (7d), γ_u the dual variable of Constraint (7e), and ε_a the dual variable of Constraint (26).

Subsequently, we prove some auxiliary results required for the proofs of the main theorems of this section.

Lemma 6.1. *It holds*

$$I - MP = \begin{bmatrix} 1 & e^\top \\ 0 & 0 \end{bmatrix}, \tag{30}$$

where e is the vector of ones in suitable dimension.

Proof. We define $B := \operatorname{diag}((B_a)_{a \in A}) \in \mathbb{R}^{|A|\times|A|}$ as a diagonal matrix consisting of the susceptances of the network's arcs in the same order as used for the node-arc incidence matrix M in (25). Then, it can be shown that

$$P := \begin{bmatrix} 0 & BM_1^\top (M_1 B M_1^\top)^{-1} \end{bmatrix} \in \mathbb{R}^{|A|\times|N|} \quad \text{with} \quad M := \begin{bmatrix} m_1^\top \\ M_1 \end{bmatrix} \quad \text{and} \quad m_1^\top \in \mathbb{R}^{|A|}$$

holds. With this, we obtain

$$MP = \begin{bmatrix} 0 & m_1^\top BM_1^\top (M_1 B M_1^\top)^{-1} \\ 0 & M_1 B M_1^\top (M_1 B M_1^\top)^{-1} \end{bmatrix} = \begin{bmatrix} 0 & m_1^\top BM_1^\top (M_1 B M_1^\top)^{-1} \\ 0 & I \end{bmatrix}$$

and

$$I - MP = \begin{bmatrix} 1 & -m_1^\top BM_1^\top (M_1 B M_1^\top)^{-1} \\ 0 & 0 \end{bmatrix}.$$

Subsequently, we analyze the structure of the first row of MP. From $M_1 B M_1^\top (M_1 B M_1^\top)^{-1} = I$ follows

$$\sum_{a \in \delta^{\text{out}}(u)} P_{a,\cdot} - \sum_{a \in \delta^{\text{in}}(u)} P_{a,\cdot} = (0, e_u^\top) \quad \text{for all } u \in N \setminus \{1\}, \tag{31}$$

where $P_{a,\cdot}$ denotes the ath row of P. As every arc $a \in A$ is ingoing and outgoing for exactly one node, it holds

$$\sum_{u \in N} \left(\sum_{a \in \delta^{\text{out}}(u)} P_{a,\cdot} - \sum_{a \in \delta^{\text{in}}(u)} P_{a,\cdot} \right) = 0. \tag{32}$$

Summing up (31) for all nodes $u \in N \setminus \{1\}$ yields, together with (32),

$$\begin{aligned} m_1^\top P &= (0, m_1^\top B M_1^\top (M_1 B M_1^\top)^{-1}) \\ &= \sum_{a \in \delta^{\text{out}}(1)} P_{a,\cdot} - \sum_{a \in \delta^{\text{in}}(1)} P_{a,\cdot} \\ &= (0, -1, \dots, -1) \in \mathbb{R}^{|N|}. \end{aligned}$$

With the latter, we finally get

$$MP = \begin{bmatrix} 0 & -e^\top \\ 0 & I \end{bmatrix} \quad \text{and} \quad I - MP = \begin{bmatrix} 1 & e^\top \\ 0 & 0 \end{bmatrix}. \qquad \square$$

Lemma 6.2. *Let $(d, y, f; \beta^\pm, \alpha, \delta^\pm, \gamma, \varepsilon)$ be an optimal primal-dual solution of Problem (27). Then,*

$$w_u - \beta_u^- + \beta_u^+ + (P^\top(\delta^+ - \delta^-))_u = w_1 - \beta_1^- + \beta_1^+ = \gamma_1 \tag{33}$$

holds for all nodes $u \in N$.

