Time to Failure under Varying Thermal Stresses

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Abstract

Varying thermal stresses influence significantly the time to failure of electric components as used, for instance, in automotive devices. For applications as autonomous driving a high reliability has to be guaranteed.

In this article we discuss how to combine probability distributions for failure. Discrete and continuous changes of the probability distribution in time are both considered. It turns out that the temporal order of the distributions, corresponding to the succession of stresses in applications, is essential. The latter observation restricts the general applicability of the widely used temperature collectives where only the total time of a temperature stress is considered neglecting the order of the stresses.

An application of our results are thermal overstress tests on electric components. We may explain yet not well understood measurements for automotive electric cables.

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1. Introduction

Due to the trend to automated and autonomous driving, high reliability of electrical components in cars has become mandatory (ISO 26262 (2018)). Moreover, along with the electrification of the powertrain and electromobility, the number of electric components in cars has multiplied in recent years, and the vehicle electrical system usually has become the heaviest part thus competing with the demand that the car as a whole should become lighter. Therefore, the prediction of the Time To Failure (TTF) of these components that allows for their optimization and secure layout has become a task of today, and it is the subject of this paper.

For this purpose the concept of Failure In Time (FIT) rates has been introduced, originating in aerospace engineering, see for instance FIDES (2010). In recent time, the concept of FIT rates is getting more and more attention in the field of automotive engineering (ZVEI-BI (2021)).

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Failure rates are measured in the unit of 1 FIT = 1 failure in 10^9 hours and, typically, a failure rate less than 1 FIT should be guaranteed. Just to give an idea, 1 FIT means that a test of 10^5 specimen over 10^4 hours (i.e. longer than a year) yields only 1 failure in average. Since measured TTF in the order of several FITs may be unavailable due to lack of time, imperceptible failure rates, and costs, other approaches than measurements in real time are in order:

1. Using experience, available measurements, and well-accepted models from the physics of failures a handbook is standardized including tables and calculation regulations, for instance MIL-HDBK-217F (1991); SN 29500 (2004); IEC 61709 (2017); FIDES (2010). In this relation, the computed failure rates are also called FIT rates.
2. Measurements under overstress, that either (i) accelerate the TTF or (ii) allow for a refined prediction by gradually detected failure on a shorter time scale.
3. By statistical assessment of collected field failures from the past, see, e.g., Ostendorf, Wielsch & Reiniger (2014).

The results developed in this article are designated to improve 2(i). For an overview of the state of the art of reliability of technical systems see, e.g., Eberlin & Hock (2014).

Primarily, in this paper we focus on the impact of stresses on electrical components caused by varying temperatures. Experimental data for some temperatures are available with the help of which expected values and variances for the TTF can be estimated. Since these tests are destructive, adapted statistical methods are required and applied. By model-based data science approaches using the Arrhenius law for the TTF depending on the temperature, see, e.g., Laidler (2013), the estimation of the TTF may be extended to a large temperature range. The phenomenological Arrhenius law describes the degradation (ageing) due to chemical reactions in general, it says that the logarithm of the TTF is proportional to the reciproke absolute temperature. It holds for monomolecular as well as for macromolecular reactions. By this means for temperature overstress significantly shorter TTF may be expected that are more easily manageable in laboratories. We will use the Arrhenius law in several forms.

As it turns out, the temporal order of different temperatures has a significant effect on the TTF of the component under consideration. For instance, cooling down instead of heating up, while having the same duration of the temperature load, yields a shorter TTF of the component accompanied with a larger variance and standard deviation, respectively. Whereas data from accelerated tests for monotonically increasing stress have been analyzed by Nelson (2001), scenarios with generally varying stresses have to the best knowledge of the authors so far not yet been considered in the generality presented here. Our result restricts the applicability of temperature (or other load) collectives common in engineering, see e.g. Eller, Binner, Reister, Widdecke & Wiedemann (2015) or Knothe & Stichel (2017), where the overall durations of each temperature stress alone are considered. But the order of the temperature stresses may not be neglected in general. By our results for a prescribed lifetime, components may be chosen optimally under the constraint of a certain use profile, coming along with varying temperatures stresses.

The paper is organized as follows: In the next chapter two deterministic approaches for the determination of the TTF of some electrical component under stresses caused by temperature changes are presented. The first one is based on the assumption that components hold a life capacity, a so called viability, which is, depending on the stresses, consumed over time. The second one is derived from the kinetic theory of chemical reactions. By use of Arrhenius' law both approaches lead to the same forms for the temperature dependence of TTFs, and with their help the above mentioned impact of temporal arrangements of different temperatures is discussed.

In Chapter 3 probability distributions for the lifetime of electrical components as well as their expected values and variances are determined, first for discrete temperature steps and thereafter for continuous temperature changes, and the results are illustrated with the help of the exponential lifetime distribution for a constant temperature. Also the results of the deterministic and stochastic approaches, again using Arrhenius' law, are compared. In Chapter 4 the results of Chapter 3 are applied to the widely accepted Weibull distribution for the lifetime. In Chapter 5 the results of some experiments on automotive cables are presented and compared with the theoretical findings. We close with a final chapter where we discuss the obtained results and where we give an overview of relevant applications in near future.

2. Deterministic models

In this chapter we present two deterministic approaches to the analysis of failure times of components under stresses. The first one uses the concept of a life capacity, a so-called viability, of a component. The second one is based on the kinetic theory of chemical reactions.

2.1. Linear consumption of viability

Let $s(t)$ be the consumption of viability at a time t (in h) under the stress T applied up to t. The unit of s depends on the application. The consumption of viability adds up like distances that are determined by velocity and time duration. We start with

Assumption 2.1. *Consumption of viability*

(A1) The consumed viability s depends linearly on the time duration t (in h),

$$s(t) = v(T)t, \tag{1}$$

where the so-called life velocity $v(T)$ depends on the applied stress T.

(A2) The nominal consumed viability $s_\#$ is the product of a life velocity $v(T_\#)$, depending on a nominal stress $T_\#$, and of the nominal Time To Failure $t_\#$:

$$s_\# = v(T_\})\, t_\#. \tag{2}$$

(A3) Let $s = s(t)$ be the consumed viability at time $t = \sum_{k=1}^{K} t_k$, where t_k, $k = 1, \ldots, K$, corresponds to time intervals under stress T_k. Then fractions s_k, $k = 1, \ldots, K$, of the

consumed viability s corresponding to intervals t_k add up linearly,

$$s = \sum_{k=1}^{K} s_k. \tag{3}$$

Combining (1) and (3) yields

$$s(t) = \sum_{k=1}^{K} v(T_k) t_k, \quad t = \sum_{k=1}^{K} t_k. \tag{4}$$

Let $\bar{t} = \bar{t}(T)$ be the Time to Failure for an applied stress (typically a temperature) profile $T(t)$, $t \geq 0$. In contrast, let $t_\#$ be a given nominal Time To Failure prescribed by a standard, or ensured by a manufacturer declaration for a nominal stress $T_\#$.

Thus, with (2) and (4)

$$\bar{s} = s(\bar{t}) = s_\# = v(T_\#) t_\# = \sum_{k=1}^{K} v(T_k) t_k, \quad \bar{t} = \sum_{k=1}^{K} t_k, \tag{5}$$

describes all the initial viability consumed at the Time To Failure $\bar{t}$.

For $K = 2$, for example, we get with $t_2 = \bar{t} - t_1$

$$s_\# = v(T_1) t_1 + v(T_2)(\bar{t} - t_1)$$

or with (2)

$$\bar{t} = \frac{v(T_\#)}{v(T_2)} t_\# + \left(1 - \frac{v(T_1)}{v(T_2)}\right) t_1. \tag{6}$$

Before proceeding, let us present an intuitive example.

Example 2.2 (Viability of a car)**.** *For a car, we consider the filled up tank volume as available (nominal) viability $s_\#$ (in ℓ) that is consumed completely at time $t_\#$ when the tank is empty. Also, let $c_\#$ (in ℓ/km) be the nominal fuel consumption corresponding to a nominal constant velocity (stress) profile $V_\#$. Then we have in analogy to (2)*

$$s_\# = c_\# \, V_\# \, t_\#. \tag{7}$$

Take for example $\bar{s} = 60\ell$, $c = 0.06\ell/km$ and $V = 100km/h$, then we get from (7) $t_\# = 10h$, corresponding to a travelled distance $S_\# = 1000km$.

Let us assume now that the car travels until t_1 with velocity V_1 and consumption c_1, and with velocity V_2 and consumption c_2 from t_1 to $\bar{t}$. Then we have in analogy to (6)

$$\bar{t} = \frac{\bar{s} + (V_2 c_2 - V_1 c_1) t_1}{V_2 c_2}. \tag{8}$$

For example, assume that the car travels $t_1 = 4h$ at $V_1 = 140km/h$ with a corresponding consumption $c_1 = 0.09\ell/km$. This yields the viability $s(t_1) = 50.4\ell$, contrary to the viability

30ℓ *in the constant velocity case. After* t_1 *let the car travel at* $V_2 = 80km/h$ *with* $c_2 = 0.04\ell/km$. *From (8) we obtain* $\bar{t} = 7h$. *We observe that the travelled distance is* $S = S_1 + S_2 = 560km + 240km = 800km$.

But if our car travels until t_1 *with* V_2 *and subsequently with* V_1 *until the tank is empty, then we end up with* $\bar{t} \approx 7.75h$ *and* $S \approx 320km + 525km = 845km$. *The TTF of a tank filling and the distance travelled is longer as in the last example. Note that* $s(t_1) = 12.8\ell$.

Assume a certain standard consumption $\tilde{c}_\# = 0.05\ell/km$ *has been stated by the the manufacturer of the car. It is not precised, how* $\tilde{c}_\#$ *depends on the velocity. If we consider the guaranteed* $\tilde{s}_\# = 59\ell$ *(though it might be actually filled up to* 60ℓ*), this prescribes a certain radius* $\tilde{S}_\# = 1180km$ *for the car to be ensured.*

Let us return to (3) and (5). For continuous changes of the stress T over time, we get from (3)

$$s(t) = \int_0^t v(T(t'))\, dt'$$

and therefore for $t = \bar{t}$ and with (5)

$$s_\# = v(T_\#)t_\# = \int_0^{\bar{t}} v(T(t'))\, dt'. \tag{9}$$

For $T(t) = T_\#$ for all $t \geq 0$ we get from (9) $\bar{t} = t_\#$ as it should be.

