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# **”Pre-Print”**: *This is a pre-print of a contribution to be published by Springer.* **On the relaxation approximation for $2 \times 2$ hyperbolic balance laws**

Martin Gugat, Michael Herty and Hui Yu

**Abstract** The relaxation approximation for systems of conservation laws has been studied intensively for example by [17, 5, 19, 24]. In this paper the corresponding relaxation approximation for  $2 \times 2$  systems of balance laws is studied. Our driving example is gas flow in pipelines described by the isothermal Euler equations. We are interested in the limiting behavior as the relaxation parameter tends to zero. We give conditions where the relaxation converges to the states of the original system and counterexamples for cases where the steady states depend on the space variable. **Keywords:** Jin-Xin relaxation, Lyapunov function, Stabilization, Steady states

## **1 Introduction**

For numerical purposes, the dynamics of hyperbolic conservation laws have been successfully approximated by relaxation schemes as for example proposed in [17]. We are interested in the effects of this relaxation on the damping of perturbations of steady states of hyperbolic balance laws. Our primary example are gas dynamics in pipelines. We use a Lyapunov function approach similar to [9, 15] to obtain exponential decay of the perturbation. The advantage of the relaxation system is its semilinear structure in contrast to the quasilinear structure of the original system, see [8] for an overview. In the case of conservation laws the limit of the relaxation

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system towards the original system has been studied by many authors, among others [20, 17, 7, 24, 5, 11, 10]. This has also been exploited intensively for numerical purposes, see among others [2, 1, 12, 22, 18]. In the case of hyperbolic balance laws we are not aware of the a similar analysis for the corresponding relaxation systems with non-zero source term. We derive decay results independent of the relaxation parameter for time towards infinity extending existing results [13]. This requires the treatment of a positive semidefinite operator in the relaxation approximation. However, the convergence towards the nonconstant steady states of the original system does not hold true in general. To illustrate the different situations we discuss two examples from gas dynamics. Similar systems have been studied recently e.g. in [11, 10]. In this paper our focus is on the limit behavior for small relaxation parameter. We study the long term behavior using an  $L^2$ -Lyapunov function whereas in [10] BV-estimates are derived. Also, in [23, 24] the relaxation limit for conservation laws has been discussed and sufficient conditions for the existence of the limiting equation as well as boundary conditions have been analysed. Note that if the stationary states of the original balance law are constants (in space), then similar results as in [23] for conservation laws hold true. However, for a general balance law it may well be that the stationary states are spatially dependent.

## 2 Natural relaxation approximation for hyperbolic balance laws

Let  $t \in \mathbb{R}^+$  and  $x \in \mathbb{R}$  denote the time and spatial variables respectively. Consider a  $2 \times 2$  hyperbolic system of balance laws in the following form:

$$\begin{cases} \rho_t + q_x = 0, & (1a) \\ q_t + f(\rho, q)_x = g(\rho, q), & (1b) \end{cases}$$

where  $\rho(t, x)$  and  $q(t, x)$  are unknown scalar functions.  $f$  and  $g$  are given functions that depend continuously differentiablely on  $\rho$  and  $q$ . Clearly, the stationary states of (1), denoted by  $(\rho_s, q_s)$  satisfies

$$q_s = \text{constant}, \quad f(\rho_s, q_s)_x = g(\rho_s, q_s). \quad (2)$$

Our driving example is gas flow in pipeline systems where  $g$  denotes the pipe wall friction and  $f$  is the momentum flux, see e.g. [4]. The equations are presented in Section 4.

With two real parameters  $\Lambda$  and  $\varepsilon > 0$  we consider the following natural semi-linear relaxation formulation of the balance law (1), where the additional relaxation variables  $u$  and  $w$  are introduced:

$$\begin{cases} \rho_t + u_x = 0, \\ q_t + w_x = g(\rho, q), \\ u_t + \Lambda^2 \rho_x = \frac{1}{\varepsilon}(q - u), \\ w_t + \Lambda^2 q_x = \frac{1}{\varepsilon}(f(\rho, q) - w). \end{cases} \quad (3)$$

The parameter  $\Lambda$  determines the slopes of the characteristic curves of the semilinear relaxed problem and hence it defines the domains of influence in the relaxation system. The value of  $\Lambda$  should be chosen sufficiently large such that the domain of dependence contains the one for the original system, and satisfies the subcharacteristic condition proposed in [20]. The second real relaxation parameter is  $\varepsilon > 0$  and it yields a stiff source term. In the case of conservation laws convergence with respect to  $\varepsilon \rightarrow 0+$  has been established by various authors, see e.g. [23, 5, 11].