Proof. Solving the dual feasibility condition (28c) for ε_a and substituting ε in the matrix formulation of (28b) yields

$$0 = w - \beta^- + \beta^+ + P^\top(\delta^+ - \delta^-) - (I - P^\top M^\top)\gamma. \tag{34}$$

With (30) it is $(I - P^\top M^\top)\gamma = e^\top \gamma_1$ and, thus, (34) reads $\gamma_1 = w_u - \beta_u^- + \beta_u^+ + (P^\top(\delta^+ - \delta^-))_u$ for all $u \in N$. Since node 1 is the reference node, the first column of the PTDF matrix P only contains zeros. Hence, $\gamma_1 = w_1 - \beta_1^- + \beta_1^+$ and the assertion follows. $\square$

Lemma 6.3. *Let $(d, y, f; \beta^\pm, \alpha, \delta^\pm, \gamma, \varepsilon)$ be an optimal primal-dual solution of Problem (27) with $-f_a^+ < f_a < f_a^+$ for all arcs $a \in A$. Then, it holds*

$$\begin{aligned} \gamma_1 &\geq w_u \quad \text{for all nodes } u \in N \text{ with } y_u = \bar{y}_u, \\ \gamma_1 &\leq w_u \quad \text{for all nodes } u \in N \text{ with } y_u = 0, \\ \gamma_1 &= w_u \quad \text{for all nodes } u \in N \text{ with } 0 < y_u < \bar{y}_u. \end{aligned}$$

Proof. As no flow bound is binding, the KKT complementarity conditions (29d), (29e) imply $\delta^\pm = 0$. Thus, (33) reads

$$w_u - \beta_u^- + \beta_u^+ = w_1 - \beta_1^- + \beta_1^+ = \gamma_1 \tag{35}$$

for all $u \in N$. If $y_u = \bar{y}_u$ holds at a node $u \in N$, we obtain $\beta_u^- = 0$ from (29a). Hence, (35) implies $w_u + \beta_u^+ = \gamma_1$ and $\gamma_1 \geq w_u$ follows directly from the non-negativity of the dual variable β_u^+. If there is no production at a node $u \in N$, i.e., $y_u = 0$, KKT complementarity (29b) yields $\beta_u^+ = 0$. Then, (35) implies $w_u - \beta_u^- = \gamma_1$ and non-negativity of β_u^- leads to $\gamma_1 \leq w_u$. At nodes $u \in N$ with production $0 < y_u < \bar{y}_u$, $\beta_u^\pm = 0$ holds by (29a) and (29b). Thus, (35) simplifies to $w_u = \gamma_1$. $\square$

In the next lemma we stay with the case of inactive flow bounds and show that productions are always in merit order in this case. Formally, we say that productions y are in merit order if for each two distinct nodes $u, v \in N$ with $w_u < w_v$ and $0 \leq y_u < \bar{y}_u$ one has $y_v = 0$.

Lemma 6.4. *Suppose Assumption 1 holds. Let (d, y, f, Θ) be a solution of Problem (7) with $-f_a^+ < f_a < f_a^+$ for all arcs $a \in A$. Then, the productions y are in merit order.*

Proof. Assume the productions y are not in merit order. Then, there are two distinct nodes $u \neq v \in N$ with $w_u < w_v$, $0 \le y_u < \bar{y}_u$, and $y_v > 0$. As (d, y, f) is also a solution of the PTDF formulation (27) and the KKT conditions of (27) are sufficient and necessary, there exist dual variables such that $(d, y, f; \beta^\pm, \alpha, \delta^\pm, \gamma, \varepsilon)$ is an optimal primal-dual solution of Problem (27). Then, by applying Lemma 6.3 we see that the dual variable γ_1 of the flow balance condition at the reference node satisfies $w_v \le \gamma_1 \le w_u$. This contradicts the assumption $w_u < w_v$. □

We are now ready to prove our first a posteriori uniqueness criterion.