Let us consider now the viability of cable insulations which is the application standing behind these theoretical considerations. According to the Arrhenius law modelling the enhancement of a chemical reaction caused by temperature stress we set

$$v(T) \sim \exp\left(-\frac{T_a}{T - T_\infty}\right) \tag{10}$$

where T_a (in K) is the activation temperature and T_∞ (in K) the solidification temperature. Note that in the classical Arrhenius law, $T_\infty = 0$ K is considered (see Schill & Liess (2017, August)). Actually, this life velocity turns out to be an acceleration factor. In the case of Arrhenius' law, we see

$$\bar{t}(T) = t_\pi \exp\left(\frac{T_a}{T - T_\infty}\right) \tag{11}$$

where the so-called reaction kinetic period t_π (in h) turns out to be a choice for $\bar{s}$, the viability that is consumed completely at failure.

Different cross sections (yielding different insulation thicknesses) or cable types have characteristic invariants T_a, T_∞, and t_π. Here the invariants are assumed to be given, e.g. by a calculated fit to experimental data.

For two temperature steps we get from (6)

$$\bar{t} = \exp\left(-\frac{T_a}{T_\# - T_\infty} + \frac{T_a}{T_2 - T_\infty}\right) t_\# + \left(1 - \exp\left(-\frac{T_a}{T_1 - T_\infty} + \frac{T_a}{T_2 - T_\infty}\right)\right) t_1. \tag{12}$$

Also, applying (10) to $T = T_{\#}$ and using (9) we get for continuous temperature changes

$$\exp\left(-\frac{T_a}{T_{\#} - T_\infty}\right) t_{\#} = \int_0^{\bar{t}} \exp\left(-\frac{T_a}{T(t') - T_\infty}\right) dt' \tag{13}$$

which represents an integral equation for the unknown Time To Failure $\bar{t}$ under time dependent stress $T(t)$, $t \geq 0$. We will return to (13) in the next section.

Note that in this case the viability s, here measured in hours, is unavailable for direct measurements. We assume $s_{\#} = 1$ and we see that this implies

$$t_\pi = t_{\#} \exp\left(-\frac{T_a}{T_{\#} - T_\infty}\right). \tag{14}$$

Thus, instead of fixing a constant nominal stress $T_{\#}$ corresponding to a prescribed nominal TTF $t_{\#}$ we may consider t_π as invariant. With (14) we get from (12)

$$\bar{t} = t_\pi \exp\left(\frac{T_a}{T_2 - T_\infty}\right) + \left(1 - \exp\left(-\frac{T_a}{T_1 - T_\infty} + \frac{T_a}{T_2 - T_\infty}\right)\right) t_1. \tag{15}$$

Let us conclude this approach with some remarks on the consequences of the arrangement of stresses.

2.2. Consequences of the arrangement of stresses

When we apply a higher stress at first and switch at a time t_1 to a lower stress, we obtain in general a shorter Time To Failure than in the other case, when we apply the lower stress at first and switch at t_1 to the higher stress. This behaviour is observed in experiments and we analyze, how this is reflected in the deterministic model proposed in this section.

We consider the general structure

$$s = v_1 t_1 + v_2(\bar{t}_{1,2} - t_1) \tag{16}$$

where s is the fixed, but possibly unknown viability of the component, $v_i > 0$ is the life velocity[1] corresponding to the stress T_i, $i = 1, 2$, $\bar{t}_{1,2}$ is the sought-after Time To Failure for applying at first stress T_1 and then T_2, and $t_1 \in [0, \bar{t}_{1,2}]$ is a given time at which we switch to the second stress. If we arrange the stresses in different order, there holds

$$s = v_2 t_1 + v_1(\bar{t}_{2,1} - t_1) \tag{17}$$

with Time To Failure $\bar{t}_{2,1}$ and $t_1 \in [0, \bar{t}_{2,1}]$. We require $t_1 \in [0, \min\{s/v_1, s/v_2\}]$.

From (16) and (17) we find

$$\bar{t}_{1,2} \gtrless \bar{t}_{2,1} \quad \Longleftrightarrow \quad \frac{s}{v_1 + v_2} \begin{matrix} \geqslant \\ \leqslant \end{matrix} t_1 \text{ for } \begin{matrix} v_1 \geq v_2, \\ v_1 \leq v_2. \end{matrix} \tag{18}$$

[1] A zero life velocity corresponds to a process without ageing and this is not in the scope of our study.

In Example 2.2 we have

$$\bar{t}_{1,2} \gtrless \bar{t}_{2,1} \quad \Longleftrightarrow \quad t_1 \gtrless \frac{60}{140 \cdot 0.09 + 80 \cdot 0.08}\text{h} \approx 3.80\text{h},$$
$$t_1 \leq \min\{60/140/0.09; 60/80/0.04\} \approx 4.76\text{h}.$$

For a graphical presentation of the changes of sign, we introduce

$$\xi_i := t_1 \frac{v_i}{s}, \quad i = 1, 2,$$

and rewrite (18) as

$$\bar{t}_{1,2} \gtrless \bar{t}_{2,1} \quad \Longleftrightarrow \quad 1 \begin{matrix} \geqslant \\ \leqslant \end{matrix} \xi_1 + \xi_2 \text{ for } \begin{matrix} \xi_1 \geq \xi_2, \\ \xi_1 \leq \xi_2. \end{matrix} \tag{19}$$

The four cases are illustrated in Fig. 1.

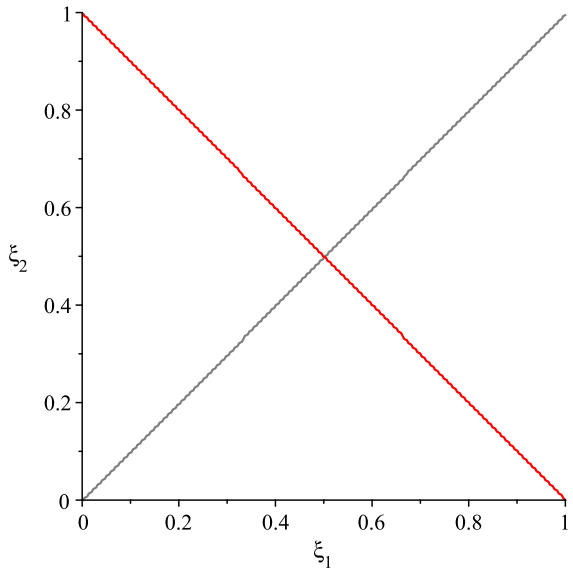

Figure 1: Graphical illustration of (19): Depending on the (dimensionless) ratio ξ_i between switching time and TTF under T_i, $i = 1, 2$, we obtain for $T_1 < T_2 \Longleftrightarrow \xi_1 < \xi_2$ (i.e. beyond the angle bisector) a larger expected value $\bar{t}_{1,2}$ (for heating up) than for $\bar{t}_{2,1}$ (for cooling down). We have symmetry w.r.t. the angle bisector (grey). For the case-by-case analysis see the text. The most interesting case for applications lies above the red curve (the latter representing identical expected values).

2.3. Reaction kinetics

Starting from the differential equation for the concentration $c(t)$ (in mol/m^3) for a certain substance [2] under a chemical reaction of order $\kappa \in [0, \infty)$

$$\dot{c}(t) = -k(t)c^{\kappa}(t), \quad t \geq 0,$$
$$c(0) = c_0,$$

[2]For instance, the concentration of the occurring polymers in the insulation of the component.

we derive an integral equation giving a deterministic description of the failure times in case of the Arrhenius model. Here c_0 (in mol/m^3) is the given initial concentration and $k(t)$ (in 1/h) is the rate constant of the reaction that is in general time-dependent. Equivalent to the differential equation we obtain in the case $\kappa = 1$ by integration

$$c(t) = c_0 \exp\left(-\int_0^t k(t')\,dt'\right), \quad t \geq 0. \tag{20}$$

According to Arrhenius, extended by the solidification temperature T_∞, there holds

$$k(t) = A_\pi \exp\left(-\frac{T_a}{T(t) - T_\infty}\right), \tag{21}$$

where A_π is the reaction frequency factor (that is in principal available), that we assume to be temperature-independent, and $T(t)$ is a given (time-dependent) temperature. So k only depends on time by T. Inserting k, i.e. (21), into (20) yields

$$c(t) = c_0 \exp\left(-A_\pi \int_0^t \exp\left(-\frac{T_a}{T(t') - T_\infty}\right)\,dt'\right). \tag{22}$$

To the temperature-dependent failure time $\bar{t}(T)$ there corresponds a critical concentration

$$\bar{c} := c(\bar{t}) = c_0 \exp\left(-A_\pi \int_0^{\bar{t}} \exp\left(-\frac{T_a}{T(t') - T_\infty}\right)\,dt'\right), \tag{23}$$

that we assume to be temperature-independent. Taking the logarithm of this expression we get

$$\frac{\ln(c_0/\bar{c})}{A_\pi} = \int_0^{\bar{t}} \exp\left(-\frac{T_a}{T(t') - T_\infty)}\right)\,dt', \tag{24}$$

here the left-hand side is temperature independent again as A_π.

As well as in (13), here $\bar{t}$ has to be determined numerically in general. For the well-posedness and the numerical determination, we refer to Appendix B.

Now let us assume $T(t) = T_\#$ for $t \geq 0$ which leads to $\bar{t} = t_\#$. Then we get from (24) and (14)

$$\frac{\ln\left(c_0/\bar{c}\right)}{A_\pi} = \exp\left(-\frac{T_a}{T_\# - T_\infty}\right) t_\# = t_\pi$$

which leads with (24) to the same integral equation as (13) for the first approach. More than that: If we assume that $T(t)$ changes only at discrete time steps, we see immediately that we get the same equations for the Time To Failure $\bar{t}$, in particular (12) for two time steps.

3. Stochastic models

We assume that the Time to Failure (TTF) of an electrical component is intrinsically subject to randomness and therefore, describe it as a random variable[3] $\mathcal{T}$. Subject to a constant temperature stress T the cumulative distribution function $F(t)$ of $\mathcal{T}$ reads for $t \geq 0$

$$w(\mathcal{T} \leq t) = F(t) = 1 - W(t) = \int_0^t f(t')dt',$$

where w represents a probability, W is the probability of survival, and f is the probability density. The expected value of $\mathcal{T}$, also called Mean Time To Failure (MTTF) is given by

$$\bar{t} = E(\mathcal{T}) = \int_0^\infty t\, f(t)\, dt.$$

Under appropriate assumptions, see Lemma A.1, this may be rewritten as

$$\bar{t} = E(\mathcal{T}) = \int_0^\infty W(t)\, dt.$$

In the following we assume first, that the component under consideration is at discrete time steps subject to constant temperatures, thereafter we consider continuous temperature changes. In both cases, we determine the overall probability distribution and their first two moments.

