

# Inscribable stacked polytopes

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## Abstract

We characterize the combinatorial types of stacked  $d$ -polytopes that are inscribable. Equivalently, we identify the triangulations of a simplex by stellar subdivisions that can be realized as Delaunay triangulations.

## 1 Introduction

*How do geometric constraints restrict the combinatorics of polytopes?*

One instance of this question asks for the combinatorial types of  $d$ -polytopes that are *inscribable*, that is, that have realizations with all vertices on a sphere. This question was raised in 1832 by Jacob Steiner [15], who asked whether all 3-dimensional polytopes are inscribable. The answer was given by Ernst Steinitz [16] nearly 100 years later: No — despite a claim to the contrary by Brückner [5, p. 163] this is not even true for all simplicial types. Indeed, the polytope obtained by “stacking” a new vertex onto each facet of a tetrahedron is not inscribable (compare Grünbaum [9, Sect. 13.5]).

General 3-polytopes are inscribable if and only if a certain associated linear program is feasible: This was proved by Rivin [10] using hyperbolic geometry; a complete combinatorial characterization for inscribability of 3-polytopes is not available up to now (compare Dillencourt & Smith [6]).

In this paper we embark on a study of inscribability for  $d$ -dimensional convex polytopes. One instance our motivating question is

*How does the condition of inscribability restrict the  $f$ -vectors of polytopes?*

In this context, we observe (Section 2.5.1) that all  $f$ -vectors of 3-polytopes (as characterized by Steinitz [16] in 1906) also occur for inscribable polytopes. Also, as the cyclic polytopes are inscribable, we get that the Upper Bound Theorem of McMullen [11] is sharp for the restricted class of inscribable polytopes (Section 2.5.2). Moreover, it is easy to see that there are stacked  $d$ -polytopes with  $d + 1 + n$  vertices that are inscribable ( $d \geq 2$ ,  $n \geq 0$ ), so we find that also

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the Lower Bound Theorem of Barnette [1] [2] is sharp for the restricted class of inscribable polytopes. One is thus naturally led to ask whether all  $f$ -vectors of convex polytopes, or at least all  $f$ -vectors of simplicial convex polytopes can be obtained from inscribable polytopes. This will be further studied in [8].

One can then proceed and try to characterize inscribability for some of these classes. This seems out of reach for neighborly polytopes, as according to Shemer [14] there are huge numbers of combinatorial types, and no combinatorial classification in sight. However, as the main result of this paper we provide a combinatorial characterization of inscribable stacked polytopes. It refers to the dual tree of a stacked polytope, which will be formally defined in Section 2.2 below; for now, we refer to Figure 1.

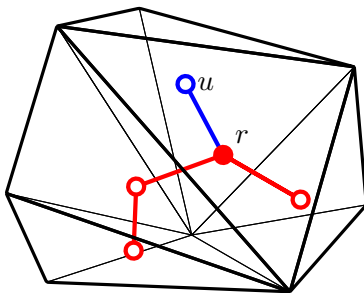


Figure 1: The dual tree of a stacked 3-polytope.

**Theorem 1.** *A stacked polytope is inscribable if and only if all nodes of its dual tree have degree at most 3.*

Thus the requirement of inscribability does not restrict the possible  $f$ -vectors of stacked polytopes. However, other combinatorial parameters are restricted. For example, in any inscribable stacked  $d$ -polytope (other than a simplex,  $d \geq 3$ ) less than half of the vertices are simple, while for general stacked  $d$ -polytopes roughly  $\frac{d-1}{d}$  of the vertices can be simple.

The study of inscribable convex  $d$ -polytopes is, via stereographic projection, equivalent to the study of  $(d-1)$ -dimensional Delaunay triangulations. (The importance of stereographic projection in this context was stressed in 1979 by Brown [4].) Under this correspondence (which is detailed in Section 2.4), the stacked  $d$ -polytopes with  $d+1+n$  vertices correspond to the Delaunay triangulations of a  $(d-1)$ -simplex generated by a sequence of  $n$  stellar subdivisions of  $(d-1)$ -faces. (The rooted tree of a multiple stellar subdivision of a simplex is discussed in Section 2.3 below; for now, we refer to Figure 2.)

**Theorem 2.** *A triangulation that is a multiple stellar subdivision of a  $(d-1)$ -simplex can be realized as a Delaunay triangulation if and only if at most two of the  $(d-1)$ -simplices generated in any single stellar subdivision are further subdivided.*

## 2 Stacked polytopes and Delaunay triangulations

### 2.1 Inscribable polytopes

**Definition 3** (inscribed polytope). A convex  $d$ -polytope is *inscribed* if its vertices lie on a  $(d-1)$ -sphere. It is *inscribable* if it is combinatorially equivalent to an inscribed polytope, that is, if it has a realization that is inscribed.

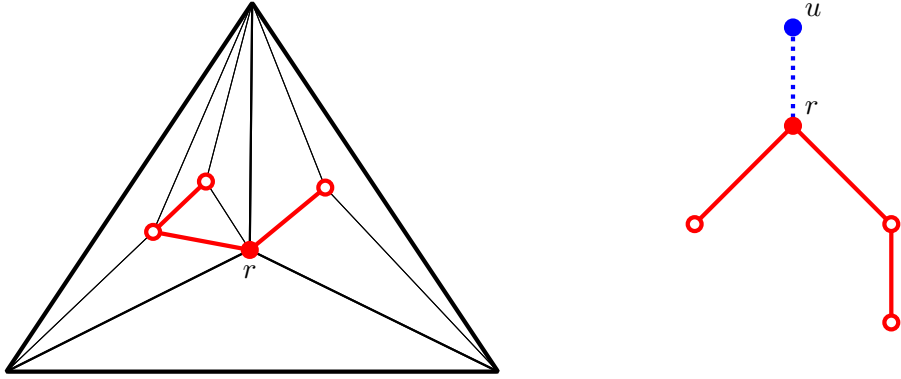


Figure 2: A stellar subdivision of a simplex and its dual rooted tree.

