# A Local-Search Algorithm for Steiner Forest 

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#### Abstract

In the Steiner Forest problem, we are given a graph and a collection of source-sink pairs, and the goal is to find a subgraph of minimum total length such that all pairs are connected. The problem is APX-Hard and can be 2-approximated by, e.g., the elegant primal-dual algorithm of Agrawal, Klein, and Ravi from 1995.

We give a local-search-based constant-factor approximation for the problem. Local search brings in new techniques to an area that has for long not seen any improvements and might be a step towards a combinatorial algorithm for the more general survivable network design problem. Moreover, local search was an essential tool to tackle the dynamic MST/Steiner Tree problem, whereas dynamic Steiner Forest is still wide open.

It is easy to see that any constant factor local search algorithm requires steps that add/drop many edges together. We propose natural local moves which, at each step, either (a) add a shortest path in the current graph and then drop a bunch of inessential edges, or (b) add a set of edges to the current solution. This second type of moves is motivated by the potential function we use to measure progress, combining the cost of the solution with a penalty for each connected component. Our carefully-chosen local moves and potential function work in tandem to eliminate bad local minima that arise when using more traditional local moves.

Our analysis first considers the case where the local optimum is a single tree, and shows optimality w.r.t. moves that add a single edge (and drop a set of edges) is enough to bound the locality gap. For the general case, we show how to "project" the optimal solution onto the different trees of the local optimum without incurring too much cost (and this argument uses optimality w.r.t. both kinds of moves), followed by a tree-by-tree argument. We hope both the potential function, and our analysis techniques will be useful to develop and analyze local-search algorithms in other contexts.


[^0]Partly

## 1 Introduction

The Steiner Forest problem is the following basic network design problem: given a graph $G=(V, E)$ with edge-lengths $d_{e}$, and a set of source-sink pairs $\left\{\left\{s_{i}, t_{i}\right\}\right\}_{i=1}^{k}$, find a subgraph $H$ of minimum total length such that each $\left\{s_{i}, t_{i}\right\}$ pair lies in the same connected component of $H$. This problem generalizes the Steiner Tree problem, and hence is APX-hard. The Steiner Tree problem has a simple 2-approximation, namely the minimum spanning tree on the terminals in the metric completion; however, the forest version does not have such obvious algorithms.

Indeed, the first approximation algorithm for this problem was a sophisticated and elegant primal-dual 2-approximation due to Agrawal, Klein, and Ravi [AKR95]. Subsequently, Goemans and Williamson streamlined and generalized these ideas to many other constrained network design problems [GW95]. These results prove an integrality gap of 2 for the natural cut-covering LP. Other proofs of this integrality gap were given in [Jai01, CS09]. No better LP relaxations are currently known (despite attempts in, e.g., KLS05, KLSvZ08]), and improving the approximation guarantee of 2 remains an outstanding open problem. Note that all known constant-factor approximation algorithms for Steiner Forest were based on linear programming relaxations, until a recent greedy algorithm [GK15]. In this paper, we add to the body of techniques that give constant-factor approximations for Steiner Forest. The main result of this paper is the following:

Theorem 1. There is a (non-oblivious) local search algorithm for Steiner Forest with a constant locality gap. It can be implemented to run in polynomial time.

The Steiner Forest problem is a basic network problem whose approximability has not seen any improvements in some time. We explore new techniques to attacking the problem, with the hope that these will give us more insights into its structure. Moreover, for many problems solved using the constrained forest approach of [GW95], the only constant factor approximations known are via the primal-dual/local-ratio approach, and it seems useful to bring in new possible techniques. Another motivation for our work is to make progress towards obtaining combinatorial algorithms for the survivable network design problem. In this problem, we are given connectivity requirements between various source-sink pairs, and we need to find a minimum cost subset of edges which provide this desired connectivity. Although we know a 2 approximation algorithm for the survivable network design problem [Jai01] based on iterative rounding, obtaining a combinatorial constant-factor approximation algorithm for this problem remains a central open problem [WS11]. So far, all approaches of extending primal-dual or greedy algorithms to survivable network design have only had limited success. Local search algorithms are more versatile in the sense that one can easily propose algorithms based on local search for various network design problems. Therefore, it is important to understand the power of such algorithms in such settings. We take a step towards this goal by showing that such ideas can give constant-factor approximation algorithms for the Steiner Forest problem.
Finally, we hope this is a step towards solving the dynamic Steiner Forest problem. In this problem, terminal pairs arrive online and we want to maintain a constant-approximate Steiner Forest while changing the solution by only a few edges in each update. Several of the approaches used for the Steiner Tree case (e.g., in [MSVW12, GGK13, LOP ${ }^{+}$15]) are based on local-search, and we hope our local-search algorithm for Steiner Forest in the offline setting will help solve the dynamic Steiner Forest problem, too.

### 1.1 Our Techniques

One of the challenges with giving a local-search algorithm for Steiner Forest is to find the right set of moves. Indeed, it is easy to see that simple-minded approaches like just adding and dropping a constant number of edges at each step is not enough. E.g., in the example of Figure 1, the improving moves must add an edge
and remove multiple edges. (This holds even if we take the metric completion of the graph.) We therefore consider a natural generalization of simple edge swaps in which we allow to add paths and remove multiple edges from the induced cycle.

Local Moves: Our first task is to find the "right" moves that add/remove many edges in each "local" step. At any step of the algorithm, our algorithm has a feasible forest, and performs one of these local moves (which are explained in more detail in $\$ 33$ ):

- edge/set swaps: Add an edge to a tree in the forest, and remove one or more edges from the cycle created.
- path/set swaps: Add a shortest-path between two vertices of a tree in the forest, and again remove edges from the cycle created.
- connecting moves: Connect some trees of the current forest by adding edges between them.

At the end of the algorithm, we apply the following post-processing step to the local optimum:

- clean-up: Delete all inessential edges. (An edge is inessential if dropping it does not alter the feasibility of the solution.)

Given these local moves, the challenge is to bound the locality gap of the problem: the ratio between the cost of a local optimum and that of the global optimum.

The Potential: The connecting moves may seem odd, since they only increase the length of the solution. However, a crucial insight behind our algorithm is that we do not perform local search with respect to the total length of the solution. Instead we look to improve a different potential $\phi$. (In the terminology of [Ali94, KMSV98], our algorithm is a non-oblivious local search.) The potential $\phi(T)$ of a tree $T$ is the total length of its edges, plus the distance between the furthest source-sink pair in it, which we call its width. The potential of the forest $\mathscr{A}$ is the sum of the potentials of its trees. We only perform moves that cause the potential of the resulting solution to decrease.

In $\$$ A. 2 we give an example where performing the above moves with respect to the total length of the solution gives us local optima with $\operatorname{cost} \Omega(\log n)$ - OPT - this example is useful for intuition for why using this potential helps. Indeed, if we have a forest where the distance between two trees in the forest is much less than both their widths, we can merge them and reduce the potential (even though we increase the total length). So the trees in a local optimum are "well-separated" compared to their widths, an important property for our analysis.

The Proof: We prove the constant locality gap in two conceptual steps.
As the first step, we assume that the local optimum happens to be a single tree. In this case we show that the essential edges of this tree $T$ have cost at most $\mathcal{O}(\mathrm{OPT})$-hence


Figure 1: The black edges (continuous lines) are the current solution. If $\ell \gg k$, we should move to the blue forest (dashed lines), but any improving move must change $\Omega(k)$ edges. Details can be found in $\$$ A. 1 the final removal of inessential edges gives a good solution. To prove this, we need to charge our edges to OPT's edges. However, we cannot hope to charge single edges in our solution to single edges in OPT-we need to charge multiple edges in our solution to edges of OPT. (We may just have more edges than OPT does. More concretely, this happens in the example from Figure 1 when $\ell=\Theta(k)$ and we are at the black tree and OPT is the blue forest.) So we consider edge/set swaps that try to swap some subset $S$ of $T$ 's edges for an edge $f$ of OPT. Since such a swap is non-improving at a local optimum, the cost of $S$ is no more than that of $f$. Hence, we would like to partition $T$ 's edges into groups and find an $O(1)$-to-1 map of groups to edges of OPT of no less cost. Even if we cannot find an explicit such map, it turns out that Hall's theorem is the key to showing its existence.

Indeed, the intuition outlined above works out quite nicely if we imagine doing the local search with respect to the total length instead of the potential. The main technical ingredient is a partitioning of our edges into equivalence classes that behave (for our purposes) "like single edges", allowing us to apply a Hall-type argument. This idea is further elaborated in $\$ 4.1$ with detailed proofs in $\$ 6$ However, if we go back to considering the potential, an edge/set swap adding $f$ and removing $S$ may create multiple components, and thus increase the width part of the potential. Hence we give a more delicate argument showing that similar charging arguments work out: basically we now have to charge to the width of the globally optimal solution as well. A detailed synopsis is presented in $\$ 4.2$ and the proofs are in $\$ 7$,

The second conceptual step is to extend this argument to the case where we can perform all possible local moves, and the local optimum is a forest $\mathscr{A}$. If OPT's connected components are contained in those of $\mathscr{A}$, we can do the above analysis for each $\mathscr{A}$-component separately. So imagine that OPT has edges that go between vertices in different components of $\mathscr{A}$. We simply give an algorithm that takes OPT and "projects" it down to another solution $\mathrm{OPT}^{\prime}$ of comparable cost, such that the new projected solution $\mathrm{OPT}^{\prime}$ has connected components that are contained in the components of $\mathscr{A}$. We find the existence of a cheap projected solution quite surprising; our proof crucially uses the optimality of the algorithm's solution under both path/set swaps and connecting moves. Again, a summary of our approach is in $\$ 5$, with proofs in $\$ 8$,

Polynomial-time Algorithm. The locality gap with respect to the above moves is at most 46. Finally, we show that the swap moves can be implemented in polynomial time, and the connecting moves can be approximated to within constant factors. Indeed, a $c$-approximation for weighted $k$-MST gives a $23(1+c)+$ $\varepsilon$-approximation. Applying a weighted version of Garg's 2-approximation [Gar05, Gar16] yields $c=2$. The resulting approximation guarantee is 69 (compared to 96 for [GK15]). Details on this can be found in $\$ \mathbb{B}$,

### 1.2 Related Work

Local search techniques have been very successful for providing good approximation guarantees for a variety of problems: e.g., network design problems such as low-degree spanning trees [FR94], min-leaf spanning trees [LR96, SO98], facility location and related problems, both uncapacitated [KPR00, AGK ${ }^{+}$04] and capacitated [PTW01], geometric $k$-means [KMN ${ }^{+}$04], mobile facility location [AFS13], and scheduling problems [PS15]. Other examples can be found in, e.g., the book of Williamson and Shmoys [WS11]. More recent are applications to optimization problems on planar and low-dimensional instances [CM15, CG15]. In particular, the new PTAS for low dimensional $k$-means in is based on local search [CAKM16, FRS16].
Local search algorithms have also been very successful in practice - e.g., the widely used Lin-Kernighan heuristic [LK73] for the travelling salesman problem, which has been experimentally shown to perform extremely well [Hel00].

Imase and Waxman [W91] defined the dynamic Steiner tree problem where vertices arrive/depart online, and a few edge changes are performed to maintain a near-optimal solution. Their analysis was improved by [MSVW12, GGK13, GK14, LOP ${ }^{+}$15], but extending it to Steiner Forest remains wide open.

## 2 Preliminaries

Let $G=(V, E)$ be an undirected graph with non-negative edge weights $d_{e} \in \mathbb{R}_{>0}$. Let $n:=|V|$. For $W \subseteq V$, let $G[W]=(W, E[W])$ be the vertex-induced subgraph, and for $F \subseteq E, G[F]=(V[F], F)$ the edge-induced subgraph, namely the graph consisting of the edges in $F$ and the vertices contained in them. A forest is a set of edges $F \subseteq E$ such that $G[F]$ is acyclic.
For a node set $W \subseteq V$ and an edge set $F \subseteq E$, let $\delta_{F}(W)$ denote the edges of $F$ leaving $W$. Let $\delta_{F}(A: B):=\delta_{F}(A) \cap \delta_{F}(B)$ for two disjoint node sets $A, B \subseteq V$ be the set of edges that go between $A$
and $B$. For forests $F_{1}, F_{2} \subseteq E$ we use $\delta_{F}\left(F_{1}: F_{2}\right):=\delta_{F}\left(V\left[F_{1}\right]: V\left[F_{2}\right]\right)$. We may drop the subscript if it is clear from the context.

Let $\mathfrak{T} \subseteq\{\{v, \bar{v}\} \mid v, \bar{v} \in V\}$ be a set of terminal pairs. Denote the shortest-path distance between $u$ and $\bar{u}$ in $(G, d)$ by $\operatorname{dist}_{d}(u, \bar{u})$. Let $n_{t}$ be the number of terminal pairs. We number the pairs according to nondecreasing shortest path distance (ties broken arbitrarily). Thus, $\mathfrak{T}=\left\{\left\{u_{1}, \bar{u}_{1}\right\}, \ldots,\left\{u_{n_{t}}, \bar{u}_{n_{t}}\right\}\right\}$ and $i<j$ implies $\operatorname{dist}_{d}\left(u_{i}, \bar{u}_{i}\right) \leq \operatorname{dist}_{d}\left(u_{j}, \bar{u}_{j}\right)$. This numbering ensures consistent tie-breaking throughout the paper. We say that $G=(V, E)$, the weights $d$ and $\mathfrak{T}$ form a Steiner Forest instance. We often use $\mathscr{A}$ to denote a feasible Steiner forest held by our algorithm and $\mathscr{F}$ to denote an optimal/good feasible solution to which we compare $\mathscr{A}$.

Width. Given a connected set of edges $E^{\prime}$, the width $w\left(E^{\prime}\right)$ of $E^{\prime}$ is the maximum distance (in the original graph) of any terminal pair connected by $E^{\prime}$ : i.e., $w\left(E^{\prime}\right)=\max _{\{u, \bar{u}\} \in \mathfrak{z}, u, \bar{u} \in V\left[E^{\prime}\right]} \operatorname{dist}_{d}(u, \bar{u})$. Notice that $w\left(E^{\prime}\right)$ is the width of the pair $\left\{u_{i}, \bar{u}_{i}\right\}$ with the largest $i$ among all pairs in $V\left[E^{\prime}\right]$. We set $\mathfrak{i n d e x}\left(E^{\prime}\right):=$ $\max \left\{i \mid u_{i}, \bar{u}_{i} \in V\left[E^{\prime}\right]\right\}$, i.e., $w\left(E^{\prime}\right)=\operatorname{dist}_{d}\left(u_{\text {inder }\left(E^{\prime}\right)}, \bar{u}_{\text {inder }\left(E^{\prime}\right)}\right)$.
For a subgraph $G[F]=(V[F], F)$ given by $F \subseteq E$ with connected components $E_{1}, \ldots, E_{l} \subseteq F$, we define the total width of $F$ to be the sum $w(F):=\sum_{i=1}^{l} w\left(E_{i}\right)$ of the widths of its connected components. Let $d(F):=\sum_{e \in F} d_{e}$ be the sum of edge lengths of edges in $F$ and define

$$
\phi(F):=d(F)+w(F) .
$$

By the definition of the width, it follows that $d(F) \leq \phi(F) \leq 2 d(F)$.

## 3 The Local Search Algorithm

Our local-search algorithm starts with a feasible solution $\mathscr{A}$, and iteratively tries to improve it. Instead of looking at the actual edge cost $d(\mathscr{A})$, we work with the potential $\phi(\mathscr{A})$ and decrease it over time.
In the rest of the paper, we say a move changing $\mathscr{A}$ into $\mathscr{A}^{\prime}$ is improving if $\phi\left(\mathscr{A}^{\prime}\right)<\phi(\mathscr{A})$. A solution $\mathscr{A}$ is <move>-optimal with respect to certain kind of move if no moves of that kind are improving.


- untouched by move
-     -         - added by move
..... removed by move
Figure 2: Our different moves.
Swaps. Swaps are moves that start with a cycle-free feasible solution $\mathscr{A}$, add some edges and remove others to get to another cycle-free feasible solution $\mathscr{A}^{\prime}$.
- The most basic swap is: add an edge e creating a cycle, remove an edge $f$ from this cycle. This is called an edgeledge swap $(e, f)$.
- We can slightly generalize this: add an edge e creating a cycle, and remove a subset $S$ of edges from this cycle $C(e)$. This is called the edge/set swap $(e, S)$. Edge/edge swaps are a special case of edge/set swaps, so edge/set swap-optimality implies edge/edge swap-optimality.
There may be many different subsets of $C(e)$ we could remove. A useful fact is that if we fix some edge $f \in C(e)$ to remove, this uniquely gives a maximal set $R(e, f) \subseteq C(e)$ of edges that can be removed along with $f$ after adding $e$ without violating feasibility. Indeed, $R(e, f)$ contains $f$, and also all edges on $C(e)$ that can be removed in $\mathscr{A} \cup\{e\} \backslash\{f\}$ without destroying feasibility. (See Lemma 13 for a formalization.)

Moreover, given a particular $R(e, f)$, we could remove any subset $S \subseteq R(e, f)$. If we were doing local search w.r.t. $d(\mathscr{A})$, there would be no reason to remove a proper subset. But since the local moves try to reduce $\phi(\mathscr{A})$, removing a subset of $R(e, f)$ may be useful. If $e_{1}, \ldots, e_{\ell}$ are the edges in $R(e, f)$ in the order they appear on $C(e)$, we only need swaps where $S$ consists of edges $e_{j}, \ldots, e_{j}$ that are consecutive in the above order. There are $\mathcal{O}\left(n^{2}\right)$ sets $S \subseteq R(e, f)$ that are consecutive ${ }^{*}$ Moreover, there are at most $n-1$ choices for $e$ and $O(n)$ choices for $f$, so the number of edge/set swaps is polynomial.

- A further generalization: we can pick two vertices $u, v$ lying in some component $T$, add a shortest-path between them (in the current solution, where all other components are shrunk down to single points, and the vertices/edges in $T \backslash\{u, v\}$ are removed). This creates a cycle, and we want to remove some edges. We now imagine that we added a "virtual" edge $\{u, v\}$, and remove a subset of consecutive edges from some $R(\{u, v\}, f) \subseteq C(\{u, v\})$, just as if we'd have executed an edge/set swap with the "virtual" edge $\{u, v\}$. We call such a swap a path/set swap $(u, v, S)$.
Some subtleties: Firstly, the current solution $\mathscr{A}$ may already contain an edge $\{u, v\}$, but the $u v$-shortestpath we find may be shorter because of other components being shrunk. So this move would add this shortest-path and remove the direct edge $\{u, v\}$-indeed, the cycle $C(u v)$ would consist of two parallel edges, and we'd remove the actual edge $\{u, v\}$. Secondly, although the cycle contains edges from many components, only edges within $T$ are removed. Finally, there are a polynomial number of such moves, since there are $O\left(n^{2}\right)$ choices for $u, v, O(n)$ choices for $f$, and $O\left(n^{2}\right)$ consecutive removal sets $S$.

Note that edge/set swaps never decrease the number of connected components of $\mathscr{A}$, but path/set swaps may increase or decrease the number of connected components.

Connecting moves. Connecting moves reduce the number of connected components by adding a set of edges that connect some of the current components. Formally, let $G_{\mathscr{A}}^{\text {all }}$ be the (multi)graph that results from contracting all connected components of $\mathscr{A}$ in $G$, deleting loops and keeping parallel edges. A connecting move (denoted $\operatorname{conn}(T)$ ) consists of picking a tree in $G_{\mathscr{A}}^{\text {all }}$, and adding the corresponding edges to $\mathscr{A}$. The number of possible connecting moves can be large, but Section B.1 discusses how to do this approximately, using a $k$-MST procedure.

Note that connecting moves cause $d\left(\mathscr{A}^{\prime}\right)>d(\mathscr{A})$, but since our notion of improvement is with respect to the potential $\phi$, such a move may still cause the potential to decrease.
In addition to the above moves, the algorithm runs the following post-processing step at the end.
Clean-up. Remove the unique maximal edge set $S \subseteq \mathscr{A}$ such that $\mathscr{A} \backslash S$ is feasible, i.e., erase all unnecessary edges. This might increase $\phi(\mathscr{A})$, but it will never increase $d(\mathscr{A})$.

Checking whether an improving move exists is polynomial except for connecting moves, which we can do approximately (see $\sqrt[s]{B .11}$ ). Thus, the local search algorithm can be made to run in polynomial time by using standard methods (see $\S$ B).

## 4 In Which the Local Optimum is a Single Tree

We want to bound the cost of a forest that is locally optimal with respect to the moves defined above. To start, let us consider a simpler case: suppose we were to find a single tree $T$ that is optimal with respect to just the edge/edge and edge/set swaps. (Recall that edge/set swaps add an edge and remove a consecutive

[^1]subset of the edges on the resulting cycle, while maintaining feasibility. Also, recall that optimality means that no such moves cause the potential $\phi$ to decrease.) Our main result of this section is the following:

Corollary 2. Let $G=(V, E)$ be a graph, let $d_{e}$ be the cost of edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a set of terminal pairs. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible Steiner forests for $(G, d, \mathfrak{T})$ with $V[\mathscr{A}]=V[\mathscr{F}]$. Assume that $\mathscr{A}$ is a tree and that $\mathscr{A}$ is swap-optimal with respect to $\mathscr{F}$ and $\phi$ under edgeledge and edge/set swaps. Denote by $\mathscr{A}^{\prime}$ the modified solution where all inessential edges have been dropped from $\mathscr{A}$. Then,

$$
d\left(\mathscr{A}^{\prime}\right) \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F}) \leq 11.5 \cdot d(\mathscr{F}) .
$$

The actual approximation guarantee is 42 for this case: indeed, Corollary 2 assumes $V[\mathscr{A}]=V[\mathscr{F}]$, which can be achieved (by taking the metric completion on the terminals) at the cost of a factor 2.

The intuition here comes from a proof for the optimality of edge/edge swaps for the Minimum Spanning tree problem. Let $\mathscr{A}$ be the tree produced by the algorithm, and $\mathscr{F}$ the reference (i.e., optimal or near-optimal) solution, with $V[\mathscr{A}]=V[\mathscr{F}]$. Suppose we were looking for a minimum spanning tree instead of a Steiner forest: one way to show that edge/edge swaps lead to a global optimum is to build a bipartite graph whose vertices are the edges of $\mathscr{A}$ and $\mathscr{F}$, and which contains edge $(e, f)$ when $f \in \mathscr{F}$ can be swapped for $e \in \mathscr{A}$ and $d_{e} \leq d_{f}$. Using the fact that all edge/edge swaps are non-improving, we can show that there exists a perfect matching between the edges in $\mathscr{A}$ and $\mathscr{F}$, and hence the cost of $\mathscr{A}$ is at most that of $\mathscr{F}$.

Our analysis is similar in spirit. Of course, we now have to (a) consider edge/set swaps, (b) do the analysis with respect to the potential $\phi$ instead of just edge-lengths, and (c) we cannot hope to find a perfect matching because the problem is NP-hard. These issues make the proofs more complicated, but the analogies still show through.

### 4.1 An approximation guarantee for trees and $d$

In this section, we conduct a thought-experiment where we imagine that we get a connected tree on the terminals which is optimal for edge/set swaps with respect to just the edge lengths, not the potential. In very broad strokes, we define an equivalence relation on the edges of $\mathscr{A}$, and show a constant-to- 1 cost-increasing map from the resulting equivalence classes to edges of $\mathscr{F}$-again mirroring the MST analysis-and hence bounding the cost of $\mathscr{A}$ by a constant times the cost of $\mathscr{F}$. The analysis of the real algorithm in $\$ 4.2$ builds on the insights we develop here.

