

Joint Chance Constrained Programming for Hydro Reservoir Management

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Abstract In this paper, we deal with a hydraulic reservoir optimization problem with uncertainty on inflows in a joint chance constrained programming setting. In particular, we will consider inflows with a persistency effect, following a causal time series model, and examine the impact of the "Gaussian" assumption for such inflows. We present an iterative algorithm for solving similarly structured joint chance constrained programming problems that requires a Slater point and the computation of gradients. Several alternatives to the joint chance constraint problem are presented. In particular, we present an individual chance constraint problem and a robust model. We illustrate the interest of joint chance constrained programming by comparing results obtained on a realistic hydro-valley with those obtained from the alternative models. Despite the fact that the alternative models often require less hypothesis on the law of the inflows, we show that they yield conservative and costly solutions. The simpler models, such as the individual chance constraint one, are shown to yield insufficient robustness and are therefore not useful. We therefore conclude that Joint Chance Constrained programming appears as a technique offering a good trade-off between cost and robustness and can be tractable for complex realistic models.

Keywords Chance-Constrained-Programming · Hydro-Reservoir-Management · Joint-Chance-Constraints · Stochastic-Inflows

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1 Introduction

An important optimization problem in energy management, known as the "Unit-Commitment Problem", aims at computing the production schedule that satisfies the offer-demand equilibrium at minimal cost. That schedule indicates production levels for each production unit in a hydro-thermal system. Each unit is subject to many complex technical constraints. This fact, together with the fact that the offer-demand equilibrium constraints are coupling constraints and link all these various and numerous units together, leads to the conclusion that the Unit commitment problem is often large-scale and difficult to solve. In order to tackle these large scale problems, the coupling constraints are often dualized, using Lagrangian techniques, leading to an effective price decomposition scheme ([2,6]). In this setting, the hydraulic valley optimization problem can be seen as a sub-problem of the general Unit-Commitment problem. Alternatively, one can interpret this problem as an optimization against market-prices. Complex dynamic constraints on watershed controls introduce combinatorial aspects in this sub-problem, making it difficult to solve. For this reason, uncertainty is often neglected, even though decisions are taken at least one day in advance.

In this paper, we deal with a hydraulic reservoir optimization problem with uncertainty on inflows in a joint chance constrained programming setting. We wish to examine a model wherein uncertainty is an important part of the model. We will however neglect the combinatorial aspect in this paper. Since introducing uncertainty in the global Unit-Commitment problem will lead to a very large scale and complex model, we will focus for the moment on the sub-problem consisting of a hydraulic valley.

Introduced by [1], probability constraints are quite an appealing tool for dealing with uncertainty. In particular, when uncertainty arises in physical constraints, since they also offer a simple interpretation. A classical introduction to the theory and numerical treatment of chance constraints can be found in [11]. In the same monograph, one can find convexity results, for uncertainty separated from decisions and for a large class of distributions including the multivariate Gaussian one. Since their first introduction, chance constraints have become quite common in hydro valley management ([9,3,4,8,10,16,15,14]), but often individual chance constraints are used and not joint chance constraints. Though a very appealing approximation, individual chance constraints unfortunately do not offer sufficient robustness (see [14]).

This paper is organized as follows. In section 2, we present our model for hydro reservoir management, where combinatorial constraints are neglected and random inflows are introduced. We give a detailed description of a real hydro valley, and present the main optimization problem. As the uncertainty on inflows is concerned, many statistical models are based on a deterministic trend (potentially dependent on explanatory variables) and a causal noise process. Since convexity results exist for specific classes of randomness and in particular Gaussian ones, it seems tempting to place ourselves in such a setting. Restricting uncertainty laws to such a setting, might seem restrictive at first. However, we will show that a large class of models is available, i.e., the class of causal time series models with Gaussian innovations ([13]).

In section 3, we derive algorithms for dealing with our model. One difficulty for solving joint chance-constraint models is to dispose of a tractable representation of the gradient of such constraints. This is the case for "Gaussian" rectangles as shown in ([14]). We present here an iterative algorithm for solving, in the Gaussian setting, joint chance constrained programming problems that requires a Slater point and the computation of gradients. Several alternatives to the joint chance constraint problem are then presented. In particular, we present an individual chance constraint problem and a robust model.

In section 4, we report results obtained when solving these various models on a realistic instance of a hydro-valley management problem. The interest of joint chance constrained programming is illustrated by comparing results obtained on this hydro-valley with those obtained from the alternative models.

An algorithmic perspective and some auxiliary lemmas are given in section 5. This section also contains a slight extension of the theoretical results obtained in [14]. Finally, conclusions are drawn in section 6.

2 Problem Description

In this section we will give a description of the hydro reservoir management problem. We will consider a discretized time horizon. To this end let τ denote the set of (homogeneous) time steps. Let Δt be this time step size expressed in hours. Occasionally T will denote the last time step. We will begin by providing problem constraints and the objective function. We will conclude with a paragraph highlighting the structure of the problem.

2.1 Problem Constraints

2.1.1 Topology

A hydro valley can be seen as a set of connected reservoirs and associated turbines. We can therefore represent this with a directed graph. Let \mathcal{N} be the set of nodes and let A (of size $|\mathcal{N}| \times |\mathcal{N}|$) be the connection matrix, i.e., $A_{n,m} = 1$ whenever water released from reservoir n will flow into reservoir m . We will assume that D is the flow duration matrix, i.e., D_m is the amount of time (measured in time steps) it takes for water to flow from reservoir m to its child. Let $\mathcal{T} := \{g^i, i = 1, \dots, N_{\mathcal{T}}\}$ denote the set of turbines and $\mathcal{P} := \{p^i, i = 1, \dots, N_{\mathcal{P}}\}$ denote the set of pumping stations. We furthermore introduce the mapping $\sigma_{\mathcal{T}} : \{1, \dots, N_{\mathcal{T}}\} \rightarrow \mathcal{N}$ ($\sigma_{\mathcal{P}} : \{1, \dots, N_{\mathcal{P}}\} \rightarrow \mathcal{N}$) attributing to each turbine (pumping station) the reservoir number to which it belongs. We will also introduce the sets $\mathcal{A}(n) = \{m \in \mathcal{N} : A_{m,n} = 1\}$ and $\mathcal{F}(n) = \{m \in \mathcal{N} : A_{n,m} = 1\}$. The set $\mathcal{A}(n)$ is empty for top reservoirs and the set $\mathcal{F}(n)$ for bottom reservoirs.

2.1.2 Controls

We will assume that each turbine (and pumping station) can be controlled for each time step. To this end we introduce the variables $x^i(t)$ for each $t \in \tau$ and $i = 1, \dots, N_{\mathcal{T}}$. In a similar way we introduce the variables $y^i(t)$ for the pumping stations. The units are in cubic meters per hour, i.e., m^3/h . Furthermore we assume that each of these variables are bounded from below by zero and from above by \bar{x}^i (\bar{y}^i respectively).

2.1.3 Water Values

For water-values we will present 2 models. One of these models provides a very detailed cost structure allowing us to integrate inter-temporal preferences. These could enforce withholding turbinning on usually calm periods and save this water for usually more perturbed periods. This model could be of more interest for larger time horizons. We will compare this last formulation with a third possible formulation. Unfortunately the latter model adds non-convexity and hence will be discarded. In order to show the difference between these formulations, we will expose the main ideas by placing ourselves in the setting of a single reservoir with a single compartment.