3.1. Probability distributions of failure times for discrete temperature changes

We start with the situation that for times $t \in [0, t_1)$ the temperature stress T_1 is applied to the components and for $t \in [t_1, \infty)$ the temperature stress T_2. t_1 is the switching time between the two temperature stresses. The corresponding cumulative distribution functions for the two temperature stresses are F_k and the densities f_k, $k = 1, 2$. The expected values are denoted by μ_k and the standard deviations by σ_k, $k = 1, 2$.

Thus the failure distribution $F(t)$ is for $t < t_1$ given by $F(t) = F_1(t)$. For $F(t)$ for $t \geq t_1$ we make the following assumptions:

1. $F(t)$ is for $t > t_1$ linear in $F_2(t)$ (what is reasonable due to $F(t) \propto F_1(t)$ for $t \leq t_1$ [4]):

$$F(t) = \gamma F_2(t) + \delta \text{ for } t > t_1. \tag{25}$$

2. $F(t)$ is continuous in t_1:

$$F(t_1) = \gamma F_2(t_1) + \delta = F_1(t_1). \tag{26}$$

[3] By general agreement in the theory of probabilities and statistics random variables are denoted by upper-case latin letters, its realizations by corresponding lower-case latin letters and distribution parameters by lower-case Greek letters. Since T denotes in our application usually a temperature, we deviate with the notation $\mathcal{T}$ for the random variable failure time from the convention.

[4] A general approach with $F(t) = g(F_2(t))$, g continuous, monotone increasing, does not allow to determine uniquely F. However, for small time intervals we may linearize yielding again the above approach.

3. $F(t)$ tends to 1 as $t \to \infty$:

$$F(\infty) = \gamma F_2(\infty) + \delta = \gamma + \delta = 1. \tag{27}$$

From the conditional equations (26) and (27) we obtain uniquely

$$\gamma = \gamma(t_1) = \frac{1 - F_1(t_1)}{1 - F_2(t_1)} \tag{28}$$

and $\delta = 1 - \gamma$ yielding together with (25)

$$F(t) = \begin{cases} F_1(t) & ; \quad \text{for } 0 \le t \le t_1, \\ \gamma F_2(t) + 1 - \gamma \; ; & \text{for } t > t_1. \end{cases} \tag{29}$$

Consequently the density corresponding to $F(t)$ is given by

$$f(t) = \frac{dF(t)}{dt} = \begin{cases} f_1(t) & ; \quad \text{for } 0 \le t \le t_1, \\ \gamma f_2(t) \; ; & \text{for } t > t_1 \end{cases}$$

and we may validate the normalisation of f.

Rewritten for the probability of survival $W_k = 1 - F_k(t)$, $k = 1, 2$, that turns out to be the essential quantity, (29) means with (28)

$$W(t) = \begin{cases} W_1(t) & ; \quad \text{for } 0 \le t \le t_1, \\ \gamma W_2(t) \; ; & \text{for } t > t_1 \end{cases} \tag{30}$$

where we have

$$\gamma = \frac{W_1(t_1)}{W_2(t_1)}.$$

For probabilities, respectively, (30) reads

$$w(\mathcal{T} > t) = \begin{cases} w_1(\mathcal{T} > t) & ; \quad \text{for } 0 \le t \le t_1, \\ w_1(\mathcal{T} > t) w_2(\mathcal{T} > t \,|\, \mathcal{T} > t_1) \; ; & \text{for } t > t_1, \end{cases} \tag{31}$$

where w_k, $k = 1, 2$, are probabilities corresponding to the distributions F_k. In other words the combined probability of survival after the switching point t_1 is the first probability, that $\mathcal{T} > t_1$, multiplied by the second probability being a conditional probability, that $\mathcal{T} > t$ under the condition that $\mathcal{T} > t_1$. This correspondence is plausible and could have been derived intuitively as well. Moreover, the first two moments of $\mathcal{T}$ are

$$E(\mathcal{T}) = \int_0^{t_1} t \, f_1(t) \, dt + \gamma \int_{t_1}^{\infty} t \, f_2(t) \, dt,$$
$$E(\mathcal{T}^2) = \int_0^{t_1} t^2 f_1(t) \, dt + \gamma \int_{t_1}^{\infty} t^2 f_2(t) \, dt.$$

We thus obtain for the addition of the contributions $E^{(k)}(\mathcal{T})$, $k = 1, 2$, on the subintervals

$I_1 = [0, t_1]$ and $I_2 = (t_1, \infty)$,

$$E(\mathcal{T}) =: E^{(1)}(\mathcal{T}) + \gamma E^{(2)}(\mathcal{T}). \tag{32}$$

Note that $E^{(k)}(\mathcal{T}) = \int_{I_k} t\, f_k(t)\, dt$, $k = 1, 2$, is no expected value, only a contribution to it. Correspondingly, we introduce $Var^{(k)}(\mathcal{T})$, $k = 1, 2$.

If the premise in Eq. (A.1) is fulfilled, then Lemma A.1 yields directly from (30)

$$E(\mathcal{T}) = \int_0^\infty W(t)\, dt = \int_0^{t_1} W_1(t)\, dt + \gamma \int_{t_1}^\infty W_2(t)\, dt, \tag{33}$$

$$E(\mathcal{T}^2) = 2\int_0^\infty tW(t)\, dt = 2\int_0^{t_1} tW_1(t)\, dt + 2\gamma \int_{t_1}^\infty tW_2(t)\, dt. \tag{34}$$

The contributions from subintervals to the variance add up as

$$Var(\mathcal{T}) = Var^{(1)}(\mathcal{T}) + \gamma \left\{ Var^{(2)}(\mathcal{T}) - 2E^{(1)}(\mathcal{T})E^{(2)}(\mathcal{T}) - (1-\gamma)E^{(2)}(\mathcal{T})^2 \right\}$$

or by means of Lemma A.1, respectively,

$$\begin{aligned} Var(\mathcal{T}) = {} & 2\int_0^{t_1} tW_1(t)\, dt - \left(\int_0^{t_1} W_1(t)\, dt \right)^2 \\ & + \gamma \left\{ 2\int_{t_1}^\infty tW_2(t)\, dt - \gamma \left(\int_{t_1}^\infty W_2(t)\, dt \right)^2 \right\} \\ & - 2\gamma \int_0^{t_1} W_1(t)\, dt \int_{t_1}^\infty W_2(t)\, dt. \end{aligned} \tag{35}$$

We find that we do not need to compute the combined density f, if the premise in (A.1) is satisfied which may be demonstrated for the Weibull distribution (and thus for the exponential distribution).

Note that we do not add directly the expected values μ_k or the standard deviation σ_k, $k = 1, 2$, respectively, since only a part of the integrals enters into the combined formulas and the relevant subintervals I_k depend on t_1.

Example 3.1 (Exponential distribution)**.** *Consider the case that the Time To Failure $\mathcal{T}$ is exponetially distributed with parameter τ,*

$$F(t) = 1 - W(t) = 1 - \exp\left(-\frac{t}{\tau}\right) \quad \textit{for } t \geq 0. \tag{36}$$

According to (28) we have

$$\gamma = \exp\left(-t_1 \left(\frac{1}{\tau_1} - \frac{1}{\tau_2} \right) \right)$$

and therefore, we get with (33) and (35)

$$E(\mathcal{T}) = \tau_1 + (\tau_2 - \tau_1)\exp\left(-\frac{t_1}{\tau_1}\right), \tag{37}$$

$$Var(\mathcal{T}) = \tau_1^2 + 2(\tau_2 + t_1)(\tau_2 - \tau_1)\exp\left(-\frac{t_1}{\tau_1}\right) - (\tau_2 - \tau_1)^2\exp\left(-2\frac{t_1}{\tau_1}\right). \tag{38}$$

We will come back to these formulas in the next section.

For $K > 2$, $K \in \mathbb{N}$, temperature loads in series we may derive analogous formulas. We consider $K-1$ switching times t_k, $k = 1, \ldots, K-1$ and set (as before) $t_0 = 0$ and formally $t_K := \infty$. The corresponding weight factors are $\gamma_k = W_k(t_k)/W_{k+1}(t_k)$, $k = 1, \ldots, K-1$ and $\gamma_0 := 1$. Then, for instance, (30) yields

$$W(t) = \begin{cases} \left(\Pi_{l=0}^{k-1}\gamma_l\right) W_k(t)\ ; & \text{for } t_{k-1} \le t \le t_k, \\ & k = 1, \ldots, K. \end{cases} \tag{39}$$

In the case $K = 2$ there holds $\gamma = \gamma_1$. We briefly consider the behaviour of γ as a function of t_1 for two arbitrary distributions. If $W_1(t_1) > W_2(t_1)$, which is typically for $T_1 < T_2$, then there holds $\gamma \in [0, 1)$. Else if $W_1(t_1) < W_2(t_1)$, which usually holds for $T_1 > T_2$, then we have $\gamma \in (1, \infty]$. If $T_1 \gg T_2$, then the expected value of $\mathcal{T}$ is dominated by $E^{(2)}$ and, thus, by T_2, and the variation is larger as for the case $T_1 \ll T_2$. In the latter case the expected value is dominated by $E^{(1)}$, i.e. by T_1.

3.2. Consequences of the arrangement of temperature loads

Let E_i and Var_i, $i = 1, 2$, denote the expected value and the variance of the Time To Failure for a temperature load T_i constant in time. If a higher temperature implies a higher stress, as in the Arrhenius law, then $T_1 < T_2$ implies for the Mean Time To Failure the relation $E_1 > E_2$ (or $\tau_1 > \tau_2$). For variances there holds that $T_1 < T_2$ implies $Var_1 > Var_2$.

Now we consider two different temperatures T_i, $i = 1, 2$, w.l.o.g. assume $T_1 < T_2$. Let $E_{1,2}$ or $Var_{1,2}$ denote the expected value and the variance, if until time t_1 temperature T_1 is applied and then T_2 (therefore a situation of "heating up"), otherwise ("cooling down") we write $E_{2,1}$ or $Var_{2,1}$, respectively.