Some Definitions. The crucial equivalence relation is defined as follows: For edges $e, f \in \mathscr{A}$, let $T_{e, f}$ be the connected component of $\mathscr{A} \backslash\{e, f\}$ that contains the unique $e-f$-path in $\mathscr{A}$. We say $e$ and $f$ are compatible w.r.t. $\mathscr{F}$ if $e=f$ or if there are no $\mathscr{F}$-edges leaving $T_{e, f}$, and denote it by $e \sim_{c p} f$. In Lemma 6 we show that $\sim_{c p}$ is an equivalence relation, and denote the set of equivalence classes by $\mathfrak{S}$.
An edge is essential if dropping it makes the solution infeasible. If $T_{1}, T_{2}$ are the connected components of $\mathscr{A} \backslash\{e\}$, then $e$ is called safe if at least one edge from $\mathscr{F}$ crosses between $T_{1}$ and $T_{2}$. Observe that any essential edge is safe, but the converse is not true: safe edges can be essential or inessential. However, it turns out that the set $S_{u}$ of all unsafe edges in $\mathscr{A}$ forms an equivalence class of $\sim_{c p}$. Hence, all other equivalence classes in $\mathfrak{S}$ contain only safe edges. Moreover, these equivalence classes containing safe edges behave like single edges in the following sense. (Proof in Lemma 14 in $\$ 6$ )
(1) Each equivalence class $S$ lies on a path in $\mathscr{A}$.
(2) For any edge $f \in \mathscr{F}$, either $S$ is completely contained in the fundamental cycle $C_{\mathscr{A}}(f)$ obtained by adding $f$ to $\mathscr{A}$, or $S \cap C_{\mathscr{A}}(f)=\emptyset$.
(3) If $(\mathscr{A} \backslash\{e\}) \cup\{f\}$ is feasible, and $e$ belongs to equivalence class $S$, then $(\mathscr{A} \backslash S) \cup\{f\}$ is feasible. (This last property also trivially holds for $S=S_{u}$.)

Charging. We can now give the bipartite-graph-based charging argument sketched above.
Theorem 3. Let $I=(V, E, \mathfrak{T}, d)$ be a Steiner Forest instance and let $\mathscr{F}$ be a feasible solution for $I$. Furthermore, let $\mathscr{A} \subseteq E$ be a feasible tree solution for I. Assume that $V[\mathscr{F}]=V[\mathscr{A}]$. Let $\Delta: \mathfrak{S} \rightarrow \mathbb{R}$ be a cost function that assigns a cost to all $S \in \mathfrak{S}$. Suppose that $\Delta(S) \leq d_{f}$ for all pairs of $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ and $f \in \mathscr{F}$ such that the cycle in $\mathscr{A} \cup\{f\}$ contains $S$. Then,

$$
\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \Delta(S) \leq \frac{7}{2} \cdot \sum_{f \in \mathscr{F}} d_{f} .
$$

Proof. Construct a bipartite graph $H=(A \cup B, E(H))$ with nodes $A:=\left\{a_{S} \mid S \in \mathfrak{S} \backslash\left\{S_{u}\right\}\right\}$ and $B:=\left\{b_{f} \mid f \in \mathscr{F}\right\}$. Add an edge $\left\{a_{S}, b_{f}\right\}$ whenever $f$ closes a cycle in $\mathscr{A}$ that contains $S$. By our assumption, if $\left\{a_{S}, b_{f}\right\} \in E(H)$ then $\Delta(S) \leq d_{f}$. Suppose that we can show that $\frac{7}{2} \cdot|N(X)| \geq|X|$ for all $X \subseteq A$, where $N(X) \subseteq B$ is the set of neighbors of nodes in $X$. By a generalization of Hall's Theorem (stated as Fact 15 in $\mathbb{\$ 6}$ ), this condition implies that there is an assignment $\alpha: E \rightarrow \mathbb{R}_{+}$such that $\sum_{e \in \delta_{H}(a)} \alpha(e) \geq 1$ for all $a \in A$ and $\sum_{e \in \delta_{H}(b)} \alpha(e) \leq \frac{7}{2}$ for all $b \in B$. Hence

$$
\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \Delta(S) \leq \sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{e \in \delta_{H}\left(a_{S}\right)} \alpha(e) \Delta(S)=\sum_{f \in \mathscr{F}} \sum_{e \in \delta_{H}\left(b_{f}\right)} \alpha(e) \Delta(S) \leq \sum_{f \in \mathscr{F}} \sum_{e \in \delta_{H}\left(b_{f}\right)} \alpha(e) d_{f} \leq \frac{7}{2} \sum_{f \in \mathscr{F}} d_{f} .
$$

It remains to show that $\frac{7}{2} \cdot|N(X)| \geq|X|$ for all $X \subseteq A$. To that aim, fix $X \subseteq A$ and define $\mathfrak{S}^{\prime}:=\left\{S \mid a_{S} \in\right.$ $X\}$. In a first step, contract all $e \in U:=\bigcup_{S \in \mathfrak{G} \backslash \mathfrak{G}^{\prime}} S$ in $\mathscr{A}$, and denote the resulting tree by $\mathscr{A}^{\prime}:=\mathscr{A} / U{ }^{\dagger}$ Note that edges in each equivalence class are either all contracted or none are contracted. Also note that all unsafe edges are contracted, as $S_{u} \notin \mathfrak{S}^{\prime}$. Apply the same contraction to $\mathscr{F}$ to obtain $\mathscr{F}^{\prime}:=\mathscr{F} / U$, from which we remove all loops and parallel edges.
Let $f \in \mathscr{F}^{\prime}$. Since $\mathscr{A}^{\prime}$ is a tree, $f$ closes a cycle $C$ in $\mathscr{A}^{\prime}$ containing at least one edge $e \in \mathscr{A}^{\prime}$. Denoting the equivalence class of $e$ by $S_{e}$, observing that all edges in $\mathscr{A}^{\prime}$ are safe, and using property (2) given above (formally, using Lemma $14(\mathrm{c})$ ), we get that cycle $C$ contains $S_{e}$. Hence the node $b_{f} \in B$ corresponding to $f$ belongs to $N\left(a_{S_{e}}\right) \subseteq N(X)$. Thus, $|N(X)| \geq\left|\mathscr{F}^{\prime}\right|$ and it remains to show that $\frac{7}{2}\left|\mathscr{F}^{\prime}\right| \geq|X|$.
We want to find a unique representative for each $a_{S} \in X$. So we select an arbitrary root vertex $r \in V\left[\mathscr{A}^{\prime}\right]$ and orient all edges in $\mathscr{A}^{\prime}$ away from $r$. Every non-root vertex now has exactly one incoming edge. Every equivalence class $S \in \mathfrak{S}^{\prime}$ consists only of safe edges, so it lies on a path (by Lemma 14(a)). Consider the two well-defined endpoints which are the outermost vertices of $S$ on this path. For at least one of them, the unique incoming edge must be an edge in $S$. We represent $S$ by one of the endpoints which has this property and call this representative $r_{S}$. Let $R \subseteq V\left[\mathscr{A}^{\prime}\right]$ be the set of all representative nodes. Since every vertex has an unique incoming edge, $S \neq S^{\prime}$ implies that $r_{S} \neq r_{S^{\prime}}$. Hence $|R|=\left|\mathfrak{S}^{\prime}\right|=|X|$. Moreover, let $R_{1}$ and $R_{2}$ be the representatives with degrees 1 and 2 in $\mathscr{A}^{\prime}$, and $L$ be the set of leaves of $\mathscr{A}^{\prime}$. As the number of vertices of degree at least 3 in a tree is bounded by the number of its leaves, the number of representatives of degree at least 3 in $\mathscr{A}^{\prime}$ is bounded by $|L|$. So $|X| \leq\left|R_{1}\right|+\left|R_{2}\right|+|L|$.
We now show that every $v \in R_{2} \cup L$ is incident to an edge in $\mathscr{F}^{\prime}$. First, consider any $v \in L$ and let $e$ be the only edge in $\mathscr{A}^{\prime}$ incident to $v$. As $e$ is safe, there must be an edge $f \in \mathscr{F}^{\prime}$ incident to $v$. Now consider any $r_{S} \in R_{2}$ and let $e_{1}, e_{2} \in \mathscr{A}^{\prime}$ be the unique edges incident to $r_{S}$. Because $r_{S}$ is the endpoint of the path corresponding to the equivalence class $S$, the edges $e_{1}$ and $e_{2}$ are not compatible. Hence there must be

[^2]an edge $f \in \mathscr{F}^{\prime}$ incident to $r_{S}$. Because $R_{2}$ and $L$ are disjoint and every edge is incident to at most two vertices, we conclude that $\left|\mathscr{F}^{\prime}\right| \geq\left(\left|R_{2}\right|+|L|\right) / 2$.
Finally, we show that $\left|\mathscr{F}^{\prime}\right| \geq \frac{2}{3}\left|R_{1}\right|$. Let $\mathcal{C}$ be the set of connected components of $\mathscr{F}^{\prime}$ in $G / U$. Let $\mathcal{C}^{\prime}:=\left\{T \in \mathcal{C}| | V[T] \cap R_{1} \mid \leq 2\right\}$ and $\mathcal{C}^{\prime \prime}:=\left\{T \in \mathcal{C}| | V[T] \cap R_{1} \mid>2\right\}$. Note that no representative $r_{S} \in R_{1}$ is a singleton as every leaf of $\mathscr{A}^{\prime}$ is incident to an edge of $\mathscr{F}^{\prime}$. We claim that $|T| \geq\left|V[T] \cap R_{1}\right|$ for every $T \in \mathcal{C}^{\prime}$. Assume by contradiction that this was not true and let $T \in \mathcal{C}^{\prime}$ with $|T|<\left|V(T) \cap R_{1}\right|$. This means that $V[T] \cap R_{1}$ contains exactly two representatives $r_{S}, r_{S^{\prime}} \in R_{1}$ and $T$ contains only the edge $\left\{r_{S}, r_{S^{\prime}}\right\}$. Let $e \in S$ and $e^{\prime} \in S^{\prime}$ be the edges of $\mathscr{A}^{\prime}$ incident to $r_{S}$ and $r_{S^{\prime}}$, respectively. As $e$ and $e^{\prime}$ are not compatible, there must be an edge $f \in \mathscr{F}^{\prime}$ with exactly one endpoint in $\left\{r_{S}, r_{S^{\prime}}\right\}$, a contradiction as this edge would be part of the connected component $T$. We conclude that $|T| \geq\left|V[T] \cap R_{1}\right|$ for every $T \in \mathcal{C}^{\prime}$. Additionally, we have that $|T| \geq|V[T]|-1 \geq \frac{2}{3}|V[T]|$ for all $T \in \mathcal{C}^{\prime \prime}$ as $|V[T]|>2$. Therefore,
$$
\left|\mathscr{F}^{\prime}\right|=\sum_{T \in \mathcal{C}}|T| \geq \sum_{T \in \mathcal{C}^{\prime}}\left|V[T] \cap R_{1}\right|+\sum_{T \in \mathcal{C}^{\prime \prime}} \frac{2}{3}\left|V[T] \cap R_{1}\right| \geq \frac{2}{3}\left|R_{1}\right| .
$$

The three bounds together imply $|X| \leq\left|R_{1}\right|+\left|R_{2}\right|+|L| \leq \frac{3}{2}\left|\mathscr{F}^{\prime}\right|+2\left|\mathscr{F}^{\prime}\right|=\frac{7}{2}\left|\mathscr{F}^{\prime}\right|$.
We obtain the following corollary of Theorem 3 ,
Corollary 4. Let $I=(V, E, \mathfrak{T}, d)$ be a Steiner Forest instance and let OPT be a solution for I that minimizes $d(O P T)=\sum_{e \in O P T} d_{e}$. Let $\mathscr{A} \subseteq E$ be feasible tree solution for I that does not contain inessential edges. Assume $V[\mathscr{A}]=V[O P T]$. If $\mathscr{A}$ is edgeledge and edge/set swap-optimal with respect to OPT and d, then it holds that $\sum_{e \in \mathscr{A}} d_{e} \leq(7 / 2) \cdot \sum_{e \in O P T} d_{e}$.
Proof. Since there are no inessential edges, $S_{u}=\emptyset$. We set $\Delta(S):=\sum_{e \in S} d_{e}$ for all $S \in \mathfrak{S}$. Let $f \in$ OPT be an edge that closes a cycle in $\mathscr{A}$ that contains $S$. Then, $(\mathscr{A} \backslash\{e\}) \cup\{f\}$ is feasible for any single edge $e \in S$ because it is still a tree. By Lemma 14\d), this implies that $(\mathscr{A} \backslash S) \cup\{f\}$ is also feasible. Thus, we consider the swap that adds $f$ and deletes $S$. It was not improving with respect to $d$, because $\mathscr{A}$ is edge/set swap-optimal with respect to edges from OPT and $d$. Thus, $\Delta(S)=\sum_{e \in S} d_{e} \leq d_{f}$, and we can apply Theorem 3 to obtain the result.

### 4.2 An approximation guarantee for trees and $\phi$

We now consider the case where a connected tree $\mathscr{A}$ is output by the algorithm when considering the edge/set swaps, but now with respect to the potential $\phi$ (instead of just the total length as in the previous section). These swaps may increase the number of components, which may have large widths, and hence edge/set swaps that are improving purely from the lengths may not be improving any more. This requires a more nuanced analysis, though relying on ideas we have developed in the previous section.
Here is the high-level summary of this section. Consider some equivalence class $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ of safe edges: these lie on a path (by Lemma 14(a)), hence look like this:

$$
\overline{w_{0}} \xrightarrow{e_{1}} v_{1} \rightarrow \cdots \rightarrow w_{1} \xrightarrow{e_{2}} v_{2} \rightarrow \cdots \rightarrow w_{i-1} \xrightarrow{e_{i}} \sqrt[v_{i} \rightarrow]{\ldots}{\overline{w_{\ell(S)-1}}}^{e_{\ell(S)}} \sqrt{v_{\ell(S)}},
$$

where there are $\ell(S)$ edges and hence $\ell(S)+1$ components formed by deleting $S$. We let $\operatorname{In}_{S}$ be set of the "inner" components (the ones containing $v_{1}, \ldots, v_{\ell(S)-1}$ ), and $\operatorname{In}_{S^{\prime}}$ be the inner components except the two with the highest widths. Just taking the definition of $\phi$, and adding and subtracting the widths of these "not-the-two-largest" inner components, we get

$$
\phi(\mathscr{A})=w(\mathscr{A})+\sum_{e \in S_{u}} d_{e}+\underbrace{\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}}\left(\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{K \in \operatorname{In}_{S^{\prime}}} w(K)\right)}_{\leq 10.5 \cdot d(\mathscr{F}) \text { by Corollary } 118}+\underbrace{\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{K \in \operatorname{In}_{S^{\prime}}} w(K)}_{\leq w(\mathscr{F}) \text { by Lemma[2] }} .
$$

As indicated above, the argument has two parts. For the first summation, look at the cycle created by adding edge $f \in \mathscr{F}$ to our solution $\mathscr{A}$. Suppose class $S$ is contained in this cycle. We prove that edge/set swap optimality implies that $\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{K \in \mathrm{In}_{S^{\prime}}} w(K)$ is at most $3 d_{f}$. (Think of this bound as being a weak version of the facts in the previous section, which did not have a factor of 3 but did not consider weights in the analysis.) Using this bound in Theorem 3 from the previous section gives us a matching that bounds the first summation by $3 \cdot(7 / 2) \cdot d(\mathscr{F})$. (A couple of words about the proofs: the bound above follows from showing that three different swaps must be non-improving, hence the factor of 3. Basically, we break the above path into three at the positions of the two components of highest width, since for technical reasons we do not want to disconnect these high-width components. Details are in $\$ 7.1$ )
For the second summation, we want to sum up the widths of all the "all-but-two-widest" inner components, over all these equivalence classes, and argue this is at most $w(\mathscr{F})$. This is where our notions of safe and compatible edges comes into play. The crucial observations are that (a) given the inner components corresponding to some class $S$, the edges of some class $S^{\prime}$ either avoid all these inner components, or lie completely within some inner component; (b) the notion of compatibility ensures that these inner components correspond to distinct components of $\mathscr{F}$, so we can get more width to charge to; and (c) since we don't charge to the two largest widths, we don't double-charge these widths. The details are in $\$ 7.3$,

## 5 In Which the Local Optimum may be a Forest

The main theorem. In the general case, both $\mathscr{A}$ and $\mathscr{F}$ may have multiple connected components. We assume that the distance function $d$ is a metric. The first thing that comes to mind is to apply the previous analysis to the components individually. Morally, the main obstacle in doing so is in the proof of Theorem 3: There, we assume implicitly that no edge from $\mathscr{F}$ goes between connected components of $\mathscr{A}$ 团 This is vacuously true if $\mathscr{A}$ is a single tree, but may be false if $\mathscr{A}$ is disconnected. In the following, our underlying idea is to replace $\mathscr{F}$-edges that cross between the components of $\mathscr{A}$ by edges that lie within the components of $\mathscr{A}$, thereby re-establishing the preconditions of Theorem 3. We do this in a way that $\mathscr{F}$ stays feasible, and moreover, its cost increases by at most a constant factor. This allows us to prove that the local search has a constant locality gap.

Reducing to local tree optima. Suppose $\mathscr{F}$ has no inessential edges to start. Then we convert $\mathscr{F}$ into a collection of cycles (shortcutting non-terminals), losing a factor of 2 in the cost. Now observe that each "offending" $\mathscr{F}$-edge (i.e., one that goes between different components of $\mathscr{A}$ ) must be part of a path $P$ in $\mathscr{F}$ that connects some $s, \bar{s}$, and hence starts and ends in the same component of $\mathscr{A}$. This path $P$ may connect several terminal pairs, and for each such pair $s, \bar{s}$, there is a component of $\mathscr{A}$ that contains $s$ and $\bar{s}$. Thus, $P$ could be replaced by direct connections between $s, \bar{s}$ within the components of $\mathscr{A}$. This would get rid of these "offending" edges, since the new connections would stay within components of $\mathscr{A}$. The worry is, however, that this replacement is too expensive. We show how to use connecting tree moves to bound the cost of the replacement.
Consider one cycle from $\mathscr{F}$, regarded as a circuit $C$ in the graph $G_{\mathscr{A}}$ where the connected components $A_{1}, \ldots, A_{p}$ of $\mathscr{A}$ are shrunk to single nodes, i.e., $C$ consists of offending edges. The graph $G_{\mathscr{A}}$ might contain parallel edges and $C$ might have repeated vertices. So $C$ is a circuit, meaning that it is a potentially non-simple cycle, or, in other words, a Eulerian multigraph. The left and middle of Figure 3 are an example.
Index the $A_{j}$ 's such that $w\left(A_{1}\right) \leq \cdots \leq w\left(A_{p}\right)$ and say that node $j$ in $G_{\mathscr{A}}$ corresponds to $A_{j}$. Suppose $C$ visits the nodes $v_{1}, \ldots, v_{|C|}, v_{1}$ (where several of these nodes may correspond to the same component $A_{j}$ ) and that component $A_{j}$ is visited $n_{j}$ times by $C$. In the worst case, we may need to insert $n_{j}$ different

[^3]$s, \bar{s}$ connections into component $A_{j}$ of $\mathscr{A}$, for all $j$. The key observation is that the total cost of our direct connections is at most $\sum_{i=1}^{|C|} n_{i} w\left(A_{i}\right)$. We show how to pay for this using the length of $C$ 团
To do so, we use optimality with respect to all moves, in particular connecting moves. The idea is simple: We cut $C$ into a set of trees that each define a valid connecting move. For each tree, the connecting move optimality bounds the widths of some components of $\mathscr{A}$ by the length of the tree. E.g., $w\left(A_{1}\right)+w\left(A_{4}\right)$ is at most the length of the tree connecting $A_{1}, A_{4}, A_{5}$ in Figure 3 Observe that we did not list $w\left(A_{5}\right)$ : Optimality against a connecting move with tree $T$ relates the length of $T$ to the width of all the components that $T$ connects, except for the component with maximum width. We say a tree pays for $A_{j}$ if it hits $A_{j}$, and also hits another $A_{j}$ of higher width. So we need three properties: (a) the trees should collectively pack into the edges of the Eulerian multigraph $C$, (b) each tree hits each component $A_{j}$ at most once, and (c) the number of trees that pay for $A_{j}$ is at least $n_{j}$.
Assume that we found such a tree packing. For circuit $C$, if $A_{j^{\star}}$ is the component with greatest width hit by $C$, then using connecting move optimality for all the trees shows that
$$
\sum_{j: A_{j} \text { hit by } C, j \neq j^{\star}} n_{j} w\left(A_{j}\right) \leq d(C)
$$

In fact, even if we have $c$-approximate connection-move optimality, the right-hand side just gets multiplied by $c$. But what about $n_{j^{\star}} w\left(A_{j^{\star}}\right)$ ? We can cut $C$ into sub-circuits, such that each subcircuit $C^{\prime}$ hits $A_{j^{\star}}$ exactly once. To get this one extra copy of $w\left(A_{j^{\star}}\right)$, we use path/set swap optimality which tells us that the missing connection cannot be more expensive than the length of $C$. Thus, collecting all our bounds (see Lemma36), adding all the extra connections to $\mathscr{F}$ increases the cost to at most $2(1+c) d(\mathscr{F})$ : the factor 2 to make $\mathscr{F}$ Eulerian, $(1+c)$ to add the direct connections, using $c$-approximate optimality with respect to connecting moves and optimality with respect to path/set swaps. 8 discusses that $c \leq 2$.
Now each component $A_{j}$ of $\mathscr{A}$ can be treated separately, i.e., we can use Corollary 2 on each $A_{j}$ and the portion of $\mathscr{F}$ that falls into $A_{j}$. By combining the conclusions for all connected components, we get that

$$
d\left(\mathscr{A}^{\prime}\right) \stackrel{\text { Cor. } 2}{\leq} 10.5 d\left(\mathscr{F}^{\prime}\right)+w\left(\mathscr{F}^{\prime}\right) \leq 11.5 d\left(\mathscr{F}^{\prime}\right) \stackrel{\text { Lem. } 36}{\leq} 23(1+c) \cdot d(\mathscr{F}) \leq 69 \cdot d(\mathscr{F})
$$

for any feasible solution $\mathscr{F}$. This proves Theorem 1 .
Obtaining a decomposition into connecting moves. It remains to show how to take $C$ and construct the set of trees. If $C$ had no repeated vertices (it is a simple cycle) then taking a spanning subtree would suffice. And even if $C$ has some repeated vertices, decomposing it into suitable trees can be easy: E.g., if $C$ is the "flower" graph on $n$ vertices, with vertex 1 having two edges to each vertex $2, \ldots, n$. Even though 1 appears multiple times, we find a good decomposition (see Figure (4). Observe, however, that breaking $C$ into simple cycles and than doing something on each simple cycle would not work, since it would only pay 1 multiple times and none of the others.