The first model values (the differential of) volume with a water value w , i.e., we value $w(V(T) - V(0))$, the latter term being a constant, it can be neglected. So it comes down to the valuation of the end-volume:

$$wV(T). \quad (1)$$

A natural extension of this first model, is the valuation of volume at each time step:

$$\sum_{t \in \tau} w_t V(t). \quad (2)$$

The effect will be that the model will favor periods, wherein water values are high with respect to periods where they are globally lower. As such the model will be incited to keep water for later troubled periods. This model therefore satisfies the above requirement of allowing us to integrate inter-temporal preferences. Whenever compartments are added, this model is convex, just like the previous model (Details are given in section 2.1.5 below).

This last model formulation can be compared with one wherein we would value the valuation of volume changes:

$$\sum_{t \in \tau} w_t (V(t) - V(t-1)). \quad (3)$$

By removing the valuation of the initial volume $w_0 V_0$ from (1) and rewriting, one can obtain the following identity

$$w_T V_T - w_0 V_0 = \sum_{t \in \tau} w_t (V(t) - V(t-1)) + \sum_{t \in \tau} (w_t - w_{t-1}) V(t-1),$$

showing that whenever changes in valuation induced by changes in water-values are compensated for, we end up with the simple above model. One can obtain model (3) from (2) by simply subtracting the current valuation of the past volume. This shows that one can move from one model to another by compensating for various features. Unfortunately, when adding compartments, model (3) will yield a non-convex model (Details can be found in section 2.1.4). Model (2) will give a simple extension of model (1). Moreover, similarly to model (3), it will allow us to incorporate temporal effects within the optimization. Since the model is convex, we believe that we should retain this model instead of the non-convex one.

2.1.4 Time and volume dependent water values

Our aim is to set up a model in which volumetric difference of adjacent time steps are valued with a water value that depends on time and volume. The goal is two-fold. On one hand, we wish to value the final volume, in order to prevent optimization to be carried out at the expense of later time periods. On the other hand, water-values may reflect temporal preferences, such as withholding turbinning on usually calm periods for more perturbed periods. This is of particular interest when the time horizon of the model increases and reflects, say a month. In this section we will show that model (3) is non-convex. We therefore suggest using model (2) instead, which preserves the above highlighted features and is a simple extension of the model detailed in section 2.1.5 below.

We subdivide the levels of each reservoir into a finite number of values from bottom to top:

$$V_0^n, \dots, V_{K_n}^n \quad \forall n. \quad (4)$$

Each compartment $[V_{i-1}^n, V_i^n)$ is assigned a water value $W_i(t)^n$ (in $\text{€}/m^3$), $i = 1, \dots, K_n$. These are assumed to be decreasing:

$$W_{i-1}^n(t) > W_i^n(t) \geq 0 \quad \forall n \quad \forall i = 1, \dots, K_n \quad \forall t. \quad (5)$$

We will reason with respect to changes in expected volume $\mathbb{E}(V^n(t))$ induced by turbinning and pumping at time $t \in \tau$. These changes will be valued against the water values.

For reservoir n , let $i_t^* = \max\{i \mid \mathbb{E}(V^n(t)) \geq V_i^n\} + 1$. Let $z_{x,i}^n(t)$ represent negative variations of volume with respect to expected volume $\mathbb{E}(V^n(t))$ (i.e., turbinning) and let $z_{y,i}^n(t)$ denote positive variations (i.e., pumping), where these variations are restricted to the interval $[V_{i-1}^n, V_i^n)$. This means:

$$V^n(t + \Delta t) = V^n(t) + \sum_{i=1}^{K_n} (z_{y,i}^n(t) - z_{x,i}^n(t)).$$

Since $z_{x,i}^n(t)$ is defined as the negative variations with respect to the expected volume $\mathbb{E}(V^n(t))$, one has $z_{x,i}^n(t) = 0$ for all $i > i_t^*$. Similarly one obtains $z_{y,i}^n(t) = 0$ for all $i < i_t^*$. Moreover, one has

$$z_{x,j}^n(t) > 0 \Rightarrow z_{x,i}^n(t) = \min \{V_i^n, \mathbb{E}(V^n(t))\} - V_{i-1}^n \quad \forall j < i \leq i_t^* \quad (6)$$

$$z_{y,j}^n(t) > 0 \Rightarrow z_{y,i}^n(t) = V_i^n - \max \{V_{i-1}^n, \mathbb{E}(V^n(t))\} \quad \forall j > i \geq i_t^*. \quad (7)$$

Therefore, the following term values this water differential:

$$\sum_{i=1}^{K_n} W_i^n(t)(z_{x,i}^n(t) - z_{y,i}^n(t)). \quad (8)$$

The following constraints

$$\begin{aligned} 0 &\leq z_{x,i}^n(t) \leq \max(\mathbb{E}(V^n(t)) - V_{i-1}^n, 0) \quad \forall i \quad \forall n \\ 0 &\leq z_{y,i}^n(t) \leq \max(V_i^n - \mathbb{E}(V^n(t)), 0) \quad \forall i \quad \forall n, \end{aligned}$$

make sure that $z_{x,i}^n(t) = 0$ for $i \geq i_t^* + 1$, and $z_{y,i}^n(t) = 0$ for $i \leq i_t^* - 1$. One can see that the requirements (6) and (7) are automatically satisfied by hypothesis (5) and objective function (8).

Now these constraints can be transformed into quadratic constraints by applying Lemma 3¹. The same lemma also allows us to conclude the non-convexity of this model (3).

2.1.5 Volume dependent water values

Our aim is to set up a model which evaluates the expected amount of water in the reservoir at the end of the optimization horizon². This is necessary in order not to carry out the optimization at the expense of later periods of time. A possible way to do so is to subdivide the levels of each reservoir into a finite number of values from bottom to top as in (4).

Each compartment $[V_{i-1}^n, V_i^n]$ is assigned a water value W_i^n (in €/m³) such that

$$W_{i-1}^n > W_i^n \geq 0 \quad \forall n \forall i = 1, \dots, K_n. \quad (9)$$

The value of the expected final water level $\mathbb{E}(V^n(T))$ of reservoir n is then simply the cumulative value of water in the compartments below:

$$\sum_{i \leq i^*} W_i^n (V_i^n - V_{i-1}^n) + W_{i^*}^n (\mathbb{E}(V^n(T)) - V_{i^*}^n), \quad i^* := \max\{i | \mathbb{E}(V^n(T)) \geq V_i^n\}.$$

Note that this value is an increasing function of the expected final level $\mathbb{E}V^n(T)$ despite the fact that water values are strictly decreasing from bottom to top.

Now, in order to avoid combinatorial arguments concerning the index i^* , we introduce auxiliary variables z_i^n indicating for each reservoir n the amount of water in compartment $[V_{i-1}^n, V_i^n]$. Of course, since all compartments have to be completely filled up to i^* , one has that

$$z_i^n = \begin{cases} V_i^n - V_{i-1}^n & i = 1, \dots, i^* \\ \mathbb{E}(V^n(T)) - V_{i^*}^n & i = i^* + 1 \\ 0 & i = i^* + 2, \dots, K_n \end{cases} \quad \forall n. \quad (10)$$

¹ In section 2.1.7 we will derive a linear equation for $\mathbb{E}(V^n(t))$ in an extended problem vector x

² In practice, one would evaluate the difference of the final and initial volume. The latter adds a constant to the objective function and can theoretically be omitted. In practice, it may generate some numerical difficulties, especially when large volumes are valued and turbinning/pumping capacity is small compared to the volume. In that case, relative changes in valuation induced by the controls are easily considered negligible. Moreover, the constant can easily be added.