Formally, we expect that  $(u, w)$  tend to  $(q, f(\rho, q))$  when  $\varepsilon \rightarrow 0+$ . In contrast to the relaxation for conservation laws, in the relaxed system (3) in the second equation the source term  $g(\rho, q)$  appears. For any fixed  $(\Lambda, \varepsilon)$ , the steady states  $(\bar{\rho}, \bar{q}, \bar{u}, \bar{w})$  of (3) solve the ordinary differential equations

$$\bar{u}_x = 0, \bar{w}_x = g(\bar{\rho}, \bar{q}), \bar{\rho}_x = \frac{1}{\varepsilon \Lambda^2} (\bar{q} - \bar{u}), \bar{q}_x = \frac{1}{\varepsilon \Lambda^2} (f(\bar{\rho}, \bar{q}) - \bar{w}).$$

In general, those states are completely different from the stationary states of (1); see the example of isothermal Euler equations in Section 4.1, where  $\rho_s$  is strictly concave and  $\bar{\rho}$  is strictly convex.

Next, we rewrite the system (3) in vector matrix notation. Define  $U = (\rho, q, u, w)^T$ ,

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Lambda^2 & 0 & 0 & 0 \\ 0 & \Lambda^2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F(\rho, q, u, w) = \begin{pmatrix} 0 \\ g(\rho, q) \\ \frac{1}{\varepsilon}(q - u) \\ \frac{1}{\varepsilon}(f(\rho, q) - w) \end{pmatrix}. \quad (4)$$

Then (3) is equivalent to

$$U_t + A U_x = F(\rho, q, u, w). \quad (5)$$

*Remark 1.* The matrix  $A$  has the eigenvalues  $-\Lambda$  and  $\Lambda$ . For each eigenvalue, two linear independent eigenvectors exist. A basis of eigenvectors is given by

$$v_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ \Lambda \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ \Lambda \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \Lambda \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 0 \\ \Lambda \\ 0 \end{pmatrix}. \quad (6)$$

Thus for the relaxed system at each boundary point we prescribe two boundary conditions instead of one for the original system. The additional boundary conditions should be chosen in such a way that they are compatible with the original system (1), that is such that the last two components of  $F$  vanish. Those conditions are independent of  $\varepsilon$ . This choice is as suggested in [17].

In order to analyze the relaxed system (3), (5) respectively, we bring it first in a form with a symmetric system matrix.

**Lemma 1.** *Let  $Y \in \mathbb{R}^{4 \times 4}$  be symmetric and such that  $AY = YA^T$ . Then we have*

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & \Lambda^2 Y_1 \end{pmatrix}, \quad (7)$$

where

$$Y_1 = \begin{pmatrix} y_a & y_b \\ y_b & y_e \end{pmatrix}, \quad Y_2 = \begin{pmatrix} y_c & y_d \\ y_d & y_g \end{pmatrix} \quad (8)$$

with  $y_a, y_b, y_c, y_d, y_e$  and  $y_g$  being real numbers. Moreover, if  $Y_1$  is positive definite and  $\Lambda$  is sufficiently large, the matrix  $Y$  is positive definite.

The matrix  $Y$  will be used to symmetrize the system in a similar way as in assumption (H1) in [6]. However, in our case since  $g \neq 0$  it is not possible to symmetrize the (linearized) source term simultaneously. Since  $g \neq 0$  the stationary solutions depend on the space variable  $x$ . If we linearize the system locally around a stationary state and make the source term at the same time as symmetric as possible, the corresponding symmetrizer  $Y$  will also depend on  $x$ , that is  $Y = Y(x)$ , and we need to take this into account in the forthcoming analysis.

Let  $f_\rho, g_\rho$  denote the partial derivatives of  $f, g$  with respect to  $\rho$  and  $f_q, g_q$  denote the partial derivatives of  $f, g$  with respect to  $q$ . Let  $\bar{U} = (\bar{\rho}, \bar{q}, \bar{u}, \bar{w})^T$  denote a continuously differentiable stationary solution of (3), that is

$$A \bar{U}_x = F(\bar{\rho}, \bar{q}, \bar{u}, \bar{w}).$$

Introduce the perturbation solution  $\tilde{U} = (\tilde{\rho}, \tilde{q}, \tilde{u}, \tilde{w})^T$  and decompose the solution of (3) around the stationary state:  $U = \bar{U} + \tilde{U}$ . We obtain the linearized system for the first order perturbation  $\tilde{U}$  as

$$\tilde{U}_t + A \tilde{U}_x = -B \tilde{U} \quad (9)$$

where with the notation  $\bar{f}_\rho = f_\rho(\bar{\rho}, \bar{q})$ , etc., the matrix  $B$  is given by

$$B = - \begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{g}_\rho & \bar{g}_q & 0 & 0 \\ 0 & \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} & 0 \\ \frac{1}{\varepsilon} \bar{f}_\rho & \frac{1}{\varepsilon} \bar{f}_q & 0 & -\frac{1}{\varepsilon} \end{pmatrix}, \quad (10)$$

and we did not indicate the dependence on  $x$  for readability. Since it is not possible to have a completely symmetric  $BY$ , we construct a symmetrizer  $Y$  such that  $BY$  is symmetric up to a  $2 \times 2$  submatrix. Such a symmetrizer  $Y$  is presented in the following lemma.