The flower graph has a property that is a generalization of a simple cycle: We say that $C$ is minimally guarded if (a) the largest vertex is visited only once (b) between two occurrences of the same (non-maximal) vertex, there is at least one larger number. The flower graph and the circuit at the top of Figure 5 have this property. We show that every minimally guarded circuit can be decomposed suitably by providing Algorithm 1. It iteratively finds trees that pay for all (non maximal) $j$ with $j \leq z$ for increasing $z$. Figure 5 shows how the set of trees $\mathfrak{M}_{5}$ is converted into $\mathfrak{M}_{6}$ in order to pay for all occurrences of 6 . Intuitively, we look where 6 falls into the trees in $\mathfrak{M}_{5}$. Up to one occurrence can be included in a tree. If there are more occurrences, the tree has to be split into multiple trees appropriately. $\$ 8.1 .2$ explains the details.

Finally, Lemma[35]shows how to go from minimally guarded circuits to arbitrary $C$ in a recursive fashion.

[^4]

Figure 3: The charging argument with four components $A_{1}, A_{4}, A_{5}$ and $A_{7}$ of $\mathscr{A}$. The area of each component corresponds to its width. On the left. A cycle in $\mathscr{F}$. In the middle. The corresponding circuit (non-simple cycle) in $G_{\mathscr{A}}$. On the right. A suitable decomposition into connecting moves.


Figure 4: A flower graph. Even though the graph is a non-simple cycle, we can easily decompose it into trees that pay for each $j \neq 6$ at least $n_{j}$ times ( 1 is payed for $7=n_{1}+1$ times).





Figure 5: Two iterations of an example run of the Algorithm 1

## 6 Proofs: Bounds on $d$ when the Local Optimum is a Tree

Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible Steiner forests with respect to a set $\mathfrak{T}$ of terminal pairs. We think of $\mathscr{F}$ as an optimum or near optimum Steiner forest and of $\mathscr{A}$ as a feasible solution computed by our algorithm. We assume that $V[\mathscr{A}]=V[\mathscr{F}]$ (this will be true when we use the results from this section later on). Throughout this section, we assume that $\mathscr{A}$ is a tree (thus, it is a spanning tree on $V[\mathscr{A}]=V[\mathscr{F}]$ ). The following definition is crucial for our analysis.

Definition 5. Let $e=\left\{s, v_{1}\right\}, f=\left\{v_{l}, t\right\} \in \mathscr{A}$ be two edges. Consider the unique (undirected) path $P=s \xrightarrow{e} v_{1} \rightarrow \cdots \rightarrow v_{l} \xrightarrow{f} t$ in $\mathscr{A}$ that connects $e$ and $f$. Let $T_{e, f}$ be the connected component in

(a) First case: Edge $f$ lies on the path from $e$ to $g$. If an $\mathscr{F}$-edge is cut by $T_{f, g}$, the edges $f$ and $g$ cannot be compatible. Likewise, $e$ and $f$ cannot be compatible if an $\mathscr{F}$-edge is cut by $T_{e, f}$. Thus, no $\mathscr{F}$-edge is cut by $T_{e, f} \cup T_{f, g}$ and $e$ and $g$ are compatible.

(b) Second case: The path connecting $f$ to the path $P$ from $e$ to $g$ touches an outer node of $P$. There can be no $\mathscr{F}$-edge between $T_{f, g}$ and $T_{f} \cup T_{e, f}$ because $e$ and $f$ are compatible. Likewise, no $\mathscr{F}$-edge can cross from $T_{f, g}$ to $T_{g}$ because $f$ and $g$ are compatible. This shows that no $\mathscr{F}$-edge is cut by $T_{f, g}$ and thus, $e$ and $g$ are compatible.

(c) Third case: The path connecting $f$ to the path $P$ from $e$ to $g$ touches an inner node $v$ of $P$. Here, no $\mathscr{F}$-edge can cross from $T_{e, f, g}$ to $T_{e}$ nor from $T_{f}$ to $T_{g}$ since $e$ and $f$ are compatible. The edges $f$ and $g$ being compatible, there also cannot be $\mathscr{F}$-edges from $T_{e, f, g}$ to $T_{g}$ nor from $T_{e}$ to $T_{f}$. Thus, no $\mathscr{F}$-edge is cut by $T_{e, f, g} \cup T_{f}$ and thus, the edges $e$ and $g$ are compatible.

Figure 6: The situation in Lemma6.
$\mathscr{A} \backslash\{e, f\}$ that contains $P \backslash\{e, f\}$. We say that $e$ and $f$ are compatible with respect to $\mathscr{F}$ if $e=f$ or if there are no $\mathscr{F}$-edges leaving $T_{e, f}$. In this case, we write $e \sim_{c p} f$.

Observe that $\sim_{c p}$ is a reflexive and symmetric relation. We show that the relation is also transitive, i.e., it is an equivalence relation.

Lemma and Definition 6. Let $\mathscr{A}$ be feasible and let $e, f \in \mathscr{A}$ and $f, g \in \mathscr{A}$ be two pairs of compatible edges. Then, $e$ and $g$ are also compatible. In particular, $\sim_{c p}$ is an equivalence relation. We denote the set of all equivalence classes of $\sim_{c p}$ by $\mathfrak{S}$.

Proof. Consider the unique path $P$ that connects $e$ and $g$ in $\mathscr{A}$. This path has the form $v_{1} \xrightarrow{e} v_{2} \rightarrow \cdots \rightarrow$ $v_{s-1} \xrightarrow{g} v_{s}$. We distinguish three cases (see Figure 6).
For the first case, suppose that $f=\left\{v_{i}, v_{i+1}\right\}$ lies on $P$, for some $i \in\{2, \ldots, s-2\}$. Let $T_{e, f}$ and $T_{f, g}$ be the connected components of $\mathscr{A} \backslash\{e, f, g\}$ that contain $v_{2}, \ldots, v_{i}$ and $v_{i+1}, \ldots, v_{s-1}$, respectively. Assume by contradiction that $\delta_{\mathscr{F}}\left(T_{e, g}\right) \neq \emptyset$. Since $\delta_{\mathscr{F}}\left(T_{e, g}\right) \neq \emptyset$, it follows that $\delta_{\mathscr{F}}\left(T_{e, f}\right) \neq \emptyset$ or $\delta_{\mathscr{F}}\left(T_{f, g}\right) \neq \emptyset$. This is a contradiction to the assumption that $e, f$ and $f, g$ are compatible, respectively.
For the second case, suppose that $f$ does not lie on $P$. Let $R$ be the unique, possibly empty, path that connects $f$ to $P$ and suppose that $R$ meets $P$ at one of the outer nodes (w.l.o.g. assume that $P$ and $R$ meet in $v_{1}$ ). Thus, we are in the situation where $R \cup P$ has the form $w_{1} \xrightarrow{f} w_{2} \rightarrow \cdots \rightarrow w_{t} \rightarrow v_{1} \xrightarrow{e} v_{2} \cdots \rightarrow$ $v_{s-1} \xrightarrow{g} v_{s}$. Consider the four connected components that $\mathscr{A} \backslash\{e, f, g\}$ decomposes into: The component $T_{f}$ contains $w_{1}$, the component $T_{f, e}$ contains $w_{2}, \ldots, w_{t}, v_{1}$, the component $T_{e, g}$ contains $v_{2}, \ldots, v_{s-1}$ and $T_{g}$ contains $v_{s}$. If $R$ is empty, the nodes $w_{2}, w_{t}$ and $v_{1}$ coincide and $T_{f, e}$ contains only $v_{1}$. Both the cuts $\delta_{\mathscr{F}}\left(T_{e, g}: T_{g}\right)$ and $\delta_{\mathscr{F}}\left(T_{e, g}: T_{f}\right)$ must be empty because $f$ and $g$ are compatible. The cut $\delta_{\mathscr{F}}\left(T_{e, g}: T_{f, e}\right)$ must be empty because $e$ and $f$ are compatible. It follows that $e$ and $g$ are compatible.

For the last remaining case, suppose that $f$ does not lie on $P$ and that the unique path $R$ connecting $f$ to


Figure 7: The situation from the proof of Lemma 12. Since $e_{1}, e_{2}$, and $e_{3}$ are pairwise compatible, no $\mathscr{F}$-edge can leave $T_{1}$.
$R$ meets $P$ in an inner node $v_{i}, i \in\{2, \ldots, s-1\}$. Here, $R$ has the form $w_{1} \xrightarrow{f} w_{2} \rightarrow \cdots \rightarrow w_{t} \rightarrow v_{i}$ and $\mathscr{A} \backslash\{e, f, g\}$ decomposes into four components: The component $T_{e}$ contains $v_{1}$, the component $T_{e, f, g}$ contains $v_{2}, \ldots, v_{s-1}$ and $w_{2}, \ldots, w_{t}$, the component $T_{g}$ contains $v_{s}$ and the component $T_{f}$ contains $w_{1}$. The cuts $\delta_{\mathscr{F}}\left(T_{f}: T_{g}\right)$ and $\delta_{\mathscr{F}}\left(T_{e, f, g}: T_{e}\right)$ must be empty because $e$ and $f$ are compatible. The cuts $\delta_{\mathscr{F}}\left(T_{f}: T_{e}\right)$ and $\delta_{\mathscr{F}}\left(T_{e, f, g}: T_{g}\right)$ must be empty because $f$ and $g$ are compatible. Thus, the edges $e$ and $g$ are compatible. This concludes the proof.

We classify the edges in $\mathscr{A}$ with respect to $\mathscr{F}$ and $\mathfrak{T}$. Recall that we assume $V[\mathscr{A}]=V[\mathscr{F}]$, and that $\mathscr{A}$ is a tree, thus, it is a spanning tree on the vertices that we are interested in.

Definition 7. An edge $e \in \mathscr{A}$ is essential if $\mathscr{A}$ is feasible with respect to $\mathfrak{T}$, but $\mathscr{A} \backslash\{e\}$ is infeasible with respect to $\mathfrak{T}$.

Definition 8. Let $e \in \mathscr{A}$ and let $T_{1}, T_{2}$ be the connected components of $\mathscr{A} \backslash\{e\}$. We say that $e$ is safe if $\delta_{\mathscr{F}}\left(T_{1}\right)=\delta_{\mathscr{F}}\left(T_{2}\right) \neq \emptyset$, i.e., if at least one $\mathscr{F}$-edge crosses between $T_{1}$ and $T_{2}$.

By definition, any essential edge is safe, however, the opposite is not true in general: Safe edges can be essential or inessential. We thus have classified the edges of $\mathscr{A}$ into three classes: safe essential edges, safe inessential edges and unsafe (and ergo, inessential) edges.

Lemma 9. Let e,f $f \in \mathscr{A}$ be two compatible edges. Then e is safe if and only if $f$ is safe.
Proof. Suppose that $e$ is safe and let $P=v_{1} \xrightarrow{e} v_{2} \rightarrow \cdots \rightarrow v_{s-1} \xrightarrow{f} v_{s}$ be the unique path between $e$ and $f$ in $\mathscr{A}$. Let $T_{e}, T_{e, f}$ and $T_{f}$ be the connected components of $\mathscr{A}$ that contain the vertex $v_{1}$, the vertices $v_{2}, \ldots, v_{s-1}$ and the vertex $v_{s}$, respectively. Since $e$ and $f$ are compatible we know that $\delta_{\mathscr{F}}\left(T_{f}: T_{e, f}\right)$ is empty. Also, since $e$ is safe, we have that $\delta_{\mathscr{F}}\left(T_{e}: T_{f}\right)=\delta_{\mathscr{F}}\left(T_{f}: T_{e}\right)$ is non-empty. It follows from $\delta_{\mathscr{F}}\left(T_{f}\right)=\delta_{\mathscr{F}}\left(T_{f}: T_{e}\right) \cup \delta_{\mathscr{F}}\left(T_{f}: T_{e, f}\right)$ that $f$ is safe. The reverse implication follows since $e$ and $f$ are exchangeable.

Lemma 10. Let e, $f \in \mathscr{A}$ be two unsafe edges. Then e and $f$ are compatible.
Proof. Let $P=v_{1} \xrightarrow{e} v_{2} \rightarrow \cdots \rightarrow v_{s-1} \xrightarrow{f} v_{s}$ be the unique path between $e$ and $f$ in $\mathscr{A}$. Let $T_{e}$, $T_{e, f}$ and $T_{f}$ be the connected components of $\mathscr{A}$ that contain the vertex $v_{1}$, the vertices $v_{2}, \ldots, v_{s-1}$ and the vertex $v_{s}$, respectively. We need to show that $\delta_{\mathscr{F}}\left(T_{e}: T_{e, f}\right)=\delta_{\mathscr{F}}\left(T_{f}: T_{e, f}\right)=\emptyset$. Observe that $\delta_{\mathscr{F}}\left(T_{e}: T_{e, f}\right) \subseteq \delta_{\mathscr{F}}\left(T_{e}\right)$ and that $\delta_{\mathscr{F}}\left(T_{f}: T_{e, f}\right) \subseteq \delta_{\mathscr{F}}\left(T_{f}\right)$. The assumption that $e$ and $f$ are both unsafe implies that $\delta_{\mathscr{F}}\left(T_{e}\right)=\delta_{\mathscr{F}}\left(T_{f}\right)=\emptyset$ and we have thus shown the claim.

We have the following summary lemma showing that $\sim_{c p}$ behaves well.

Lemma 11. Let $e, f \in \mathscr{A}$ be compatible edges. Then

1. $e$ is essential if and only if $f$ is essential and
2. e is safe if and only $f$ is safe.

Furthermore, if two edges $e, f \in \mathscr{A}$ are unsafe, then they are compatible. Thus, the set

$$
S_{u}:=\{e \in \mathscr{A} \mid e \text { is unsafe }\}
$$

forms an equivalence class of $\sim_{c p}$. If $S \in \mathfrak{S}$ is an equivalence class of $\sim_{c p}$, then either all edges $e \in S$ are essential or none. Also, if $S \neq S_{u}$, then all edges in $S$ are safe.

Proof. We show Statement 1.
Statement 2 and the compatibility of unsafe edges are shown in Lemma 9 and Lemma 10 Let $P=v_{1} \xrightarrow{e}$ $v_{2} \rightarrow \cdots \rightarrow v_{s-1} \xrightarrow{f} v_{s}$ be the unique path between $e$ and $f$ in $\mathscr{A}$. Let $T_{e}, T_{e, f}$ and $T_{f}$ be the connected components of $\mathscr{A}$ that contain the vertex $v_{1}$, the vertices $v_{2}, \ldots, v_{s-1}$ and the vertex $v_{s}$, respectively. Suppose that $e$ is essential. We show that there must be a terminal pair $u, \bar{u}$ with $u \in T_{e} \cup T_{e, f}$ and $\bar{u} \in T_{f}$. It then follows that $\mathscr{A} \backslash\{f\}$ is infeasible.
Since the edge $e$ is essential, there must be a terminal pair $u, \bar{u}$ with $u \in T_{e}$ and $\bar{u} \in T_{e, f} \cup T_{f}$. Since $e$ and $f$ are compatible, however, we know that $\delta_{\mathscr{F}}\left(T_{e}: T_{e, f}\right)=\delta_{\mathscr{F}}\left(T_{f}: T_{e, f}\right)=\emptyset$ which implies that $\delta_{\mathscr{F}}\left(T_{e, f}\right)=\emptyset$. Thus, if $\bar{u} \in T_{e, f}$, the forest $\mathscr{F}$ would be infeasible. It follows that $\bar{u} \in T_{f}$ and thus, that $f$ is essential.

We can now show that the compatibility classes in $\mathfrak{S} \backslash S_{u}$ behave as if they were single edges.
Lemma 12. Let $K:=\left\{e_{1}, \ldots, e_{l}\right\} \subseteq \mathscr{A}$ with $l \geq 2$ be a set of pairwise compatible edges. Furthermore, suppose that $e_{i} \in K$ is safe for all $i=1, \ldots, l$. Then there is a path $P \subseteq \mathscr{A}$ with $K \subseteq P$.

Proof. For $e, f \in K$ let $P_{e, f}$ be the unique path in $\mathscr{A}$ starting with $e$ and ending with $f$. Define $R:=$ $\bigcup_{e, f \in K} P_{e, f}$. Observe that $K \subseteq R \subseteq \mathscr{A}$ and that $R$ is a tree whose leaves are all incident to edges in $K$. By contradiction assume $R$ is not a path. Then there exists a vertex $v_{0}$ with $\left|\delta_{R}\left(v_{0}\right)\right| \geq 3$. In particular, there are three leaves $v_{1}, v_{2}, v_{3}$ of $R$ such that the unique $v_{0}-v_{i}$-paths in $R$ for $i \in\{1,2,3\}$ are pairwise edge-disjoint. Let $e_{1}, e_{2}, e_{3} \in K$ be the edges incident to $v_{1}, v_{2}, v_{3}$ in $R$, respectively. Let $T_{0}, T_{1}, T_{2}, T_{3}$ be the four components of $\mathscr{A} \backslash\left\{e_{1}, e_{2}, e_{3}\right\}$ with $v_{i} \in T_{i}$ for $i \in\{0,1,2,3\}$, see Figure 7. Observe that $\delta_{\mathscr{F}}\left(T_{1}: T_{0} \cup T_{3}\right)=\delta_{\mathscr{F}}\left(T_{1}: T_{0} \cup T_{2}\right)=\emptyset$ because $e_{1} \sim_{c p} e_{2}$ and $e_{1} \sim_{c p} e_{3}$. We deduce that $\delta_{\mathscr{F}}\left(T_{1}\right)=\emptyset$, contradicting the fact that $e_{1}$ is safe.

Lemma 13. Let $S \in \mathfrak{S}$ be an equivalence class of $\sim_{c p}$ and let $e \in S$. Let $f \in \mathscr{F}$ such that $\mathscr{A} \backslash\{e\} \cup\{f\}$ is feasible. Then, $\mathscr{A} \backslash S^{\prime} \cup\{f\}$ is feasible for all $S^{\prime} \subseteq S$.

Proof. If $e$ is inessential, then all edges in $S$ are inessential by Lemma 11. Thus, $\mathscr{A} \backslash S$ is feasible even without $f$ and we are done. Otherwise, $e$ is essential, which implies that all edges in $S$ are essential by Lemma 11. So, they are also safe and we can apply 14(b) from Lemma 14, Let $T_{0}, \ldots, T_{l}$ be the connected components of $\mathscr{A} \backslash S$ in the order they are traversed by the path $P$ containing $S$. Statement 14(b) implies that $\delta_{\mathscr{F}}\left(T_{i}\right)=\emptyset$ for all $i \in\{1, \ldots, l-1\}$. Therefore, for every terminal pair $\{u, \bar{u}\} \in \mathfrak{T}$ there either is an $i \in\{0, \ldots, l\}$ with $u, \bar{u} \in T_{i}$, or $u \in T_{0}$ and $\bar{u} \in T_{l}$ (w.l.o.g.). Hence the only terminal pairs that are disconnected by the removal of $S$ are those with $u \in T_{0}$ and $\bar{u} \in T_{l}$. Statement 14(b) also implies that either $f \in \delta_{\mathscr{F}}\left(T_{0}: T_{l}\right)$ or both endpoints of $f$ are contained in one of the $T_{i}$. In the former case, $\mathscr{A} \backslash S \cup\{f\}$ is feasible since $T_{0} \cup T_{l}$ is a connected component of this solution. In the latter case, the connected components of $\mathscr{A} \backslash\{e\} \cup\{f\}$ and $\mathscr{A} \backslash\{e\}$ are the same, which would imply that $A \backslash\{e\}$ is feasible by our assumption, which is a contradiction to $e$ being essential.

Lemma 14. Let $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ be an equivalence class of safe edges. Let $f \in \mathscr{F}$ be an edge.
a. Let $K \subseteq S$. Then there is a path $P \subseteq \mathscr{A}$ with $K \subseteq P$.
b. Let $P \subseteq \mathscr{A}$ be the unique minimal path containing $S$ and let $T_{0}, \ldots, T_{l}$ be the components of $\mathscr{A} \backslash S$ in the order they are traversed by $P$. Then either $f \in \delta_{\mathscr{F}}\left(T_{0}: T_{l}\right)$, or there is an $i \in\{0, \ldots, l\}$ such that $T_{i}$ contains both endpoints of $f$.
c. Let $C$ be the unique cycle in $\mathscr{A} \cup\{f\}$. Then $S \subseteq C$ or $S \cap C=\emptyset$.
d. If $\mathscr{A} \backslash\{e\} \cup\{f\}$ is feasible for some edge $e \in S$, then $\mathscr{A} \backslash S^{\prime} \cup\{f\}$ is feasible for all $S^{\prime} \subseteq S$. This also holds for $S=S_{u}$.

Proof. The proof of 14(a) is done in Lemma 12, For 14(b), denote the edges in $S$ by $e_{1}, \ldots, e_{l}$ such that $e_{i}$ is the edge between $T_{i-1}$ and $T_{i}$ for all $i \in\{1, \ldots, l\}$. Since for any $i \neq j \in\{1, \ldots, l\}$, the edges $e_{i}$ and $e_{j}$ are pairwise compatible, no $\mathscr{F}$-edge can cross between $T_{i}$ and $T_{j}$, for any $i \neq j \in\{1, \ldots, l\}$. Nor can there be $\mathscr{F}$-edges from $T_{0}$ to any $T_{i}$, for $i \in\{1, \ldots, l\}$, since $e_{1}$ and $e_{i+1}$ are compatible. Therefore either both endpoints of $f$ lie within the same component $T_{i}$, or $f \in \delta_{\mathscr{F}}\left(T_{0}: T_{l}\right)$.
Property 14 (c) follows from 14(b) for $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$. Notice that for any $f \in \mathscr{F}$, all edges on $C$ are automatically safe. Thus, the statement is true for $S_{u}$ as well. Statement 14(d) is proven in Lemma 13,

The following fact is a straightforward generalization of Hall's theorem and an easy consequence of max flow/min cut (see, e.g., [CIM09]).

Fact 15. Let $G=(A \cup B, E)$ be a bipartite graph. For any $A^{\prime} \subseteq A$, let $N\left(A^{\prime}\right) \subseteq B$ be the set of neighbors of the nodes in $A^{\prime}$. If

$$
\text { for all } A^{\prime} \subseteq A:\left|N\left(A^{\prime}\right)\right| \geq\left|A^{\prime}\right| / c
$$

then there exists an assignment $\alpha: E \rightarrow \mathbb{R}_{+}$such that $\sum_{e \in \delta(a)} \alpha(e) \geq 1$ for all $a \in A$ and $\sum_{e \in \delta(b)} \alpha(e) \leq c$ for all $b \in B$.

## 7 Proofs: Bounds on $\phi$ when the Local Optimum is a Tree

Let's recall (and formalize) some notation from $\mathbb{\$ 4 . 2}$ Let equivalence class $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ of safe edges contain $\ell(S)$ edges. By Lemma 14(a), the edges of $S=\left\langle e_{S, 1}, \ldots, e_{S, \ell(S)}\right\rangle$ lie on a path: let

$$
w_{S, 0} \xrightarrow{e_{S, 1}} v_{S, 1} \rightarrow \cdots \rightarrow w_{S, 1} \xrightarrow{e_{S, 2}} v_{S, 2} \rightarrow \cdots \rightarrow w_{S, i-1} \xrightarrow{e_{S, i}} v_{S, i} \rightarrow \ldots w_{S, \ell(S)-1} \xrightarrow{e_{S, \ell(S)}} v_{S, \ell(S)}
$$

be this path, where each $e_{S, i}=\left\{w_{S, i-1}, v_{S, i}\right\}$. When the context is clear (as it is in this subsection), we use $e_{1}, \ldots, e_{\ell(S)}$ and $v_{1}, \ldots, v_{\ell(S)}, w_{0}, \ldots, w_{\ell(S)-1}$ instead. Notice that $w_{S, i}=v_{S, i}$ is possible, but we always have $w_{S, i-1} \neq v_{S, i}$.
Removing $S$ decomposes $\mathscr{A}$ into $\ell(S)+1$ connected components. Let the connected component containing $w_{i}$ be ( $V_{S, i}, E_{S, i}$ ) and the connected component that contains $v_{\ell(S)}$ be $\left(V_{S, \ell(S)}, E_{S, i}\right)$. As in the rest of the paper, we will associate each components by its edge set. We think of $E_{S, 0}, E_{S, \ell(S)}$ as the outside of the path that $S$ lies on, and $E_{S, 1}, \ldots, E_{S, \ell(S)-1}$ as the inner components. Note that these components form vertex-disjoint subtrees of $\mathscr{A}$.