Then, the value of the final water level in reservoir n equals

$$\sum_{i=1}^{K_n} W_i^n z_i^n \quad \forall n. \quad (11)$$

We claim that the relations (10) for variables z_i^n can be replaced by the following relations in which the crucial index i^* is absent:

$$\sum_{i=1}^{K_n} z_i^n = \mathbb{E}(V^n(T)) - V_0^n \quad \forall n \quad (12)$$

$$0 \leq z_i^n \leq V_i^n - V_{i-1}^n \quad \forall n \forall i = 1, \dots, K_n. \quad (13)$$

The argument is as follows: as part of the overall objective function in our problem as introduced in (18) below, we shall maximize the value of the final water level (11). Given the strictly decreasing order of water levels in (9) (from bottom to top), it is clear from (12) that the upper inequality in (13) will be satisfied as an equality as long as possible and that only the most upper compartment may not be completely filled. This of course is equivalent with (10) but avoiding the explicit description of that most upper compartment.

Since the initial volume $V^n(0)$ is known in advance, one can define variables $z_{0,i}^n$ in a similar way as z_i^n . It then follows that

$$\sum_{i=1}^{K_n} W_i^n (z_{0,i}^n - z_i^n)$$

is the cost of used water. The valuation induced by $\sum_{i=1}^{K_n} W_i^n z_{0,i}^n$ is in fact a constant and can be omitted.

2.1.6 Random Inflows

We will assume that inflows (in m^3/h) in reservoirs are the result of some stochastic process. Let $A^n(t)$ denote this stochastic process for reservoir n . Not all reservoirs will have stochastic inflows, some of them will have deterministic inflows. This can be explained by the fact that top reservoirs have random inflows due to the melting of snow in the high mountains, whereas rain can be neglected for lower reservoirs. Let $\mathcal{N}^r \subseteq \mathcal{N}$ denote the set of reservoirs receiving random inflows. We will assume that the stochastic inflow process is the sum of a deterministic trend s_t^n and a causal process ([13]) generated by Gaussian innovations. To this end let $\zeta^n(t)$ be a Gaussian white noise process, where $(\zeta^{k_1}(t), \dots, \zeta^{k_l}(t))$ is a Gaussian random vector of zero average and variance-covariance matrix $\Sigma(t)$ ($\{k_1, \dots, k_l\} = \mathcal{N}^r$). We will assume independence between time steps of the ζ vector. Since $A^n(t)$ is a causal process, we can write it as follows

$$A^n(t) = s_t^n + \sum_{j=0}^{\infty} \psi_j^n \zeta^n(t-j) = s_t^n + \sum_{j=t}^{\infty} \psi_j^n \zeta^n(t-j) + \sum_{j=0}^{t-1} \psi_j^n \zeta^n(t-j),$$

for some coefficient vector ψ^n . We will assume that randomness before $t = 0$ is known and as such we can assume w.l.o.g. that the random inflow process can be written as

$$A^n(t) = s_t^n + \sum_{j=0}^{t-1} \psi_j^n \zeta^n(t-j). \quad (14)$$

2.1.7 Flow constraints and Volume bounds

Each reservoir is subject to flow constraints induced by pumping and turbinning. The following equilibrium constraint applies

$$\begin{aligned} V^n(t) = & V^n(t-1) + \sum_{m \in \mathcal{A}(n)} \sum_{i \in \sigma_{\mathcal{T}}^{-1}[m]} x^i(t - D_m) \Delta t - \sum_{i \in \sigma_{\mathcal{T}}^{-1}[n]} x^i(t) \Delta t \\ & + \sum_{m \in \mathcal{F}(n)} \sum_{i \in \sigma_{\mathcal{P}}^{-1}[m]} y^i(t) \Delta t - \sum_{i \in \sigma_{\mathcal{P}}^{-1}[n]} y^i(t) \Delta t + s_t^n \Delta t + \sum_{j=0}^{t-1} \psi_j^n \zeta^n(t-j) \Delta t. \end{aligned} \quad (15)$$

The above equation is entirely deterministic except for the reservoirs $n \in \mathcal{N}^r$. In order to deal with this randomness and reservoir bounds we will therefore add the following constraints

$$\mathbb{P}[V_{min}^n(t) \leq V^n(t) \leq V_{max}^n(t) \forall t \in \tau, n \in \mathcal{N}^r] \geq p \quad (16)$$

$$V_{min}^n(t) \leq V^n(t) \leq V_{max}^n(t) \forall t \in \tau, n \in \mathcal{N} \setminus \mathcal{N}^r, \quad (17)$$

where \mathbb{P} is a probability measure and p a security level. Constraint (16) is a joint chance constraint. This means that we wish to satisfy all linear inequalities of the stochastic system simultaneously with high enough probability. This can be compared to a model with individual chance constraints, which is a model wherein we wish to satisfy each inequality with high enough probability, but taken separately. We will see in this paper that the latter model offers insufficient robustness.

2.2 Objective function

Often, in reality, each reservoir only has a single turbine. The power output of turbinning x , in cubic meters per second m^3/s , is given by a function $\rho(x)$. This function is strictly increasing and concave, i.e., $\rho'(x) \geq 0$ and $\rho''(x) \leq 0$. In our model we have split this range into several subsections (hence several turbines), each with efficiency $\rho_i = \rho'(s_i^*)/3600$ (MWh/m^3) for some s_i^* in each section. We can thus remark that for any two turbines i_1 and i_2 belonging to the same reservoir we have $\rho_{i_1} \geq \rho_{i_2}$ whenever $i_1 \leq i_2$. This approximation comes down to approximating $\rho(x)$ by a piece-wise linear function.

We assume given a time dependent price signal $\lambda(t)$ (in €/MWh). The following objective function has to be minimized, when considering the model (12),(13):

$$\sum_{n \in \mathcal{N}} \sum_{i=1}^{K_n} (W_i^n (z_{0,i}^n - z_{F,i}^n) - \sum_{t \in \tau} \lambda(t) \Delta t (\sum_{i=1}^{N_{\mathcal{T}}} \rho_i(t) x^i(t) - \sum_{i=1}^{N_{\mathcal{P}}} \frac{1}{\theta_i(t)} y^i(t))), \quad (18)$$

where the first part corresponds to the cost of using water expressed by the water-values, and $\theta^i(t)$ is the efficiency of pumping.

2.3 Matrix formulation

In this section we show that (16) can be written as bilateral joint chance constraint. This means that the model we are interested in is a bilateral joint chance constrained program with linear objective function and some polyhedral constraints.

We define the following general matrix mappings \mathfrak{D}^n and \mathfrak{C}^n as follows. For any matrix Ψ and any sequence of matrices $\psi := (\psi_0, \dots, \psi_{n-1})$ the matrix $\mathfrak{D}^n(\Psi)$ is the block diagonal matrix containing n

copies of Ψ , whereas

$$\mathfrak{C}^n(\psi) = \begin{pmatrix} \psi_0 & 0 & 0 & \cdots & 0 \\ \psi_0 + \psi_1 & \psi_0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \sum_{j=0}^{|\mathcal{T}|-1} \psi_j & \cdots & \cdots & \cdots & \psi_0 \end{pmatrix}.$$

We furthermore define \mathfrak{I}^k as the k dimensional identity matrix and we extend the definition of \mathfrak{C}^n to a sequence ψ shorter than n by appending with zero matrices.