Let us switch to the variables $\xi_i := t_1/\tau_i$, $i = 1, 2$. Then we get from (37) and (38)

$$\frac{E(\mathcal{T})}{t_1} = \frac{1}{\xi_1} + \left(\frac{1}{\xi_2} - \frac{1}{\xi_1}\right)\exp(-\xi_1),$$

$$\frac{Var(\mathcal{T})}{t_1^2} = \frac{1}{\xi_1^2} + 2\left(\frac{1}{\xi_2} + 1\right)\left(\frac{1}{\xi_2} - \frac{1}{\xi_1}\right)\exp(-\xi_1) - \left(\frac{1}{\xi_2} - \frac{1}{\xi_1}\right)^2\exp(-2\xi_1)$$

ans therefore we get

$$\begin{aligned} E_{1,2} \gtrless E_{2,1} \quad &\Longleftrightarrow \quad \left(\frac{1}{\xi_1} - \frac{1}{\xi_2}\right)(1 - \exp(-\xi_1) - \exp(-\xi_2)) \gtrless 0 \\ &\Longleftrightarrow \quad (\xi_2 - \xi_1)\,(1 - \exp(-\xi_1) - \exp(-\xi_2)) \gtrless 0 \end{aligned} \tag{40}$$

and furthermore,

$$\begin{aligned} & Var_{1,2} \gtrless Var_{2,1} \\ \iff \quad & \xi_2^2 - \xi_1^2 \gtrless (\xi_2 - \xi_1)\Big\{ (2\xi_1(1+\xi_2) + (\xi_2 - \xi_1)\exp(-\xi_1))\exp(-\xi_1) \\ & \qquad + (2\xi_2(1+\xi_1) + (\xi_1 - \xi_2)\exp(-\xi_2))\exp(-\xi_2)\Big\}. \end{aligned} \tag{41}$$

Moreover, we get for $\xi_1 < \xi_2$

$$\begin{aligned} & Var_{1,2} \gtrless Var_{2,1} \\ \iff \quad & \xi_2 + \xi_1 \gtrless \quad (2\xi_1(1+\xi_2) + (\xi_2 - \xi_1)\exp(-\xi_1))\exp(-\xi_1) \\ & \qquad + (2\xi_2(1+\xi_1) + (\xi_1 - \xi_2)\exp(-\xi_2))\exp(-\xi_2). \end{aligned}$$

After transition to the variables ξ_i, we may undertake a graphical analysis in 2D, see Fig. 2 which corresponds to Fig. 1. There exists an intersection point between the level sets of the equation $E_{1,2} = E_{2,1}$ (red curve) and the equation $Var_{1,2} = Var_{2,1}$ (blue curve) that we may determine numerically. But this intersection point lies in such a way that ξ_1 is between 0 and the computational accuracy whereas ξ_2 is around 30000 (w.l.o.g. we consider $\xi_1 \leq \xi_2$). Thus the latter is irrelevant in applications, since this corresponds to an unrealistic large T_2 and an unrealistic small T_1 being applied to the component.

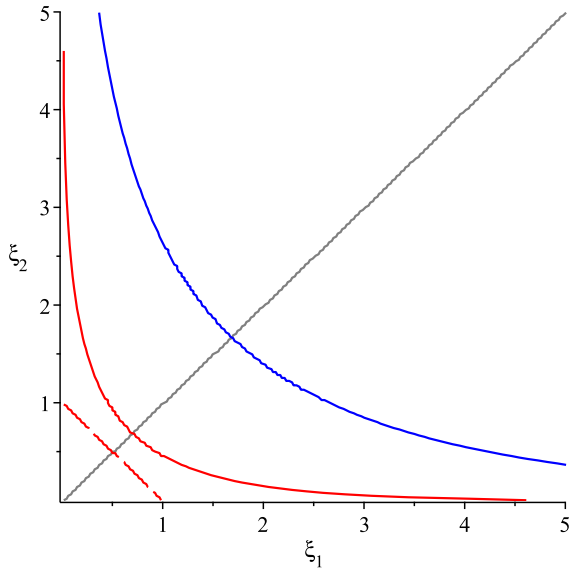

Figure 2: Graphical illustration of (40) and (41): Depending on the (dimensionless) ratio ξ_i between switching time and MTTF under T_i, $i = 1, 2$, we obtain for $T_1 < T_2 \iff \xi_1 < \xi_2$ (i.e. beyond the angle bisector) a larger expected value $E_{1,2}$ (for heating up) or $E_{2,1}$ (for cooling down) and a larger variance $Var_{1,2}$ (for heating up) or $Var_{2,1}$ (for cooling down). We have symmetry w.r.t. the angle bisector (grey). For the case-by-case analysis see the text. The most interesting case for applications lies between the red curve (identical expected values) and the blue curve (identical variances). For comparison, the dashed red line indicates the identical TTF in the deterministic model.

From theory we obtain the following regions that are represented in Fig. 2:

I) Below the red curve: $E_{1,2} < E_{2,1}$ and $Var_{1,2} < Var_{2,1}$: When cooling down we have a larger Mean Time To Failure with larger standard deviation than for heating up. Interpretation: t_1 is too short for that the temperature applied at first plays a significant role. This situation corresponds quasi to constant temperature loads: $E_{1,2} \approx E_2$ and $E_{2,1} \approx E_1$ as well as $Var_{1,2} \approx Var_2$ and $Var_{2,1} \approx Var_1$.

II) Between red and blue curve: $E_{1,2} > E_{2,1}$ and $Var_{1,2} < Var_{2,1}$: When cooling down we have a shorter MTTF with larger standard deviation than for heating up. This corresponds to experimental observations in typical applications: Cooling down shortens the MTTF in contrast to heating up, while the fluctuation margin is increased; the intersection point of the red curve with the angle bisector is at $\xi_1 = \xi_2 = \ln 2 \approx 0.6931$.

III) Above blue curve: $E_{1,2} > E_{2,1}$ and $Var_{1,2} > Var_{2,1}$: When cooling down we have a shorter MTTF with smaller standard deviation than for heating up. Interpretation: t_1 is too large for that the temperature applied secondly plays a significant role, since too many failures have already occurred. Corresponds quasi to constant temperature loads; the intersection point of the blue curve with the angle bisector is at $\xi_1 = \xi_2 \approx 1.678$.

IV) Not visible in the figure is the mentioned case (irrelevant to applications) $E_{1,2} < E_{2,1}$ and $Var_{1,2} > Var_{2,1}$.

Moreover, we have the special cases on the curves.

3.3. Probability distributions of failure times for continuous temperature changes

Now we consider a temperature stress that may change continuously, i.e. we consider the situation that at time $t \in [0, \infty)$ a temperature load $T(t)$ acts on the components. In generalization of the case of discrete temperature changes we denote by $F(T(t), t)$ and $f(T(t), t)$ the cumulative distribution function of the failure time and its density for the case that for any $t > 0$ the temperature of the component is T. Analogously, we denote by $W(T(t), t) = 1 - F(T(t), t)$ the corresponding probability of survival. As before we write $F(t)$, $W(t)$, and $f(t)$ for the distribution function, the probability of survival, and the density of the failure time for an arbitrary temperature profile.

To derive a general formula we start with the special case that W is a step function with K temperature changes as it is discussed in the last section. We write

$$H_k(t) := \ln(W_k(t)), \quad k = 1, \ldots, K.$$

From the probability of survival for K temperature steps, Eq. (39), together with $\gamma_0 = 1$, $\gamma_l = W_l(t_l)/W_{l+1}(t_l)$, $l = 1, \ldots, K-1$, and by $H(t) := \ln(W(t))$, we obtain

$$H(t) = \begin{cases} H_1(t) & 0 \le t \le t_1, \\ \sum_{l=1}^{k-1} \left(H_l(t_l) - H_{l+1}(t_l)\right) + H_k(t) \; ; & t_{k-1} \le t \le t_k, k = 2, \ldots, K-1, \\ \sum_{l=1}^{K-1} \left(H_l(t_l) - H_{l+1}(t_l)\right) + H_K(t) \; ; & t_K \le t. \end{cases}$$

We take $H_i(t) = H(T(t_i), t_i)$ and we expand the summands in the last formula by

$$(\Delta T)_i := T(t_{i+1}) - T(t_i) \text{ and } (\Delta t)_i := t_{i+1} - t_i,$$

i.e. we rewrite the summands in the form

$$\frac{H(T(t_{i+1}), t_i) - H(T(t_i), t_i)}{(\Delta T)_i} \frac{(\Delta T)_i}{(\Delta t)_i} (\Delta t)_i.$$

A formal limit process from differences to differentials and from the sum to an integral yields:

$$H(t) = -\int_0^t \frac{\partial H(T(s), s)}{\partial T} \frac{dT(s)}{ds}\, ds + H(T(t), t). \tag{42}$$

We can reformulate formula (42) for W. With

$$\frac{\partial H}{\partial t} = \frac{\partial \ln(W)}{\partial T} = \frac{1}{W} \frac{\partial W}{\partial T}$$

we get from (42)

$$W(t) = \exp\left(-\int_0^t \frac{1}{W(T(s), s)} \frac{\partial W(T(s), s)}{\partial T} \frac{dT(s)}{ds}\, ds \right) W(T(t), t). \tag{43}$$

Then the Mean Time To Failure reads

$$E(\mathcal{T}) = \int_0^\infty \exp\left(-\int_0^t \frac{1}{W(T(s), s)} \frac{\partial W(T(s), s)}{\partial T} \frac{dT(s)}{ds}\, ds \right) W(T(t), t)\, dt.$$

Example 3.2 (Exponential distribution for continuous temperature changes)**.** *Like in Section 3.1 we consider the case that the Time to Failure $\mathcal{T}$ is exponentially distributed. For ease of presentation, we write $\lambda(T(t)) = 1/\tau(T(t))$, thus, there holds for $t \geq 0$*

$$F(T, t) = 1 - \exp\left(-\lambda(T(t))\right) = 1 - W(T, t).$$

In (43) we replace

$$-\frac{1}{W(T(s), s)} \frac{\partial W(T(s), s)}{\partial T} \frac{dT(s)}{ds} = \frac{\partial \lambda(T(s))}{\partial T} \frac{dT(s)}{ds} s = \frac{d\lambda(T(s))}{ds} s.$$

Thus with (43)

$$W(t) = \exp\left(\int_0^t \frac{d\lambda(T(s))}{ds} s\, ds - \lambda(T(t)) t \right)$$

and by integration by parts

$$W(t) = \exp\left(-\int_0^t \lambda(T(s))\, ds \right). \tag{44}$$

Note that with the definition of an averaged failure rate

$$\bar{\lambda} = t^{-1} \int_0^t \lambda(T(s))\, ds$$

this can be rewritten as

$$W(t) = \exp\left(-\bar{\lambda} t\right). \tag{45}$$

Using the Arrhenius law we assume

$$\lambda(T(t)) = \frac{1}{t_\pi} \exp\left(-\frac{T_a}{T(t) - T_\infty}\right) \tag{46}$$

with t_π to be justified later, and obtain

$$W(t) = \exp\left(-\frac{1}{t_\pi} \int_0^t \exp\left(-\frac{T_a}{T(s) - T_\infty}\right)\, ds\right). \tag{47}$$

Moreover (44) yields

$$E(\mathcal{T}) = \int_0^\infty \exp\left(-\int_0^t \lambda(T(s))\, ds\right)\, dt$$

or, respectively,

$$E(\mathcal{T}) = \int_0^\infty \exp\left(-\frac{1}{t_\pi} \int_0^t \exp\left(-\frac{T_a}{T(s) - T_\infty}\right)\, ds\right)\, dt.$$

We observe in passing that in case of discrete time steps the latter formulas reduce to the formulas from Section 3.1.