Observation 16. Let $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ and $i, i^{\prime} \in\{0, \ldots, \ell(S)\}, i \neq i^{\prime}$. Then $E_{S, i} \cap E_{S, i^{\prime}}=\emptyset$. Also, $\mathscr{A}$ is the disjoint union of $E_{S, 0}, \ldots, E_{S, \ell(S)}$ and $S$.

Next, we set $\mathfrak{m}(S)$ and $\mathfrak{n}(S)$ to be the index of the inner components with the largest and second-largest widths, respectively. (We use the indexing $\mathfrak{i n d e x}$ from $\$ 2$ to break ties, so $\mathfrak{m}(S):=\arg \max _{i=1, \ldots, \ell(S)-1} \mathfrak{i n d e x}\left(E_{S, i}\right)$, and $\mathfrak{n}(S):=\arg \max _{i \in\{1, \ldots, \ell(S)-1\} \backslash\{\mathfrak{m}(S)\}} \mathfrak{i n d} \mathfrak{e x}\left(E_{S, i}\right)$.) Without loss of generality, we assume that the orientation of the path is such that $\mathfrak{m}(S)<\mathfrak{n}(S)$. Let $\operatorname{In}_{S}:=\{1,2, \ldots, \ell(S)-1\}$ be the indices of the inner components, and $\operatorname{In}_{S^{\prime}}:=\operatorname{In}_{S} \backslash\{\mathfrak{m}(S), \mathfrak{n}(S)\}$. Now we split the cost of the solution $\mathscr{A}$ into three terms. It holds that

$$
\begin{aligned}
& \phi(\mathscr{A})=w(\mathscr{A})+\sum_{e \in \mathscr{A}} d_{f}=w(\mathscr{A})+\sum_{e \in S_{u}} d_{e}+\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{e \in S} d_{e} \\
&=\left(w(\mathscr{A})+\sum_{e \in S_{u}} d_{e}\right)+\left(\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}}\left(\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{i \in \ln S} w\left(E_{S, i}\right)\right)\right) \\
&+\left(\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In}_{S}} w\left(E_{S, i}\right)\right) .
\end{aligned}
$$

### 7.1 Bounding the Middle Term

The next lemma and the corollary bound the cost of the middle term.
Lemma 17. Let $I=(V, E, \mathfrak{T}, d)$ be a Steiner Forest instance and let $\mathscr{F}$ be a feasible solution for $I$. Furthermore, let $\mathscr{A} \subseteq E$ be a feasible tree solution for I that is edge/set swap-optimal with respect to $\mathscr{F}$ and $\phi$. Set $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ be an equivalence class of safe edges. Let $f \in \mathscr{F}$ be an edge that closes a cycle in $\mathscr{A}$ that contains $S$. Then

$$
d_{f} \geq \frac{1}{3}\left(\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{i \in \operatorname{In}_{S}} w_{i}\right) .
$$

Proof. Since $f$ closes a cycle in $\mathscr{A}$ that contains $S$, any edge $e$ in $S$ satisfies that $\mathscr{A} \backslash\{e\} \cup\{f\}$ is feasible. By Lemma 14 (d) this implies that $\mathscr{A} \backslash S \cup\{f\}$ is feasible as well. Thus, adding $f$ and removing any subset $S^{\prime} \subseteq$ $S$ is a feasible edge/set swap. We consider three subsets $S_{1}^{\prime}:=\left\{e_{1}, \ldots, e_{\mathfrak{m}(S)}\right\}, S_{2}^{\prime}:=\left\{e_{\mathfrak{m}(S)+1}, \ldots, e_{\mathfrak{n}(S)}\right\}$ and $S_{3}^{\prime}:=\left\{e_{\mathfrak{n}(S)+1}, \ldots, e_{\ell(S)}\right\}$. Since $\mathscr{A}$ is edge/set swap-optimal with respect to $\mathscr{F}$ and $\phi$, we know that adding $f$ and removing $S_{1}^{\prime}, S_{2}^{\prime}$ or $S_{3}^{\prime}$ is not an improving swap. When adding $f$, we pay $d_{f}$. When removing $S_{1}^{\prime}$, we gain $\sum_{i=1}^{\mathfrak{m}(S)} d_{e_{i}}$, but we have to pay the width of the connected components that we create. Notice that we detach the connected components $E_{S, 1}, \ldots E_{S, \mathfrak{m}(S)-1}$ from the tree. Since $E_{S, \mathfrak{m}(S)}$ has the maximum width on the path, the increase in $\phi$ that originates from widths has to be $\sum_{i=1}^{\mathfrak{m}(S)-1} w\left(E_{S, i}\right)$. Thus, the fact that the swap is not improving yields that

$$
d_{f}+\sum_{i=1}^{\mathfrak{m}(S)-1} w\left(E_{S, i}\right) \geq \sum_{i=1}^{\mathfrak{m}(S)} d_{e_{i}} \Leftrightarrow d_{f} \geq \sum_{i=1}^{\mathfrak{m}(S)} d_{e_{i}}-\sum_{i=1}^{\mathfrak{m}(S)-1} w\left(E_{S, i}\right)
$$

Similarly, we gain that

$$
d_{f} \geq \sum_{i=\mathfrak{m}(S)+1}^{\mathfrak{n}(S)} d_{e_{i}}-\sum_{i=\mathfrak{m}(S)+1}^{\mathfrak{n}(S)-1} w\left(E_{S, i}\right) \quad \text { and } \quad d_{f} \geq \sum_{i=\mathfrak{n}(S)+1}^{\ell(S)} d_{e_{i}}-\sum_{i=\mathfrak{n}(S)}^{\ell(S)-1} w\left(E_{S, i}\right)
$$

from the fact that $\left(f, S_{2}^{\prime}\right)$ and $\left(f, S_{3}^{\prime}\right)$ are not improving. The statement of the lemma follows by adding the inequalities and dividing the resulting inequality by three.

Corollary 18. Let $I=(V, E, \mathfrak{T}, d)$ be a Steiner Forest instance and let $\mathscr{F}$ be a feasible solution for $I$. Furthermore, let $\mathscr{A} \subseteq E$ be a feasible tree solution for I that is edgeledge and edge/set swap-optimal with
respect to $\mathscr{F}$ and $\phi$. Then,

$$
\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}}\left(\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{i \in \operatorname{In} S} w\left(E_{S, i}\right)\right) \leq 10.5 \cdot \sum_{e \in \mathscr{F}} d_{e} .
$$

Proof. We set $\Delta(S):=\frac{1}{3}\left(\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{i \in \operatorname{In}_{S}} w\left(E_{S, i}\right)\right)$. The corollary then follows by Lemma 17 and Theorem 3 .

### 7.2 Bounding the First Term

The following lemma bounds the first term of the cost.
Lemma 19. Let $I=(V, E, \mathfrak{T}, d)$ be a Steiner Forest instance and let $\mathscr{F}$ be a feasible solution for $I$ with $c c(\mathscr{F})$ connected components $F_{1}, \ldots, F_{c c(\mathscr{F})}$. Furthermore, let $\mathscr{A} \subseteq E$ be a feasible tree solution for $I$ and recall that $S_{u}$ is the equivalence class of unsafe edges in $\mathscr{A}$. Assume that $\mathscr{A}$ is removing swap-optimal. It holds that

$$
w(\mathscr{A})+\sum_{e \in S_{u}} d_{e} \leq \sum_{i=1}^{c c(\mathscr{F})} w\left(F_{i}\right) .
$$

Proof. $\mathscr{A} \backslash S_{u}$ contains $\left|S_{u}\right|+1$ connected components $E_{1}, \ldots, E_{\left|S_{u}\right|+1}$, number them by decreasing width, i.e., $w\left(E_{1}\right) \geq w\left(E_{2}\right) \geq \ldots \geq w\left(E_{\left|S_{u}\right|+1}\right)$. Since $\mathscr{F}$ does not connect them by definition of $S_{u}$, its number of connected components satisfies $c c(\mathscr{F}) \geq\left|S_{u}\right|+1$. Furthermore, any component $E_{i}$ contains a component $F_{j}$ with $w\left(F_{j}\right)=w\left(E_{i}\right)$. Assume that the components are numbered such that $w\left(F_{i}\right)=w\left(E_{i}\right)$ for $i=1, \ldots,\left|S_{u}\right|+1$.

Consider the removing swap that removes all edges in $S_{u}$. Before the swap, $\mathscr{A}$ paid all edges and $w(\mathscr{A})$. After the swap, $\mathscr{A}$ would pay all edges except those in $S_{u}$ plus $\sum_{i=1}^{\left|S_{u}\right|+1} w\left(E_{i}\right)=\sum_{i=1}^{\left|S_{u}\right|+1} w\left(F_{i}\right)$. Since the swap was not improving, we know that

$$
w(\mathscr{A})+\sum_{e \in \mathscr{A}} d_{e} \leq \sum_{i=1}^{\left|S_{u}\right|+1} w\left(F_{i}\right)+\sum_{e \in \mathscr{A}} d_{e}-\sum_{e \in S_{u}} d_{e}
$$

Thus, $w(\mathscr{A})+\sum_{e \in S_{u}} d_{e} \leq \sum_{i=1}^{\left|S_{u}\right|+1} w\left(F_{i}\right) \leq \sum_{i=1}^{c c(\mathscr{F})} w\left(F_{i}\right)$.

### 7.3 Bounding the Third Term: Sum of Widths

For the third term, we need a bit more work. The main step is to show that $\mathfrak{i n d e x}\left(E_{S, i}\right)$ is an injective function of $S, i$ when restricting to $i \in \operatorname{In}_{S}$. The following lemma helps to prove this statement.

Lemma 20. Let $S, S^{\prime} \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ with $S \neq S^{\prime}$. Exactly one of the following two cases holds:

- $S^{\prime} \subseteq E_{S, 0} \cup E_{S, \ell(S)}$, i.e., $S^{\prime}$ lies in the outside of the path of $S$
- There exists $i \in\{1, \ldots, \ell(S)-1\}$ with $S^{\prime} \subseteq E_{S, i}$.

Proof. Notice that $S \cap S^{\prime}=\emptyset$ by definition because $S$ and $S^{\prime}$ are different equivalence classes. Let $e^{\prime}, f^{\prime} \in$ $S^{\prime}$, i.e., $e^{\prime}, f^{\prime} \notin S$. Assume that $e^{\prime} \in E_{S, i}$ for an $i \in\{1, \ldots, \ell(S)-1\}$, but $f^{\prime} \notin E_{S, i}$. Following the notation on page 15, we denote the unique edges in $S$ that touch $E_{S, i}$ by $e_{S, i}$ and $e_{S, i+1}$ and their adjacent vertices in $E_{S, i}$ by $v_{i}$ and $w_{i}$. Removing $e_{S, i}$ and $e_{S, i+1}$ creates three connected components in $\mathscr{A}$. The first component contains $E_{S, 0}, \ldots, E_{S, i-1}$ and the edges $e_{S, 1}, \ldots, e_{S, i-1}$ and we name it $E_{S,<}$. The second

(a) First case: Removing $e^{\prime}$ disconnects $v_{i}$ and $w_{i}$. Thus, $e_{i}, e^{\prime}$ and $e_{i+1}$ lie on a path. No $\mathscr{F}$-edge can leave $E_{S, i}=E_{S, i}^{1} \cup E_{S, i}^{2}$. Also, no $\mathscr{F}$-edge can go between $E_{S, i}^{1}$ and $E_{S, i}^{2}$ since $e^{\prime}$ and $f^{\prime}$ are compatible. Thus, $e^{\prime}$ is compatible to $e_{i}$ and $e_{i+1}$.

(b) Second case: The nodes $v_{i}$ and $w_{i}$ lie in the same connected component when removing $e^{\prime}$. Again, no $\mathscr{F}$-edge can leave $E_{S, i}$. Also, no edge can go between $E_{S, i}^{1}$ and $E_{S, i}^{2}$ because $e^{\prime}$ and $f^{\prime}$ are compatible. Thus, $e^{\prime}$ is compatible to $e_{i}$ and $e_{i+1}$.

Figure 8: Two cases that occur in Lemma20.
component is $E_{S, i}$. The third component contains $E_{S, i+1}, \ldots, E_{S, \ell(S)}$ and the edges $e_{S, i+1}, \ldots, e_{S, \ell(S)}$ and we name it $E_{S,>}$. Since $f^{\prime} \notin E_{S, i}$, and $f^{\prime} \notin S$, we know that $f^{\prime} \in E_{S,<} \cup E_{S,>}$.
Notice that $e^{\prime}$ and $f^{\prime}$ are compatible, $e_{S, i}$ and $e_{S, i+1}$ are compatible, but $e^{\prime}$ is not compatible to either $e_{S, i}$ or $e_{S, i+1}$. If we remove $e^{\prime}$ in addition to $e_{S, i}$ and $e_{S, i+1}$, then $E_{S, i}$ decomposes into two connected components, $E_{S, i}^{1}$ and $E_{S, i}^{2}$. We assume without loss of generality that $v_{S, i} \in E_{S, i}^{1}$. The vertex $w_{S, i}$ can either be separated from $v_{S, i}$ when removing $e^{\prime}$, i.e., $w_{S, i} \in E_{S, i}^{2}$, or not, i.e., $w_{S, i} \in E_{S, i}^{1}$. The two cases are depicted in Figure 8 ,
If $w_{S, i} \in E_{S, i}^{2}$, then the edges $e_{S, i}, e^{\prime}$ and $e_{S, i+1}$ lie on a path. No $\mathscr{F}$-edge can leave $E_{S, i}=E_{S, i}^{1} \cup E_{S, i}^{2}$ because $e_{S, i}$ and $e_{S, i+1}$ are compatible. Thus, any $\mathscr{F}$-edge leaving $E_{S, i}^{1}$ has to end in $E_{S, i}^{2}$. On the other hand, consider removing $e^{\prime}$ and $f^{\prime}$ from $\mathscr{A}$, which creates three components $T_{e^{\prime}}, T_{e^{\prime}, f^{\prime}}, T_{f^{\prime}}$. As both $E_{S, i}^{1}$ and $E_{S, i}^{2}$ are incident to $e^{\prime}$, they are contained in two different components, $T_{e^{\prime}}$ and $T_{e^{\prime}, f^{\prime}}$. Since no edge can leave $T_{e^{\prime}, f^{\prime}}$ because $e^{\prime}$ and $f^{\prime}$ are compatible, no edge can cross between $E_{S, i}^{1}$ and $E_{S, i}^{2}$. Thus, no $\mathscr{F}$-edges leave $E_{S, i}^{1}$ or $E_{S, i}^{2}$, which means that $e^{\prime}$ is compatible to $e_{S, i}$ and $e_{S, i+1}$.
If $w_{S, i} \in E_{S, i}^{1}$, the argument is similar. Still, no $\mathscr{F}$-edge can leave $E_{S, i}=E_{S, i}^{1} \cup E_{S, i}^{2}$ because $e_{S, i}$ and $e_{S, i+1}$ are compatible. Also, $E_{S, i}^{1} \subseteq T_{e^{\prime}, f^{\prime}}$ and $E_{S, i}^{2} \subseteq T_{e^{\prime}}$ still are in different connected components when $e^{\prime}$ and $f^{\prime}$ are removed. Thus, no $\mathscr{F}$-edge can connect them, and thus $e^{\prime}$ is compatible to $e_{S, i}$ and $e_{S, i+1}$.
Both cases end in a contradiction since $e^{\prime}$ cannot be compatible to any edge in $S$ because $S$ and $S^{\prime}$ are different equivalence classes. Thus, $e^{\prime} \in E_{S, i}$ implies that $S \subseteq E_{S, i}$. Only if no edge in $S^{\prime}$ lies in any $E_{S, i}$ for $i \in\{1, \ldots, \ell(S)-1\}$, then it is possible that edges of $S^{\prime}$ lie in $E_{S, 0}$ or $E_{S, \ell(S)}$. However, then $S^{\prime} \subseteq\left(E_{S, 0} \cup E_{S, \ell(S)}\right)$.
We set $\mu(S, i)=\mathfrak{i n d e x}\left(E_{S, i}\right)$ for all $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ and $i \in \operatorname{In}_{S}$.
Lemma 21. Let $S, S^{\prime} \in \mathfrak{S} \backslash\left\{S_{u}\right\}, i \in \operatorname{In}_{S}, i^{\prime} \in \operatorname{In}_{S^{\prime}}$. Then

$$
\mu(S, i)=\mu\left(S^{\prime}, i^{\prime}\right) \Rightarrow S=S^{\prime} \text { and } i=i^{\prime}
$$

i.e., the mapping $\mu$ is injective.

Proof. Let $\mu(S, i)=\mu\left(S^{\prime}, i^{\prime}\right)$ and denote by $u^{*}:=u_{\mu(S, i)}=u_{\text {inneer }\left(E_{S, i}\right)}$ and $\bar{u}^{*}:=\bar{u}_{\mu(S, i)}=\bar{u}_{\text {inder }\left(E_{S, i}\right)}$ the corresponding terminal pair. By the definition of $\mathfrak{i n d e x}$, it follows that $u^{*}, \bar{u}^{*} \in V_{S, i} \cap V_{S^{\prime}, i^{\prime}}$ and in particular, we know that $V_{S, i} \cap V_{S^{\prime}, i^{\prime}} \neq \emptyset$. If $S=S^{\prime}$, then either $i=i^{\prime}$ (and we are done) or $i \neq i^{\prime}$ implies that $V_{S, i} \cap V_{S^{\prime}, i^{\prime}}=\emptyset$ which is a contradiction. Thus, assume in the following that $S \neq S^{\prime}$.
Let $P$ be the unique $u^{*}-v_{S, i}$-path in $E_{S, i}$. If $P \cap S^{\prime}=\emptyset$, then $P \subseteq E_{S^{\prime}, i^{\prime}}$ because $u^{*} \in V_{S^{\prime}, i^{\prime}}$. Thus, $v_{S, i} \in V_{S^{\prime}, i^{\prime}}$, which also implies that $e_{S, i} \in E_{S^{\prime}, i^{\prime}}$. By Lemma20, we get $S \subseteq E_{S^{\prime}, i^{\prime}}$. On the other hand, if $P \cap S^{\prime} \neq \emptyset$, then $S^{\prime} \subseteq E_{S, i}$ by Lemma 20. Thus $S \subseteq E_{S^{\prime}, i^{\prime}}$ or $S^{\prime} \subseteq E_{S, i}$. W.l.o.g., we assume the latter. Now let $j \in\left\{\mathfrak{m}\left(S^{\prime}\right), \mathfrak{n}\left(S^{\prime}\right)\right\}$ be such that $E_{S^{\prime}, j} \cap S=\emptyset$ (note that $j$ exists due to Lemma 20). Because $e_{S^{\prime}, j}, e_{S^{\prime}, j+1} \in S^{\prime} \subseteq E_{S, i}$ and $E_{S^{\prime}, j} \cap S=\emptyset$, we deduce that $E_{S^{\prime}, j} \subseteq E_{S, i}$. This implies that $\mu(S, i) \geq \mathfrak{i n d e x}\left(E_{S^{\prime}, j}\right)>\mu\left(S^{\prime}, i^{\prime}\right)$, again a contradiction.

We can now bound the third term of our objective function (see page 16).
Lemma 22. As previously, let $\mathfrak{S}$ denote the set of all equivalence classes of the compatibility relation $\sim_{c p}$, let $\mathscr{A}$ be a feasible tree and denote by $E_{S, i}$ the $i$-th inner connected component on the path that contains $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$. Let $\mathscr{F}$ be a feasible Steiner Forest with cc( $\left.\mathscr{F}\right)$ connected components $F_{1}, \ldots, F_{c c}(\mathscr{F})$. Then,

$$
\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In} S} w\left(E_{S, i}\right) \leq \sum_{i=1}^{c c(\mathscr{F})} w\left(F_{i}\right)
$$

Proof. As previously, we set $\mu(S, i)=\mathfrak{i n d e x}\left(E_{S, i}\right)$ for all $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ and $i \in \operatorname{In}_{S}$. We also recall that by definition, $w\left(E_{S, i}\right)=d_{G}\left(u_{\text {inder }(S, i)}, \bar{u}_{\text {inder }(S, i)}\right)=d_{G}\left(u_{\mu(S, i)}, \bar{u}_{\mu(S, i)}\right)$ for all $S \in \mathfrak{S} \backslash\left\{S_{u}\right\}$ and all $i \in \operatorname{In}_{S}$. We thus have

$$
\begin{equation*}
\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In}_{S}} w\left(E_{S, i}\right)=\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In}_{S}} d_{G}\left(u_{\mu(S, i)}, \bar{u}_{\mu(S, i)}\right) . \tag{1}
\end{equation*}
$$

Let $\chi(S, i)$ denote the index of the connected component $F_{\chi(S, i)}$ containing the terminal pair $u_{\text {inder }(S, i)}, \bar{u}_{\text {inder }(S, i)}$ in $\mathscr{F}$. We claim that $\chi$ is injective. To see this, consider $S, S^{\prime} \in \mathcal{S}$ and $i \in \operatorname{In}_{S}, i^{\prime} \in \operatorname{In}_{S^{\prime}}$ with $\chi(S, i)=\chi\left(S^{\prime}, i^{\prime}\right)$. Since $\delta_{\mathscr{F}}\left(V_{S, i}\right)=\delta_{\mathscr{F}}\left(V_{S^{\prime}, i^{\prime}}\right)=\emptyset$ and $F_{\chi(S, i)}$ is connected, we deduce that $V\left[F_{\chi(S, i)}\right] \subseteq$ $V_{S, i} \cap V_{S^{\prime}, i^{\prime}}$. This implies that $u_{\mu(S, i)}, \bar{u}_{\mu(S, i)} \in V_{S^{\prime}, i^{\prime}}$ and that $u_{\mu\left(S^{\prime}, i^{\prime}\right)}, \bar{u}_{\mu\left(S^{\prime}, i^{\prime}\right)} \in V_{S, i}$. Hence $\mu(S, i)=$ $\mu\left(S^{\prime}, i^{\prime}\right)$, which implies $S=S^{\prime}$ and $i=i^{\prime}$ by Lemma 21 ,
Since $d_{G}\left(u_{\mu(S, i}, \bar{u}_{\mu(S, i)}\right) \leq w\left(F_{\chi(S, i)}\right)$, we can now continue (1) to see that

$$
\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In}_{S}} d_{G}\left(u_{\mu(S, i)}, \bar{u}_{\mu(S, i)}\right) \leq \sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In}_{S}} w\left(F_{\chi(S, i)}\right) \quad \leq \sum_{i=1}^{c c(\mathscr{F})} w\left(F_{i}\right) .
$$

Here, the last inequality follows from our argument that $\chi$ is injective.