Theorem 6.5. *Suppose Assumptions 1 and 2 hold. If there exists a solution of Problem (7) in which $-f_a^+ < f_a < f_a^+$ for all arcs $a \in A$ holds, then the solution of Problem (7) is unique.*

Proof. With Theorem 3.3 all solutions of Problem (7) have the same demands. Assume that $z := (d, y, f, \Theta)$ and $z' := (d, y', f', \Theta')$ are two solutions of Problem (7) with $y \neq y'$ and $-f_a^+ < f_a < f_a^+$ for all $a \in A$. We partition the node set $N = N_< \cup N_= \cup N_>$ in pairwise disjoint node sets $N_< := \{u \in N : y'_u < y_u\}$, $N_= := \{u \in N : y'_u = y_u\}$, and $N_> := \{u \in N : y'_u > y_u\}$. Since $y \neq y'$, Lemma 3.5 implies $|N_<| + |N_>| \ge 3$ and as z and z' have the same demands, summing up the flow balance conditions (7e) over all nodes yields $\sum_{u \in N} y_u = \sum_{u \in N} y'_u$. Consequently, there is at least one node $i \in N_<$ and one node $j \in N_>$, i.e., $y'_i < y_i$ and $y'_j > y_j$. This implies

$$0 < y_i \le \bar{y}_i \quad \text{and} \quad 0 \le y_j < \bar{y}_j. \tag{36}$$

Without loss of generality, let i be the node with largest variable production costs in $N_<$. Lemma 6.4 yields that the productions y are in merit order as in z no flow bounds are active. Hence, to satisfy (36) the variable production costs at node i have to be less than at node j, i.e., $w_i < w_j$. Further, there exists no node $k \in N_> \setminus \{j\}$ with $w_k < w_i$ as the merit order production and $y_i > 0$ imply that $y_k = \bar{y}_k$ holds. Then, $y'_k > y_k = \bar{y}_k$ cannot hold because y' fulfills the production bounds (7b). Likewise, there exists no node $k \in N_< \setminus \{i\}$ with $w_j < w_k$ because the merit order production together with $y_j < \bar{y}_j$ yields $y_k = 0$. Thus, it is not possible to have a non-negative production y'_k with $y'_k < y_k = 0$. Consequently, $w_k > w_i$ for all nodes $k \in N_>$ and $w_k \le w_i < w_j$ for all nodes $k \in N_<$. Since the solutions z and z' of Problem (7) have the same demands, their objective function difference reads

$$\begin{aligned} 0 &= \sum_{u \in N} w_u (y_u - y'_u) \\ &= \sum_{u \in N_<} w_u (y_u - y'_u) + \sum_{u \in N_=} w_u (y_u - y'_u) + \sum_{u \in N_>} w_u (y_u - y'_u). \end{aligned} \tag{37}$$

With the definition of $N_=$ it is $\sum_{u \in N_=} w_u (y_u - y'_u) = 0$. At each node $u \in N_>$ we have $y_u - y'_u < 0$ and also $w_u > w_i$, which gives the inequality

$$\sum_{u \in N_>} w_u (y_u - y'_u) < \sum_{u \in N_>} w_i (y_u - y'_u).$$

Furthermore, as $y_u - y'_u > 0$ holds at all nodes $u \in N_<$ with $w_u \le w_i$, we also have

$$\sum_{u \in N_<} w_u (y_u - y'_u) \le \sum_{u \in N_<} w_i (y_u - y'_u).$$

Altogether, (37) reads

$$\begin{aligned} 0 &= \sum_{u \in N_<} w_u(y_u - y'_u) + \sum_{u \in N_>} w_u(y_u - y'_u) \\ &< \sum_{u \in N_<} w_i(y_u - y'_u) + \sum_{u \in N_>} w_i(y_u - y'_u) \\ &= \sum_{u \in N_< \cup N_>} w_i(y_u - y'_u) = w_i \sum_{u \in N} (y_u - y'_u) = 0, \end{aligned}$$

where we used $|N_>| \geq 1$. This yields the contradiction $0 < 0$. □

Informally speaking, the latter theorem states that in the case of over-dimensioned networks, market equilibria are always unique. In other words, for scenarios with moderate load one can expect unique solutions.