3.4. Comparison of approaches

In the following we compare first our results of the deterministic and of the stochastic approaches for discrete changes of the temperature load and thereafter for continuous changes. In so doing, we compare the formulas of $\bar{t}$ for the deterministic approach with $E(\mathcal{T})$ for the stochastic one. Furthermore, we restrict our comparison in case of the stochastic approach to exponentially distributed Times To Failure.

Let us consider first discrete temperature changes. From (37) we obtain

$$E(\mathcal{T}) = \tau_1 + (\tau_2 - \tau_1) \exp\left(-\frac{t_1}{\tau_1}\right).$$

Moreover, for the Arrhenius law, we have $\tau_i = t_\pi \exp(T_a/(T_i - T_\infty))$ and thus

$$\begin{aligned} E(\mathcal{T}) = {} & t_\pi \exp\left(\frac{T_a}{T_1 - T_\infty}\right) + t_\pi \left(\exp\left(\frac{T_a}{T_2 - T_\infty}\right) - \exp\left(\frac{T_a}{T_1 - T_\infty}\right)\right) \\ & \times \exp\left(-\frac{t_1}{t_\pi} \exp\left(-\frac{T_a}{T_1 - T_\infty}\right)\right). \end{aligned}$$

If not too many components fail before t_1, then we may assume $t_1 \ll \tau_1 = t_\pi \exp(-T_a/(T_1 - T_\infty))$. This yields for (37) approximately

$$E(\mathcal{T}) \approx \tau_2 - \left(\frac{\tau_2}{\tau_1} - 1\right) t_1 + \mathcal{O}\left(\left(\frac{t_1}{\tau_1}\right)^2\right) \tag{48}$$

$$= t_\pi \exp\left(\frac{T_a}{T_2 - T_\infty}\right) + \left(1 - \exp\left(-\frac{T_a}{T_1 - T_\infty} + \frac{T_a}{T_2 - T_\infty}\right)\right) t_1 + \mathcal{O}\left(\left(\frac{t_1}{t_\pi} \exp\left(-\frac{T_a}{T_1 - T_\infty}\right)\right)^2\right). \tag{49}$$

Thus, with (15) we realize $\bar{t} = E(\mathcal{T})$ in good approximation in case the Arrhenius law holds in the form (46) which in turn justifies the factor t_π of the exponential form.

In the case of continuous temperature changes we write (47) for $t = E(\mathcal{T})$ in the form

$$t_\pi = \frac{1}{\ln(1/W(E(\mathcal{T})))} \int_0^{E(\mathcal{T})} \exp\left(-\frac{T_a}{T(t') - T_\infty}\right) dt' \tag{50}$$

in case that $W(t) < 1$ for any $t \geq 0$.

Comparing (50) with (13) and (14), we see that in case of $W(\bar{t}) = \exp(-1)$ both equations are the same. In the limit $t_1/\tau_1 \to 0$ and in particular in case of constant temperature stress this holds indeed, thus, in that sense, the stochastic approach generalizes the deterministic one.

Considering the arrangement of stresses, we compare Fig. 1 for the deterministic model and Fig. 2 for the stochastic one with exponential distribution. In both cases we observe a shorter MTTF for $T_1 > T_2$ (cooling down in case of temperature stresses) than for $T_1 < T_2$ (heating up in case of temperature stresses). This mathematical result is plausible from the experimenter's point of view.

Though the general structure is similar, the stochastic approach exhibits a different level set, representing the change of sign in dependence of t_1. Note that in the stochastic approach there is no natural upper limit on t_1 and, hence, the latter level set does not cross any of the both axes.

4. Weibull distribution

In this chapter we rephrase previous results yet obtained for the exponential distribution to the Weibull distribution. A standard assumption for components of all kind is that the Mean Time To Failure $\mathcal{T}$ obeys a Weibull distribution. It is defined for $t \geq 0$ and its cumulative distribution function $F(t)$ as well as its survival probability $W(t)$ are given by

$$F(t) = 1 - W(t) = 1 - \exp\left(-\left(\frac{t}{\tau}\right)^\beta\right) \tag{51}$$

with parameters τ, $\beta \in (0, \infty)$. $\lambda = 1/\tau$ and β are called scale and shape parameters.

For the probability of survival there holds $\lim_{t\to\infty} t^k W(t) = 0$ for any $k \in \mathbb{N}$. Thus according to Lemma A.1 the expected value is

$$\mu(T) := E(\mathcal{T}) = \frac{\tau}{\beta}\Gamma\left(\frac{1}{\beta}\right) = \tau\Gamma\left(1+\frac{1}{\beta}\right), \tag{52}$$

and the standard deviation

$$\sigma(T) := \frac{\tau}{\beta}\left(2\beta\,\Gamma\left(\frac{2}{\beta}\right) - \Gamma^2\left(\frac{1}{\beta}\right)\right)^{1/2} = \tau\left(\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)\right)^{1/2}. \tag{53}$$

where

$$\Gamma(z) := \int_0^\infty x^{z-1}\exp(-x)\,dx, \quad z \in \mathbb{C},\ Re(z) \geq 0,$$

denotes Euler's gamma function (that is available in tables).

For the consecutive combination of different temperature loads in a statistical model, we require the parameters $\tau(T)$ and $\beta(T)$ as a function of time. By a regression calculation on measured data let $\mu(T)$ and $\sigma(T)$ be given.

For approximations of τ and β in case of sufficiently large β, we use the following expansion of the gamma function for sufficiently small x (i.e. $x \ll 1$) at point 1:

$$1/\Gamma\left(1+x\right) \approx 1 + \tilde{\gamma}x. \tag{54}$$

Here $\tilde{\gamma} \approx 0.577$ is the Euler-Mascheroni constant. From (52) and (53) we get

$$\tau(T) = \frac{\mu(T)}{\Gamma\left(1+\frac{1}{\beta(T)}\right)} = \mu(T)\left(1+\mathcal{O}\left(\frac{1}{\beta(T)}\right)\right) \approx \mu(T), \tag{55}$$

$$\beta(T) \approx \frac{\pi}{\sqrt{6}}\frac{\mu(T)}{\sigma(T)}. \tag{56}$$

Example 4.1 (Two temperature steps in case of a Weibull distribution)**.** *For two Weibull distributions with* (τ_1, β_1) *and* (τ_2, β_2) *corresponding to two temperatures, we find with the switching time* t_1

$$\gamma = \exp\left(-\left(\frac{t_1}{\tau_1}\right)^{\beta_1} + \left(\frac{t_1}{\tau_2}\right)^{\beta_2}\right)$$

and thus

$$W(t) = \begin{cases} \exp\left(-\left(\frac{t}{\tau_1}\right)^{\beta_1}\right); & \text{für } 0 \leq t \leq t_1, \\ \exp\left(-\left(\frac{t_1}{\tau_1}\right)^{\beta_1} + \left(\frac{t_1}{\tau_2}\right)^{\beta_2} - \left(\frac{t}{\tau_2}\right)^{\beta_2}\right); & \text{für } t > t_1. \end{cases}$$

This yields

$$E^{(1)}(\mathcal{T}) = \tau_1 \left\{ \Gamma\left(1+\frac{1}{\beta_1}\right) - \hat{\Gamma}\left(1+\frac{1}{\beta_1}, \left(\frac{t_1}{\tau_1}\right)^{\beta_1}\right)\right\},$$
$$E^{(2)}(\mathcal{T}) = \tau_2 \hat{\Gamma}\left(1+\frac{1}{\beta_2}, \left(\frac{t_1}{\tau_2}\right)^{\beta_2}\right),$$

where $\hat{\Gamma}(z,a) := \int_a^\infty x^{z-1}\exp(-x)\,dx$ *is the incomplete upper gamma function. The moments may be computed numerically only. For instance, by (32) we obtain*

$$E(\mathcal{T}) = \tau_1 \left\{ \Gamma\left(1+\frac{1}{\beta_1}\right) - \hat{\Gamma}\left(1+\frac{1}{\beta_1}, \left(\frac{t_1}{\tau_1}\right)^{\beta_1}\right)\right\} + \tau_2 \exp\left(-\left(\frac{t_1}{\tau_1}\right)^{\beta_1} + \left(\frac{t_1}{\tau_2}\right)^{\beta_2}\right) \hat{\Gamma}\left(1+\frac{1}{\beta_2}, \left(\frac{t_1}{\tau_2}\right)^{\beta_2}\right). \tag{57}$$

By (35) the variance could be stated correspondingly.