### 7.3.1 Wrapping Things Up

We can now prove the main result of this section.
Theorem 23. Let $G=(V, E)$ be a graph, let $d_{e}$ be the cost of edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a terminal set. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible solutions for $(G, d, \mathfrak{T})$ with $V[\mathscr{A}]=V[\mathscr{F}]$. Furthermore, suppose that $\mathscr{A}$ is a tree and that $\mathscr{A}$ is edge/set swap optimal with respect to $\mathscr{F}$ and $\phi$. Then,

$$
\phi(\mathscr{A})=\sum_{e \in \mathscr{A}} d_{e}+w(\mathscr{A}) \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F})+\sum_{e \in S_{u}} d_{e}+w(\mathscr{A}) .
$$

In particular, $d(\mathscr{A}) \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F})+\sum_{e \in S_{u}} d_{e}$.

Proof. Rewrite $\phi(\mathscr{A})$ as on page 16 to

$$
\phi(\mathscr{A})=w(\mathscr{A})+\sum_{e \in S_{u}} d_{e}+\underbrace{\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}}\left(\sum_{i=1}^{\ell(S)} d_{e_{i}}-\sum_{i \in \operatorname{In}_{S}} w\left(E_{S, i}\right)\right)}_{\leq 10.5 \cdot d(\mathscr{F}) \text { by Corollary }[18}+\underbrace{\sum_{S \in \mathfrak{S} \backslash\left\{S_{u}\right\}} \sum_{i \in \operatorname{In}}{ }_{S} w\left(E_{S, i}\right)}_{\leq w(\mathscr{F}) \text { by Lemma[2] }} .
$$

This proves the theorem.
Additionally applying Lemma 19 yields the following reformulation of Theorem 23 ,
Corollary 24. Let $G=(V, E)$ be a graph, let $d_{e}$ be the cost of edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a terminal set. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible solutions for $(G, d, \mathfrak{T})$ with $V[\mathscr{A}]=V[\mathscr{F}]$. Furthermore, suppose that $\mathscr{A}$ is a tree and that $\mathscr{A}$ is optimal with respect to $\mathscr{F}$ and $\phi$ under edgeledge, edge/set and removing swaps. Then,

$$
\phi(\mathscr{A}) \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F})+\sum_{e \in S_{u}} d_{e}+w(\mathscr{A}) \leq 10.5 \cdot d(\mathscr{F})+2 w(\mathscr{F}) \leq 10.5 \cdot \phi(\mathscr{F})
$$

If we want to bound the original objective function of the Steiner Forest problem, we do not need removing swaps.

Corollary 2. Let $G=(V, E)$ be a graph, let $d_{e}$ be the cost of edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a set of terminal pairs. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible Steiner forests for $(G, d, \mathfrak{T})$ with $V[\mathscr{A}]=V[\mathscr{F}]$. Assume that $\mathscr{A}$ is a tree and that $\mathscr{A}$ is swap-optimal with respect to $\mathscr{F}$ and $\phi$ under edgeledge and edge/set swaps. Denote by $\mathscr{A}^{\prime}$ the modified solution where all inessential edges have been dropped from $\mathscr{A}$. Then,

$$
d\left(\mathscr{A}^{\prime}\right) \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F}) \leq 11.5 \cdot d(\mathscr{F}) .
$$

Proof. Let $S_{i s}$ be the set of safe but inessential edges. We have $d(\mathscr{A}) \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F})+\sum_{e \in S_{u}} d_{e}$ by Theorem 23, That implies

$$
\sum_{e \in \mathscr{A}^{\prime}} d_{e}=d(\mathscr{A})-\sum_{e \in S_{u}} d_{e}-\sum_{e \in S_{i s}} d_{e} \leq 10.5 \cdot d(\mathscr{F})+w(\mathscr{F})-\sum_{e \in S_{i s}} d_{e} .
$$

## 8 Proofs: Bounds when the Local Optimum is a Forest

The aim of this section is to transform (a.k.a. "project") a pair $(\mathscr{A}, \mathscr{F})$ of arbitrary solutions into a pair $\left(\mathscr{A}, \mathscr{F}^{\prime}\right)$ of solutions to which the results from the previous section apply. I.e., each connected component of $\mathscr{F}^{\prime}$ is contained within a connected component of $\mathscr{A}$. The main lemma is

Lemma 36. Let $G=(V, E)$ be a complete graph, let $d: E \rightarrow \mathbb{R}_{\geq 0}$ be a metric that assigns a cost $d_{e}$ to every edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a set of terminal pairs. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible Steiner Forest solutions for $(G, \mathfrak{T})$. Furthermore, suppose that $\mathscr{A}$ is edgeledge, edge/set and path/set swap-optimal with respect to $E$ and $\phi$, that $\mathscr{A}$ is c-approximate connecting move optimal and that $\mathscr{A}$ only uses edges between terminals. Then there exists a feasible solution $\mathscr{F}^{\prime}$ with $d\left(\mathscr{F}^{\prime}\right) \leq 2(1+c) \cdot d(\mathscr{F})$ that satisfies $\mathscr{F}^{\prime}=\mathscr{F}^{\prime}{ }_{0}$ such that $\mathscr{A}$ is edgeledge and edgelset swap-optimal with respect to $\mathscr{F}^{\prime}$.

We use the notation $\mathscr{F}_{\circlearrowleft}$ to denote the set of edges in $\mathscr{F}$ that go within components of $\mathscr{A}$ and $\mathscr{F}_{\leftrightarrow}$ to denote the set of all edges between different components, and $c$ is the approximation guarantee of the approximate connecting moves. Formally, $c$-approximate tree move optimality is defined as follows:

Definition 25. A $c$-approximate connecting move for some constant $c \geq 1$ is a connecting move conn $(T)$ applied to the current solution $\mathscr{A}$ using a tree $T$ in $G_{\mathscr{A}}^{\text {all }}$ such that $c \cdot d(T) \leq \bar{w}(\mathscr{A})-\bar{w}(\mathscr{A} \cup T)$. A solution is $c$-approximate connecting move optimal, if there are no c-approximate connecting moves.

Lemma 36 shows that for any solution $\mathscr{F}$, we can find a solution $\mathscr{F}^{\prime}$ with

$$
d\left(\mathscr{F}^{\prime}\right) \leq 2(1+c) \cdot d(\mathscr{F})
$$

which does not contain edges between different components of $\mathscr{A}$, i.e., $\mathscr{F}_{\circlearrowright}^{\prime}=\mathscr{F}^{\prime}$. With Section B.2, we know that $c$ is at most 2 , i.e., we can find $\mathscr{F}^{\prime}$ with $d\left(\mathscr{F}^{\prime}\right) \leq 6 \cdot d(\mathscr{F})$. Every connected component $A_{j}$ of $\mathscr{A}$ can now be treated separately by using Corollary 2 on $A_{j}$ and the part of $\mathscr{F}^{\prime}$ that falls into $A_{j}$. By combining the conclusions for all connected components, we get that

$$
d\left(\mathscr{A}^{\prime}\right) \leq 11.5 \cdot d\left(\mathscr{F}^{\prime}\right) \leq 23(1+c) \cdot d(\mathscr{F}) \leq 69 \cdot d(\mathscr{F})
$$

for any feasible solution $\mathscr{F}$. This proves the main theorem.
Theorem 1. There is a (non-oblivious) local search algorithm for Steiner Forest with a constant locality gap. It can be implemented to run in polynomial time.

Proof outline The forest case depends on path/set swaps and connecting moves. Exploiting connecting move-optimality is the main effort of the section, while path/set swap-optimality is only used to handle one specific situation.

The goal is to replace $\mathscr{F}_{\leftrightarrow}$ by edges that go within components of $\mathscr{A}$.
We first convert $\mathscr{F}$ into a collection of disjoint cycles at the expense of a factor of 2 in the edge costs. Let $F_{i}$ be one of the cycles. We want to replace $\left(F_{i}\right)_{\leftrightarrow}$. To do so, we look at $F_{i}$ in $G_{\mathscr{A}}$ - the graph where the edges in $\mathscr{F}_{\circlearrowright}$ are contracted and loops are removed. In this graph $G_{\mathscr{A}}$, the set $\left(F_{i}\right)_{\leftrightarrow}$ is a circuit (i.e., a possibly non-simple cycle). In a first step, we use Algorithm 1 to cope with the case that $\left(F_{i}\right)_{\leftrightarrow}$ has a special structure that we call minimally guarded. The second step inductively ensures that this structure is present.
An example run of Algorithm 1 is visualized in Figure 12, The algorithm partitions $\left(F_{i}\right)_{\leftrightarrow}$ into trees in $G_{\mathscr{A}}$. These trees define connecting moves, so (approximate) connecting move-optimality gives us a lower bound on the total edge weight of each tree.
Let $n_{j}$ be the number of times that $F_{i}$ passes through $A_{j}$, the $j$-th connected component of $\mathscr{A}$. Then the lower bound that we get is

$$
d\left(\left(F_{i}\right)_{\leftrightarrow}\right) \geq\left(\sum_{j=1}^{p} n_{j} w\left(A_{j}\right)\right)-n_{j_{m}} w\left(A_{j_{m}}\right) .
$$

Here, $j_{m}$ is the component with the largest width among all components that $F_{i}$ touches. In a minimally guarded circuit, this component is only visited once. The lower bound (for minimally guarded circuits) results from Lemma 29 and Corollary 34 (notice that $n_{j}$ is defined differently in the actual proof, but we need less detail here). The part between the two statements establishes invariants of Algorithm 1 that we need to show that it computes trees with the correct properties.
The lower bound means that we can do the following. Assume that we delete $\left(F_{i}\right)_{\leftrightarrow}$. Now some vertices in $\mathscr{A}$ do no longer have adjacent $\mathscr{F}$-edges which is a problem for applying Corollary 2 , We fix this by inserting a direct connection to their terminal partner, which has to be in the same component of $\mathscr{A}$. In order to keep the modified solution feasible, we insert the new connections a bit differently, but with the same result. This
connection is paid for by $\left(F_{i}\right)_{\leftrightarrow}$ which - due to our construction - can give each of these vertices a budget equivalent to the width of its component. This enables us to use the results from the tree case.

This argument does not work for the vertices in the largest-width component. If $F_{i}$ is minimally guarded, it visits the largest width component exactly once and there are exactly two problematic vertices. To reconnect these vertices, we use path/set swap-optimality and charge $\left(F_{i}\right)_{\leftrightarrow}$ again to pay for directly connecting them (this argument comes later, in the proof of Lemma35, when Corollary 34 is applied).
If otherwise $F_{i}$ is not minimally guarded, we need a second step. This is taken care of in Lemma 35 which extends Corollary 34 to guarded circuits that are not necessarily minimal. This is done by applying Corollary 34 to subcircuits and removing these until the minimality criterion is met. The proof is by induction. Lemma 35 outputs a broken solution $\mathscr{F}^{\prime}$ which is not feasible, but is equipped with the widths from the lower bound. Figures 13 up to 21 show the recursive process for an example circuit and visualize the broken solution that comes out of it. The solution is then repaired in Lemma $\sqrt{36}$, giving the final reduction result.

### 8.1 Details: Getting a Good Tree Packing

While the previous sections operated on arbitrary graphs, we now consider the metric case of the Steiner forest problem. Consequently, we assume that $G=(V, E)$ is the complete graph on $V$ and that the cost $d_{e}=d_{v w}$ of each edge $e=\{v, w\} \in E$ is given by a metric $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$. Together with a set of terminal pairs $\mathfrak{T} \subseteq V$, the graph $G$ and the metric $d$ define an instance of the metric Steiner Forest problem.
The more important change in our setting is, however, that we no longer assume that our feasible solution $\mathscr{A} \subseteq E$ is a tree; rather, $\mathscr{A}$ can be an arbitrary feasible forest in $G$. We write its connected components as $A_{1}, \ldots, A_{p} \subseteq \mathscr{A}$, where the numbering is fixed and such that $w\left(A_{1}\right) \leq w\left(A_{2}\right) \leq \cdots \leq w\left(A_{p}\right)$ holds. As in the previous sections, we compare $\mathscr{A}$ to another solution $\mathscr{F}$.

The connected components of $\mathscr{A}$ are trees and it is a natural idea to apply the results from the previous sections to these trees by considering each connected component of $\mathscr{A}$ individually. Morally, the main obstacle in doing so is the following: In the proof of Theorem 3, we assume implicitly that no $\mathscr{F}$-edge crosses between connected components of $\mathscr{A}$ This is vacuously true in the case where $\mathscr{A}$ is a tree; however, if $\mathscr{A}$ is a forest, this assumption is not justified in general. In the following, our underlying idea is to replace $\mathscr{F}$-edges that cross between the components of $\mathscr{A}$ by edges that lie within the components of $\mathscr{A}$, thereby re-establishing the preconditions of Theorem 3, We show how to do this such that $\mathscr{F}$ stays feasible and such that its cost is increased by at most a constant factor.

### 8.1.1 Notation

We start with some normalizing assumptions on $\mathscr{F}$. As before, we denote the connected components of $\mathscr{F}$ by $F_{1}, \ldots, F_{q} \subseteq \mathscr{F}$. First we can assume that each $F_{i}$ has no inessential edges. Then, since we are in the metric case, we can convert each $F_{i}$ into a simple cycle-this can be done with at most a factor 2 loss in the cost of $\mathscr{F}$ by taking an Euler tour and short-cutting over repeated vertices and non-terminals. This implies that that $V[\mathscr{F}] \subseteq V[\mathscr{A}]$, since now $\mathscr{F}$ only has terminals, which are all covered by $\mathscr{A}$. Assume that $V[\mathscr{A}]$ only contains terminals. Then, $V[\mathscr{F}]$ and $V[\mathscr{A}]$ are equal. Recall that $\mathfrak{T}$ is the set of terminal pairs, let $V_{\mathfrak{T}}$ be the set of all terminals. Henceforth, we assume that $V=V[\mathscr{F}]=V[\mathscr{A}]=V_{\mathfrak{x}}$.

Observation 26. Let $\mathscr{F}, \mathscr{A}$ be feasible solutions and assume that $V[\mathscr{A}]=V_{\mathfrak{x}}$. Then there exists a solution $\mathscr{F}^{\prime}$ of cost $2 \cdot \sum_{e \in \mathscr{F}} d_{e}+\sum_{i=1}^{q} w\left(F_{q}\right)$ whose connected components are node disjoint cycles and which satisfies $V\left[\mathscr{F}^{\prime}\right]=V[\mathscr{A}]=V_{\mathfrak{T}}$.

[^5]
$T_{1}$

Figure 9: A simple cycle $F_{i}$ in $G$ (depicted in blue and black) that induces the blue circuit $C_{i}$ in $G_{\mathscr{A}}$. Notice that $C_{i}$ is not simple. The rounded rectangles represent connected components of $\mathscr{A}$, and their size indicates their width.

Next, we define a convenient notation for the connected components of $\mathscr{A}$ and the $\mathscr{F}$-cycles that pass through them. For each $v \in V$, we set $\xi(v)=j$ for the unique $j \in\{1, \ldots, p\}$ that satisfies that $v \in A_{j}$, i.e., $\xi(v)$ is the index of the connected component of $\mathscr{A}$ that contains $v$. Using this notation, we define a graph $G_{\mathscr{A}}=G /\left\{A_{1}, \ldots, A_{p}\right\}=\left(V_{\mathscr{A}}, E_{\mathscr{A}}\right)$ which results from contracting the connected components of $\mathscr{A}$ in $F$. We set

$$
V_{\mathscr{A}}:=\{1, \ldots, p\} \quad \text { and } \quad E_{\mathscr{A}}:=\left\{e_{f} \mid f=\{v, w\} \in \mathscr{F}, \xi(v) \neq \xi(w)\right\} .
$$

It is important that this definition removes all loops induced by the contraction, but it retains possible parallel edges. In this way, the edges of $G_{\mathscr{A}}$ correspond to the edges of $\mathscr{F}$ that are crossing between the connected components of $\mathscr{A}$, while the nodes correspond to the components. Notice that $G_{\mathscr{A}}$ can be seen as a subgraph of $G_{\mathscr{A}}^{\text {all }}$ defined in the preliminaries for connecting moves. Thus, every tree in $G_{\mathscr{A}}$ induces a connecting move. We extend $d$ to $G_{\mathscr{A}}$ by setting $d_{e_{f}}=d_{f}$ for all $e_{f} \in E_{\mathscr{A}}$. We also consider the graph $\hat{G}_{\mathscr{A}}$ on $V_{\mathscr{A}}$ that is the transitive closure of $G_{\mathscr{A}}$. For all pairs $j_{1}, j_{2} \in V_{\mathscr{A}}$, it contains an additional edge $e_{j_{1} j_{2}}^{\prime}$ whose weight $d_{e_{j_{1} j_{2}}^{\prime}}$ is given by the length of a shortest $j_{1}-j_{2}$-path in $G_{\mathscr{A}}$.
For each $i \in\{1, \ldots, q\}$, the simple cycle $F_{i}$ in $G$ induces a circuit $C_{i}$ in $G_{\mathscr{A}}$. Figure 9 shows a cycle $F_{i}$ and its induced circuit $C_{i}$. The edges of $C_{i}$ correspond to those edges of $F_{i}$ that we want to replace. Observe that $C_{i}$ is indeed not necessarily simple: Whenever $F_{i}$ revisits the connected component $A_{j}$ of $\mathscr{A}$, the induced circuit $C_{i}$ revisits the same node $j \in V_{\mathscr{A}}$. Assume $C_{i}$ visits exactly $s$ distinct vertices. Then we name them $\xi_{1}, \ldots, \xi_{s}$ and assume without loss of generality that $\xi_{1}>\cdots>\xi_{s}$. Since the connected components of $\mathscr{A}$ are numbered according to their width, we know that $w\left(A_{\xi_{1}}\right) \geq \cdots \geq w\left(A_{\xi_{s}}\right)$, and thus the $\xi_{i}$ are ordered according to the widths of the components as well. Finally, let $n_{\ell}$ be the number of times that $C_{i}$ visits $\xi_{\ell}$. In Figure $9, \xi_{1}=7, \xi_{2}=5, \xi_{3}=4, \xi_{4}=1$, and $n_{1}=1, n_{2}=1, n_{3}=2$ and $n_{4}=1$.
The crucial idea for the replacement of $C_{i}$ is to use the connecting move optimality $\uparrow$ of $\mathscr{A}$ to lower bound $d\left(C_{i}\right)$. Any subgraph of $C_{i}$ that is a tree in $G_{\mathscr{A}}$ induces a connecting move. For an example, consider

[^6]

Figure 10: A partitioning of the blue circuit in Figure 9 into trees $T_{1}, T_{2}, T_{3}$ in $G_{\mathscr{A}}$. If none of the three induced connecting moves is improving, then $d\left(T_{1}\right) \geq w\left(A_{4}\right), d\left(T_{2}\right) \geq w\left(A_{4}\right)+w\left(A_{1}\right)$ and $d\left(T_{3}\right) \geq$ $w\left(A_{4}\right)+w\left(A_{1}\right)$. Thus, we get $d\left(C_{i}\right) \geq 3 w\left(A_{4}\right)+2 w\left(A_{1}\right)$.

Figure 10. We partitioned the edges in $C_{i}$ from Figure 9 into three trees. For any $T_{i}$ among the three trees, connecting move optimality guarantees that the sum of the edges $d\left(T_{i}\right)$ is at least as expensive as the sum of the widths of the components that get connected, except for the largest. For example, when adding $T_{1}$ to the solution, the edge cost increases by $c\left(T_{1}\right)$, but the width cost decreases by $w\left(A_{4}\right)$. Thus, $d\left(T_{1}\right) \geq w\left(A_{4}\right)$ when $\mathscr{A}$ is connecting move optimal. Since $T_{1}, T_{2}$ and $T_{3}$ are a edge-disjoint partitioning of $C_{i}$, it holds that $d\left(C_{i}\right)=d\left(T_{1}\right)+d\left(T_{2}\right)+d\left(T_{3}\right)$ and thus we get $d\left(C_{i}\right) \geq 3 w\left(A_{4}\right)+2 w\left(A_{1}\right)$ by considering all three connecting moves.

Now consider Figure 11. Here, we partitioned the edges of $C_{i}$ into a different set of trees. It turns out that this partitioning provides a better lower bound on $d\left(C_{i}\right)$, namely $w\left(A_{5}\right)+2 w\left(A_{4}\right)+2 w\left(A_{1}\right)$. In fact, this lower bound contains $w\left(A_{\xi_{\ell}}\right)$ at least $n_{\ell}$ times for all $\ell \in\{2,3,4\}$. We observe a sufficient condition for guaranteeing that such a partitioning exists.

Definition 27. We say that a tree pays for $\xi_{\ell}$ (once) if it contains $\xi_{\ell}$ and at least one vertex $\xi_{\ell^{\prime}}$ with $\xi_{\ell^{\prime}}>\xi_{\ell}$.

Definition 28. Let $C=\left(e_{1}, \ldots, e_{|C|}\right)$ be a circuit in $G_{\mathscr{A}}$ that visits the nodes $v_{1}, \ldots, v_{|C|+1}=v_{1}$ in this order. We say that $C$ is guarded if we have $v_{i}<v_{1}$ for all $i \in\{2, \ldots,|C|\}$. A circuit $C$ is minimally guarded if it is guarded and no subcircuit $\left(v_{i_{1}}, \ldots, v_{i_{2}}\right)$ with $i_{1}, i_{2} \in\{2, \ldots,|C|\}, i_{1}<i_{2}$ and $v_{i_{1}}=v_{i_{2}}$ is guarded.

Notice that in any guarded circuit, the highest component number only appears once. In Figure $9, C_{i}$ is minimally guarded because the only component visited between the two visits of $A_{4}$ is $A_{5}$, which has a higher index.