Next, we derive our second a posteriori uniqueness criterion. To this end, we again need some auxiliary results and the sufficient and necessary first-order optimality conditions of Problem (7) given below. We denote with α_u the dual variable of Constraint (7c), with $\beta_u^\pm$ the dual variables of Constraints (7b), with $\delta_a^\pm$ the dual variables of Constraints (7d), with γ_u the dual variable of Constraint (7e), and with ε_a the dual variable of Constraint (7f). Then, the KKT conditions of Problem (7) comprise dual feasibility conditions

$$p_u(d_u) + \alpha_u - \gamma_u = 0, \quad u \in N, \tag{38a}$$

$$-w_u - \beta_u^+ + \beta_u^- + \gamma_u = 0, \quad u \in N, \tag{38b}$$

$$\delta_a^- - \delta_a^+ - \gamma_u + \gamma_v + \varepsilon_a = 0, \quad a = (u, v) \in A, \tag{38c}$$

$$\sum_{a \in \delta^{\text{in}}(u)} \varepsilon_a B_a - \sum_{a \in \delta^{\text{out}}(u)} \varepsilon_a B_a = 0, \quad u \in N, \tag{38d}$$

primal feasibility conditions (7b)–(7f), non-negativity of inequality multipliers

$$\alpha_u, \beta_u^\pm \geq 0, \quad u \in N, \qquad \delta_a^\pm \geq 0, \quad a \in A, \tag{39}$$

and KKT complementarity conditions

$$\beta_u^- y_u = 0, \quad u \in N, \tag{40a}$$

$$\beta_u^+ (y_u - \bar{y}_u) = 0, \quad u \in N, \tag{40b}$$

$$\alpha_u d_u = 0, \quad u \in N, \tag{40c}$$

$$\delta_a^- (-f_a^+ - f_a) = 0, \quad \delta_a^+ (f_a - f_a^+) = 0, \quad a \in A. \tag{40d}$$

The first auxiliary result is a statement on the production y_u at a node $u \in N$ depending on the relation of the market price $p_u(d_u)$ and the variable production costs w_u at this node.

Lemma 6.6. *Let $(d, y, f, \Theta; \beta^\pm, \alpha, \delta^\pm, \gamma, \varepsilon)$ be an optimal primal-dual solution of Problem (7). Then, for $u \in N$*

- *$w_u < p_u(d_u)$ implies $y_u = \bar{y}_u$,*
- *$w_u > p_u(d_u)$ implies either $y_u = 0$ or $d_u = 0$,*
- *$w_u = p_u(d_u)$ implies $\alpha_u - \beta_u^+ = 0$ and $\beta_u^- = 0$.*

Proof. Let $u \in N$ be an arbitrary node. Summing up the dual feasibility conditions (38a) and (38b) for u yields

$$p_u(d_u) - w_u + \alpha_u - \beta_u^+ + \beta_u^- = 0. \tag{41}$$

In case of $w_u < p_u(d_u)$ we obtain $\alpha_u - \beta_u^+ + \beta_u^- < 0$ and due to (39) the dual variable β_u^+ is positive. Hence, $y_u = \bar{y}_u$ follows from (40b). On the other hand, if $w_u > p_u(d_u)$, (41) implies $\alpha_u - \beta_u^+ + \beta_u^- > 0$. As $\alpha_u, \beta_u^\pm \geq 0$, either $\alpha_u > 0$

or $\beta_u^- > 0$ holds. This together with the KKT complementarity conditions (40a) and (40c) yields $d_u = 0$ or $y_u = 0$. The last assertion in this lemma is again a consequence of (41) because in case of $w_u = p_u(d_u)$ we have

$$\alpha_u - \beta_u^+ + \beta_u^- = 0. \tag{42}$$

Assume that $\beta_u^- > 0$. Then, the KKT complementarity conditions (40a), (40b), and $\bar{y}_u > 0$ imply $\beta_u^+ = 0$. Since (39) holds, we cannot satisfy (42). Consequently, we have $\beta_u^- = 0 = \alpha_u - \beta_u^+$. □

Now we obtain uniqueness of the production at special nodes as a direct consequence of Theorem 3.3 and Lemma 6.6.

Corollary 6.7. *Suppose Assumption 2 holds. Let d be the demands in a solution of Problem (7). Then, the production at node $u \in N$ is unique if either (i) $w_u > p_u(d_u)$ and $d_u > 0$ or (ii) $w_u < p_u(d_u)$ holds.*

We are ready to state and prove our second uniqueness criterion.