Example 4.2 (Continuous temperature changes in case of Weibull distributions)**.** *Here we have for* $t \geq 0$ *and* $1/\tau = \lambda(T(t))$*, assuming* β *being independent from* T*,*

$$W(T(t),t) = \exp\left(-\left(\lambda(T(t))t\right)^\beta\right)$$

and we obtain with

$$\frac{1}{W(T(s),s)}\frac{\partial W(T(s),s)}{\partial T} = -\frac{\partial\left(\lambda(T(s))s\right)^\beta}{\partial T}$$

from (43)

$$W(t) = \exp\left(\int_0^t \frac{\partial\left(\lambda(T(s))s\right)^\beta}{\partial T}\frac{dT(s)}{ds}\,ds - \left(\lambda(T(t))t\right)^\beta\right).$$

By integration by parts we end up with

$$W(t) = \exp\left(-\beta\int_0^t (\lambda(T(s)))^\beta s^{\beta-1}\,ds\right).$$

As expected value we find

$$E(\mathcal{T}) = \int_0^\infty \exp\left(-\beta\int_0^t (\lambda(T(s)))^\beta s^{\beta-1}\,ds\right)\,dt.$$

However, since the MTTF for a Weibull distribution is given by $\lambda^{-1}\Gamma(1+1/\beta)$ *this yields for the Arrhenius model with* $\Gamma(1+z) = z\Gamma(z)$

$$E(\mathcal{T}) = \int_0^\infty \exp\left(-\frac{\beta^{\beta+1}}{t_\pi^\beta\Gamma(1/\beta)^\beta}\int_0^t \exp\left(-\frac{\beta T_a}{T(s)-T_\infty}\right)s^{\beta-1}\,ds\right)\,dt.$$

In case $\beta = 1$ *we recover the result for the exponential distribution.*

We compare the result of the deterministic model, (19), with the outcome of the stochastic model in case of the Weibull distribution. If the shape parameter β is sufficiently close to 1, we may expect that the Weibull distribution is well approximated by the exponential distribution with same scale parameter τ (and $\beta = 1$) for which we find agreement between the stochastic and deterministic model (see Sect. 3.4). In fact, similarly as in Section 3.4 for the exponential distribution, in case of a Weibull distribution under constant stress we get by a Taylor expansion w.r.t. β at 1

$$W(\bar{t}) = \exp(-(\Gamma(1+1/\beta))^\beta) = \exp(-1)(1 + (1-\tilde{\gamma})(\beta-1) + \mathcal{O}((\beta-1)^2)).$$

Thus in the case of continuous stress the stochastic approach implies the continuous approach in the approximation $\beta \approx 1$.

Thus we consider in the following only the case $\beta \gg 1$. We focus on the interesting situation $t_1 \ll \min_{k=1,2} \tau_k$, i.e. we switch between the temperature stresses sufficiently early before we reach the MTTFs corresponding to constant stresses. As in Example 4.1, we suppose f_k, $k = 1, 2$, are the densities of Weibull distributions with parameters τ_k and β_k, each depending on T_k.

Complementary to the upper incomplete gamma function we introduce the lower incomplete gamma function $\check{\Gamma}(x,z) := \Gamma(x) - \hat{\Gamma}(x,z)$ for $x > 0$ and $z \geq 0$. We need the following expansions of the lower and upper incomplete gamma functions (see e.g. Dingle (1973)), respectively, for $z \ll 1$:

$$\check{\Gamma}(x,z) = \Gamma(x) z^x \exp(-z) \left(\frac{1}{\Gamma(x+1)} + \frac{z}{\Gamma(x+2)} + \dots \right), \tag{58}$$

$$\hat{\Gamma}(x,z) = \Gamma(x) \left(1 - z^x \exp(-z) \left(\frac{1}{\Gamma(x+1)} + \frac{z}{\Gamma(x+2)} + \dots \right) \right), \tag{59}$$

where in our situation $x = 1 + 1/\beta_k$ and $z = (t_1/\tau_k)^{\beta_k}$, $k = 1$ or 2.

We consider $k = 1$ in (58) and $k = 2$ in (59), yielding

$$\check{\Gamma}\left(1 + \frac{1}{\beta_1}, \left(\frac{t_1}{\tau_1}\right)^{\beta_1}\right) \approx \frac{\Gamma\left(1+\frac{1}{\beta_1}\right)}{\Gamma\left(2+\frac{1}{\beta_1}\right)} \left(\frac{t_1}{\tau_1}\right)^{1+\beta_1} \exp\left(-\left(\frac{t_1}{\tau_1}\right)^{\beta_1}\right),$$

$$\hat{\Gamma}\left(1 + \frac{1}{\beta_2}, \left(\frac{t_1}{\tau_2}\right)^{\beta_2}\right) \approx \frac{\Gamma\left(1+\frac{1}{\beta_2}\right)}{\Gamma\left(2+\frac{1}{\beta_2}\right)} \left(1 - \left(\frac{t_1}{\tau_2}\right)^{1+\beta_2} \exp\left(-\left(\frac{t_1}{\tau_2}\right)^{\beta_2}\right)\right).$$

Exploiting the structure of the factor γ, we obtain from (57)

$$E(\mathcal{T}) \approx \left(\frac{\beta_1}{\beta_1+1} \tau_1 \left(\frac{t_1}{\tau_1}\right)^{1+\beta_1} + \frac{\beta_2}{\beta_2+1} \tau_2 \left(1 - \left(\frac{t_1}{\tau_2}\right)^{1+\beta_2}\right)\right) \exp\left(-\left(\frac{t_1}{\tau_1}\right)^{\beta_1}\right).$$

Exploiting again that t_1/τ_1 is small and $\beta_k \geq 1$, $k = 1, 2$,

$$\begin{aligned} E(\mathcal{T}) &\approx \left(\frac{\beta_1}{\beta_1 + 1} \tau_1 \left(\frac{t_1}{\tau_1} \right)^{1+\beta_1} + \frac{\beta_2}{\beta_2 + 1} \tau_2 \left(1 - \left(\frac{t_1}{\tau_2} \right)^{1+\beta_2} \right) \right) \left(1 - \left(\frac{t_1}{\tau_1} \right)^{\beta_1} \right) \\ &= \frac{\beta_1}{\beta_1 + 1} t_1 \left(\frac{t_1}{\tau_1} \right)^{\beta_1} + \frac{\beta_2}{\beta_2 + 1} \tau_2 \left(1 - \left(\frac{t_1}{\tau_1} \right)^{\beta_1} \right) - \frac{\beta_2}{\beta_2 + 1} t_1 \left(\frac{t_1}{\tau_2} \right)^{\beta_2} \\ &\quad + \mathcal{O}\left(t_1 \max_{k=1,2} \left(\frac{t_1}{\tau_k} \right)^{2\beta_k} \right). \end{aligned}$$

For β large the latter simplifies to

$$E(\mathcal{T}) \approx -t_1 \ln \gamma + \tau_2.$$

If $-\ln \gamma$ is of order $1 - \mu_2^{\beta_2}/\mu_1^{\beta_1}$, this equation resembles (48) that links the stochastic to the deterministic model.

Recall that we have $\mu_k \approx \tau_k$ for large β and in case of Arrhenius $\mu_k = t_\pi \exp\left(-\frac{T_a}{T_k - T_\infty}\right)$, $k = 1, 2$. Since

$$\begin{aligned} -\ln \gamma = \frac{t_1^{\beta_1}}{\mu_1^{\beta_1}} - \frac{t_1^{\beta_2}}{\mu_2^{\beta_2}} &= \left(\frac{t_1}{\mu_2} \right)^{\beta_2} \left(t_1^{\beta_1 - \beta_2} \frac{\mu_2^{\beta_2}}{\mu_1^{\beta_1}} - 1 \right) \\ &\overset{\beta_1 = \beta_2}{=} - \left(\frac{t_1}{\mu_2} \right)^{\beta_2} \left(1 - \left(\frac{\mu_2}{\mu_1} \right)^{\beta_2} \right), \end{aligned}$$

we see that the assumption on γ is realistic in applications, e.g. for $\beta_1 \approx \beta_2$.

Under reasonable approximations, we observe that for discrete steps the stochastic approach generalizes the deterministic one for sufficiently large β as well.

5. Technical application: TTF of automotive electric cables

As application we present thermal overstress tests for electric components as used in automobiles. In order to guarantee the reliability of a component over the lifetime of a car of about 15 years, the component has to be tested for its operating time of about 8 000 h.

Chemical degradation of the insulation enveloping the cable has turned out to be the failure mechanism that occurs first in general. As failure criterion we consider winding tests (according to standard IEC 60216 (2021)).Let us describe shortly this test procedure: Several identical specimens of the cable are stored in a climate cabinet at a fixed temperature. After a specific time some of the specimens are taken out, winded up at another prescribed test temperature, and high voltage is applied. If the insulation is damaged after this procedure, then a short-circuit occurs. The later is defined as the failure criterion for the winding test.

Thus the winding test is destructive and we encouter a type I censoring (see, e.g., Ch. 5 in Mann, Schafer & Singpurwalla (1974)) of the data.

Since laboratory experiments for temperatures, typically occuring under regular use, are long (8 000 h and more) as well as expensive and since field data (e.g. from customer com-

plaints) is not available (if at all) before serial use, it is widely accepted to consider these so called accelerated experiments by applying temperature overstresses. However, great care is appropriate that the overstress does not induce another chemical, mechanical etc. behaviour that is not present at normal stress, for instance cf. Song, Yuan & Koch (2018).

We provide numerical simulations in order to obtain characteristic values for cables from measured data for different cross-sections. These characteristic parameters allow for a specification of the cables by extrapolation to further temperature values. Based on these reference data, we may predict the failure times for discrete and continuous time-dependent thermal stresses.

Even for uniform test conditions the measured failure times are subject to a statistic scattering. If $\bar{t}$ is interpreted as expected value, then (11) yields a deterministic failure time for a fixed temperature T with a confidence range related to an observed standard deviation $\bar{\sigma}$. A modified Arrhenius law, corresponding to the next-dominant reactions, postulates

$$\bar{\sigma}(T) = \sigma_\pi \exp\left(\left(\frac{T_a}{T - T_\infty}\right)^d\right) \tag{60}$$

where σ_π is a fitting parameter corresponding to the reaction-kinetic period of this side reaction. Physico-chemically $d \in (0, \infty)$ may be motivated as an interaction of chemical reactions of different order, the model for the expected value (11) corresponding to the dominant single chemical reaction of first order. Our approach (60) is justified by fits to experimental data, for details see Kimmerle & Liess (2019). For a discussion of the temperature dependence of the standard deviation, various approaches for it, and its importance we refer to Chan (1991).