Let T be the $|\mathcal{N}| \times N_{\mathcal{T}}$ matrix with

$$T_{ni} = \begin{cases} 1 & \text{if } i \in \sigma_{\mathcal{T}}^{-1}[n] \forall n \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

and let P be defined as

$$P_{ni} = \begin{cases} 1 & \text{if } i \in \sigma_{\mathcal{P}}^{-1}[n] \forall n \in \mathcal{N} \\ 0 & \text{otherwise.} \end{cases}$$

Let x be the vector containing all $x^i(t)$ ranged over i first.

The following cumulated flows leave each node for each time step, counted from the beginning

$$\mathfrak{D}^{|\mathcal{T}|}(T)\mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{T}}})x$$

Let $R_s(d)$ be the right shift matrix over size d , i.e., $R_s(d)_{i,j} = \mathbb{1}_{i-d=j}$. Now define the rearrangement matrix $(nm \times nm)$ $\mathfrak{R}^{n,m}$ as $\mathfrak{R}_{i,j}^{n,m} = \mathbb{1}_{j=(\text{mod}(i-1,n)m+1+\text{div}(i-1,n))}$. We remark that $\mathfrak{R}^{m,n}$ is the inverse of $\mathfrak{R}^{n,m}$. Let R_s be the diagonal block matrix containing $|\mathcal{N}|$ blocks, where the n th block is $R_s(D_n)$. Now

$$\mathfrak{D}^{|\mathcal{T}|}(A)^{\top} \mathfrak{D}^{|\mathcal{T}|}(T) \mathfrak{R}^{N_{\mathcal{T}},|\mathcal{T}|} R_s \mathfrak{R}^{|\mathcal{T}|,N_{\mathcal{T}}} \mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{T}}})x$$

are the flows arriving in each reservoir for each time step.

For pumping these are respectively $\mathfrak{D}^{|\mathcal{T}|}(A)^{\top} \mathfrak{D}^{|\mathcal{T}|}(P)\mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{P}}})y$ leaving and $\mathfrak{D}^{|\mathcal{T}|}(P)\mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{P}}})y$ arriving. We define $\Pi_n^{\mathcal{T}} = (0, \dots, 0, I, 0, \dots, 0) \mathfrak{R}^{|\mathcal{T}|,|\mathcal{N}|}$, where the identity matrix I is in the n th position as the reservoir contraction matrix ($\Pi_n^{\mathcal{P}}$ is defined similarly). Moreover, we define the matrix C^z as $(|\mathcal{N}| |\mathcal{T}| \times |\mathcal{T}| \sum_{n \in \mathcal{N}} K_n)$ by setting $C_{|\mathcal{N}|(i-1)+j,k}^z = \mathbb{1}_{k-(i-1)\sum_{n \in \mathcal{N}} K_n - \sum_{n=1}^{j-1} K_n \in \{1, \dots, K_j\}}$.

Equation (15) can be written as

$$\begin{aligned} V_n &= V_0 + \Delta t \Pi_n^{\mathcal{T}} (\mathfrak{D}^{|\mathcal{T}|}(A)^{\top} \mathfrak{D}^{|\mathcal{T}|}(T) \mathfrak{R}^{N_{\mathcal{T}},|\mathcal{T}|} R_s \mathfrak{R}^{|\mathcal{T}|,N_{\mathcal{T}}} \mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{T}}})x - \mathfrak{D}^{|\mathcal{T}|}(T)\mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{T}}})x) \\ &\quad + \Delta t \Pi_n^{\mathcal{P}} (\mathfrak{D}^{|\mathcal{T}|}(P)\mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{P}}})y - \mathfrak{D}^{|\mathcal{T}|}(A)^{\top} \mathfrak{D}^{|\mathcal{T}|}(P)\mathfrak{C}^{|\mathcal{T}|}(\mathfrak{I}^{N_{\mathcal{P}}})y) \\ &\quad + \Delta t \mathfrak{C}^{|\mathcal{T}|}(1)s + \Delta t \mathfrak{C}^{|\mathcal{T}|}(\psi^n)\zeta^n, \end{aligned} \tag{19}$$

where s is the vector formed from the deterministic trend s_t^n of equation (14).

Equations (12),(13) can be written easily in linear form by extracting the last line from equation (19) without the term in ζ^n . In the case of time dependent water values and model (2), several lines have to be extracted, but equations are easily formed.

3 Models for dealing with uncertainty

In this section we will provide our main model, which is a joint chance constrained programming problem (JCCP). We will also provide several alternative models. In a later section these models will be compared in a numerical experiment. From now on, we will assume that we are using water values that are only dependent on the volume.

3.1 A Joint Chance Constraint Model (JCCP)

When combining equations (12), (19) and relations (13), (17) we know that we can find some extended decision vector (also noted $x \in \mathbb{R}^n$) containing (x, y, z) and some matrix A , vector b such that the system $Ax \leq b$ models all these deterministic constraints (including bounds on x).

Finally by combining equation (19) with (16), we can see that the above problem can be cast into the following form, where η is a Gaussian random vector with variance-covariance matrix Σ and zero mean (we have explicitly extracted the non-zero average in eq.(19)):

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & p \leq \mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x]. \end{aligned} \quad (20)$$

In fact the feasible set of (20) is convex due to the Gaussian character of η and a theorem by Prékopà ([11]). This makes the previous optimization problem a convex one. Let $\varphi(x)$ denote the chance constraint of problem (20).

In order to solve problem (20) we use the cutting planes algorithm of which convergence in a finite number of steps is a well-known result. We repeat the algorithm for completeness.

1. Let x_0 be the solution of (22), x_s a Slater point for (20). Set $A_0 = A$, $b_0 = b$ and $k = 0$ and pick some tolerance tol , e.g., $tol = 10^{-2}$.
2. Find λ^* such that $x_k^* = (1 - \lambda^*)x_k + \lambda^*x_s$ and $|\varphi(x_k^*) - p| < \varepsilon$.
3. Add constraint $-\nabla\varphi(x_k^*)^\top x \leq -\nabla\varphi(x_k^*)^\top x_k^*$ to the matrix.
4. Solve

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & A_k x \leq b_k \end{aligned}$$

to find x_{k+1} .

5. If $\frac{c^\top(x_k^* - x_{k+1})}{c^\top x_{k+1}} < tol$ then stop, else move $k = k + 1$ and go in step 2.

For the previous algorithm to function we require a Slater point, i.e., some x_s such that $Ax_s \leq b$, and $\varphi(x_s) > p$. It can be obtained by solving the "max-p" problem (see section 3.2). Moreover, we should be able to efficiently evaluate φ and $\nabla\varphi$. As shown in corollary 1 below and theorem 1 of [14], evaluating the gradient can be analytically reduced to computing function values in lesser dimension. Finally computing function values such as $\varphi(x)$ can be done by using the code of Genz ([5]). Evaluating φ and $\nabla\varphi$ requires $2n + 1$ calls to Genz' code.

3.2 Max-P Problem

We define the "max-p" problem as the following optimization problem:

$$\begin{aligned} \max_{x \geq 0} \quad & \varphi(x) := \mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x] \\ & Ax \leq b. \end{aligned} \quad (21)$$

Clearly any solution x_s of the previous problem with objective function value strictly bigger than p is a Slater point for problem (20). This "max-p" problem is not only an auxiliary problem for obtaining Slater points, but can also be interpreted as the problem of a decision-maker looking for maximum robustness, regardless of the costs. As a matter of fact if the optimal solution of (21) is strictly below one, then almost surely satisfying the "random" physical constraints (16) is not possible. The "max-p" problem therefore also provides us with information on the maximum robustness level p that is "reasonable".