Lemma 29. Let $C=\left(e_{1}, \ldots, e_{|C|}\right)$ be a guarded circuit in $G_{\mathscr{A}}$ that visits the nodes $v_{1}, \ldots, v_{|C|+1}=v_{1}$ in this order. Assume that $v_{1}=v_{|C|+1} \geq v_{i}$ for all $j \in\{2, \ldots,|C|\}$ and that $\left\{v_{1}, \ldots, v_{|C|}\right\}$ consists of $s$ disjoint elements $\xi_{1}>\xi_{2}>\ldots>\xi_{s}$ (this means that $v_{1}=\xi_{1}$ ). Furthermore, let $n_{\ell}$ be the number of times that $C$ visits node $\xi_{\ell}$, for all $\ell=1, \ldots$,s. If $\mathscr{A}$ is $c$-approximate connecting move optimal and there exists a set of trees $\mathfrak{M}$ in $G_{\mathscr{A}}$ that satisfies that

1. all trees in $\mathfrak{M}$ are edge-disjoint and only contain edges from $C$ and
2. for all $\ell \in\{2, \ldots, s\}$, there are at least $n_{\ell}$ trees in $\mathfrak{M}$ that pay for $\xi_{\ell}$,
then it holds that

$$
\sum_{i=2}^{|C|} w\left(A_{v_{i}}\right)=\sum_{\ell=2}^{s} n_{\ell} w\left(A_{\xi_{\ell}}\right) \leq c \cdot \sum_{i=1}^{|C|} d_{e_{i}}=c \cdot d(C)
$$

Recall that $A_{v_{i}}$ is the connected component of $\mathscr{A}$ that corresponds to the index $v_{i}$.
Proof. By the first precondition we know that

$$
\sum_{T \in \mathfrak{M}} \sum_{e \in T} d_{e} \leq \sum_{i=1}^{|C|} d_{e_{i}}
$$

Now notice that every tree in $G_{\mathscr{A}}$ and thus every tree in $\mathfrak{M}$ defines a connecting move. Since $\mathscr{A}$ is $c$ approximate connecting move optimal, it holds that

$$
\sum_{v \in V[T]} w\left(A_{v}\right)-\max _{v \in V[T]} w\left(A_{v}\right) \leq c \cdot \sum_{e \in T} d_{e}
$$

for every tree $T$ in $\mathfrak{M}$. Let $\operatorname{low}(T)=V[T] \backslash\left\{\max _{\xi_{i} \in T} \xi_{i}\right\}$. Then, we have

$$
\begin{aligned}
\sum_{i=2}^{|C|} w\left(A_{v_{i}}\right) & =\sum_{\ell=2}^{s} n_{\ell} w\left(A_{\xi_{\ell}}\right) \\
& 2^{2 .} \sum_{\ell=2}^{s} \sum_{T \in \mathfrak{M}} \mathbb{1}_{l o w(T)}\left(\xi_{\ell}\right) w\left(A_{\xi_{\ell}}\right) \\
& =\sum_{T \in \mathfrak{M}} \sum_{\ell=2}^{s} \mathbb{1}_{l o w(T)}\left(\xi_{\ell}\right) w\left(A_{\xi_{\ell}}\right) \\
& =\sum_{T \in \mathfrak{M}} \sum_{v \in \operatorname{low}(T)} w\left(A_{v}\right) \\
& =\sum_{T \in \mathfrak{M}}\left(\sum_{v \in V[T]} w\left(A_{v}\right)-\max _{v \in V[T]} w\left(A_{v}\right)\right) \\
& \leq c \cdot \sum_{T \in \mathfrak{M}} \sum_{e \in T} d_{e}
\end{aligned}
$$

and this proves the lemma.

### 8.1.2 The Partitioning Algorithm

We show how to partition minimally guarded circuits by providing Algorithm 1 It computes a sequence of sets $\mathfrak{M}_{k}$ of trees. For $k \in\{2, \ldots, s\}, \mathfrak{M}_{k}$ contains a partitioning with $n_{i}$ tree that each pay for $\xi_{i}$ once, for all $i \in\{1, \ldots, k\}$. The output of the algorithm is $\mathfrak{M}_{s}$.
The algorithm maintains a partitioning of $C$ into a set of sub-paths $\mathfrak{P}_{k}$. While these paths are not necessarily simple, they are at all times edge-disjoint. The algorithm iteratively splits non-simple subpaths into simple subpaths. At the same time, it needs to make sure that the subpaths can be combined to trees that satisfy the conditions of Lemma 29, This is accomplished by building the trees of $\mathfrak{M}_{k}$ in the transitive closure $\hat{G}_{\mathscr{A}}$ of $G_{\mathscr{A}}$ : In this way, we can ensure that any edge of each tree in $\mathfrak{M}_{k}$ corresponds to a path in the current partitioning $\mathfrak{P}_{k}$. More precisely, if a tree in $\mathfrak{M}_{k}$ contains an edge $(v, w)$, then the partitioning contains a subpath $(v, \ldots, w)$. This is why we say that a tree $T \in \mathfrak{M}_{k}$ claims a sub-path $p$ of $C$ if one of the edges in $T$ corresponds to $p$. Each time the algorithm splits a sub-path, it also splits the corresponding edge of

```
Algorithm 1: A charging algorithm
    input : A minimally guarded circuit \(C=\left(v_{1}, \ldots, v_{|C|+1}\right)\) in \(G_{\mathscr{A}}\) with \(v_{1}=v_{|C|+1}\).
            Let \(\left\{\xi_{1}, \ldots, \xi_{s}\right\}\) be the set of disjoint vertices on \(C\), w.l.o.g. \(\xi_{1}>\cdots>\xi_{s}\).
    output: A set \(\mathfrak{M}\) of edge disjoint trees in \(G_{\mathscr{A}}\) consisting of edges from \(C\)
    Initialization: Observe that \(\xi_{2}\) can only occur once on \(C\). If \(v_{i_{1}}, v_{i_{2}} \in C\) with \(i_{1} \neq i_{2}\), but
    \(v_{i_{1}}=v_{i_{2}}=\xi_{2}\), then the subcircuit \(\left(v_{i_{1}}, \ldots, v_{i_{2}}\right)\) certifies that \(C\) is not minimally guarded.
    let \(v\) be the unique node in \(C\) with \(v=\xi_{2}\).
    let \(T=\left\{\left\{v_{1}, v\right\}\right\}\) and let \(\mathfrak{M}_{2}=\left\{\left(T, v_{1}\right)\right\} \quad T\) is stored with root \(v_{1}\)
    let \(\mathfrak{P}_{2}=\left\{\left(v_{1}, \ldots, v\right),\left(v, \ldots, v_{|C|-1}\right)\right\}\). the second part of \(C\) is unclaimed
    Main loop: Iteration \(k\) computes \(\mathfrak{M}_{k}\) and \(\mathfrak{P}_{k}\)
    foreach \(k=3, \ldots, s\) do
        let \(\mathfrak{M}_{k}=\mathfrak{M}_{k-1}\) and let \(\mathfrak{P}_{k}=\mathfrak{P}_{k-1}\)
        We need to process all occurrences of \(\xi_{k}\) on \(C\), so we store their indices in \(I\)
        let \(I=\left\{j \in\{2, \ldots,|C|-1\} \mid v_{j}=\xi_{k}\right\}\).
        Find the path \(P_{j}\) that \(j\) lies on. We assume that \(v_{P_{j}}\) occurs before \(w_{P_{j}}\) on \(C\)
        let \(P_{j}=\left(v_{P_{j}}, \ldots, w_{P_{j}}\right)\) be the path in \(\mathfrak{P}_{k-1}\) with \(j\) as inner node, for all \(j \in I\).
        First case: Treats all occurences of \(\xi_{k}\) in unclaimed parts of \(C\) by creating new trees.
        foreach \(j \in I\) with \(\pi_{k-1}\left(P_{j}\right)=\perp\) do
            let \(T=\left\{\left\{v_{P_{j}}, v_{j}\right\}\right\}\) and let \(\mathfrak{M}_{k}=\mathfrak{M}_{k} \cup\left\{\left(T, v_{P_{j}}\right)\right\} \quad\) new tree claims left part
            let \(\mathfrak{P}_{k}=\mathfrak{P}_{k} \backslash\left\{P_{j}\right\} \cup\left\{\left(v_{P_{j}}, \ldots, v_{j}\right),\left(v_{j}, \ldots, w_{P_{j}}\right)\right\} \quad\) path \(P_{j}\) is split at \(v_{j}\)
        end
        Second case: Treats all occurrences of \(\xi_{k}\) that fall on edges of trees in \(\mathfrak{M}_{k}\). This is done by
        iterating through all trees and processing all occurrences in the same tree together.
        foreach \((T, r) \in \mathfrak{M}_{k-1}\) do
            let \(I_{T}=\left\{j \in I \mid \pi_{k-1}\left(P_{j}\right) \in T\right\}\)
            if \(I_{T}=\emptyset\) then continue If \(I_{T}\) is empty, then \(T\) remains unchanged
            select \(j^{*} \in I_{T}\) such that the path from \(j^{*}\) to \(r\) in \(T\) contains no \(j \in I_{T} \backslash\left\{j^{*}\right\}\)
            \(T\) is modified to include \(j^{*}\). Notice that \(\pi_{k-1}\left(P_{j^{*}}\right)=\left\{v_{P_{j}^{*}}, w_{P_{j}^{*}}\right\}\).
            let \(T=T \backslash\left\{\left\{v_{P_{j}^{*}}, w_{P_{j}^{*}}\right\}\right\} \cup\left\{\left\{v_{P_{j}^{*}}, v_{j^{*}}\right\},\left\{v_{j^{*}}, w_{P_{j}^{*}}\right\}\right\}\)
            let \(\mathfrak{P}_{k}=\mathfrak{P}_{k} \backslash\left\{P_{j^{*}}\right\} \cup\left\{\left(v_{P_{j}^{*}}, \ldots, v_{j^{*}}\right),\left(v_{j^{*}}, \ldots, w_{P_{j}^{*}}\right)\right\}\)
            foreach \(j \in I_{T} \backslash\left\{j^{*}\right\}\) do
                Any edge containing a \(j \neq j^{*}\) is split: Half of the edge becomes a new tree, and the other
                half is used to keep the tree connected. Notice that \(\pi_{k-1}\left(P_{j}\right)=\left\{v_{P_{j}}, w_{P_{j}}\right\}\).
                let \(T=T \backslash\left\{\left\{v_{P_{j}}, w_{P_{j}}\right\}\right\} \cup\left\{\left\{v_{j}, w_{P_{j}}\right\}\right\}\)
                let \(T^{\prime}=\left\{\left\{v_{P_{j}}, v_{j}\right\}\right\}\) and let \(\mathfrak{M}_{k}=\mathfrak{M}_{k} \cup\left\{\left(T^{\prime}, v_{P_{j}}\right)\right\}\)
                    let \(\mathfrak{P}_{k}=\mathfrak{P}_{k} \backslash\left\{P_{j}\right\} \cup\left\{\left(v_{P_{j}}, \ldots, v_{j}\right),\left(v_{j}, \ldots, w_{P_{j}}\right)\right\}\)
            end
        end
    end
    return \(\mathfrak{M}_{s}\)
```



Figure 11: A different partitioning of the blue circuit in 9 into trees $T_{1}, T_{2}, T_{3}$ in $G_{\mathscr{A}}$. If none of the three induced connecting moves is improving, then $d\left(T_{1}\right) \geq w\left(A_{5}\right)+w\left(A_{4}\right), d\left(T_{2}\right) \geq w\left(A_{4}\right)+w\left(A_{1}\right)$ and $d\left(T_{3}\right) \geq w\left(A_{1}\right)$. Thus, we get $d\left(C_{i}\right) \geq w\left(A_{5}\right)+2 w\left(A_{4}\right)+2 w\left(A_{1}\right) \geq n_{2} w\left(A_{5}\right)+n_{3} w\left(A_{4}\right)+n_{4} w\left(A_{1}\right)$.
a tree in $\mathfrak{M}_{k}$. To represent the correspondence of trees and subpaths, we define the mapping $\pi_{k}: \mathfrak{P}_{k} \rightarrow$ $\cup_{T \in \mathfrak{M}_{k}} T \times\{\perp\}$ that maps a path $p=(v, \ldots, w) \in \mathfrak{P}_{k}$ to an edge $e \in \cup_{T \in \mathfrak{M}_{k}} T$ if and only if $e \cap p=\{v, w\}$. If no such edge exists, then $\pi_{k}(P)=\perp$. This mapping is well-defined. The trees in the final set $\mathfrak{M}_{s}$ do not contain transitive edges, i.e., they are subgraphs of $G_{\mathscr{A}}$. They also leave no part of $C$ unclaimed.

We now describe the algorithm in more detail and simultaneously observe its main property:
Invariant 30. For all $k=2, \ldots, s$, it holds after iteration $k$ that for all $i \in\{2, \ldots, k\}$ there are at least $n_{i}$ trees in $\mathfrak{M}_{k}$ that pay for $\xi_{i}$.

For presentation purposes, we assume that we already know that the following invariants are true and prove them later in Lemma 33. An example run of the algorithm to accompany the explanation can be found in Figure [12, Notice that the trees in $\mathfrak{M}_{k}$ are rooted, i.e., we store each tree as a tuple consisting of the actual tree plus a root. The trees in connecting moves are unrooted, the roots in $\mathfrak{M}_{k}$ are only needed for the computation.

Invariants 31. For all $k=2, \ldots, s$, the following holds:

1. The trees in $\mathfrak{M}_{k}$ are edge disjoint.
2. The paths in $\mathfrak{P}_{k}$ are edge-disjoint and it holds that $\bigcup_{p \in \mathfrak{P}_{k}} p=C \backslash\left\{\left\{v_{|C|}, v_{|C|+1}\right\}\right\}$.
3. If $v$ is an outer node of some $p \in \mathfrak{P}_{k}$, then $v \in\left\{\xi_{1}, \ldots, \xi_{k}\right\}$. If $v$ is an inner node, then $v \in$ $\left\{\xi_{k+1}, \ldots, \xi_{s}\right\}$.
4. For any $e \in T, T \in \mathfrak{M}_{k}, \pi_{k}^{-1}(e)$ consists of one path from $\mathfrak{P}_{k}$.
5. If $\left\{v_{j_{1}}, v_{j_{2}}\right\}$ with $j_{1}<j_{2}$ is an edge in $T,(T, r) \in \mathfrak{M}_{k}$, then $v_{j_{1}}$ is closer to $r$ than $v_{j_{2}}$.

The initialization consists of setting $\mathfrak{M}_{2}$ and $\mathfrak{P}_{2}$. Observe that $\xi_{2}$ is visited exactly once by $C$ : If there were $v_{i}, v_{j} \in C$ with $i \neq j$, but $v_{i}=v_{j}=\xi_{2}$, the sub-circuit $\left(v_{i}, \ldots, v_{j}\right)$ would be such that $v_{i}>v_{j}$ for all $j \in\{i+1, \ldots, j-1\}$, because $\xi_{1}$ only occurs at $v_{1}$ and $v_{|C|+1}$. This would be a contradiction to the assumption that $C$ is minimally guarded.
The algorithm splits $\left(v_{1}, \ldots, v_{|C|}\right)$ at the unique occurrence $v$ of $\xi_{2}$ on $C$. This is done by setting $\mathfrak{P}_{2}$ to consist of the paths $\left(v_{1}, \ldots, v\right)$ and $\left(v, \ldots, v_{|C|}\right)$ and by inserting the tree $T=\left\{\left\{v_{1}, v\right\}\right\}$ with root $v_{1}$ into

$\mathfrak{P}_{3} \longmapsto \longmapsto \longmapsto$




Figure 12: An example for Algorithm [1. On top, we see a circuit $v_{1}, \ldots, v_{s}$ drawn in a path form with the only occurrences of $\xi_{1}=7$ at the endpoints. Below that, we see how $\mathfrak{M}_{k}$ and $\mathfrak{P}_{k}$ develop through the iterations $k=2, \ldots, 6$, after which the algorithm stops. Iteration $k=5$ is the first where two occurrences of $\xi_{k}$ fall into the same tree, which changes the structure of the tree, because the edge between $v_{5}$ and $v_{8}$ is split and distributed between two trees: the edge $v_{7}$ and $v_{8}$ stays in the tree, and the edge between $v_{5}$ and $v_{7}$ forms a new tree. Notice that connectivity is maintained by this operation.
$\mathfrak{M}_{2}$. Notice that now $\pi_{2}\left(\left(v_{1}, \ldots, v\right)\right)=\left\{v_{1}, v\right\}$ in $T$ and $\pi_{2}\left(\left(v, \ldots, v_{|C|}\right)\right)=\perp$. Invariant 30 is true because there is now one tree paying for $\xi_{2}$.

For $k \geq 3$, we assume that the properties are true for $k-1$ by induction. The algorithm starts by setting $\mathfrak{M}_{k}=\mathfrak{M}_{k-1}$ and $\mathfrak{P}_{k}=\mathfrak{P}_{k-1}$. Then it considers the set $I=\left\{j \mid v_{j}=\xi_{k}\right\}$ of all occurrences of $\xi_{k}$ on $C$. For all $j \in I$ it follows from Properties 31-2 and 31-3 that there is a unique path $P_{j}=\left(v_{P_{j}}, \ldots, w_{P_{j}}\right)$ in $\mathfrak{P}_{k-1}$ with $j \in P_{j}$. The algorithm defines $P_{j}$ in Line 7 We know that the paths $P_{j}$ are different for all $j \in I$ : By Property 31-3, all inner nodes are $\xi_{k}$ or a $\xi_{\ell}$ with higher index. If at least two occurrences of $\xi_{k}$ would fall on the same $P_{j}$, take the two that are closest together: All nodes between them would be equal to a $\xi_{j^{\prime}}$ with $j^{\prime}>k$, which contradicts the assumption that $C$ is minimally guarded (because the $\xi_{\ell}$ are sorted decreasingly). Thus, all $j \in I$ have a distinct $P_{j}$ that they lie on. During the whole algorithm, the endpoints of $P_{j}$ are always named $v_{P_{j}}$ and $w_{P_{j}}$, where $v_{P_{j}}$ occurs first on $C$.
Suppose that $\pi_{k-1}\left(P_{j}\right) \neq \perp$. The algorithm deals with all $j$ that satisfy this in the for loop that starts in Line 12. For any $T$ with occurences of $\xi_{k}$, it considers $I_{T}=\left\{j \in I \mid \pi_{k-1}\left(P_{j}\right) \in T\right\}$, the set of all occurrences of $\xi_{k}$ that fall into the same tree $T \in \mathfrak{M}_{k-1}$. It select a node $j^{*} \in I_{T}$ whose unique path to the root of $T$ does not contain any other $j \in I_{T}$. This node must exist because $T$ is a tree. The algorithm updates $T$, and adds a new tree for any $j \in I_{T} \backslash\left\{j^{*}\right\}$. The idea is that the edge that $j^{*}$ falls on is divided into two edges that stay in $T$, while all other edges are split into an edge that stays in $T$ and an edge that forms a new tree. Since all $v_{j}$ with $j \in I_{T}$ represent the same connected component, $T$ stays connected.
More precisely, for $j^{*}$ the edge $\left\{v_{p_{j^{*}}}, w_{p_{j^{*}}}\right\}$ is divided in $T$ and is replaced by two edges $\left\{v_{p_{j^{*}}}, v_{j^{*}}\right\}$ and $\left\{v_{j^{*}}, w_{p_{j^{*}}}\right\}$. For all $j \in I_{T} \backslash\left\{j^{*}\right\}$, the algorithm replaces $\left\{v_{p_{j}}, w_{p_{j}}\right\}$ by $\left\{v_{j}, w_{P_{j}}\right\}$ in $T$. By Property 31+5, we know that $v_{P_{j}}$ is closer to the root $r$ of $T$. Thus, removing $\left\{v_{p_{j}}, w_{p_{j}}\right\}$ disconnects the subtree at $w_{p_{j}}$ from $r$. However, adding $\left\{v_{j}, w_{P_{j}}\right\}$ reconnects the tree because $v_{j}$ and $v_{j^{*}}$ are the same node and the algorithm assured that $v_{j^{*}} \in V[T]$. Thus, $T$ stays connected. The algorithm also adds the new tree $\left\{\left\{v_{P_{j}}, v_{j}\right\}\right\}$ to $\mathfrak{M}_{k}$ which it can do because this part of $C$ is now free.

The algorithm also updates $\mathfrak{P}_{k}$ by removing $P_{j}$, inserting $\left\{v_{P_{j}}, \ldots, v_{j}\right\}$ and $\left\{v_{j}, \ldots, w_{P_{j}}\right\}$ instead, thus splitting $P_{j}$ at $v_{j}$ for all $j \in I_{T}$ including $j^{*}$.
The algorithm also processes all $j \in I$ where $\pi_{k-1}\left(P_{j}\right)=\perp$. This is done in the for loop in Line 8. In this case, the path $P_{j}$ is split at node $j$ by removing $P_{j}$ from $\mathfrak{P}_{k}$. Then, the algorithm inserts its two parts $\left(v_{P_{j}}, \ldots, v_{j}\right)$ and $\left(v_{j}, \ldots, w_{P_{j}}\right)$ into $\mathfrak{P}_{k}$ to $\mathfrak{P}_{k}$. It also adds a new tree $\left\{\left\{v_{P_{j}}, v_{j}\right\}\right\}$ with root $v_{P_{j}}$ to $\mathfrak{M}_{k}$.
Notice that all trees that are created satisfy that there is a vertex with a higher number than $\xi_{k}$ : if a vertex is added into a tree, then the other vertices in the tree have higher value, and if a new tree is created, then it consists of an edge to a vertex which previously was an endpoint of a path, and these have numbers in $\left\{\xi_{1}, \ldots, \xi_{k-1}\right\}$. For every $j \in I$, it either happens that a tree is updated or that a new tree is created. Thus, the set $\mathfrak{M}_{k}$ satisfies Invariant 30 when the iteration is completed.

## Lemma 32. Invariant 30 holds.

Verifying the other invariants consists of checking all updates on $\mathfrak{M}_{k}$ and $\mathfrak{P}_{k}$.

## Lemma 33. Invariants 31 hold.

Proof. It is easy to verify that all properties hold for $k=2$ since the algorithm sets $\mathfrak{M}_{2}=\left(\left\{v_{1}, v\right\}, v_{1}\right)$ and $\mathfrak{P}_{2}=\left\{\left(v_{1}, \ldots, v\right),\left(v, \ldots, v_{|C|-1}\right)\right\}$. Property 31-1] is also true for the new tree created in Line 9 because it is only executed if $P_{j}$ is unclaimed. Line 16 subdivides an edge into two. Lines 19 and 20 split an existing edge and distributes it among $T$ and $T^{\prime}$. Thus, Property 31-1 is preserved. For Property 31-2, consider Lines 10,17 and 21 to verify that paths are only split into subpaths and no edges are lost. Property 31-3 stays true because iteration $k$ processes all occurrences of $\xi_{k}$ and always executes one of the Lines 10,17 and 21
thus splitting the corresponding paths such that $\xi_{k}$ becomes an outer node. Lines $9,10,16,17,19,20,21$ affect Property 31-4, In all cases, $\mathfrak{M}_{k}$ and $\mathfrak{P}_{k}$ are adjusted consistently.
Finally, consider Property 31-5, Line 9 creates a new tree by claiming $\left(v_{P_{j}}, \ldots, v_{j}\right)$ of the unclaimed edges on $\left(v_{P_{j}}, \ldots, w_{P_{j}}\right)$. Notice that we assume that $v_{P_{j}}$ occurs on $C$ before $v_{P_{j}}$. Thus, assuming that Property $31-5$ holds for all trees existing before Line 9 we see that it also holds for the new tree. Line 16 modifies a tree $T$ by inserting $v_{j^{*}}$ into an edge. Since $v_{j^{*}}$ is an inner node of $\left(v_{P_{j^{*}}}, \ldots, w_{j^{*}}\right)$, Property 31-5 is preserved. Line 19 splits edge $\left\{v_{P_{j}}, w_{P_{j}}\right\}$. Again, recall that $v_{P_{j}}$ occurs before $w_{P_{j}}$ on $C$ and thus inductively is closer to $r$. Further notice that $v_{j}$ occurs before $w_{P_{j}}$ and that $w_{P_{j}}$ gets disconnected from $r$ when $\left\{v_{P_{j}}, w_{P_{j}}\right\}$ is removed. It is then reconnected to $r$ by adding $\left\{v_{j}, w_{P_{j}}\right\}$. This means that $v_{j}$ is closer to $r$ than $w_{P_{j}}$. Line 20 creates a new tree that satisfies Property 31-5]because $v_{P_{j}}$ occurs on $C$ before $v_{j}$.