Theorem 6.8. *Suppose Assumptions 1 and 2 hold. Let $z := (d, y, f, \Theta)$ be a solution of Problem (7). If z satisfies at least at $|N| - 2$ nodes $u \in N$ either (i) $w_u > p_u(d_u)$ and $d_u > 0$ or (ii) $w_u < p_u(d_u)$, then the solution of Problem (7) w.r.t. a reference node with fixed phase angle is unique.*

Proof. By Theorem 3.3 the demands are unique in all solutions of Problem (7). At all nodes $u \in N$ with $w_u < p_u(d_u)$, it is $y_u = \bar{y}_u$ and $y_u = 0$ whenever both $w_u > p_u(d_u)$ and $d_u > 0$ hold. This is the assertion of Corollary 6.7. Thus, the production is at least at $|N| - 2$ nodes uniquely determined. Due to Lemma 3.5, two solutions with different productions differ at least at three nodes. Consequently, the productions are the same in all solutions of (7) and the claim follows with Theorem 3.2. □

In the case without a network it is well known that market clearing prices equal marginal cost of production. That means there is at most one swing producer. The latter theorem states that in the case of a DC network, if at most two "local" swing producers exist, the market equilibrium is unique.

In the remainder of this section we provide a third and last a posteriori uniqueness criterion for Problem (7) on general networks. To this end, we need the following assumption.

Assumption 5. *Let $z := (d, y, f, \Theta)$ be a solution of Problem (7). Then, for its flow-induced partition $\{G^i := (N^i, A^i)\}_{i \in I}$ it holds:*

(a) $|\{u \in N^i : 0 < y_u < \bar{y}_u\}| \leq 2$ for all $i \in I$,
(b) $|\{i \in I : \exists u \neq v \in N^i,\ 0 < y_u < \bar{y}_u,\ 0 < y_v < \bar{y}_v\}| \leq 1$.

Informally speaking, this assumption states the following. Every flow-induced component has at most one producer that is not restricted by its production bounds—except for one component which may have two of these producers.

The following theorem yields that there exist no two solutions with the same binding flow and production pattern satisfying the latter assumption.

Theorem 6.9. *Suppose Assumption 1 and 2 hold. Let z and z' be solutions of Problem (7) w.r.t. the same reference node. If z and z' have the same binding production and flow bounds (7b), (7d) and if both z and z' satisfy Assumption 5, then $z = z'$ holds.*

Proof. Due to Theorem 3.3 the demands are unique. Assume that $(d, y, f, \Theta) =: z \neq z' := (d, y', f', \Theta')$ are two solutions of (7) satisfying Assumption 5, which have the same binding production and flow bounds, and $y \neq y'$. Thus, z and

z' have the same flow-induced components $G^i = (N^i, A^i)$, $i \in I \subseteq \mathbb{N}$. Summing up (7e) for all nodes in component G^i yields $\sum_{u \in N^i} y_u = \sum_{u \in N^i} y'_u$ for all $i \in I$. As a consequence, the difference in productions y and y' only occurs within the components G^i, $i \in I$. Lemma 3.5 states that y and y' differ in at least three nodes and by Assumption 5(a) we know that there are at most two nodes $u \in N^i$ for which $0 < y_u, y'_u < \bar{y}_u$ holds for all components G^i. Thus, there are at least two components in which the production differs (and is not binding) in y and y' at least at two nodes. This contradicts Assumption 5(b). Hence, $y \neq y'$ cannot be true and the theorem follows with Theorem 3.2. □

Note that both Conditions (a) and (b) in Assumption 5 are necessary for Theorem 6.9. In the example of Section 6.1 both solutions have the same binding flow and production bounds, satisfy part (a) of Assumption 5, but violate part (b). The following example shows the existence of multiple solutions for which part (a) of Assumption 5 fails to hold but part (b) is satisfied and the binding structure is the same.