3.3 Expectation model

In the deterministic variant of (20) we assume that constraint (16) is satisfied in expectation, i.e., $\eta = \mathbb{E}(\eta) = 0$ in that variant. This gives:

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & -A^r x \leq b^r \\ & A^r x \leq -a^r. \end{aligned} \tag{22}$$

This model can be identified with the model considered in a classical deterministic unit commitment setting. The chance constrained model can be seen as an extension of this model since it takes into account the available stochastic information on the distribution of randomness, whereas this simple model only uses a single parameter. The following Lemma shows that any feasible solution of (20) is feasible for (22). Physically this can be explained by the fact that a "robust" control has to work well in the average situation.

Lemma 1 *Assume that $p > 0.5$ and that η is a symmetric law. The feasible set of (20) is contained in the feasible set of (22). As a consequence the optimal value of (22) is lower than that of (20).*

Proof Assume that x is not feasible for (22), for instance not $a + A^r x \leq 0$, i.e., there is at least one strictly positive component. By rearranging we may assume that this is the first one. Now

$$\mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x] \leq \mathbb{P}[a^r + A^r x \leq \eta] \leq \mathbb{P}[e_1^\top (a^r + A^r x) \leq e_1^\top \eta] \leq \mathbb{P}[0 < \xi] < 0.5,$$

where ξ is some centered one dimensional Gaussian random variable, and e_1 is a standard unit-vector. This shows that x can't be feasible for (20).

One should mention here that this expectation model is a simple linear program. It is therefore much easier to solve than problem (20). Despite this fact and the fact that it yields solutions with low optimal values, it will be shown later in this paper that the solutions are useless since they violate constraints almost surely.

3.4 Individual Chance Constraint Model (ICCP)

We consider a simplification of the joint chance constrained model (20) by transforming each stochastic inequality into individual chance constraints of type $\mathbb{P}[\langle a_1, x \rangle \leq b] \geq p$ and $\mathbb{P}[b \leq \langle a_2, x \rangle] \geq p$ for well chosen vectors a_1, a_2 and some random variable b . An exact formulation is:

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & \mathbb{P}[e_i^\top (a^r + A^r x) \leq \eta_i] \geq p \quad \forall i \\ & \mathbb{P}[\eta_i \leq e_i^\top (b^r + A^r x)] \geq p \quad \forall i, \end{aligned} \tag{23}$$

where e_i is the i -th standard unit vector.

As a matter of fact, model (23) can be reduced to a simple linear program since the inverse of $F_{\eta_i}(z) = \mathbb{P}[\eta_i \leq z]$ can be evaluated easily. It also offers improved robustness with respect to the expectation model (22) that offered none. However it can't guarantee a probability level of p for the whole stochastic inequality system and therefore offers far less robustness as the joint model (20) ([14]). This will again become apparent in the numerical experience.

3.5 A Robust Model

We would like to identify an uncertainty set \mathcal{E}_p for our random inflow process η in such a way that the probability of η falling in this set is approximately p . We will then enforce the constraints to hold for all η in this set. In fact, we will use a specific ellipsoidal form for the uncertainty set and show that the above optimization problem then boils down to a linear problem, once two conic quadratic problems have been solved.

In order to determine \mathcal{E}_p , let $LL^\top = \Sigma$ be the Cholesky decomposition of Σ . Let $y = L^{-1}\eta$ and assume that we dispose of a statistical estimate of $\mathbb{E}(y_i^4)$ (in the Gaussian case these are known exactly). Whenever the law of η is unknown, we can use the variance covariance matrix Σ by statistically estimating it. By construction, y is uncorrelated, we will make the (wrong) approximation that this is the same as independence. Now by the Lindeberg-Feller Central Limit Theorem ([12]) we obtain that

$$y^\top y \approx \mathcal{N}(n, \sigma_C),$$

with $\sigma_C = \sqrt{\sum_{i=1}^n \mathbb{E}(y_i^4) - n}$. We now define $\mathcal{E}_p = \{x : x^\top \Sigma^{-1} x \leq n + \Phi^{-1}(p)\sigma_C\}$. It follows in the case that η follows a multivariate Gaussian law that $\mathbb{P}[x \in \mathcal{E}_p] = p$. This will be true approximately in the general case.

We therefore consider the following robust version of problem (20):

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & a^r + A^r x \leq \inf \mathcal{E}_p \\ & b^r + A^r x \geq \sup \mathcal{E}_p, \end{aligned} \tag{24}$$

where $\inf \mathcal{E}_p$ is defined to be the largest vector x^i in \mathbb{R}^n having $x^i \leq y \forall y \in \mathcal{E}_p$ ($\sup \mathcal{E}_p$ is defined similarly). Both $\inf \mathcal{E}_p$ and $\sup \mathcal{E}_p$ are solutions of a conic quadratic optimization problem. Indeed this model is equivalent with

$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & a^r + A^r x \leq \xi \leq b^r + A^r x \forall \xi \in \mathcal{E}_p. \end{aligned}$$

Since this model (24) basically looks at the smallest rectangle containing \mathcal{E}_p and requires satisfaction of constraints for all elements in the rectangle, one could also look at alternatively ways to obtain such a rectangle. Basically, we are looking for some $\underline{\eta}$ and $\bar{\eta}$ such that $\mathbb{P}[\underline{\eta} \leq \eta \leq \bar{\eta}] \approx p$. These would then give better bounds than $\inf \mathcal{E}_p$ and $\sup \mathcal{E}_p$ as above, since in general $\mathbb{P}[\inf \mathcal{E}_p \leq \eta \leq \sup \mathcal{E}_p] > p$. In the Gaussian case considered here we can exactly evaluate the probabilistic contents of such rectangles and hence fine-tune the rectangle. Clearly any feasible solution of problem (20) will also provide such vectors. This last way of obtaining those vectors offers no computational advantage to (24) other than prematurely ending the algorithm that solves (20). An alternative would be to take some $q < p$, such that $\mathbb{P}[\inf \mathcal{E}_q \leq \eta \leq \sup \mathcal{E}_q] \approx p$. This is computationally not intensive, but requires evaluations of probabilistic contents. In order to investigate the impact of the choice of this rectangle we have made some runs with model (24) wherein the rectangle was made to fit perfectly. In practice, we have obtained $\underline{\eta}$ and $\bar{\eta}$ by taking some ad-hoc convex combinations between the Slater point and the solution of (22). These results will be referred to as Robust-Calibrated (Robust-Calib) or (24)-Calib.

4 Numerical Example

In this section we consider a numerical example from the industry. The instance size is moderate but realistic. The nominal inflows are considered constant through time. Finally, the water values are not

assumed to depend on the volume, and thereby correspond to the V_0 level. We will consider 24 time steps of 2 hours each. Figures 1 and 2 show further data of our example. This implies the following dimensions for our problem: the Gaussian vector dimension is 48, the decision vector has 700 elements and the polyhedral constraints are defined by about 1000 linear inequalities.

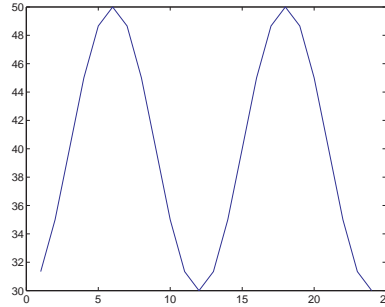


Fig. 1 The price signal

We assume that the innovations of the inflows have a standard deviation equal to 20% of the nominal values, whenever they are considered random ($0.3 \text{ m}^3/\text{s}$ for reservoir 2). In a first stage we consider that inflows follow an AR(1) process with coefficient 0.9. We assume that inflows are random on the first two reservoirs. The required probability level is taken to be 0.8. A second instance is one wherein we assume that inflows on reservoir 2 follow an AR(3) process with coefficients (0.9, 0.7, -0.7).