**Lemma 2.** For  $\varepsilon > 0$ , consider the symmetrizer  $Y(x, \varepsilon)$  of  $A$  given by

$$Y_1 = \begin{pmatrix} -\bar{g}_q & \bar{g}_\rho \\ \bar{g}_\rho & y_e \end{pmatrix}, \quad Y_2 = \begin{pmatrix} \bar{g}_\rho & y_d \\ y_d & y_g \end{pmatrix}, \quad (11)$$

where

$$y_d = \bar{f}_q \bar{g}_\rho - \bar{f}_\rho \bar{g}_q, \quad y_e = y_d + \varepsilon \left( \bar{g}_q y_d + \bar{g}_\rho^2 \right), \quad y_g = \frac{\bar{f}_\rho \bar{g}_\rho + \bar{f}_q y_e - \varepsilon \bar{g}_\rho y_d}{1 + \varepsilon \bar{g}_q}.$$

If  $Y_1$  is positive definite and  $\Lambda$  is sufficiently large, the matrix  $Y$  is positive definite.

Note that  $Y_1$  positive definite in particular implies  $\bar{g}_q < 0$ . Using the above  $Y$ , we have

$$BY = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\bar{g}_q y_e - \bar{g}_\rho^2 & -\bar{g}_q y_d - \bar{g}_\rho^2 & -\bar{g}_q y_g - \bar{g}_\rho y_d \\ 0 & -\bar{g}_q y_d - \bar{g}_\rho^2 & \frac{-(\Lambda^2 - \bar{f}_\rho) \bar{g}_q - \bar{g}_\rho \bar{f}_q}{\varepsilon} & \frac{\Lambda^2 \bar{g}_\rho - y_g}{\varepsilon} \\ 0 & -\bar{g}_q y_g - \bar{g}_\rho y_d & \frac{(\Lambda^2 - \bar{f}_\rho) \bar{g}_\rho - \bar{f}_q y_d}{\varepsilon} & \frac{\Lambda^2 y_e - \bar{f}_\rho y_d - \bar{f}_q y_g}{\varepsilon} \end{pmatrix}. \quad (12)$$

Hence, only the (3,4)–th and (4,3)–th components are left to be symmetrized. Denote the symmetric part of the  $3 \times 3$  submatrix in the bottom right corner of  $BY$  by  $B_3$ , i.e.,

$$B_3 = \begin{pmatrix} -\bar{g}_q y_e - \bar{g}_\rho^2 & -\bar{g}_q y_d - \bar{g}_\rho^2 & -\bar{g}_q y_g - \bar{g}_\rho y_d \\ -\bar{g}_q y_d - \bar{g}_\rho^2 & \frac{-(\Lambda^2 - \bar{f}_\rho) \bar{g}_q - \bar{g}_\rho \bar{f}_q}{\varepsilon} & \frac{(2\Lambda^2 - \bar{f}_\rho) \bar{g}_\rho - \bar{f}_q y_d - y_g}{2\varepsilon} \\ -\bar{g}_q y_g - \bar{g}_\rho y_d & \frac{(2\Lambda^2 - \bar{f}_\rho) \bar{g}_\rho - \bar{f}_q y_d - y_g}{2\varepsilon} & \frac{\Lambda^2 y_e - \bar{f}_\rho y_d - \bar{f}_q y_g}{\varepsilon} \end{pmatrix}. \quad (13)$$

**Lemma 3.** *The matrix  $BY$  given by (12) is positive semidefinite in the sense that  $z^T BYz \geq 0$  for any  $z \in \mathbb{R}^4$  if  $B_3$  is positive semidefinite. The matrix  $B_3$  is positive definite if all the leading principal minors are positive. The sufficient conditions are given as follows: for  $\varepsilon$  small enough and  $\Lambda$  large enough,*

$$\bar{g}_q < 0, \quad \bar{f}_q \bar{g}_\rho > \bar{f}_\rho \bar{g}_q > 0, \quad y_d > -\frac{\bar{g}_\rho^2}{\bar{g}_q}, \quad \Lambda^2 > -\frac{y_d}{\bar{g}_q}, \quad \text{and } \varepsilon < -\frac{1}{\bar{g}_q}. \quad (14)$$

If (14) holds, (9) is symmetrized in such a way that the matrix of the source term has a one-dimensional kernel and is positive semidefinite. Starting from this representation the system matrix  $A$  can be diagonalized by an orthogonal transformation in such a way that the eigenvalues of the symmetric part of the source term are not changed. In other words, the source term remains positive semidefinite with a one-dimensional kernel even after diagonalization of  $A$ .