5.1. Experimental data and statistical processing

By experiments for a specified cable subject to a constant temperature T_k (where always N specimens are taken out at a prescribed time) we obtain so-called reference pairs $(T_k, \bar{t}_k)$ and $(T_k, \bar{\sigma}_k)$, $k = 1, \ldots, K$, where $\bar{t}_k$ are statistically determined failure times with empirical standard deviations $\bar{\sigma}_k$. For details of the underlying so-called filtering methods (maximum-likelihood-method, empirical cumulative distribution function) and dealing with censored data see, for instance, Aggarwala (2001) or (Mann, Schafer & Singpurwalla, 1974, Sect. 5.2 & 9.3).

For each constant temperature stress T_k, $k = 1, ..., K$, this allows to fit a Weibull distribution with parameters τ_k and β_k to the statistically obtained failure times $\bar{t}_k$. Equivalently, we may fit the expected values μ_k and σ_k, $k = 1, ..., K$. Note that for any temperature, by using (11) and (60) the approximations (55) and (56) read

$$\tau(T) \approx \mu(T) = t_\pi \exp(x(T)) \tag{61}$$

$$\beta(T) \approx \frac{\pi}{\sqrt{6}} \frac{t_\pi}{\sigma_\pi} \exp\left(x(T) - x(T)^d\right), \tag{62}$$

where we set $x(T) = T_a/(T - T_\infty)$.

Thus, using the Arrhenius models, we may obtain by an extrapolation over $\bar{\mu}_k$ and $\bar{\sigma}_k$ the expected value μ and the standard deviation σ as functions of the temperature T. More

precisely, we interpret for our realization (data) μ as a statistical fit for $\bar{t}$ and σ as fit for $\bar{\sigma}$. For the extrapolation we determine the five parameters t_π, T_a, T_∞, σ_π, and d by a least-square-fit.

By this means we obtain the expected value and the standard deviation as functions of temperature within a certain interval. This yields also the corresponding Weibull parameters τ and β as a function on a temperature interval, see Fig. 3.

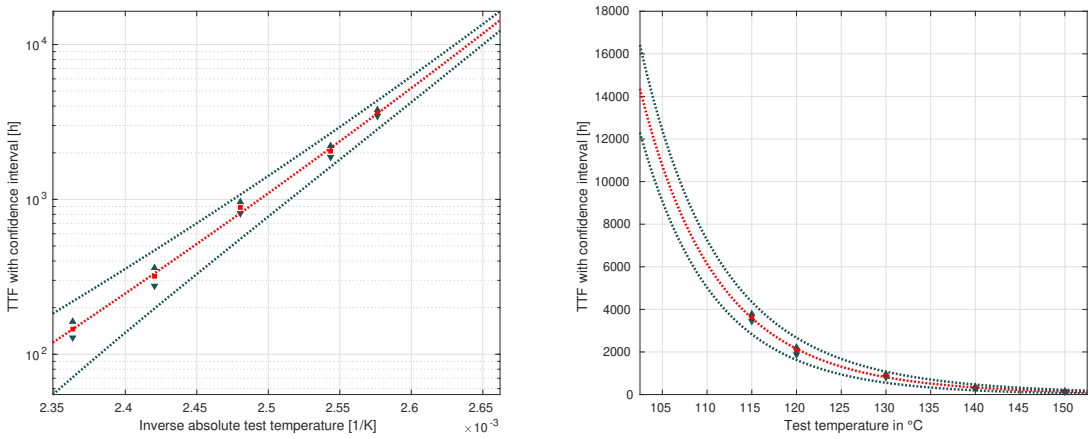

Figure 3: Extrapolation of TTF over a temperature interval for a PVC/T105 cable (i.e. a PVC cable for regular use up to 105°C) with cross-section 0.5mm^2 for various constant temperature stresses. The fitting curves are computed in a software tool from Physical Software Solutions (Kimmerle, Nolet & Dvorsky (2019)). Expected values to measurements (red dots) with confidence interval (blue triangles), extrapolated center curve (in red) expected value μ, upper and lower curve (both in blue) $\mu \pm 3\sigma$. Left-hand side: semilogarithmic scale, at which the regression is computed. Right-hand side: rescaled Arrhenius-type curve.

The analysis of the experimental data suggests that the parameters t_π, σ_π, and T_a, depend significantly on the cable cross-section A by means of the related thickness of the insulation shell, whereas for T_∞ and d only a weak dependence on the cross-section is observed. The parameters t_π, σ_π, and T_a may be interpreted as characteristic invariants for a certain type of cable insulation, while the parameters T_∞ and d may be considered as being characteristic for the chemical degradation mechanism in general.

5.2. Application to real-life problems with varying temperature stresses

Note that in our experiments the temperature stress is kept constant. However, in real-life applications at different times different temperature loads apply. Typically for the simulation of automotive components is a certain use profile of temperature loads dependent on time that is periodically repeated. This is where we apply our results from Chapter 4. For the numerical evaluation of the method described in the last section, a commercial software package, the *L-module* (Kimmerle, Nolet & Dvorsky (2019)), has been set up using MATLAB. The estimated parameters, calculated internally by the software from provided experimental reference pairs, are handed over to the outer module that returns the resulting MTTF together with quantitative upper and lower bounds. The latter serves as a risk measure being either a factor of the standard deviation or a confidence interval for a confidence level, e.g. 95%, selected by the user.

Example 5.1 (PVC insulation under two temperature loads)**.** *For two temperature steps according to Example 4.1 with a prescribed switching time t_1, we obtain the survival probability depicted in Fig. 4. The data has been provided by a renowned cable manufacturer*[5]*.*

We observe that the choice of the points in time $\tilde{t}_k$, $k = 1, \ldots, K$, where N specimens are taken out of the climate cabinet, is crucial: We should not start too early, where no failure is observed, and not too late, when all specimens have failed. The number K depends on the number of specimens taken out at each $\tilde{t}_k$ (here $N = 3$) and the capacity of climate cabinets available.

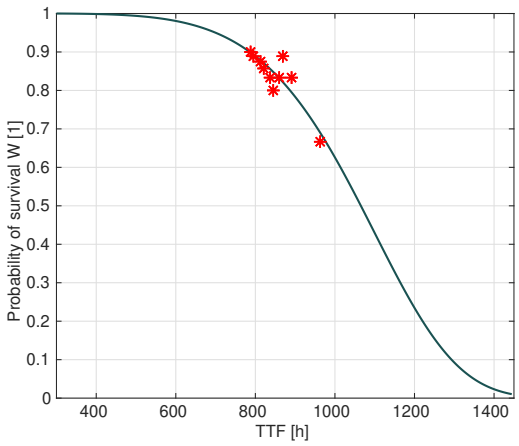

Figure 4: Measured interval-censored failure times (red crosses) of a PVC/T105 cable with cross section 6.0mm^2 subject to two temperature steps, switching from $T_1 = 140°$C to $T_2 = 120°$C at $t_1 = 300$h. Fit by use of the empirical cumulative distribution function normalized to 1 on the time interval $(t_1, 1500\text{h})$.

For another cross-section as a prediction we consider the Weibull distribution under two temperature stresses (with a different switching time t_1), see Fig. 5.

Moreover, we observe that the temporal order of temperatures is crucial and the data may be explained by our novel approach from Chapter 4.

Another application of our data analysis from Section 5.1 is the extrapolation of standards that describe minimal TTF to be guaranteed only at certain fixed temperatures.

Example 5.2 (Continuous extrapolation of standard values)**.** *The standardized TTFs to be guaranteed for PVC/T105 are according to the standard IEC 60216 (2021) and certain specifications of original equipment manufacturers (OEM)*

$$(T(^\circ C), TTF(h)) : \ (105, 3000), (130, 240), (155, 6)\,.$$

As a first approximation they are independent from cross section and conducting material in the cable.

[5]Name of the brand is suppressed for sake of confidentiality.

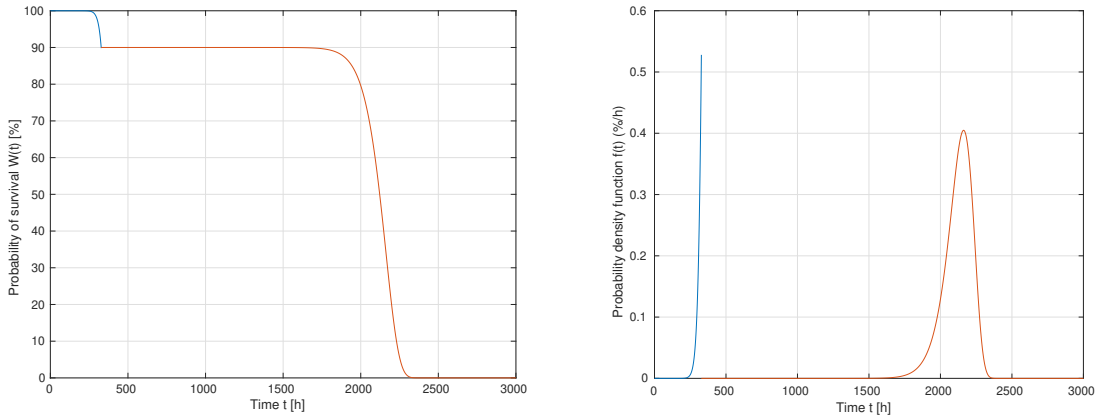

Figure 5: Combined distribution of the TTF for a PVC cable insulation under two temperature stresses. $T_1 = 140°\mathrm{C}$ is applied until $t_1 = 447.4$ h (blue curves) and $T_2 = 120°\mathrm{C}$ afterwards (orange curves). The fitted parameters are $\tau(140°\mathrm{C}) = 371,3\mathrm{h}$, $\beta(140°\mathrm{C}) = 17,97$, $\tau(120°\mathrm{C}) = 2165,1\mathrm{h}$ and $\beta(120°\mathrm{C}) = 26,45$. This case corresponds to the interesting case $\gamma_1 = 0.9$ where we have some failures before t_1 and some after. Left-hand side: probability of survival W. Right-hand side: density $f(t)$.

According to the standard, as failure criterion the winding test with winding at room temperature (RT) is considered for the long time ageing over 3 000h and for the overstress for 6h, whereas the short time ageing over 240h is subject to the cold winding test with winding at $-25°C$. The material invariants might differ for the winding tests with different test temperatures. Thus, an option is to convert the characteristic values obtained by a cold winding test to the winding test at RT, provided that conversion formulae may be derived from sufficient data sets. The best-before dates of the standard may be extrapolated to other temperatures as well as the measured data from experiments with both winding tests.

Note that we observe that the standard (as well as other OEM specifications) are very conservative in comparison to our calculated values, see Fig. 6.