Solving the problems introduced in Section 3, we obtain the results as given in Table 1 and figures 3 and 4. We have set a tolerance of 10^{-2} for the cutting planes algorithm for joint chance constrained programming. It should however be stated that the true optimal solution of problem (20) for instance 2 gives a cost, only 0.6% away from the deterministic cost. Indeed the price of chance-constrained robustness is cheap here.

Table 1 shows optimal costs and number of violations. In order to compute the latter information, we have made an a posteriori check of empirical probabilities by generating 100 scenarios and counting the number of violations. The volume trajectories resulting from these scenarios are shown in figure 3. Clearly we observe the advantage of using joint chance constrained programming. The additional cost with respect to the deterministic solution is only small, but robustness can be fine tuned. A full robust solution turns out quite costly. Finally individual chance constrained programming can not be used to mimic joint chance constraints as we have no control over the number of violations over a period of time.

Table 1 Comparison of costs and number of violations

Inst.	Item / Problem	Det (22)	JCCP (20)	ICCP (23)	Robust (\mathcal{E}_p) (24)-1	Robust (Calib) (24)-Calib	MaxP (21)
1	nbViolation	100	20	29	0	1	0
1	Cost (€)	$-1.0478e^5$	$-1.0395e^5$	$-1.0443e^5$	$-1.0355e^5$	$-1.0099e^5$	$-9.9176e^4$
2	nbViolation	100	20	35	4	21	2
2	Cost (€)	$-1.0478e^5$	$-1.0340e^5$	$-1.0422e^5$	$-1.0282e^5$	$-1.0251e^5$	$-9.9176e^4$

When comparing the turbined volumes in fig.4, one can observe that they are quite similar for most solutions (except for max-p which does not see the cost vector and is hence only incited to turbine if this allows us to improve robustness) and most reservoirs, except for "Saut Mortier". The solution (20) turbines a bit less in the beginning to avoid violations in time steps (8-10), a bit more during time steps 12-15 to avoid violations there and stops earlier to avoid violations for the last time steps. Solution (23) offers an intermediate solution. The solution (24) heavily increases turbining during steps 10-15 and drastically reduces during steps 15-20 for additional robustness. Indeed, even though the uncertainty \mathcal{E}_p

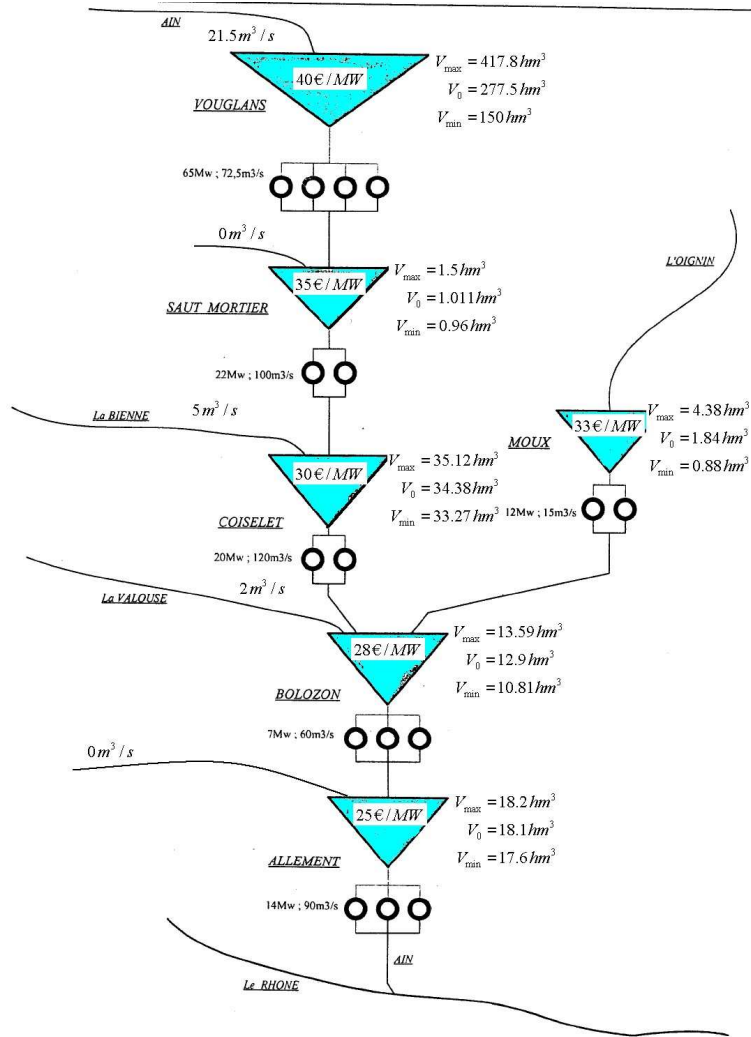


Fig. 2 The hydro Valley

is very well calibrated, the solution is over-robust. Unfortunately for larger values of p (in fact $p > 0.85$) this will lead to an empty feasible set of problem (24), whereas solutions of (20) can be found. It also shows the difficulty of getting the robust rectangle well calibrated for problem (24)-Calib. Indeed, even though the rectangle is calibrated to give exactly the same probabilistic contents in both instances, one gives over-robust results (3.6% away from deterministic solution), whereas the other gives more reasonable results as the number of violations is concerned, but still at a large cost (2.2 % away from deterministic solution).

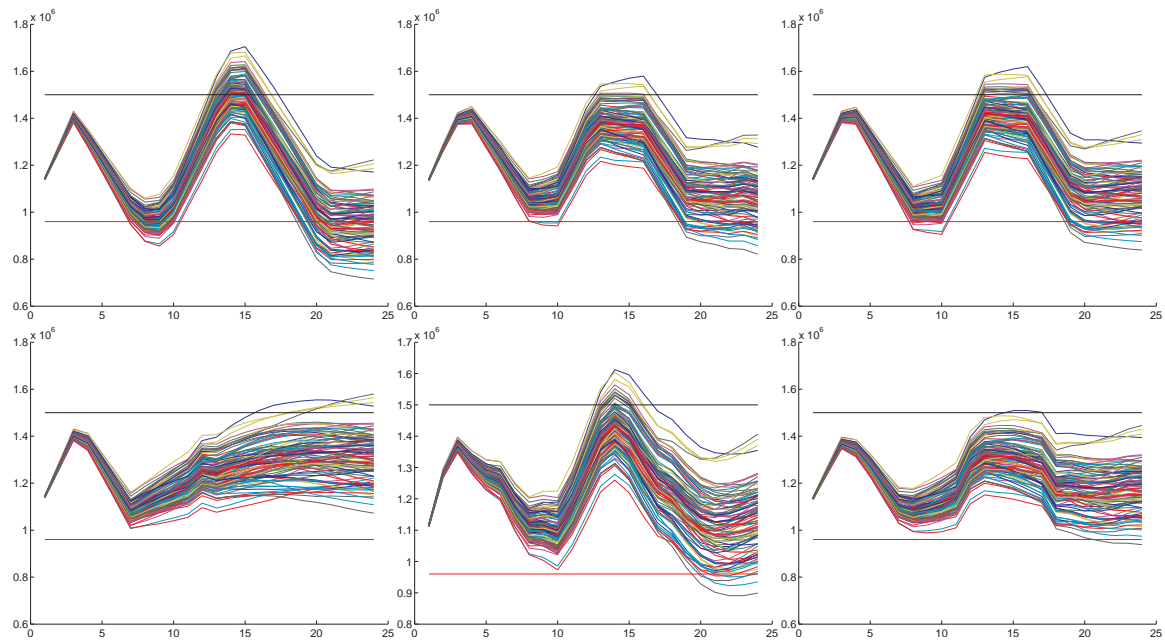


Fig. 3 Trajectories of filling levels in reservoir "Saut Mortier" and instance 2 for 100 simulated inflow scenarios. From top left to bottom right, solutions of problems (22), (20), (23), (24), (24)-Calib and (21)