For our analysis, the limit behavior of the matrices  $A$  and  $Y$  as functions of  $\varepsilon$  are essential. Moreover,  $Y$  may dependent on the spatial variable  $x$  if the stationary states  $(\bar{\rho}, \bar{q})$  are nonconstant. Let  $Y(x, \varepsilon)$  denote  $Y$  defined as in Lemma 2. Note that  $Y(x, \varepsilon)$  remains *uniformly bounded* as  $\varepsilon$  tends to zero. In fact, we have

$$Y_0(x) := \lim_{\varepsilon \rightarrow 0^+} Y(x, \varepsilon) = \begin{pmatrix} -\bar{g}_q & \bar{g}_\rho & \bar{g}_\rho & y_d \\ \bar{g}_\rho & y_d & y_d & \bar{f}_\rho \bar{g}_\rho + \bar{f}_q y_d \\ \bar{g}_\rho & y_d & -\Lambda^2 \bar{g}_q & \Lambda^2 \bar{g}_\rho \\ y_d & \bar{f}_\rho \bar{g}_\rho + \bar{f}_q y_d & \Lambda^2 \bar{g}_\rho & \Lambda^2 y_d \end{pmatrix} \quad (15)$$

and  $Y_0(x)$  is positive definite. This implies that if the conditions from Lemma 2 hold, the square roots  $Y^{1/2}(x, \varepsilon)$  and  $Y_0^{1/2}(x)$  also exist and  $\lim_{\varepsilon \rightarrow 0^+} Y(x, \varepsilon)^{1/2} = Y_0^{1/2}(x)$ , see

[16]. Since  $A$  is a constant matrix, we obtain

$$\lim_{\varepsilon \rightarrow 0^+} AY(x, \varepsilon) = AY_0(x) = \begin{pmatrix} \bar{g}_\rho & y_d & -\Lambda^2 \bar{g}_q & \Lambda^2 \bar{g}_\rho \\ y_d & \bar{f}_\rho \bar{g}_\rho + \bar{f}_q y_d & \Lambda^2 \bar{g}_\rho & \Lambda^2 y_d \\ -\Lambda^2 \bar{g}_q & \Lambda^2 \bar{g}_\rho & \Lambda^2 \bar{g}_\rho & \Lambda^2 y_d \\ \Lambda^2 \bar{g}_\rho & \Lambda^2 y_d & \Lambda^2 y_d & \Lambda^2 (\bar{f}_\rho \bar{g}_\rho + \bar{f}_q y_d) \end{pmatrix}.$$

Similar to [19, 6] we consider the symmetric matrix

$$\bar{A}(x, \varepsilon) = Y(x, \varepsilon)^{-1/2} AY(x, \varepsilon)^{1/2}.$$

Since the matrix  $\bar{A}(x, \varepsilon)$  is similar to  $A$ , it has the same eigenvalues  $\pm\Lambda$  that do *not* depend on the spatial variable  $x$ . A basis of eigenvectors of  $\bar{A}(x, \varepsilon)$  is given by  $w_i = \frac{Y(x, \varepsilon)^{-1/2} v_i}{\|Y(x, \varepsilon)^{-1/2} v_i\|}$  for  $i \in \{1, 2, 3, 4\}$  with  $v_i$  defined in (6).

Since  $\bar{A}(x, \varepsilon)$  is symmetric, it can be diagonalized by a real orthogonal matrix  $Q(x, \varepsilon)$  to a constant diagonal matrix  $D$  with the diagonal  $(-\Lambda, -\Lambda, \Lambda, \Lambda)$ , that is