Corollary 34. Assume that $\mathscr{A}$ is c-approximate connecting move optimal. Let $C=\left(v_{1}, \ldots, v_{l}\right)$ be a circuit in $G_{\mathscr{A}}$ with edges $\left(e_{1}, \ldots, e_{l}\right)$. If $C$ is minimally guarded, then

$$
\sum_{i=2}^{l-1} w\left(T_{v_{i}}\right) \leq c \cdot \sum_{i=1}^{l} d_{e_{i}}=c \cdot d(C)
$$

Proof. For all $T \in \mathfrak{M}$ and all edges $e \in T \subseteq \mathfrak{M}_{s}$, Property 31-4 says that there is a unique path $\pi^{-1}(e) \in$ $\mathfrak{P}_{s}$. Property $31-3$ for $k=s$ means that paths can no longer have inner nodes. Thus, $\pi^{-1}(e)$ is a single edge, and therefore, $e$ also exist in $G_{\mathscr{A}}$. Thus, all trees in $\mathfrak{M}_{s}$ are trees in $G_{\mathscr{A}}$. By Property 31-4, the trees are edge disjoint. Lemma 32 ensures that they satisfy the precondition of Lemma 29. The corollary then follows.

As before, we set $\xi(v)=j$ for the unique $j \in\{1, \ldots, p\}$ with $v \in A_{j}$, for all $v \in V$. For any edge set $F$ in $G$, we define $F_{\circlearrowright}:=\{e=\{u, v\} \in F \mid \xi(u)=\xi(v)\}$ and $F_{\leftrightarrow}:=\{e=\{u, v\} \in F \mid \xi(u) \neq \xi(v)\}$ as the subset of edges of $F$ within components of $\mathscr{A}$ or between them, respectively. Furthermore, if an edge set $F^{\prime}$ in $G$ satisfies $V\left[F^{\prime}\right] \subseteq V\left[A_{j}\right]$ for a $j \in\{1, \ldots, p\}$, then we set $\xi\left(F^{\prime}\right)=j$. Notice that in this case, $F^{\prime}=F_{0}^{\prime}$.

Lemma 35. Let $\bar{F}$ be a simple path in $G$ that starts and ends in the same connected component $T_{j^{*}}$ of $\mathscr{A}$ and satisfies that $\xi(v) \leq j^{*}$ for all $v \in V[\bar{F}]$. Assume that $\bar{F} \neq \bar{F}_{0}$. Assume that $\mathscr{A}$ is edge/set and path/set swap-optimal with respect to $\bar{F}_{\leftrightarrow}$ and that $\mathscr{A}$ is c-approximate connecting move optimal.
Then, there exists a set $R$ of edges on the vertices $V\left[\bar{F}_{\leftrightarrow}\right]$ with $\left(\bar{F}_{\circlearrowright} \cup R\right)_{\leftrightarrow}=\left(\bar{F}_{\circlearrowright} \cup R\right)$ that satisfies the properties listed below. Let $F_{1}^{\prime}, \ldots, F_{x}^{\prime}$ be the connected components of $\bar{F}_{\circlearrowright} \cup R$ in $\left(V\left[F^{\prime}\right], E\left[F^{\prime}\right]\right)$.

1. $\mathscr{A}$ is edge/set swap-optimal with respect to $R$.
2. It holds that $d(R) \leq d\left(\bar{F}_{\leftrightarrow}\right)$ and $\sum_{\ell=2}^{x} w\left(T_{\xi\left(F_{\ell}^{\prime}\right)}\right) \leq c \cdot d\left(\bar{F}_{\leftrightarrow}\right)$.
3. For all $F_{\ell}^{\prime}$, there exists an index $j$ such that $V\left[F_{\ell}^{\prime}\right] \subseteq V\left[A_{j}\right]$ (thus, $\xi\left(F_{\ell}^{\prime}\right)=j$ ).
4. There is only one $F_{\ell}^{\prime}$ with $\xi\left(F_{\ell}^{\prime}\right)=j^{*}$, assume w.l.o.g. that $\xi\left(F_{1}^{\prime}\right)=j^{*}$.

Proof. Let $\bar{F}=\left(s, \ldots, v_{1}, w_{1}, \ldots, w_{2}, v_{2}, \ldots, t\right)$ where $\left(s, \ldots, v_{1}\right)$ and $\left(v_{2}, \ldots, t\right)$ are the prefix and suffix of $\bar{F}$ lying in $T_{j^{*}}$, i.e., we assume that $s, \ldots, v_{1} \in T_{j^{*}}, v_{2}, \ldots, t \in T_{j^{*}}$ and $w_{1}, w_{2} \notin T_{j^{*}}$. The nodes $s$ and $v_{1}$ may coincide as well as $v_{2}$ and $t$. Let $\bar{C}$ be the circuit that $\bar{F}$ induces in $G_{\mathscr{A}}$.
We do induction on the inclusion-wise hierarchy of guarded circuits. Thus, our base case is that $\bar{C}$ is minimally guarded. In this case, we know that all vertices $v$ from $\left(w_{1}, \ldots, w_{2}\right)$ satisfy $\xi(v)<j^{*}$. We set $R=\{\bar{e}\}$ with $\bar{e}:=\left\{v_{1}, v_{2}\right\}$ and $d(\bar{e}):=d\left(\bar{F}_{\leftrightarrow}\right)$ and show that $R$ satisfies Properties 35-1 35-4, For

Property 35-1, we need path/set swap optimality. Picking $v_{1}$ and $v_{2}$ uniquely defines a set of edges $X$ which form a shortest path from $v_{1}$ to $v_{2}$ in the contracted graph and which every $v_{1}-v_{2}$-based path move adds. Let $(\bar{e}, S)$ be a edge/set swap that adds $\bar{e}$. We argue that this move cannot be improving because otherwise, the path/set swap $(X, S)$ was improving. Let $\operatorname{loss}(X)$ be the increase that adding $X$ to $\mathscr{A}$ incurs in $\phi$, and let $\operatorname{gain}(S)$ be the amount by which deleting $S$ decreases $\phi$. Assume that $(\bar{e}, S)$ is improving, i.e., $d(\bar{e})=d\left(\bar{F}_{\leftrightarrow}\right)<\operatorname{gain}(S)$. Notice that $\bar{F}_{\leftrightarrow}$ is a path from $v_{1}$ to $v_{2}$ in the contracted graph. Thus, $d\left(\bar{F}_{\leftrightarrow}\right) \geq d(X)$. Furthermore, notice that $\operatorname{loss}(X) \leq d(X)$. Thus, $\operatorname{loss}(X) \leq d\left(\bar{F}_{\leftrightarrow}\right)$, such that our assumption implies $\operatorname{loss}(X)<\operatorname{gain}(S)$. That is a contradiction to path/set swap optimality. Property $35+1$ holds.

Property $35-2$ is true because $d\left(F^{\prime}\right)=d\left(\bar{F}_{\leftrightarrow}\right)$ and by Corollary 34 since $d\left(\bar{F}_{\leftrightarrow}\right)=d(\bar{C})$. Now we look at the connected components of $\bar{F}_{\circlearrowright} \cup R$. They are equal to the connected components of $\bar{F}_{\circlearrowright}$ except that we add the edge $e$. Notice that $\bar{F}_{\circlearrowright}$ only contains edges that go within the same component, and that that $e$ connects two vertices from the same component. Thus, Property $35-4$ holds. Furthermore, notice that $\bar{F}_{\circlearrowright}$ has exactly two connected components consisting of vertices from $T_{j^{*}}$ : The vertices on the prefix and suffix of $\bar{F}$. These components are connected by $e=\left\{v_{1}, v_{2}\right\}$. Thus, Property 35-2 holds.
Now assume that $\bar{C}$ is not minimally guarded. First assume that $\bar{C}$ is not guarded. Define $v_{1}$ and $v_{2}$ as before. Since $\bar{C}$ is not guarded, $\bar{F}$ has to visit $T_{j^{*}}$ again between $v_{1}$ and $v_{2}$. Let $v_{3}$ and $v_{4}$ be the first and last vertex of one arbitrary visit to $T_{j^{*}}$ between $v_{1}$ and $v_{2}$. It is possible that $v_{3}=v_{4}$, otherwise, notice that $v_{3}$ and $v_{4}$ are connected in $\bar{F}_{\circlearrowright}$. We split $\bar{F}$ into two paths $P_{1}:=\left(v_{1}, \ldots, v_{3}\right)$ and $P_{2}:=\left(v_{4}, \ldots, v_{2}\right)$ and obtain two sets $F^{1^{\prime}}$ and $F^{2 \prime}$ by using the induction hypothesis on the two inclusionwise smaller paths. By induction hypothesis, $v_{1}, v_{3}$ and all other occurrences of vertices from $T_{j^{*}}$ in $P_{1}$ have to be connected in $\bar{F}_{\circlearrowright} \cup F^{1^{\prime}}$. Also, $v_{4}, v_{2}$ and all other occurrences of vertices from $T_{j^{*}}$ in $P_{2}$ have to be connected in $\bar{F}_{\circlearrowright} \cup F^{2^{\prime}}$. Thus, all occurrences of vertices from $T_{j^{*}}$ on $\bar{F}$ are connected in $\bar{F}_{\circlearrowright} \cup F^{1^{\prime}} \cup F^{2^{\prime}}$ because $v_{3}$ is connected to $v_{4}$ in $\bar{F}_{\circlearrowright}$. So, Property 351-4 holds. Furthermore, since $\bar{F}$ is a simple path, no other components of $F_{1}{ }^{\prime}$ and $F_{2}{ }^{\prime}$ can contain the same vertex because only $v_{3}$ is in both $P_{1}$ and $P_{2}$. Thus, Property $35-3$ holds for $F^{1^{\prime}} \cup F^{2^{\prime}}$ since it holds for $F^{1^{\prime}}$ and $F^{2^{\prime}}$ individually. If $\mathscr{A}$ is edge/set swap-optimal with respect to a set $A$ and also with respect to a set $B$, then it is edge/set swap-optimal with respect to $A \cup B$, thus Property 35-1 holds for $F^{1^{\prime}} \cup F^{2^{\prime}}$. Similarly, Property $35-2$ holds because it holds for $F_{1}^{1^{\prime}}$ and ${F_{1}^{2 \prime}}^{\prime}$ individually with respect to disjoint parts of $\bar{F}_{\leftrightarrow}$.
Finally, assume that $\bar{F}$ is guarded, but not minimally guarded. Define $v_{1}$ and $v_{2}$ as before. Since $\bar{F}$ is guarded, but not minimally guarded, it visits a connected component $T_{j}$ with $j<j^{*}$ twice between $v_{1}$ and $v_{2}$, and between these two visits, it never visits a component $T_{j^{\prime}}$ with $j^{\prime}>j$. Pick a $j$ and two visits of $T_{j}$ with this property, and let $v_{3}$ be the last vertex in the first of these visits of $T_{j}$ and let $v_{4}$ be the first vertex of the second visit of $T_{j}$. Again, $v_{3}=v_{4}$ is possible and otherwise, $v_{3}$ and $v_{4}$ are connected in $\bar{F}_{\circlearrowright}$. We apply the induction to the path $\bar{F}^{\prime}$ which is the subpath $\left(v_{3}, \ldots, v_{4}\right)$ and obtain a set $F^{\prime}$ by the induction hypothesis. Additionally, we create $\bar{F}^{\prime \prime}$ from $\bar{F}$ by replacing the subpath $\left(v_{3}, \ldots, v_{4}\right)$ by the edge $\left(v_{3}, v_{4}\right)$. Since this path is shorter, we can apply the induction hypothesis to $\bar{F}^{\prime \prime}$ to obtain a set $F^{\prime \prime}$. We claim that $F^{\prime} \cup F^{\prime \prime}$ satisfies all properties. Notice that $\bar{F}_{\leftrightarrow}=\bar{F}_{\leftrightarrow}^{\prime} \cup F_{\leftrightarrow}^{\prime \prime}$. Again, Property 35F1 is true because $F^{\prime} \cup F^{\prime \prime}$ is the union of two sets that satisfy Property $35-1$.
What are the connected components of $\bar{F}_{\circlearrowright} \cup F^{\prime} \cup F^{\prime \prime}$ ? Let $\mathcal{C C}$ be the set of connected components of $\bar{F}_{\circlearrowright}$, and define $\mathcal{C} C^{\prime}$ and $\mathcal{C} C^{\prime \prime}$ to be the connected components of $\bar{F}_{\circlearrowright} \cup F^{\prime}$ and $\bar{F}_{\circlearrowright} \cup F^{\prime \prime}$, respectively. All edges in $F^{\prime} \cup F^{\prime \prime}$ go between different components in $\mathcal{C C}$, and $F^{\prime}$ and $F^{\prime \prime}$ are defined on vertex sets that are disjoint with the exception of $v_{3}$ and $v_{4}$. Both $\mathcal{C}^{\prime}$ and $\mathcal{C}^{\prime \prime}$ contain exactly one connected component which contains both $v_{3}$ and $v_{4}$ (notice that $v_{3}$ and $v_{4}$ are connected in $\bar{F}^{\prime \prime}$ and $\bar{F}_{\circlearrowright}$ ). In $\mathcal{C}^{\prime}$, it is the connected component with the highest width (i.e.,, $F_{1}^{\prime}$ ), but in $\mathcal{C} C^{\prime \prime}$, the component with the highest width is the component that contains $v_{1}$ and $v_{2}$. Thus, $v_{1}, v_{2} \in F_{1}^{\prime \prime}$ and $v_{3}, v_{4} \in F_{j^{\prime \prime}}^{\prime \prime}$ with $j^{\prime \prime} \neq 1$. In $F_{\circlearrowright} \cup F^{\prime} \cup F^{\prime \prime}, F_{1}^{\prime}$ and $F_{j^{\prime \prime \prime}}$ are merged into
one connected component because they both contain $v_{3}$ and $v_{4}$, meaning that $w\left(F_{1}^{\prime}\right)=w\left(F_{j^{\prime \prime \prime}}\right)$ is counted twice when applying Property $35-2$ for $F^{\prime}$ and $F^{\prime \prime}$ compared to applying Property $35-2$ to $F^{\prime} \cup F^{\prime \prime}$. All other components in $\mathcal{C}^{\prime}$ and $\mathcal{C}^{\prime \prime}$ are defined on disjoint vertex sets and thus $\bar{F}_{\circlearrowright} \cup F^{\prime} \cup F^{\prime \prime}$ is the disjoint union of $\mathcal{C} \mathcal{C}^{\prime} \backslash\left\{F_{1}^{\prime}\right\}, \mathcal{C} C^{\prime \prime} \backslash\left\{F_{j^{\prime \prime \prime}}^{\prime}\right\}$ and $\left\{F_{1}^{\prime} \cup F_{j^{\prime \prime}}^{\prime}\right\}$. We see that Property $35+3$ and $35+4$ carry over to $F^{\prime} \cup F^{\prime \prime}$ from holding for $F^{\prime}$ and $F^{\prime \prime}$. Name the connected components of $\bar{F}_{\circlearrowright} \cup F^{\prime} \cup F^{\prime \prime}$ as $F_{1}^{\prime \prime \prime}, \ldots, F_{y}^{\prime \prime \prime}$. The component with the highest width among these, $F_{1}^{\prime \prime \prime}$, is the one that contains $v_{1}$ and $v_{2}$, i.e., $F_{1}^{\prime \prime \prime}=F_{1}^{\prime \prime}$. We thus have

$$
\sum_{\ell=2}^{y} w\left(T_{\xi\left(F_{\ell}^{\prime \prime \prime}\right)}\right)=\sum_{\ell=2}^{x^{\prime}} w\left(T_{\xi\left(F_{\ell}^{\prime}\right)}\right)+\sum_{\ell=2}^{x^{\prime \prime}} w\left(T_{\xi\left(F_{\ell}^{\prime \prime}\right)}\right) \leq c \cdot d\left(F_{\leftrightarrow}^{\prime}\right)+c \cdot d\left(F_{\leftrightarrow}^{\prime \prime}\right)=c \cdot d\left(\bar{F}_{\leftrightarrow}\right)
$$

Since $\bar{F}_{\leftrightarrow}=\bar{F}_{\leftrightarrow}^{\prime} \cup F_{\leftrightarrow}^{\prime \prime}$ and $d\left(F^{\prime}\right) \leq d\left(\bar{F}_{\leftrightarrow}^{\prime}\right)$ and $d\left(F^{\prime \prime}\right) \leq d\left(\bar{F}_{\leftrightarrow}^{\prime \prime}\right)$ by the induction hypothesis, we have $d\left(F^{\prime} \cup F^{\prime \prime}\right) \leq d\left(\bar{F}_{\leftrightarrow}\right)$ and thus $F^{\prime} \cup F^{\prime \prime}$ satisfies Property 35-2,

Figure 13,21 illustrate how $F^{\prime}$ is constructed recursively.


Figure 13: The rounded rectangles visualize the connected components of $\mathscr{A}$, their size indicates their width. We see a path $\bar{F}$ that corresponds to a circuit in $G_{\mathscr{A}}$ which is guarded, but not minimally guarded. We pick $v_{1}$ and $v_{2}$ as described.


Figure 14: We identify suitable $v_{3}$ and $v_{4}$ and split the path into two paths: The green path, and the black path that now contains the dashed edge.


Figure 15: These are the two paths for which we use the induction hypothesis. Both are guarded, but not minimally guarded.


Figure 16: We identify $v_{3}$ and $v_{4}$ for the two paths in Figure 15 and split the path into the green part and the black part with the dashed edge.


Figure 17: These are the two black paths. The left one is now minimally guarded, but the right one is split again.


Figure 18: The five base cases that occur.


Figure 19: We get that the blue edges cost at most the same as the black edges in Figure 18, and that summing the widths for all blue components costs at most the cost of the black edges in Figure 18 as well.


Figure 20: The result of the induction for the paths in Figure 15


Figure 21: Final set $F^{\prime}$ and connected components of $F^{\prime} \cup \bar{F}_{\circlearrowright}$.

### 8.1.3 Wrapping Up

Lemma 36. Let $G=(V, E)$ be a complete graph, let $d: E \rightarrow \mathbb{R}_{\geq 0}$ be a metric that assigns a cost $d_{e}$ to every edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a set of terminal pairs. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible Steiner Forest solutions for $(G, \mathfrak{T})$. Furthermore, suppose that $\mathscr{A}$ is edgeledge, edge/set and path/set swap-optimal with respect to $E$ and $\phi$, that $\mathscr{A}$ is c-approximate connecting move optimal and that $\mathscr{A}$ only uses edges between terminals. Then there exists a feasible solution $\mathscr{F}^{\prime}$ with $d\left(\mathscr{F}^{\prime}\right) \leq 2(1+c) \cdot d(\mathscr{F})$ that satisfies $\mathscr{F}^{\prime}=\mathscr{F}^{\prime} 0$ such that $\mathscr{A}$ is edgeledge and edge/set swap-optimal with respect to $\mathscr{F}^{\prime}$.

Proof. By Observation [26, we know that by accepting a factor of 2 in the cost, we can assume that the connected components $F_{1}, \ldots, F_{q}$ of $\mathscr{F}$ are node disjoint cycles and that $V[\mathscr{F}]=V[\mathscr{A}]$ equals the set of all terminals. The connected components of $\mathscr{A}$ are $T_{1}, \ldots, T_{p}$. Recall that we use $\xi(v)$ for the index of the component $T_{j}$ that $v$ lies in, and we use $F_{\circlearrowright}:=\{e=\{u, v\} \in F \mid \xi(u)=\xi(v)\}$ and $F_{\leftrightarrow}:=\{e=$ $\{u, v\} \in F \mid \xi(u) \neq \xi(v)\}$ for any edge set $F$ for the set of edges within components of $\mathscr{A}$ or between them, respectively. Also, if an edge set $F^{\prime}$ in $G$ satisfies $V\left[F^{\prime}\right] \subseteq V\left[T_{j}\right]$ for a $j \in\{1, \ldots, p\}$, then we use $\xi\left(F^{\prime}\right)=j$. We want to replace all cycles $F_{i}$ by sets $F_{i}^{\prime}$ which satisfy $F_{i}^{\prime}=\left(F_{i}^{\prime}\right)_{0}$ while keeping the solution feasible and within a constant factor of $\phi(\mathscr{F})$. Let $F=F_{i}$ be one of the cycles.

Let $j^{*}:=\max _{v \in V[F]} \xi(v)$ be the index of the component with the highest width among the components that $F$ visits. There have to be at least two vertices on $F$ from $T_{j^{*}}$ (every vertex is a terminal by our assumption, and since the cycles are disjoint, a lone vertex would not be connected to its mate, but $\mathscr{F}$ is a feasible solution). If the two vertices are adjacent in $F$, then we can delete the edge that connects them and obtain a path that satisfies the preconditions of Lemma 35, We get an edge set $F^{\prime}$. Otherwise, let $v_{1}$ and $v_{2}$ be two vertices from $T_{j^{*}}$ that are not connected by an edge in $F$. Then the cycle is partitioned into two paths, both with endpoints $v_{1}$ and $v_{2}$, that both satisfy the preconditions of Lemma 35, In this case, we get two solutions $F_{l}^{\prime}$ and $F_{r}^{\prime}$ and set $F^{\prime}:=F_{l}^{\prime} \cup F_{r}^{\prime}$. Notice that, either way, we get a set of edges $F^{\prime}$ on the vertices $V[F]$ inducing connected components $F_{1}^{\prime}, \ldots, F_{x}^{\prime}$ of $F_{\circlearrowright} \cup F^{\prime}$ with the following properties:

1. $\mathscr{A}$ is edge/set swap-optimal with respect to $F^{\prime}$
2. For all $F_{\ell}^{\prime}$, there exists an index $j$ such that $V\left[F_{\ell}^{\prime}\right] \subseteq V\left[T_{j}\right]$ (thus, $\xi\left(F_{\ell}^{\prime}\right)=j$ ). When $F^{\prime}=F_{l}^{\prime} \cup F_{r}^{\prime}$, then notice that the connected components of $F_{l}^{\prime}$ and $F_{r}^{\prime}$ are disjoint with the exception of those containing $v_{1}$ and $v_{2}$. Thus, no components with vertices from different $T_{j}$ will get connected.
3. There is only one $F_{\ell}^{\prime}$ with $\xi\left(F_{\ell}^{\prime}\right)=j^{*}$, assume w.l.o.g. that $\xi\left(F_{1}^{\prime}\right)=j^{*}$. When $F^{\prime}=F_{l}^{\prime} \cup F_{r}^{\prime}$, then notice that all occurrences of vertices from $T_{j^{*}}$ are connected to $v_{1}$ and $v_{2}$ in either $F_{l}^{\prime}$ or $F_{r}^{\prime}$, thus, they are all in the same connected component of $F^{\prime}$.
4. It holds that $d\left(F^{\prime}\right) \leq d\left(F_{\leftrightarrow}\right)$ and $\sum_{\ell=2}^{x} w\left(T_{\xi\left(F_{\ell}^{\prime}\right)}\right) \leq c \cdot d\left(F_{\leftrightarrow}\right)$. When $F^{\prime}=F_{l}^{\prime} \cup F_{r}^{\prime}$, notice that $d\left(F^{\prime}\right)=d\left(F_{l}^{\prime}\right)+d\left(F_{r}^{\prime}\right) \leq d\left(F_{\leftrightarrow}\right)$, and that $\sum_{\ell=2}^{x} w\left(T_{\xi\left(F_{\ell}^{\prime}\right)}\right)$, which does not include the components with $v_{1}$ and $v_{2}$, can be split according to the 'side' of the cycle that the components belong to.