6. Conclusion

In our study we have shown how to combine probability distributions for TTF depending on different stresses. The stresses may change discretely or continuously with time. We discussed deterministic as well as stochastic approaches.

In case of an exponential distribution, we may show for two discrete stress steps that the stochastic approach generalizes the deterministic approach, assuming a sufficiently early switching time t_1 compared to the TTF τ_1 or assuming a constant stress. For the Arrhenius model a similar result may be demonstrated relating the deterministic and stochastic approach for the continuous case.

It turns out that the temporal order of the stresses has a crucial influence on the TTF. For the arrangement of stresses and its consequences for the deterministic approach we refer to the end of Section 2.2, for the stochastic approach we refer to the end of Section 3.4.

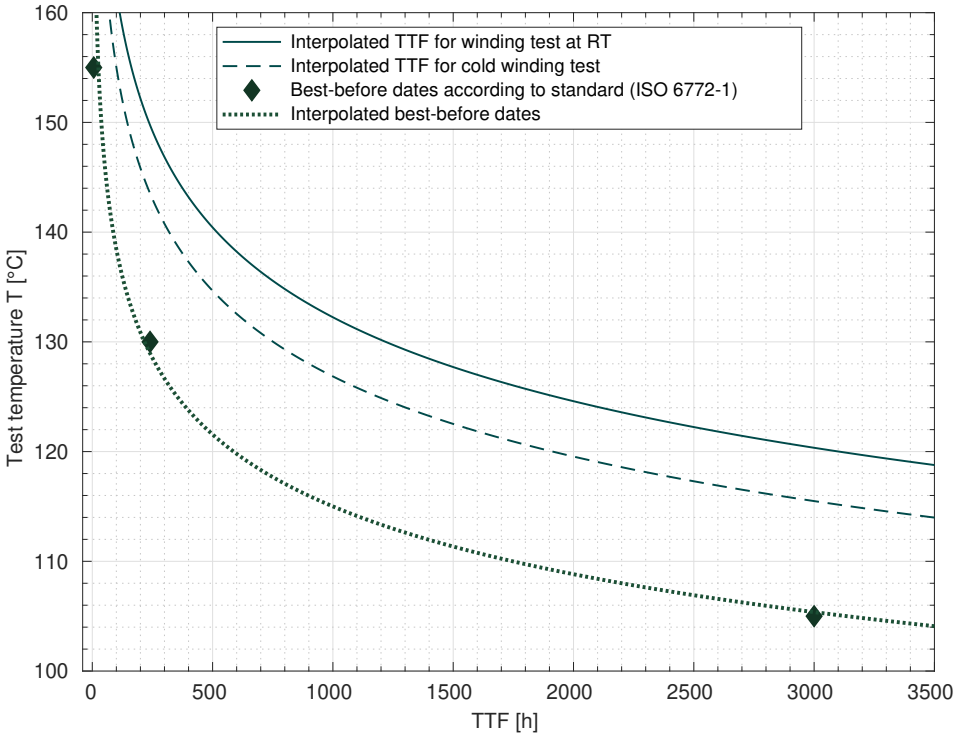

Figure 6: Extrapolation (lower curve) of best-before dates to be guaranteed by standards. Computed reference curves for a PVC/T105 cable subject to cold-winding test (center curve) and winding test at room temperature (upper curve), both tests according to the standard.

As technical application we have considered the Arrhenius law and, particularly, the chemical degradation of cable insulations. However other physical stresses might be considered as well. For instance, for mechanical stresses the Wöhler law could be examined.

We provide simulations in order to obtain characteristic values for cable insulations from measured TTF. Based on these reference data, we may predict the failure times for various time-dependent thermal loads using a MTTF analysis. For the evaluation of the complicated mathematical formulae a computing tool (Kimmerle, Nolet & Dvorsky (2019)) has been developed.

By this means our method could be validated successfully for this technical application with varying stresses that yet has not been well understood. It is expected that cable insulations relying on silicone that are common for higher electric currents may be treated similarily.

Another application (see 5.2) is the possible extension from a discrete set of given stress levels, prescribed by a standard, to a continuous stress function.

As an outlook, having examined cables, we will turn to other electric automotive components as connectors, terminal lugs, fuses, grounding bolts, and then consider finally the wiring harness as a whole. Moreover, the influence of failures by interaction of components

in the wiring harness are a further challenge. Our work provides an example for a tool-based development of the wiring harness. Moreover, it illustrates that what can be measured can be simulated. Here the nearby goal is to shift testing and validation from the road to test benches and, finally, to simulations - allowing for shorter development cycles at reduced costs.

For applications the quantities $\bar{\mu}-\bar{\sigma}$ or $\bar{\mu}-3\bar{\sigma}$ are suitable measures of risk corresponding to a certain probability of non-failure each.

The reaction kinetic frequence A_π as well as the critical concentration $\bar{c}$ may be weakly dependent on temperature. The question remains what is the influnce on our analysis and whether this might increase the coincidence with experiments.

However, further experiments and data related to failure times will be helpful for further validation and extension of our models, also to other applications.

Appendix A. Probability distributions of failure times

Lemma Appendix A.1. *Let $F : \mathbb{R}^+ \to [0,1]$ the distribution function of a probability distribution with density $f : \mathbb{R}^+ \to \mathbb{R}$ and with existing k-th moment, $k \in \mathbb{N}$. Designate $W(t) := 1 - F(t)$, $t \in \mathbb{R}^+$, the corresponding survival probability.*

If the survival probability converges faster as polynomial with degree k to zero as time goes to infinity, i.e. we have

$$\lim_{t\to\infty} t^k W(t) = 0, \tag{A.1}$$

then there holds for the k-th moment of a random variable $\mathcal{T}$ subject to this distribution

$$E(\mathcal{T}^k) = k\int_0^\infty t^{k-1}W(t)\,dt.$$

In particular, this means that for the expected value

$$E(\mathcal{T}) = \int_0^\infty W(t)\,dt$$

and for the variance (the square of the standard deviation σ)

$$Var(\mathcal{T}) = 2\int_0^\infty tW(t)\,dt - \left(\int_0^\infty W(t)\,dt\right)^2 \quad \left(= \int_0^\infty (2t - E(\mathcal{T}))W(t)\,dt\right).$$

Proof. From the definition of the expected value we obtain by integration by parts

$$E(\mathcal{T}^k) = \int_0^\infty t^k f(t)\,dt = \lim_{s\to\infty}\left(\Big[t^k F(t)\Big]_0^s - k\int_0^s t^{k-1}F(t)\,dt\right).$$

Exploiting the assumptions on the behavior for t towards infinity and that $E(\mathcal{T}^k) < \infty$, we

find

$$E(\mathcal{T}^k) = \lim_{s\to\infty}\left(\left[t^k\right]_0^s - k\int_0^s t^{k-1}F(t)\,dt\right)$$
$$= \lim_{s\to\infty}\left(k\int_0^s t^{k-1}(1-F(t))\,dt\right) = k\int_0^\infty t^{k-1}W(t)\,dt.$$

□

Appendix B. Well-posedness of the integral equation appearing in reaction kinetics and its numerical solution

Both (13) and (24) come down to the equation

$$\hat{t}_\pi = \int_0^{\bar{t}} \exp\left(-\frac{T_a}{T(t') - T_\infty}\right)\,dt', \tag{B.1}$$

For the numerical solution of this equation, we propose the following algorithm:

Algorithm Appendix B.1. *Determination of failure time in the continuous deterministic approach*

Suppose the time is discretized in small time intervals of length $h > 0$. We consider a sufficiently large $t_{max} > 0$. Let $T(t)$ be given for $t \in [0, t_{max}]$.

1. *Set $i = 0$ and $t_0 = 0$.*
2. *Set $N_i = N(t_i) = t_i h$. Compute $\tilde{t}_\pi(t_i) = \sum_{j=0}^{N(t_i)} \exp\left(-\frac{T_a}{T(t_j)-T_\infty}\right) h$. (Other, e.g. non-equidistant, discretizations of the integral on the right-hand side of (B.1) may be considered as well.)*
3. *Check (i) $\tilde{t}_\pi(t_i) < \hat{t}_\pi$ and (ii) $t_i < t_{max}$:*
   a) *If (i) and (ii) are true, then set $i \to i+1$, increase time to $t_{i+1} = t_i + h$ and go to 2.*
   b) *If (i) is true and (ii) is false, then output error message $t_{krit} > t_{max}$ and terminate. Restart the algorithm with an increased t_{max}, if necessary.*
   c) *If (i) is false and (ii) is true, then set $t = \bar{t}$ and terminate.*
   d) *If (i) and (ii) are false, then set $t = t_{max}$ and terminate.*

We establish the existence and uniqueness for a solution $\bar{t}$ of (B.1), determined by Algorithm B.1 under reasonable assumptions. Moreover, the following lemma yields an estimate for the failure time.

Lemma Appendix B.2. *Well-posedness of the failure time in the continuous deterministic approach*

Let $\hat{t}_\pi > 0$ and $0 < T_{min} \le T(t) \le T_{max}$ for all $t \in [0, \infty)$.

Then Eq. (B.1) has a unique solution

$$\bar{t} \in \left[\hat{t}_\pi \exp(T_a/T_{max}), \hat{t}_\pi \exp(T_a/T_{min})\right]$$

that is found by Algorithm B.1 for sufficiently small $h > 0$.

Proof. For given h, let $\tilde{t}_\pi(t)$ denote a piecewise continuously differentiable approximation of the $\tilde{t}_\pi$ produced by the algorithm.

By definition of $\tilde{t}_\pi(t)$, we have $\tilde{t}_\pi(0) = 0$ and $\tilde{t}_\pi(t)$ is strictly monotone increasing. Furthermore, there holds

$$0 < \exp(-T_a/T_{min}) \leq \tilde{t}'_\pi(t) \leq \exp(-T_a/T_{max})$$

and by integration

$$\exp(-T_a/T_{min})t \leq \tilde{t}_\pi(t) \leq \exp(-T_a/T_{max})t$$

and by the intermediate value theorem the existence and uniqueness follows.Moreover, this illustrates that $\bar{t}$ depends continuously on T_{min}, T_{max}, T_a and $\hat{t}_\pi$. □

In case $T = const$ we recover (11) for $t_\pi = \hat{t}_\pi$. This justifies the definition of $\hat{t}_\pi$ in (24) as done within the continuous deterministic approach.

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