$$Q^T(x, \varepsilon) \bar{A}(x, \varepsilon) Q(x, \varepsilon) = D.$$

To make sure that  $Q(x, \varepsilon)$  can be chosen in such a way that it remains uniformly bounded with respect to  $\varepsilon$ , we assume that the first column of  $Q(x, \varepsilon)$  that contains an eigenvector for the eigenvalue  $-\Lambda$  is given by  $w_1$  and the third column of  $Q(x, \varepsilon)$  that contains an eigenvector for the eigenvalue  $\Lambda$  is given by  $w_3$ . By orthogonalization we define the second column in the following way: Let  $\tilde{w}_2 = v_2 - (v_2^T w_1) w_1$ , and define the second column of  $Q(x, \varepsilon)$  as  $\frac{\tilde{w}_2}{\|\tilde{w}_2\|}$ . Similarly, we obtain the fourth column. Then,  $Q(x, \varepsilon)$  is uniquely determined and remains uniformly bounded with respect to  $\varepsilon$ . Moreover, the limit  $\lim_{\varepsilon \rightarrow 0^+} Q(x, \varepsilon) =: Q_0(x)$  exists. Using the transformation matrix  $Y(x, \varepsilon)^{1/2} Q(x, \varepsilon)$  that is uniformly bounded with respect to  $\varepsilon$ , we transform our system to a diagonal form. Through this transformation, the source term that corresponds to the matrix  $B(x, \varepsilon) Y(x, \varepsilon)$  remains positive semidefinite, but it is not necessarily uniformly bounded with respect to  $\varepsilon$ .

### 3 Uniform exponential decay

Now we consider the problem of boundary stabilization of (9) on a finite space interval  $[0, L]$ . We show that for appropriate boundary conditions (for which (17) holds) the solution decays exponentially. An example is  $z_1 = z_2 = 0$  at  $x = L$  and  $z_3 = z_4 = 0$  at  $x = 0$ . For the proof, we analyze the exponential decay of a suitably defined Lyapunov function. The decay rate is independent of  $\varepsilon$ . This is related to the results presented in [13]. Define the positive semidefinite matrix

$$\tilde{B}(x, \varepsilon) = Q(x, \varepsilon)^T Y(x, \varepsilon)^{-1/2} B(x, \varepsilon) Y(x, \varepsilon)^{1/2} Q(x, \varepsilon).$$

Note that  $\tilde{B}(x, \varepsilon)$  has a one-dimensional kernel. Define the unit vectors  $e_1, e_2, e_3, e_4$  as  $e_1 = (1, 0, 0, 0)^T$  etc. Then  $\ker BY(x, \varepsilon) = \langle e_1 \rangle$  is the vector space spanned by  $e_1$ . Hence,  $\ker \tilde{B}(x, \varepsilon) = \langle Q(x, \varepsilon)^T Y(x, \varepsilon)^{1/2} e_1 \rangle$ . We define

$$R(x, \varepsilon) = DQ_x^T Q + Q^T (Y^{-1/2})_x Y^{1/2} Q,$$

where the right-hand side depends on  $x$  and  $\varepsilon$ . We use the classical transformation

$$z(t, x) = Q^T(x, \varepsilon) Y(x, \varepsilon)^{-1/2} \tilde{U}(t, x)$$

that yields the system in diagonal form

$$z_t + Dz_x = (-\tilde{B} + R)z. \quad (16)$$

Note that  $z$  depends on the relaxation parameter  $\varepsilon$  as well. According to the previous discussion, we see that the limit  $\lim_{\varepsilon \rightarrow 0^+} R(x, \varepsilon) =: R_0(x)$  exists. We use the notation  $\lambda_1 = \lambda_2 = -\Lambda$ ,  $\lambda_3 = \lambda_4 = \Lambda$ . We consider the system in a finite space interval  $[0, L]$ , where  $L > 0$  is some given positive constant. To examine the behavior of solutions  $z$ , we start by introducing the Lyapunov functions in terms of  $z$  and  $\partial_t z$ . Let

$$J_0(t) = \frac{1}{2} \int_0^L e^{\mu_1 x} z_1^2(t, x) + e^{\mu_2 x} z_2^2(t, x) + e^{-\mu_3 x} z_3^2(t, x) + e^{-\mu_4 x} z_4^2(t, x) dx,$$

$$J_1(t) = \frac{1}{2} \int_0^L \sum_{i=1}^4 e^{-\text{sgn}(\lambda_i) \mu_i x} (\partial_t z_i(t, x))^2 dx$$

where the numbers  $\mu_i$  are positive constants to be determined to ensure the exponential decay. Recall that  $\tilde{B}(x, \varepsilon)$  is positive semidefinite and that  $R_0(x)$ , the limit of  $R(x, \varepsilon)$  as  $\varepsilon \rightarrow 0$ , exists. Hence, there exists a constant parameter  $C_0$  that is independent of  $\varepsilon$  such that

$$z^T R(x, \varepsilon) z \leq C_0 |z|^2 \quad \text{for all } z \in \mathbb{R}^4.$$