Example 6.10. We consider the network given in Figure 4. All arcs have susceptance 1. The arc and production capacities read

$$f_a^+ := \begin{cases} 5, & \text{if } a = (2,3), \\ 10, & \text{otherwise,} \end{cases} \qquad \bar{y}_u := \begin{cases} 5, & \text{if } u = 1, \\ 10, & \text{otherwise,} \end{cases}$$

and the variable production costs and the inverse demand functions are depicted in Figure 4. Here, the parameters of the cycle $(1,2,3)$ fulfill the assumptions of Theorem 5.4, i.e., restricted to the cycle we have a unique solution. We use the reference node 1 and set $\Theta_1 = 0$. Then, one solution $z := (d, y, f, \Theta)$ of Problem (7) is the first solution listed in Table 4. The flow-induced partition of z consists of one component since the only saturated arc is $(2,3)$. At the three nodes 2, 3, and 4 no production bounds are binding and hence the solution z violates Assumption 5(a). By modifying these productions we can construct another solution $z' := (d, y', f', \Theta')$. For $\Delta y_2, \Delta y_3 \in \mathbb{R}$ we define the productions at a node $u \in N$ and the flow on arc $a \in A$ by

$$y'_u := \begin{cases} y_1, & \text{if } u = 1, \\ y_2 + \Delta y_2, & \text{if } u = 2, \\ y_3 + \Delta y_3, & \text{if } u = 3, \\ y_4 - \Delta y_2 - \Delta y_3, & \text{if } u = 4, \end{cases} \qquad f'_a := \begin{cases} f_{12} - \Delta y_2, & \text{if } a = (1,2), \\ f_{23}, & \text{if } a = (2,3), \\ f_{31} + \Delta y_3, & \text{if } a = (3,1), \\ f_{41} - \Delta y_2 - \Delta y_3, & \text{if } a = (4,1). \end{cases}$$

Hence, z' satisfies the flow balance conditions (7e). Using $\Theta'_1 = \Theta_1 = 0$ and the definition of y' and f' yield feasible phase angles if $\Delta y_3 = B_{31}/B_{12} \Delta y_2$ holds. For z' being feasible for Problem (7), it also has to satisfy the production and flow bounds (7b) and (7d). This implies

$$-1.53 \leq \Delta y_2 \leq 2.05. \tag{43}$$

The difference of the objective function values of z and z' expressed by Δy_2 reads $[(w_2 - w_4) + (w_3 - w_4) B_{31}/B_{12}] \Delta y_2$. This product is always zero because

$$(w_2 - w_4) B_{12} = (w_4 - w_3) B_{31}$$

holds and consequently we obtain multiple solutions by choosing Δy_2 in accordance to (43). For instance $\Delta y_2 = 1.72$ yields the second solution given in Table 4.

Moreover, equality of the binding flow and production patterns of solutions is important in Theorem 6.9. In the example of Section 6.1, choosing the boundary values of the interval given in (24) yields the two solutions in Table 5. Both of these solutions satisfy Assumption 5 but differ in the binding productions.

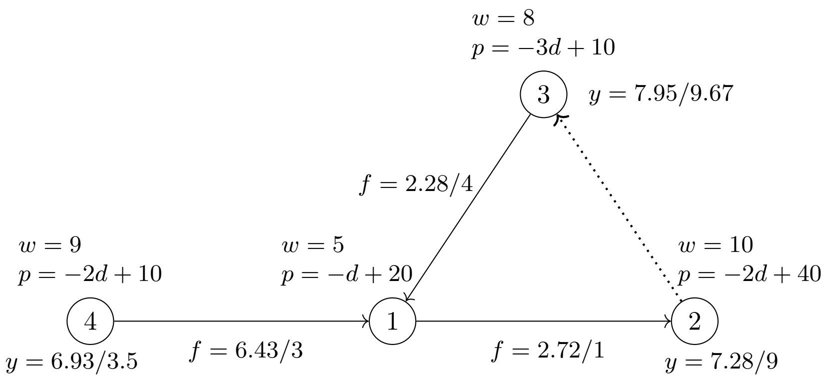

FIGURE 4. Network with multiple solutions of Problem (7). Variable costs are denoted by w, inverse demand functions by p, and flow and production values of the two solutions listed in Table 4 by f and y. Deleting the dotted arc yields the flow-induced partition of the first solution in Table 4.

TABLE 4. Two different solutions of Problem (7) for the scenario of Ex. 6.10.