The solution $\mathscr{F}^{\prime}$ that arises from substituting $F_{\leftrightarrow}$ by $F^{\prime}$ is not necessarily feasible because $F_{\circlearrowright} \cup F^{\prime}$ can consist of multiple connected components. We need to transform $F^{\prime}$ such that all terminal pairs in $F_{\circlearrowright} \cup F^{\prime}$ are connected. Notice that a terminal pair $u, \bar{u}$ always satisfies $\xi(u)=\xi(\bar{u})$ because $\mathscr{A}$ is feasible, i.e., we do not need to connect connected components with vertices from different $T_{j}$. Furthermore, all vertices in $V[F]$ from $T_{j^{*}}$ are already connected because of 3 . Fix a $j<j^{*}$ and consider all connected components $F_{\ell}^{\prime}$ with $\xi\left(F_{\ell}^{\prime}\right)=j$. Notice that $j<j^{*}$ implies that the widths of these components are part of $\sum_{\ell=2}^{x} w\left(T_{\xi\left(F_{\ell}^{\prime}\right)}\right)$. Start with an arbitrary $F_{\ell^{\prime}}^{\prime}$. If there is a terminal $u \in F_{\ell}^{\prime}$ whose partner $\bar{u}$ is in $F_{\ell^{\prime}}^{\prime}, \ell^{\prime} \neq \ell$, then connect $u$ to $\bar{u}$. Since $u, \bar{u} \in T_{j}$, their distance is at most $w\left(T_{j}\right)$. Since $w\left(T_{\xi\left(F_{\ell^{\prime}}\right)}\right)=w\left(T_{j}\right)$, the contribution of $F_{\ell^{\prime}}$ to the
width sum is large enough to cover the connection cost. Now, $F_{\ell}^{\prime}$ and $F_{\ell^{\prime}}^{\prime}$ are merged into one component, we keep calling it $F_{\ell}^{\prime}$. Repeat the process until all terminals in $F_{\ell}^{\prime}$ are connected to their partner, while always spending a connection cost that is bounded by the contribution of the component that gets merged into $F_{\ell}^{\prime}$. When $F_{\ell}^{\prime}$ is done, pick a component that was not merged and continue in the same fashion. Repeat until all components are merged or processed. In the end, $F^{\prime}$ is a feasible solution, and the money spent for the additional edges is bounded by $\sum_{\ell=2}^{x} w\left(T_{\xi\left(F_{\ell}^{\prime}\right)}\right) \leq c \cdot d\left(F_{\leftrightarrow}\right)$. Thus, the cost of the new solution is at most $(1+c) \cdot d\left(F_{\leftrightarrow}\right)$.
We process all $F_{i}$ with $F_{i} \neq\left(F_{i}\right)_{\circlearrowright}$ in this manner to obtain a solution $\mathscr{F}^{\prime}$ with $\mathscr{F}^{\prime}=\mathscr{F}_{\circlearrowright}^{\prime}$. Notice that $\mathscr{A}$ is edge/edge and edge/set swap-optimal with respect to $\mathscr{F}^{\prime}$. This holds for the new edges because they are from $G$ and $\mathscr{A}$ is swap-optimal with respect to $G$, and it holds for the edges that we get from Lemma 35 by property 1 .
Thus, we have found $\mathscr{F}^{\prime}$ with the necessary properties.
We can now apply Corollary 2 to bound the cost of $\mathscr{A}$.
Corollary 37. Let $G=(V, E)$ be a complete graph, let $d: E \rightarrow \mathbb{R}_{\geq 0}$ be a metric that assigns a cost $d_{e}$ to every edge $e \in E$ and let $\mathfrak{T} \subseteq V \times V$ be a terminal set. Let $\mathscr{A}, \mathscr{F} \subseteq E$ be two feasible Steiner Forest solutions for $(G, \mathfrak{T})$. Furthermore, suppose that $\mathscr{A}$ is edgeledge, edge/set and path/set swap-optimal with respect to $E$ and $\phi$, that $\mathscr{A}$ is $c$-approximate connecting move optimal and that $\mathscr{A}$ only uses edges between terminals. Then $d\left(\mathscr{A}^{\prime}\right) \leq 23(1+c) \cdot d(\mathscr{F})$.

Proof. Lemma 36 ensures that there is a solution $\mathscr{F}^{\prime}$ with $d\left(\mathscr{F}^{\prime}\right) \leq 2(1+c) \cdot d(\mathscr{F})$ that satisfies $\mathscr{F}_{\circlearrowright}^{\prime}=\mathscr{F}^{\prime}$. Every connected component $A_{j}$ of $\mathscr{A}$ can now be treated separately by using Corollary 2 on $A_{j}$ and the part of $\mathscr{F}^{\prime}$ that falls into $A_{j}$. By combining the conclusions for all connected components, we get that

$$
d\left(\mathscr{A}^{\prime}\right) \leq 11.5 \phi\left(\mathscr{F}^{\prime}\right) \leq 23(1+c) \cdot d(\mathscr{F})
$$

for any feasible solution $\mathscr{F}$.
Theorem 1 follows directly. Since the 2-approximation for $k$-MST [Gar05] can be adapted to the weighted case [Gar16], $c=2$ is possible and we can achieve an approximation guarantee of 69 .

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## A Notes on simpler local search algorithms

## A. 1 Adding an edge and removing a constant number of edges

Let $\ell$ and $k<\ell$ be integers and consider Figure 22] Notice that adding a single edge and removing $k$ edges does not improve the solution. However, the current solution costs more than $\ell^{2} / k$ and the optimal solution costs less than $2 \ell$, which is a factor of $\ell /(2 k)$ better.


Figure 22: A bad example for edge/set swaps that remove a constant number of edges.

## A. 2 Regular graphs with high girth and low degree

Assume that $G$ is a degree-3 graph with girth $g=c \log n$ like the graph used in Chen et al. [CRV10]. Such graphs can be constructed, see [Big98]. Select a spanning tree $\mathscr{F}$ in $G$ which will be the optimal solution. Let $E^{\prime}$ be the non-tree edges, notice that $\left|E^{\prime}\right| \geq n / 2$, and let $M$ be a maximum matching in $E^{\prime}$. Because of the degrees, we know that $|M| \geq n / 10$. The endpoints of the edges in $M$ form the terminal pairs $\mathfrak{T}$. Set the length of all edges in $\mathscr{F}$ to 1 and the length of the remaining edges to $g / 4$. The solution $\mathscr{F}$ is feasible and costs $n-1$. The solution $M$ costs $\Omega(\log n)$.
Assume we want to remove an edge $e=\{v, w\} \in M$ and our swap even allows us to add a path to reconnect $v$ and $w$ (in the graph where $M \backslash\{e\}$ is contracted). Let $P$ be such a path. Since $M$ is a matching, at most every alternating edge on $P$ is in $M$. Thus, we have to add $|P| / 2-1 \geq g / 2-1$ edges of length one at a total cost that is larger than the cost $g / 4$ of $e$. Thus, no $d$-improving swap of this type exists (note that, in particular, path/set swaps are not $d$-improving for $M$ ). As a consequence, any oblivious local search with constant locality gap needs to sport a move that removes edges from multiple components of the current solution. In order to restrict to local moves that only remove edges from a single component, we therefore introduced the potential $\phi$.

## B Making the Algorithm Polynomial

So far, we have shown that any locally optimal solution is also within a constant factor of the global optimum. In order to ensure that the local search can find such a local optimum in polynomial time, two issues have to be addressed.

First, we need to show that each improving move can be carried out in polynomial time (or it can be decided in polynomial time that no such move exists). While it is easy to see that improving edge/edge, edge/set, and path/set swaps can be found in polynomial time, finding an improving connecting move is NP-hard in general. However, as we saw in Section 8.1 it is sufficient to restrict the neighborhood of the local search to $c$-approximate connecting moves. In Section B.1, we show that the task of finding an approximate connecting move reduces to approximating the weighted $k$-MST problem. In Section B. 2 we discuss constant factor approximation algorithms for this problem. In particular, we get the following theorem:

Theorem 38. Let $\varepsilon>0$. There exists a polynomial time algorithm, called Improving-Connecting-Move, such that given a Steiner forest $\mathscr{A}$ and a metric distance $d: E \rightarrow \mathbb{R}_{\geq 0}$ on the edges then either: (i) the algorithm finds an improving connecting move with respect to $\phi$, or (ii) it guarantees that there is no
$c(1+\varepsilon)$-approximate connecting move, that is, for every tree $T$ of $G_{\mathscr{A}}$ it holds that

$$
\sum_{e \in \mathscr{A})} d(e) \geq c(1+\varepsilon) \cdot\left(\sum_{i \in V_{\mathscr{A}}} w\left(A_{i}\right)-\max _{i \in V_{\mathscr{A}}} w\left(A_{i}\right)\right)
$$

where $\left\{A_{1}, \ldots, A_{p}\right\}$ is the set of connected components of $\mathscr{A}$.
The second thing we need to show for guaranteeing polynomiality of the local search is that the total number of improving moves is bounded by a polynomial in the input size. This can easily be achieved done via a standard rounding technique incurring only a loss of a factor of $1+\varepsilon$ over the original guarantee for local optima, for an arbitrarily small $\varepsilon>0$; see Section B. 3 for details.
We finally get the following theorem:
Theorem 39. For every $\varepsilon>0$, there is a local search algorithm that computes in polynomial time a solution $\mathscr{A}$ to Steiner Forest such that $d(\mathscr{A}) \leq(1+\varepsilon) 69 \cdot O P T$.

## B. 1 How to ensure approximate connecting move optimality

Assume that we are given an algorithm Tree-Approx that computes a $c$-approximation for the following minimization problem. We call the problem weighted (rooted) $k$-MST problem, and approximating it is further discussed in Section B. 2 .

Given $G=(V, E)$ with a root $r$, a metric $d: V \times V \rightarrow \mathbb{R}^{+}$, a function $\gamma: V \rightarrow \mathbb{R}^{+}$with $\gamma(r)=0$, and a lower bound $\Gamma$, find a tree $T$ in $G$ with $r \in V[T]$ and $\sum_{v \in V[T]} \gamma(v) \geq \Gamma$ that minimizes $\sum_{e \in T} d(e)$.

We see how to use Tree-Approx to ensure $((1+\varepsilon) \cdot c)$-approximate connecting move optimality. We apply Tree-Approx to $G_{\mathscr{A}}^{\text {all }}$. Recall that the vertices in $G_{\mathscr{A}}^{\text {all }}$ are $\{1, \ldots, p\}$, corresponding to the components $A_{1}, \ldots, A_{p}$ of our solution. We try $\left|V\left(G_{\mathscr{A}}^{\text {all }}\right)\right|$ possibilities for the component with the largest width in the connecting move. After choosing the largest component to be the one with index $i$, we delete all vertices from $G_{\mathscr{A}}^{\text {all }}$ with indices larger than $i$. Then we set $\gamma(i):=0$ and $\gamma(j):=w\left(A_{j}\right)$ for $j<i$. We can collect prices between $w_{\min }:=\min \left\{w\left(A_{i}\right) \mid i \in\{1, \ldots, p\}, w\left(A_{i}\right)>0\right\}$, the smallest strictly positive width of any component, and $\sum_{j=1}^{i-1} w\left(A_{j}\right)<p \cdot w\left(A_{p}\right)$. We call Tree-Approx for $\Gamma=(1+\varepsilon / 2)^{\ell} w_{\min }$ for all $\ell \geq 1$
 number of calls of Tree-Approx is bounded by $p \cdot \log _{1+\varepsilon / 2} p \frac{w\left(A_{p}\right)}{w_{\min }} \leq n \cdot \log _{1+\varepsilon / 2} n \Delta$, where $\Delta$ is the largest distance between a terminal and its partner divided by the smallest such distance that is non-zero.
Assume that Tree-Approx returns a solution $T$ with $\sum_{e \in E[T]} d(e)>\sum_{v \in T} \gamma(v)$ for all calls, which means that it does not find an improving connecting move. Furthermore, assume that there exists a $((1+\varepsilon) \cdot c)$ approximate connecting move $T^{*}$ that we should have found, i.e., which satisfies that $\sum_{e \in E\left[T^{*}\right]} d(e) \leq$ $\frac{1}{(1+\varepsilon) c} \sum_{v \in T^{*}} \gamma(v)$. Set $\Gamma^{*}:=\sum_{v \in T^{*}} \gamma(v)$. Let $\ell$ be the index that satisfies $(1+\varepsilon / 2)^{\ell} w_{\text {min }} \leq \Gamma^{*}<$ $(1+\varepsilon / 2)^{\ell+1} w_{\text {min }}$ and consider the run with the correct choice of the largest width and the lower bound $\Gamma^{\prime}:=(1+\varepsilon / 2)^{\ell} w_{\text {min }}$.

Notice that $T^{*}$ is a feasible solution for this run: $w\left(T^{*}\right)=\Gamma^{*} \geq \Gamma^{\prime}$ satisfies the lower bound. Thus, the optimal solution to the input has a cost of at most $\sum_{e \in E\left[T^{*}\right]} d(e)$. Tree-Approx computes a $c$-approximation,
i.e., a solution $\hat{T}$ with $\sum_{v \in V[\hat{T}]} \gamma(v) \geq \Gamma^{\prime}$ and

$$
\begin{aligned}
\sum_{e \in \hat{T}} d(e) \leq c \sum_{e \in E\left[T^{*}\right]} d(e) & \leq \frac{c}{(1+\varepsilon) c} \sum_{v \in V\left[T^{*}\right]} \gamma(v) \\
& \leq(1+\varepsilon / 2) \frac{c}{(1+\varepsilon) c} \Gamma^{\prime}<\Gamma^{\prime} \leq \sum_{v \in V[\hat{T}]} \gamma(v)
\end{aligned}
$$

which means that Tree-Approx computes an improving connecting move.

## B. 2 Weighted $k$-MST

This section is about ways to provide the algorithm Tree-Approx. The problem we want to solve is a weighted version of the rooted $k$-MST problem. Given $G=(V, E)$ with a root $r$, a metric $d: V \times V \rightarrow \mathbb{R}^{+}$and a lower bound $k \in \mathbb{N}$, the rooted $k$-MST problem is to compute a tree $T$ in $G$ with $r \in V[T]$ and $|V[T]| \geq k$ that minimizes $\sum_{e \in E[T]} d(e)$. The unrooted $k$-MST problem is defined verbatim except that no distinguished root has to be part of the tree.

Work on $k$-MST. Fischetti et al. [FHJM94] show that the unrooted $k$-MST problem is NP-hard. Any algorithm for the rooted $k$-MST problem transfers to an algorithm for the unrooted case with the same approximation guarantee by testing all possible nodes and returning the best solution that was found. This in particular holds for optimal algorithms, so the rooted $k$-MST problem is also NP-hard.
As for example observed by Garg [Gar05], we can also use algorithms for the unrooted $k$-MST problem to compute solutions for the rooted $k$-MST problem with the same approximation guarantee. To do so, create $n$ vertices with zero distance to the designated root vertex and search for a tree with $n+k$ vertices. Any such tree has to include at least one copy of the root, and at least $k-1$ other vertices. Thus, any solution for the unrooted $k$-MST problem is a feasible solution for the rooted $k$-MST problem, and the cost is the same. Thus, the rooted and unrooted version of the $k$-MST problem are equivalent.

Blum, Ravi and Vempala [BRV96] develop the first constant-factor approximation for the $k$-MST problem, the factor is 17. Subsequently, Garg [Gar96] gave a 3-approximation, Arya and Ramesh [AR98] developed a 2.5-approximation, Arora and Karakostas AK00] proposed a $(2+\varepsilon)$-approximation, and, finally, Garg [Gar05] published a 2-approximation algorithm for the $k$-MST problem. Chudak, Roughgarden and Williamson [CRW04] show that an easier 5-approximation also proposed by Garg [Gar96] bears resemblances to the primal dual algorithm by Jain and Vazirani [JV01] for the $k$-median problem, in particular to the utilization of Lagrangean relaxation.

Connection to weighted $k$-MST. Johnson, Minkoff and Phillips [JMP00] observe the following reduction from the weighted $k$-MST problem to the unweighted $k$-MST problem, assuming all $\gamma(v)$ are integers. To create the unweighted instance $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, start with $V^{\prime}=V$. Then, for any vertex $v$, add $2 \gamma(v) n-1$ vertices at distance zero of $v$ (thus, there are $2 \gamma(n) n$ vertices 'at' $v$ ), and set $k$ to $2 n \Gamma$. Any solution $T$ for the weighted $k$-MST problem can be interpreted as a solution $T^{\prime}$ for the modified unweighted instance with $\sum_{v \in V[T]} 2 n \gamma(v)=2 n \Gamma$ vertices. Given a solution for the unweighted input, we can first change the solution thus that for any $v \in V$, either $v^{\prime} \in V^{\prime}$ is not picked or $v^{\prime}$ is picked and its $2 \gamma(v) n-1$ copies are picked as well. This is possible since picking more vertices at the same location incurs no additional cost. After this step, the solution can be transformed into a weighted solution with enough weight by picking the corresponding vertices in $V$.
This reduction constructs an input for the unweighted $k$-MST problem that is of pseudo-polynomial size. Johnson et al. [JMP00] note that algorithms for the unweighted $k$-MST problem can typically be adapted to handle the clouds of vertices at the same location implicitly without incurring a super-polynomial running
time. They specifically state that this is true for the 3 -approximation algorithm by Garg [Gar96] for the rooted $k$-MST problem. The more recent 2 -approximation algorithm by Garg [Gar05] can also be adapted for the weighted case such that the running time remains independent of the weights [Gar16]. This yields a polynomial 2 -approximation algorithm for weighted $k$-MST.

## B. 3 Convergence

It is easy to see that a straightforward execution of the local search algorithm discussed in this paper runs in pseudo-polynomial time, as each improving move decreases the potential at least by the length of the shortest edge. We apply a standard rounding technique to make the algorithm polynomial.

## Algorithm X

1. Set $i:=0$ and let $\mathscr{A}_{0}=\mathfrak{T}$.
2. $\beta:=\frac{\varepsilon \cdot \max _{\{v, \bar{v}\} \in \mathfrak{Z}} d(v, \bar{v})}{|E|}, \quad d_{\beta}(e):=\left\lceil\frac{d(e)}{\beta}\right\rceil \beta$.
3. While $\mathscr{A}_{i}$ admits an improving path/set swap w.r.t to $\phi_{\beta}$, or Improving-Connecting-Move finds an improving connecting move w.r.t. $\phi_{\beta}$, set $\mathscr{A}_{i+1}$ to be the resulting solution after applying the move, and $i:=i+1$.
4. Return the solution $\mathscr{A}^{\prime}$ obtained by dropping all inessential edges of $\mathscr{A}_{i}$.

Lemma 40. Assuming that the locality gap for swap-optimal and c-approximate connecting move optimal solutions is $C$, Algorithm $X$ computes in polynomial time $a(1+\varepsilon) C$-approximation to Steiner Forest.

Proof. We first observe that the algorithms runs in polynomial time. To see this, first note that $d_{\beta}(e) \geq \beta$ for all $e \in E$. Therefore, every improving path/set swap and every successful run of Improving-ConnectingMove decreases the potential by at least $\beta$, i.e., $\phi_{\beta}\left(\mathscr{A}_{i+1}\right) \leq \phi_{\beta}\left(\mathscr{A}_{i}\right)-\beta$. As $\phi_{\beta}\left(\mathscr{A}_{0}\right)=2 \sum_{\{v, \bar{v}\} \in \mathfrak{T}} d_{\beta}(v, \bar{v}) \leq$ $\frac{2 n_{t}|E|}{\varepsilon} \beta$, we conclude that the algorithm terminates after at most $2 \frac{n_{t}|E|}{\varepsilon}$ iterations, each of which can be executed in polynomial time.

Now consider the output $\mathscr{A}^{\prime}$ of Algorithm X. As it is path/set swap optimal and $c$-approximately connecting move optimal w.r.t. $d_{\beta}$, our assumption on the locality gap implies that $d_{\beta}\left(\mathscr{A}^{\prime}\right) \leq C d_{\beta}\left(\mathscr{F}_{\beta}\right)$, where $\mathscr{F}_{\beta}$ is the optimal solution of the Steiner Forest instance defined by the metric $d_{\beta}$. Furthermore $d_{\beta}\left(\mathscr{F}_{\beta}\right) \leq d_{\beta}(\mathscr{F})$, where $\mathscr{F}$ is an optimal solution to the original Steiner Forest instance defined by the metric $d$. We hence observe that

$$
\begin{aligned}
d\left(\mathscr{A}^{\prime}\right) & \leq d_{\beta}\left(\mathscr{A}^{\prime}\right) \leq C d_{\beta}\left(\mathscr{F}_{\beta}\right) \leq C d_{\beta}(\mathscr{F})=C \cdot \sum_{e \in \mathscr{F}}\left\lceil\frac{d(e)}{\beta}\right\rceil \beta \\
& \leq C \cdot \sum_{e \in \mathscr{F}}\left(\frac{d(e)}{\beta}+1\right) \beta \leq C \cdot(d(\mathscr{F})+|\mathscr{F}| \beta) \leq(1+\varepsilon) C \cdot d(\mathscr{F}),
\end{aligned}
$$

where the last inequality follows from $|\mathscr{F}| \leq|E|$ and $d(\mathscr{F}) \geq \max _{\{v, \bar{u}\} \in \mathfrak{T}} d(v, \bar{v})$.
Note that Corollary 37 asserts that we can choose $C=23(1+c)$, and [Gar05, Gar16] yields that $c=2$ is a feasible choice. Thus Lemma 40 yields a polynomial-time $(1+\varepsilon) \cdot 69$-approximation local search algorithm, proving Theorem 1 .


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[^1]:    ${ }^{*}$ In fact, we only need five different swaps $(e, S)$ for the following choices of consecutive sets $S$ : The set $S=\{f\}$, the complete set $S=R(e, f)$, and three sets of the form $S=\left\{e_{1}, \ldots, e_{i}\right\}, S=\left\{e_{i+1}, \ldots, e_{j}\right\}$ and $S=\left\{e_{j+1}, \ldots, e_{\ell}\right\}$ for specific indices $i$ and $j$. How to obtain the values for $i$ and $j$ is explained in Section4.1

[^2]:    ${ }^{\dagger}$ Formally, we define the graph $G[T] / e=(V[T] / e, T / e)$ for a tree $T$ by $V[T] / e:=V[T] \cup\{u v\} \backslash\{u, v\}$ and $T / e:=$ $T \backslash \delta(\{u, v\}) \cup\{\{w, u v\} \mid\{u, w\} \in T \vee\{v, w\} \in T\}$ for an edge $e=\{u, v\} \in E$, then set $G / U:=G / e_{1} / e_{2} / \ldots / e_{k}$ for $U=\left\{e_{1}, \ldots, e_{k}\right\}$ and let $T / U$ be the edge set of this graph. If $U \subseteq T$, then the contraction causes no loops or parallel edges, otherwise, we delete all loops or parallel edges.

[^3]:    ${ }^{\ddagger}$ More precisely, we need the slightly weaker condition that for each node $t \in V[\mathscr{A}]$, there is an $\mathscr{F}$-edge incident to $t$ that does not leave the connected component of $\mathscr{A}$ containing $t$.

[^4]:    ${ }^{\S}$ We also need to take care of the additional width of the modified solution, but this is the easier part.

[^5]:    ${ }^{\boldsymbol{\top}}$ More precisely, we need the slightly weaker condition that for each node $t \in V[\mathscr{A}]$, there is an $\mathscr{F}$-edge incident to $t$ that does not leave the connected component of $\mathscr{A}$ containing $t$.

[^6]:    ${ }^{\uparrow}$ We will later use approximate moves, but for simplicity, we forget about approximate optimality during this explanation.