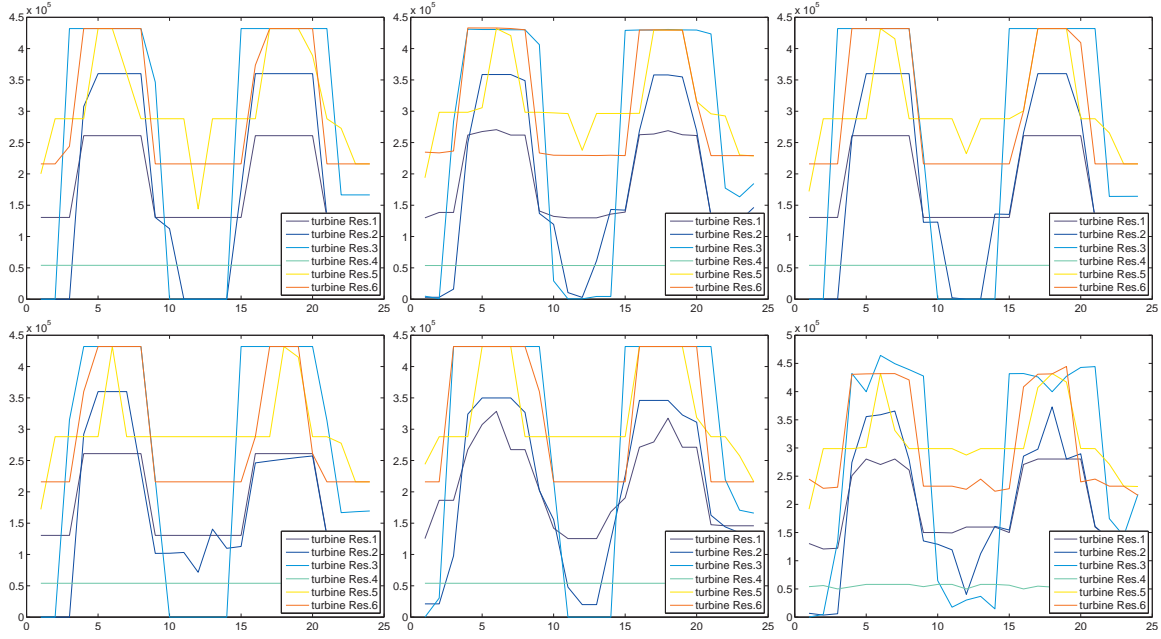


Fig. 4 Turbined volumes (m^3) for instance 2. From top left to bottom right, solutions of problems (22), (20), (23), (24), (24)-Calib. and (21)

5 Algorithmic perspectives / Second order methods

If one is interested in applying second order solution methods to increase the efficiency of the solution process, one has to work out second derivatives of the probability function φ on the basis of the gradients obtained in theorem 1 of [14]. This is done in the following lemma.

Lemma 2 *Let ξ be an n -dimensional Gaussian random vector with mean μ and variance-covariance matrix Σ . We define the mapping $F_\xi(a, b) = \mathbb{P}[a \leq \xi \leq b]$ for any rectangle, i.e., $a \leq b$. Let D_n^i denote the n -th order identity matrix from which the i th row has been deleted. For each $y \in \mathbb{R}^n$, $1 \leq i \leq n$ and $z \in \mathbb{R}$ we define $y^{c^{i,n}(z, \Sigma_i)} = D_n^i(y + \Sigma_{i,i}^{-1}(z - y_i)\Sigma_i) \in \mathbb{R}^{n-1}$, where Σ_i is the i th column of Σ . We will occasionally abbreviate this with $y^{c_1^i(z)}$. We also define*

$$y^{c_2^{i,j}(z, w)} = (y^{c_1^{i,n}(z, \Sigma_i)})^{c_2^{j, n-1}(w, \Sigma_j^{c_n(i)})},$$

where we have defined $\Sigma^{c_n(i)} = D_n^i(\Sigma - \Sigma_{i,i}^{-1}\Sigma_i\Sigma_i^\top)(D_n^i)^\top$. We define $\xi^{c_1^i(z)}$ as the Gaussian random variable with mean $\mu^{c_1^i(z)}$ and covariance matrix $\Sigma^{c_n(i)}$. In a similar way, we define $\xi^{c_2^{i,j}(z, w)}$ as the Gaussian random variable with mean $\mu^{c_2^{i,j}(z, w)}$ and covariance matrix $\Sigma^{c_2^{i,j}} := D_{n-1}^j(\Sigma^{c_n(i)} - (\Sigma_{j,j}^{c_n(i)})^{-1}\Sigma_j^{c_n(i)}(\Sigma_j^{c_n(i)})^\top)(D_{n-1}^j)^\top$, where $\Sigma_j^{c_n(i)}$ denotes the j -th column of $\Sigma^{c_n(i)}$. The following holds, for $j = \hat{j}$ if $\hat{j} < i$ and $j = \hat{j} - 1$ if $\hat{j} > i$:

$$\begin{aligned} \frac{\partial^2}{\partial a_j \partial a_i} F_\xi(a, b) &= f_{\mu_j^{c_1^i(a_i)}, \Sigma_j^{c_n(i)}}(a_j) f_{\mu_i, \Sigma_{i,i}}(a_i) F_{\xi^{c_2^{i,j}(a_i, a_j)}}(D_{n-1}^j D_n^i a, D_{n-1}^j D_n^i b) \quad \forall \hat{j} \neq i \\ \frac{\partial^2}{\partial b_j \partial a_i} F_\xi(a, b) &= \begin{cases} -f_{\mu_j^{c_1^i(a_i)}, \Sigma_j^{c_n(i)}}(b_j) f_{\mu_i, \Sigma_{i,i}}(a_i) F_{\xi^{c_2^{i,j}(a_i, b_j)}}(D_{n-1}^j D_n^i a, D_{n-1}^j D_n^i b) & \forall \hat{j} \neq i \\ 0 & \hat{j} = i \end{cases} \\ \frac{\partial^2}{\partial b_j \partial b_i} F_\xi(a, b) &= f_{\mu_j^{c_1^i(b_i)}, \Sigma_j^{c_n(i)}}(b_j) f_{\mu_i, \Sigma_{i,i}}(b_i) F_{\xi^{c_2^{i,j}(b_i, b_j)}}(D_{n-1}^j D_n^i a, D_{n-1}^j D_n^i b) \quad \forall \hat{j} \neq i, \end{aligned}$$

where $f_{\mu,\sigma}(x)$ is the Gaussian density with mean μ and variance σ . Moreover, whenever $j = i$ and z is a or b we have:

$$\begin{aligned} \frac{\partial}{\partial z_i}(b_i - a_i) \frac{\partial^2}{\partial z_i^2} F_\xi(a, b) &= -\frac{z_i - \mu_i}{\Sigma_{i,i}} f_{\mu_i, \Sigma_{i,i}}(z_i) F_{\xi^{c_i^1}(z_i)}(D_n^i a, D_n^i b) \\ &\quad - f_{\mu_i, \Sigma_{i,i}}(z_i) (D_n^i \Sigma_{i,i}^{-1} \Sigma_i)^\top (\nabla_{\tilde{a}} F_{\xi^{c_i^1}(z_i)}(\tilde{a}, \tilde{b}) + \nabla_{\tilde{b}} F_{\xi^{c_i^1}(z_i)}(\tilde{a}, \tilde{b})), \end{aligned}$$

where $\tilde{a} = D_n^i a - \mu^{c_i^1}(z_i)$, $\tilde{\xi}^{c_i^1}(z_i) = \xi^{c_i^1}(z_i) - \mu^{c_i^1}(z_i)$ and \tilde{b} is defined similarly.