<table>
<tr><th>Solution</th><th>d_1; p_1</th><th>d_2; p_2</th><th>d_3; p_3</th><th>d_4</th><th>y_1</th><th>y_2</th><th>y_3</th><th>y_4</th></tr>
<tr><td>1</td><td rowspan="2">11; 9</td><td rowspan="2">15; 10</td><td rowspan="2">0.67; 8</td><td rowspan="2">0.5; 9</td><td rowspan="2">5</td><td>7.28</td><td>7.95</td><td>6.93</td></tr>
<tr><td>2</td><td>9</td><td>9.67</td><td>3.5</td></tr>
<tr><td></td><td>f_{12}</td><td>f_{23}</td><td>f_{31}</td><td>f_{41}</td><td>Θ_1</td><td>Θ_2</td><td>Θ_3</td><td>Θ_4</td></tr>
<tr><td>1</td><td>2.72</td><td rowspan="2">-5</td><td>2.28</td><td>6.43</td><td rowspan="2">0</td><td>-2.72</td><td>2.28</td><td>6.43</td></tr>
<tr><td>2</td><td>1</td><td>4</td><td>3</td><td>-1</td><td>4</td><td>3</td></tr>
</table>

TABLE 5. Two different solutions of Problem (7) for the scenario of Section 6.1.

<table>
<tr><th>Solution</th><th>d_1</th><th>d_2</th><th>d_3</th><th>d_4</th><th>d_5</th><th>d_6</th><th>d_7</th><th>d_8</th><th>d_9</th><th>d_{10}</th><th>d_{11}</th></tr>
<tr><td>1/2</td><td>3</td><td>0</td><td>0.33</td><td>10</td><td>0</td><td>2</td><td>4</td><td>0</td><td>13</td><td>13.33</td><td>0</td></tr>
<tr><td></td><td>p_1</td><td>p_2</td><td>p_3</td><td>p_4</td><td>p_5</td><td>p_6</td><td>p_7</td><td>p_8</td><td>p_9</td><td>p_{10}</td><td>p_{11}</td></tr>
<tr><td>1/2</td><td>11</td><td>5</td><td>8</td><td>10</td><td>5</td><td>11</td><td>8</td><td>3</td><td>11</td><td>10</td><td>5</td></tr>
<tr><td></td><td>y_1</td><td>y_2</td><td>y_3</td><td>y_4</td><td>y_5</td><td>y_6</td><td>y_7</td><td>y_8</td><td>y_9</td><td>y_{10}</td><td>y_{11}</td></tr>
<tr><td>1</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">0</td><td>5.82</td><td>0</td><td>10</td><td>3.85</td><td rowspan="2">13</td><td rowspan="2">0</td><td rowspan="2">0</td><td rowspan="2">13</td></tr>
<tr><td>2</td><td>9.67</td><td>3.85</td><td>6.15</td><td>0</td></tr>
<tr><td></td><td>$f_{1,2}$</td><td>$f_{2,3}$</td><td>$f_{3,4}$</td><td>$f_{3,7}$</td><td>$f_{4,5}$</td><td>$f_{4,10}$</td><td>$f_{5,1}$</td><td>$f_{5,11}$</td><td>$f_{6,1}$</td><td>$f_{7,8}$</td><td>$f_{9,6}$</td></tr>
<tr><td>1</td><td rowspan="2">-5</td><td rowspan="2">-5</td><td>7.51</td><td>-12.85</td><td rowspan="2">-10</td><td rowspan="2">13.33</td><td>3</td><td rowspan="2">-13</td><td>-5</td><td rowspan="2">-13</td><td rowspan="2">-13</td></tr>
<tr><td>2</td><td>3.67</td><td>-9</td><td>6.85</td><td>-8.85</td></tr>
<tr><td></td><td>Θ_1</td><td>Θ_2</td><td>Θ_3</td><td>Θ_4</td><td>Θ_5</td><td>Θ_6</td><td>Θ_7</td><td>Θ_8</td><td>Θ_9</td><td>Θ_{10}</td><td>Θ_{11}</td></tr>
<tr><td>1</td><td rowspan="2">0</td><td rowspan="2">1.67</td><td rowspan="2">6.67</td><td>-0.85</td><td>3</td><td>-5</td><td>19.51</td><td>32.51</td><td>-18</td><td>-14.18</td><td>16</td></tr>
<tr><td>2</td><td>3</td><td>6.85</td><td>-8.85</td><td>15.67</td><td>28.67</td><td>-21.85</td><td>-10.33</td><td>19.85</td></tr>
</table>

