

# Optimality Conditions for Switching Operator Differential Equations

Fabian Rüffler <sup>\*</sup>      Falk M. Hante <sup>†</sup>

FAU Erlangen-Nürnberg

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We consider optimal switching of hybrid abstract evolution equations. The framework includes switching semilinear partial differential equations of parabolic or hyperbolic type, discontinuous state resets, switching costs and allows switching of the principle parts of the equations. We present adjoint-based formulae for the gradient of the cost functional with respect to position and number of switching time points that lead to first order necessary conditions. Moreover, we discuss an alternate-direction approach for implementing descent methods. As an application we consider optimal open/close-switching of valves and on/off-switching control of compressors in a gas network modelled by a graph with simplified euler equations on edges and suitable coupling conditions at nodes.

We present a framework for the optimal switching on a set of abstract differential equations. For each element  $j_n$  in a given finite set of modes  $\mathcal{M}$ , we define a continuous evolution by a generator  $A^{j_n}$  of a  $C^0$ -semigroup and a nonlinear perturbation  $f^{j_n}$ . Moreover, we allow for state transitions given by maps  $g^{j_m, j_n}$  for  $j_m, j_n \in \mathcal{M}$  each time the mode changes. Each sequence  $j = (j_1, \dots, j_N) \subseteq \mathcal{M}$  along with a sequence of switching times  $\tau = (\tau_0, \dots, \tau_N)$  then defines a hybrid system

$$\begin{aligned} \dot{z}(t) &= A^{j_n} z(t) + f^{j_n}(t, z(t)), & n \in \{0, \dots, N\}, t \in (\tau_n, \tau_{n+1}), \\ z(\tau_n) &= g^{j_{n-1}, j_n}(z^-(\tau_n)), & n \in \{1, \dots, N\}, \\ z(\tau_0) &= z_0. \end{aligned} \tag{1}$$

Under fairly mild regularity assumptions on the given operators, we can prove the existence of a unique classical solution  $z$  to (1) on a time interval  $[0, T]$  uniform in  $(j, \tau)$ .

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<sup>\*</sup>e-mail: [fabian.rueffler@fau.de](mailto:fabian.rueffler@fau.de), phone: +49 9131 85-67142, fax: +49 9131 85-67134

<sup>†</sup>e-mail: [falk.hante@fau.de](mailto:falk.hante@fau.de), phone: +49 9131 85-67128, fax: +49 9131 85-67134

For certain running costs  $l$  and switching costs  $l^{j_{n-1}, j_n}$ , we now can ask for an optimal choice for  $j$  and  $\tau$  in order to solve the minimization problem

$$\begin{aligned} \min_{j, \tau} \quad & J(j, \tau, z) = \int_0^T l(t, z(t)) dt + \sum_{n=1}^N l^{j_{n-1}, j_n}(\tau_n, z^-(\tau_n)) \\ \text{s.t.} \quad & z \text{ solves (1), } j \subseteq \mathcal{M}, \tau \in \mathcal{T}(0, T), \end{aligned} \quad (2)$$

where  $\mathcal{T}(0, T) = \{\tau = (\tau_0, \dots, \tau_{N+1}) \mid 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_N \leq \tau_{N+1} = T\}$  is an ordering cone. Formal arguments of Lagrange calculus used to find necessary optimality conditions lead us to the *adjoint system*

$$\begin{aligned} \dot{p}(t) &= -(A^{j_n})^* p(t) - [f_z^{j_n}(t, z(t))]^* p(t) + l_z(t, z(t)), \quad t \in (\tau_n, \tau_{n+1}), \quad n \in \{0, \dots, N\}, \\ p(\tau_n) &= [g_z^{j_{n-1}, j_n}(z^-(\tau_n))]^* p^+(\tau_n) - l_z^{j_{n-1}, j_n}(\tau_n, z^-(\tau_n)), \quad n \in \{1, \dots, N\}, \\ p(T) &= 0. \end{aligned} \quad (3)$$

In contrast to the solution  $z$  to the original problem, only a mild solution  $p$  can be expected for system (3). Nevertheless, the reduced cost functional  $\Phi(j, \tau) = J(j, \tau, z(j, \tau))$  can then shown to be continuously differentiable with respect to the switching time  $\tau_k$  and the *switching time gradient* is given by

$$\frac{\partial \Phi(j, \tau)}{\partial \tau_k} = l(\tau_k, z^-(\tau_k)) - l(\tau_k, z(\tau_k)) + l_\tau^{j_{k-1}, j_k}(\tau_k, z^-(\tau_k)) - \langle p^+(\tau_k), \dot{z}(\tau_k) \rangle_{Z^*, Z}. \quad (4)$$

This formulation allows us to calculate gradients for all switching times by solving for  $z$  and  $p$  once. Moreover, setting  $a(\tau, n) = \min\{m \in \{0, \dots, n\} \mid \tau_m = \tau_n\}$  and  $b(\tau, n) = \max\{m \in \{n, \dots, N+1\} \mid \tau_m = \tau_n\}$ , the following necessary optimality condition holds for fixed  $j$ : if  $\tau$  is a local minimum of  $\Phi$ , then

$$\sum_{l=a(\tau, k)}^k \frac{\partial \Phi}{\partial \tau_l}(\tau) \leq 0 \quad \text{and} \quad \sum_{l=k}^{b(\tau, k)} \frac{\partial \Phi}{\partial \tau_l}(\tau) \geq 0 \quad \text{for all } k \in \{1, \dots, N\}. \quad (5)$$