Proof The formula for the cross derivatives follow from a straight-forward second application of theorem 1 in [14]. The diagonal terms are more subtle to derive and require the following reformulation:

$$\begin{aligned} F_{\xi^{c_i^1}(z_i)}(D_n^i a, D_n^i b) &= \mathbb{P}\left(D_n^i a \leq \xi^{c_i^1}(z_i) \leq D_n^i b\right) = \mathbb{P}\left(D_n^i a - \mu^{c_i^1}(z_i) \leq \xi^{c_i^1}(z_i) - \mu^{c_i^1}(z_i) \leq D_n^i b - \mu^{c_i^1}(z_i)\right) \\ &= F_{\tilde{\xi}^{c_i^1}(z_i)}(\tilde{a}, \tilde{b}). \end{aligned}$$

In particular one obtains for $\tilde{a}(z_i) = \tilde{a}$

$$\begin{aligned} \tilde{a} &= D_n^i a - \mu^{c_i^1}(z_i) = D_n^i a - D_n^i(\mu + \Sigma_{i,i}^{-1}(z_i - \mu_i)\Sigma_i) \\ &= D_n^i(a - \mu + \Sigma_{i,i}^{-1}\mu_i\Sigma_i) - D_n^i\Sigma_{i,i}^{-1}z_i\Sigma_i, \end{aligned}$$

which together with the following identity

$$\frac{\partial}{\partial z_i} F_{\tilde{\xi}^{c_i^1}(z_i)}(\tilde{a}(z_i), \tilde{b}(z_i)) = \nabla_{\tilde{a}} F_{\tilde{\xi}^{c_i^1}(z_i)}(\tilde{a}, \tilde{b}) D_{z_i} \tilde{a}(z_i) + \nabla_{\tilde{b}} F_{\tilde{\xi}^{c_i^1}(z_i)}(\tilde{a}, \tilde{b}) D_{z_i} \tilde{b}(z_i),$$

an application of the chain-rule and the already established formula for 1st derivatives gives the proposition.

The following corollary deals with gradients and Hessians of our probability function $\varphi : x \mapsto \mathbb{P}[a + Ax \leq \xi \leq Bx + b]$. These follow easily from Lemma 2 and Theorem 1 of [14] upon noting that $\varphi(x) = F_\xi(Ax, Bx)$ with F_ξ as introduced in Lemma 2.

Corollary 1 *Let ξ be a Gaussian Random variable of dimension n . Let x, A, B, a, b be vectors and matrices of appropriate dimension. Now consider the mapping $\varphi : x \mapsto \mathbb{P}[a + Ax \leq \xi \leq Bx + b]$. We have:*

$$\begin{aligned} \nabla \varphi &= \nabla_a F_\xi(a, b)^\top A + \nabla_b F_\xi(a, b)^\top B \\ \nabla^2 \varphi &= A^\top \nabla_{aa}^2 F_\xi(a, b) A + A^\top \nabla_{ab}^2 F_\xi(a, b) B + B^\top \nabla_{ba}^2 F_\xi(a, b) A + B^\top \nabla_{bb}^2 F_\xi(a, b) B. \end{aligned}$$

The following result might be useful for the detailed water value model 2.1.4.

Lemma 3 *The following problems are equivalent: $\min_{x \in \mathbb{R}^n} \{f(x) : g(x) \leq [h(x)]^+\}$ and $\min_{x \in \mathbb{R}^n, \lambda \geq 0} \{f(x) : (g(x) - h(x) + \lambda)g(x) \leq 0\}$. If moreover, both g and h are linear, the complementarity constraint is quadratic. In the particular case, when $g(x, z) = z$ and $h(x) = \sum a_i x_i$, $x \in \mathbb{R}^n, z \in \mathbb{R}$, the matrix defining the quadratic form of the inequality is given by*

$$Q = \begin{pmatrix} 0 & -0.5a & 0 \\ -0.5a^\top & 1 & 0.5 \\ 0 & 0.5 & 0 \end{pmatrix}$$

and has n zero eigenvalues and 2 non-zero ones, $0.5 \pm 0.5\sqrt{\|a\|^2 + 2}$. The resulting quadratic constraint is non-convex.

Proof For each feasible x for the first problem, one can find some $\lambda \geq 0$ such that (x, λ) is feasible for the second one. This can be seen by case distinction on the sign of $h(x), g(x)$. By case distinction on the sign of $g(x)$ one can see that each feasible (x, λ) for problem 2, yields a feasible x for problem 1. It is easily seen that the quadratic constraint is $y^T Q y \leq 0$, with $y := (x, z, \lambda)$. Clearly any vector $(b, 0, 0)$ with $b^T a = 0$ is an eigenvector with zero eigenvalue. Moreover by solving the system $z\mu^2 - z\mu - (\frac{1}{4}a^T a + \frac{1}{4}) = 0$ one obtains the other two eigenvalues. Finally since $\text{Tr}(Q) = 1$, the last eigenvalue has to be zero too. The non-convexity follows since one eigenvalue of Q is negative.

6 Conclusions

In this paper, we have set up a joint chance constrained programming approach for dealing with uncertainty on inflows in hydro-valley optimization. We have derived a detailed model for a real hydro-valley, but one wherein combinatorial constraints are neglected. In order to have a more realistic description of inflows, we have considered a causal time series setting with Gaussian innovations. The latter choice allows us to preserve convexity and have a more realistic model on inflows. In order to solve this JCCP problem we have used a cutting planes method that requires a Slater point and gradients. The probability functions and gradients can be efficiently computed using Genz' code.

In order to highlight the interest of joint chance constraint programming, we have also investigated alternative models. Indeed, we have considered a model based on individual constraints and a robust model. The obtained results have been compared on a realistic hydro-valley. Hence, despite the fact that the alternative models often require less hypothesis on the law of the inflows, they provide conservative and costly solutions. The simpler models, such as the individual chance constraint one, are shown to yield insufficient robustness. The robust model induces an important extra cost, despite the well calibrated "uncertainty set" and moreover often leads to empty feasible sets. Joint Chance Constrained programming appears as a technique offering a good trade-off between cost and robustness and can be tractable for complex realistic models. In addition, we have shown that in principle we can handle a real size valley within reasonable computation time.

Future perspectives consist in working on model realism and on the algorithm for solving the chance constraint problem. Indeed, from a modeling perspective, we could integrate the combinatorial constraints on the decision variables, potentially without many difficulties. From an algorithmic perspective, instead of using a cutting planes idea, one could use a bundle method to hopefully improve computation times and stability. A second point that needs investigations is an improved use of Genz' code. We could combine the use of Genz' code with Prekopa's linear programming estimation method of probability measures in order to either increase the size of the model or improve the speed.

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