Using Assumption 5 and Lemma 3.4 yields that it is sufficient for the uniqueness of a solution of (7) to determine conditions such that every possible solution satisfies Assumption 5. This means, we have to ensure that in each component of the flow-induced partition of the network at no more than two nodes both production bounds (7b) are strictly satisfied and that there is at most one such component in which two nodes do not produce on the lower or upper production bound. Due to Assumption 4 we already have such conditions for cycles; see Lemma 5.2 and Lemma 5.3. For cycle-free networks these facts are direct consequences of the merit order production in each flow-induced component if all variable production costs are pairwise distinct because then, at most at one node in each flow-induced component, no production bound is binding.

Remark 6.11. Besides the presented results above we state a further result for networks with arc-disjoint cycles, i.e., each arc is at most contained in one cycle in the network. We omit the technical and long proofs here and refer to Krebs (2017). For such a network let $z := (d, y, f, \Theta)$ be a solution of Problem (7) and $\{G^i := (N^i, A^i)\}_{i \in I}$, $I \subseteq \mathbb{N}$, its flow-induced partition. Then, the productions y_u, $u \in N^i$, for each $i \in I$ in which all arcs $a \in \{(u, v) \in A \colon u \in N^i \vee v \in N^i, |f_{uv}| = f^+_{uv}\}$ are not contained in cycles in the network, are in merit order. The example in Section 6.1 shows the necessity of the latter condition for this claim. In the flow-induced component of the first solution in Table 3, containing the nodes 2, 3, 4, 7, 8, and 10, it is $y_7 < \bar{y}_7$ and $0 < y_4$ but the variable costs satisfy $w_7 < w_4$, i.e., the productions in this component are not in merit order.

Remark 6.12. In this section, we have shown different problem-specific a posteriori criteria for a solution of Problem (7) being unique. We note that Problem (7) can also be reduced to a linear optimization problem using that the demands are always unique. This also allows to apply the existing a posteriori uniqueness criteria for linear problems given in Mangasarian (1979). However, our problem-specific criteria are much easier to evaluate. In particular, they do not require to solve any additional optimization problems or any systems of linear equations and inequalities as it is the case in Mangasarian (1979).

7. Conclusion

In this paper we considered uniqueness and multiplicity of short-run market equilibria, where trading is constrained by a capacitated lossless DC power network. The power producers face fluctuating demand and their production decisions are restricted by given generation capacities. For the setting of perfect competition we proved uniqueness of equilibria on tree networks under very mild conditions and also derived a priori criteria for uniqueness of equilibria on cycle networks. Moreover, we showed that uniqueness fails to hold on general networks, presented simple examples that possess multiple solutions, and specified properties of unique solutions by proving different a posteriori uniqueness results.

This work extends the findings of Grimm, Schewe, et al. (2017). It enhances the physical modeling by integrating the lossless DC approximation into the flow model but, on the other hand, only considers the short run, whereas the long run is considered in Grimm, Schewe, et al. (2017). Since multiplicities already occur for perfectly competitive power markets on DC networks in the short run, one cannot expect uniqueness of equilibria for the long run. Moreover, we refrained from considering the case of non-competitive market models. It is very likely that uniqueness of equilibria fails to hold for such non-competitive models. This has already been observed for corresponding settings without transport networks; cf., e.g., Zöttl (2010). Since we already observe multiplicities in the competitive case on

DC networks, cf. our findings in Section 6, we do not think that there is much hope for uniqueness in the networked and non-competitive case.

Finally, the only general way that we see for obtaining a unique solution of the considered model is to use strictly convex cost functions. In this case, uniqueness follows from standard optimization theory and also carries over directly to the long run.

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