**Theorem 1.** *Define the positive constant*

$$\beta_0 = \min_{1 \leq i \leq 4} |\lambda_i| \mu_i.$$

*Choose  $\mu_i$  sufficiently large such that  $\beta_0 - C_0 > 0$ . Assume that*

$$\sum_{i=1}^4 \lambda_i [z_i^2(t, 0) - e^{-\text{sgn}(\lambda_i) \mu_i L} z_i^2(t, L)] \leq 0. \quad (17)$$

*Then,  $J_0$  decays exponentially to zero with the rate  $\nu_0 := \beta_0 - C_0$  that is independent of  $\varepsilon$ , that is for all  $t \geq 0$  we have the inequality*

$$J_0(t) \leq \exp(-\nu_0 t) J_0(0). \quad (18)$$



Further, assume that

$$\sum_{i=1}^4 \lambda_i [(\partial_t z_i(t, 0))^2 - e^{-\text{sgn}(\lambda_i)\mu_i L} (\partial_t z_i(t, L))^2] \leq 0.$$

Then,  $J_1$  decays exponentially to zero with the same rate  $\nu_0$ .

Since the Lyapunov function  $J_0$  is equivalent to the  $L^2$ -norm of  $z$ , Theorem 1 also yields the exponential  $L^2$  decay of  $z$  (and  $z_t$ , respectively).

*Proof.* The time derivative of  $J_0$  is given by

$$\begin{aligned} J_0'(t) &= \int_0^L \sum_{i=1}^4 e^{-\text{sgn}(\lambda_i)\mu_i x} z_i \partial_t z_i dx \\ &= \sum_{i=1}^4 \int_0^L e^{-\text{sgn}(\lambda_i)\mu_i x} z_i \left( -\lambda_i \partial_x z_i + \sum_{j=1}^4 (-\tilde{B} + R)_{ij} z_j \right) dx \\ &= -\frac{1}{2} \sum_{i=1}^4 \int_0^L e^{-\text{sgn}(\lambda_i)\mu_i x} \lambda_i \partial_x (z_i^2) dx + \sum_{1 \leq i, j \leq 4} \int_0^L e^{-\text{sgn}(\lambda_i)\mu_i x} z_i (-\tilde{B} + R)_{ij} z_j dx. \end{aligned}$$

Integration by parts yields

$$\begin{aligned} J_0'(t) &= -\frac{1}{2} \sum_{i=1}^4 |\lambda_i| \mu_i \int_0^L e^{-\text{sgn}(\lambda_i)\mu_i x} z_i^2 dx + \frac{1}{2} \sum_{i=1}^4 \lambda_i e^{-\text{sgn}(\lambda_i)\mu_i x} z_i^2(t, x) \Big|_L^0 \\ &\quad + \sum_{1 \leq i, j \leq 4} \int_0^L e^{-\text{sgn}(\lambda_i)\mu_i x} z_i (-\tilde{B} + R)_{ij} z_j dx \\ &\leq -(\beta_0 - C_0) J_0(t) + \frac{1}{2} \sum_{i=1}^4 \lambda_i e^{-\text{sgn}(\lambda_i)\mu_i x} z_i^2(t, x) \Big|_L^0 \leq -\nu_0 J_0(t). \end{aligned}$$

Gronwall's inequality implies that  $J_0(t)$  decays with an exponential rate  $\nu_0$ . Similarly, for  $J_1$ , we have

$$J_1'(t) \leq -(\beta_0 - C_0) J_1(t) + \frac{1}{2} \sum_{i=1}^4 \lambda_i e^{-\text{sgn}(\lambda_i)\mu_i x} (\partial_t z_i(t, x))^2 \Big|_L^0 \leq -\nu_0 J_1(t).$$

## 4 Application to the isothermal Euler equations

To exemplify the theoretical results we apply the relaxation model to the isothermal Euler equations describing gas flow in pipelines. The isothermal Euler equations are a well-established model for realistic high-pressure gas flow in pipes; see e.g. [21, 4]. Denote by  $d > 0$  the diameter of the pipe,  $\lambda_{fric} > 0$  the friction coefficient and  $\varphi \in (-\pi, \pi]$  the slope of the pipe. Define  $\xi = \sin(\varphi)$  and  $\theta = \frac{\lambda_{fric}}{d}$ . Let  $\mathbf{g}$  denote

the gravitational constant. Let  $\rho(t, x)$  denote the gas density,  $p$  the gas pressure and  $q(t, x)$  the flow rate. We consider ideal gas where

$$p := p(\rho) = R_u T \rho. \quad (19)$$

Here  $R_u$  is the gas constant and  $T$  is the constant temperature. The isothermal Euler equations for a sloped pipe with friction are then given by