In fact, it turns out that a similar gradient can be found for the introduction of new switching time points: suppose a new mode  $j_m$  is added to the mode sequence together with two new switching time points  $\tau' = \tau'' = t$  at the same time point  $t$ , one switching to the new mode and the other immediately switching back. Then formula (4) applies for  $\tau''$  and, by its sign, determines whether the newly added mode interval should be expanded or discarded. This thus leads to a *mode insertion gradient* for each mode at each time point

$$\begin{aligned} \frac{\partial \Phi(j, \tau, t)}{\partial j_m} &= l(t, z(t)) - l(t, g^{j(t), j_m}(z(t))) + (l_\tau^{j(t), j_m}(t, z(t))) \\ &\quad + \left\langle p(t), g_z^{j(t), j_m}(z(t)) \left( A^{j(t)} z(t) + f^{j(t)}(t, z(t)) \right) \right. \\ &\quad \left. - \left( A^{j_m} g^{j(t), j_m}(z(t)) + f^{j_m}(t, g^{j(t), j_m}(z(t))) \right) \right\rangle. \end{aligned} \quad (6)$$

The gradients (4) and (6) for the number and position of mode transitions can be incorporated in a numerical descent method for optimization. In detail, an alternate-direction-type approach can be used where, in each iteration, first the current switching time positions are optimized by a projected-gradient method and then a new mode is added at a time point determined by the mode insertion gradient. In [2], a detailed description along with a convergence analysis to points satisfying (5) with (6) being non-negative can be found that, though formulated for a hybrid system of ODEs, translates straightforward to our case.

Conceivable applications are, for instance, hybrid systems of ordinary, delay or partial differential equations. Specifically, here we examine the optimization of switching signals for valves and compressors in gas networks. We model the density  $\rho$  and flow  $q$  of gas in a pipe by the 1D semilinear simplification of the isothermal Euler equations

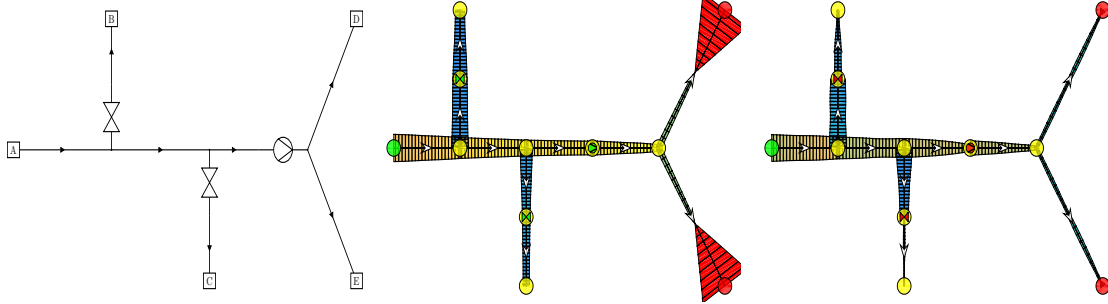
$$\begin{aligned}\frac{\partial \rho}{\partial t}(t, x) + \frac{\partial q}{\partial x}(t, x) &= 0, \\ \frac{\partial q}{\partial t}(t, x) + c^2 \frac{\partial \rho}{\partial x}(t, x) &= -\frac{\lambda}{2D} \frac{q(t, x)|q(t, x)|}{\rho(t, x)},\end{aligned}$$

that includes the conservation of mass and a balance law for the flow including a friction term on the right-hand side. Here,  $c \sim 340 \frac{\text{m}}{\text{s}}$  denotes the speed of sound,  $D \sim 1 \text{ m}$  the pipe diameter and  $\lambda \sim 0.01$  the pipe roughness. The system is supplemented by coupling conditions in each node  $v$  of the network: the gas density has to be continuous across nodes and the net flow of in- and outgoing edges has to coincide with a given node inflow  $q_v$ , i.e.

$$\rho_v(t) = \begin{cases} \rho_e(t, L_e), & \text{if } e \in \delta^+v, \\ \rho_e(t, 0), & \text{if } e \in \delta^-v, \end{cases} \quad \text{and} \quad \sum_{e \in \delta^+v} q_e(t, L_e) - \sum_{e \in \delta^-v} q_e(t, 0) = q_v(t),$$

for an appropriate choice of  $\rho_v: [0, T] \rightarrow \mathbb{R}$ , where  $L_e$  denotes the length of pipe  $e$  and  $\delta^\pm v$  denotes the set of ingoing/outgoing edges at  $v$ . Valves behave like normal nodes if open and impose a zero-flux boundary condition on each incident edge if closed. Compressors are nodes with one ingoing and one outgoing edge, behave like normal nodes if inactive and amplify the ingoing density by a given factor if active. Similar problems appear in the control of fresh and waste water, see [3] for details.

The example shown in Fig. 1 represents a tree network with 10 pipes of length 10 km, 2 valves and a compressor observed over a time of 30 minutes. Gas is supplied at node (A) with constant inflow and could be extracted at several customer nodes (B)-(E). In the particular simulation, gas is initially distributed uniformly over the network, nodes (B) and (C) are inactive and a gas demand emerges at (D) and (E). Valves and compressor are initially open/inactive. The width of the bars in Fig. 2 and Fig. 3 represents the density, the color represents the flow. Without any action, not enough gas can be transported to these nodes and the simulation shows the violation of lower pressure bounds, marked red in the final state shown in Fig. 2. However, the transport can be optimized by defining a cost functional penalizing negative values for the density and introducing switching time points for closing the valves, thereby cutting off the inactive branches of the network, and activating the compressor. Fig. 3 shows the final state after applying the algorithm for the optimization of the switching sequence.



**Fig. 1.** Topology of the network

**Fig. 2.** Final state without action

**Fig. 3.** Final state after optimization

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