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{1}{2}\theta \frac{q|q|}{\rho} - \mathbf{g} \xi \rho. \end{cases} \quad (20)$$

The velocity of the gas flow is given by  $v = \frac{q}{\rho}$  and the sound speed  $c$  is given by  $c = \sqrt{p'(\rho)} = \sqrt{R_u T}$ . For the Mach number  $M$ , this yields  $M = \frac{v}{c} = \sqrt{R_u T} \frac{q}{p}$ . Stationary states for the case of an ideal gas have been considered in [14]. We consider the case of subsonic flow where the absolute value of the velocity of the gas is strictly less than speed of sound in the gas, that is  $|M| < 1$ . Stationary states for the original and relaxation systems for different choices of parameters will be investigated. The asymptotic behavior of the relaxation model incorporated with the suitable boundary conditions proposed in Theorem 1 will show the convergence of  $\tilde{U}$  to zero in time. We will consider both a horizontal pipe with  $\xi = 0$  and a pipe with non-zero slope.

#### 4.1 Horizontal pipe with non-zero gas velocity

We show that the system (20) and its relaxation system in the form of (3) possess different stationary states by examination of the sign of  $\rho_{xx}$ . Suppose  $q_s \geq 0$  and a horizontal pipe,  $\xi = 0$ . Note that  $|M| < 1$  and therefore  $\frac{q}{\rho} < \sqrt{R_u T}$ . For the steady states  $(\rho_s, q_s)$  of the system (20), we have

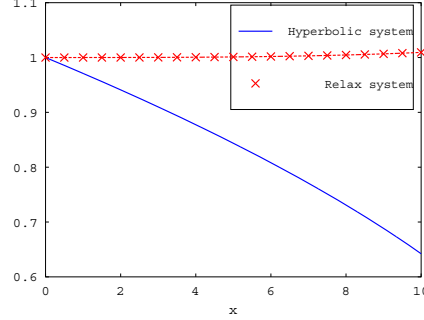
$$\partial_{xx}\rho_s(x) = \frac{g\rho f_\rho - g f_\rho \rho}{f_\rho^2} \partial_x \rho_s(x) = \frac{\theta R_u T q_s^2 g}{2\rho_s^2 f_\rho^3} =: m < 0.$$

For the steady states  $(\bar{\rho}, \bar{q})$  of the corresponding relaxation system (3), we obtain

$$\partial_{xx}\bar{\rho} = \frac{1}{\varepsilon^2 \Lambda^4} \left( f(\bar{\rho}, \bar{q}) - \int_0^x g(\bar{\rho}, \bar{q}) dy \right) > 0.$$

Hence, the relaxation approximation does not give to the correct qualitative behavior of the steady states in this case. In particular, the difference  $\bar{\rho} - \rho_s$  is strongly convex, since we have  $(\bar{\rho} - \rho_s)_{xx} \geq |m| > 0$  where  $m$  is independent of the relaxation parameter  $\varepsilon$ . For  $L > 0$  we have  $\int_0^L ((\bar{\rho} - \rho_s)_{xx})^2(x) dx \geq \int_0^L m^2(x) dx > 0$ . (Note that for the case of conservation laws, we have  $m = 0$  and thus  $\bar{\rho} - \rho_s$  can converge to zero.)

Numerically, Figure 1 shows the discrepancy where the blue curve is the steady state  $\rho_s$  for the system (1) and the red curve with crosses is  $\bar{\rho}$  for the system (3). The different behaviors of the steady-states are clearly visible.



**Fig. 1** Consider the domain  $[0, 10]$  for  $x$ . The number of mesh for the discretization in space is  $N_x = 1000$ . The steady states of the systems (1) and (3) are solved using a second order Runge-Kutta method. The blue curve is  $\rho_s(x)$  for (1) and the red curve with crosses is  $\bar{\rho}(x)$  for (3).

Hence we obtain a negative result regarding the convergence to equilibrium: in fact we have that  $\|\bar{\rho} - \rho_s\|_{L^2(0,L)} \geq C_1 > 0$  for some constant  $C_1$  independent of  $\varepsilon$ .

For the system with appropriate boundary conditions such that (17) holds, Theorem 1 implies that the transient solution  $\rho(t, x)$  to system (3) does not converge to  $\rho_s(x)$  for  $\varepsilon \rightarrow 0+$ . This can be seen as follows. Theorem 1 implies that there exists a constant  $C_2 > 0$  such that for all  $t \geq 0$  we have  $\|\rho(t, \cdot) - \bar{\rho}(\cdot)\|_{L^2(0,L)} \leq C_2 \exp(-v_0 t)$ . Provided that  $t$  is sufficiently large, we have the inequality  $C_2 \exp(-v_0 t) \leq \frac{C_1}{2}$  which implies

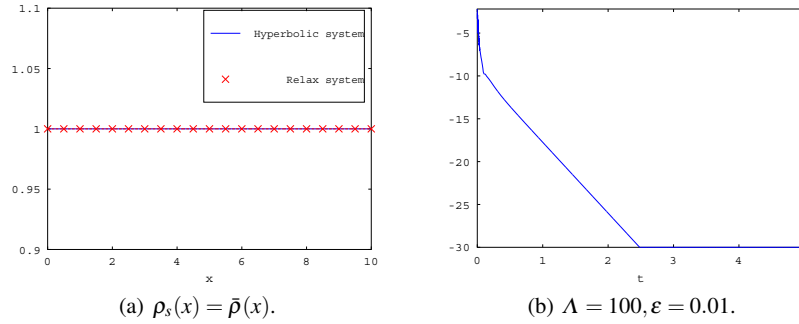
$$\begin{aligned} \|\rho(t, \cdot) - \rho_s(\cdot)\|_{L^2(0,L)} &\geq \|\bar{\rho}(\cdot) - \rho_s(\cdot)\|_{L^2(0,L)} - \|\rho(t, \cdot) - \bar{\rho}(\cdot)\|_{L^2(0,L)} \\ &\geq C_1 - C_2 \exp(-v_0 t) \geq \frac{C_1}{2} > 0. \end{aligned}$$

Since  $v_0$  is independent of  $\varepsilon$  the previous inequality holds for all  $\varepsilon > 0$ .

## 4.2 Constant stationary states

Now, we consider the case of constant stationary states  $(\rho_s, q_s)$ . This implies that  $\rho_s^2 = -\frac{1}{2g\xi} \theta q_s |q_s|$  for  $\xi \neq 0$ . If  $\xi = 0$ , then the only constant stationary state is  $q_s = 0$  and  $\rho_s = \text{const}$ . For the relaxation system (3) we have  $g(\bar{\rho}, \bar{q}) = 0$ . Further, we have  $\bar{u} = \bar{q}$  and  $\bar{w} = f(\bar{\rho}, \bar{q})$ . In this case  $Y, Q$  and  $\bar{B}$  do not depend on  $x$ . Therefore,  $R(x, \varepsilon) = 0$  which yields the simpler diagonal form  $z_t + Dz_x = -\bar{B}z$ . Since  $\bar{B}$  is positive semidefinite, the same estimates as in [6, 23] can be applied.

For the parameters in the system (20), we have  $R_u T = 400$ ,  $\theta = 1$ ,  $\mathbf{g} = 9.80665$ ,  $\xi = -1$ . The constant steady states  $(\rho_s, q_s) = (\bar{\rho}, \bar{q}) = (1, 4.4287)$  for both of the systems (1) and (3). The profiles of  $\rho_s = \bar{\rho}$  are given in Figure 2(a). In order to illustrate the convergence result we perturb the constant steady states  $(\rho_s, q_s) = (1, 4.4287)$  using a sinusoidal function at the initial time  $t = 0$ . Consider the zero boundary condition which satisfies the assumption in Theorem 1 and let  $\Lambda = 100$ ,  $\varepsilon = 0.01$ . We expect the perturbation  $\tilde{U}$  is decreasing to zero in time, as shown in Figure 2(b). As expected we also observe the exponential decay over time.



**Fig. 2** The domain for  $x$  is  $[0, 10]$ . The number of meshes for the spatial discretization is  $N_x = 1000$ . The time step is  $\Delta t = 10^{-6}$ . The final time is  $t = 5$ . (a):  $\rho_s(x)$  is given by the blue curve and  $\bar{\rho}(x)$  is given by the red curve with crosses. (b): The natural logarithm of the  $L^2$  norm of  $\bar{\rho}(x, t)$  as a function of  $t$  which is decaying to zero in time.

## 5 Conclusion

Nonconstant stationary states appear in many practical applications. Relaxation systems provide a useful numerical method for solving conservation and balance laws with constant stationary states. In the case of nonconstant stationary states we could show exponential decay of the relaxation solution over time for suitable boundary conditions. The rate of convergence is independent of the relaxation parameter. However, the relaxation solution does not necessarily converge towards the stationary state of the original system in the case of nonconstant stationary states.

**Acknowledgements** This work was supported by DFG in the framework of the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, project C03, as well as by DFG Cluster of Excellence ‘Integrative Production in High-Wage Countries’ HE5386/13,14,15-1, and NSF RNMS (KI-Net) grant 11-07444.

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