## 2.2 Stacked polytopes

**Definition 4** (stacked polytope). A polytope is *stacked* if it can be built from a  $d$ -simplex by a sequence of stacking operations: A *stacking operation* is performed onto a facet by taking the convex hull of the polytope with a new point that lies beyond the selected facet but beneath all other facets of the polytope.

A stacking operation can also be imagined as gluing a simplex onto a facet. A simplicial  $d$ -polytope  $P$  is stacked if and only if it has a triangulation with only interior faces of dimension  $d$  and  $d - 1$ . In dimension at least 3 such a triangulation is unique if it exists: Its simplices are given by the cliques (complete subgraphs) of the graph of  $P$ .

The “claim to fame” of stacked polytopes is the Lower Bound Theorem [1] [2] [3]: Among all simplicial  $d$ -polytopes with  $d + 1 + n$  vertices, the stacked polytopes have the minimal number of facets (and indeed, the minimal number of  $k$ -faces, for all  $k$ ). Moreover, for  $d \geq 4$  the stacked polytopes are the only polytopes with these parameters.

**Definition 5** (dual tree of a stacked polytope). For  $d \geq 3$ , the *dual tree*  $T_P$  of a stacked  $d$ -polytope  $P$  is the dual graph of its triangulation that has only interior  $d$ - and  $(d - 1)$ -faces: Every  $d$ -face in the triangulation corresponds to a node and every interior  $(d - 1)$ -face corresponds to an edge of the tree.

The graph  $T_P$  given by Definition 5 is indeed a tree if  $P$  is stacked.

We choose any node of  $T_P$  as a root and assign an order to the rest of the nodes such that a child is always greater than its parent. Any such order implies an iterative construction of  $P$  via stackings in the following way: The root represents the initial simplex. Every child has one vertex that it does not share with its parent. This is used to stack the  $(d - 1)$ -face that child and parent share. Assuming that  $T_P$  has at least two nodes, we see that the leaves of the tree are responsible for the simple vertices of  $P$ , but no further simple vertices are possible, except if the root has exactly one child, then there is an additional simple vertex that only the root  $d$ -simplex contains.

Clearly the dual tree  $T_P$  of a stacked  $d$ -polytope on  $d + 1 + n$  vertices has maximal degree

$$\max \deg T_P \leq \min\{d + 1, n\}$$

and stacked polytopes with these parameters exist for all  $d \geq 3$  and  $n \geq 0$ .

## 2.3 Delaunay triangulations

For any affinely spanning finite set of points  $V \subset \mathbb{R}^{d-1}$ , the *Delaunay subdivision*  $\mathcal{D}(V)$  is the unique subdivision of  $\text{conv}(V)$  into inscribed  $(d-1)$ -polytopes  $P_i = \text{conv}(V_i)$ ,  $V_i \subset V$ , that are given by the *empty circumsphere condition*: There exists a  $(d-2)$ -sphere that passes through all vertices of  $P_i$  but all other vertices of  $\mathcal{D}(V)$  lie outside this sphere. If the points in  $V$  are in sufficiently general position (which is satisfied in particular if no  $d+1$  points lie on a sphere), then the Delaunay subdivision is indeed a triangulation, known as *the Delaunay triangulation* of  $V$ .

One way to construct the Delaunay subdivision/triangulation is to derive it from an inscribed  $d$ -polytope by using the (inverse) stereographic projection, as discussed below.

We will employ the following very elegant criterion in order to test whether a given triangulation is the Delaunay triangulation. For this we call a face of a triangulation *Delaunay* if there exists a *supporting sphere* of the face, that is, a  $(d-2)$ -sphere passing through the vertices of the face such that all other vertices of the triangulation lie outside the sphere. Each interior  $(d-2)$ -face  $F$  is contained in exactly two  $(d-1)$ -faces,  $\text{conv}(F \cup \{v_1\})$  and  $\text{conv}(F \cup \{v_2\})$ ; we call it *locally Delaunay* if there exists a  $(d-2)$ -sphere passing through the vertices of  $F$  such that the vertices  $v_1$  and  $v_2$  lie outside this sphere.

**Lemma 6** (Delaunay Lemma). *Let  $V \subset \mathbb{R}^{d-1}$  be a finite, affinely spanning set of points. A triangulation  $\mathcal{T}$  of  $\text{conv}(V)$  with vertex set  $V$  is the Delaunay triangulation if and only if one of the following equivalent statements hold:*

- (1) *All  $(d-1)$ -faces of  $\mathcal{T}$  are Delaunay.*
- (2) *All faces of  $\mathcal{T}$  are Delaunay.*
- (3) *All  $(d-2)$ -faces of  $\mathcal{T}$  are Delaunay.*
- (4) *All interior  $(d-2)$ -faces of  $\mathcal{T}$  are locally Delaunay.*

*Proof.* The first statement is the definition of a Delaunay triangulation. It implies the second statement: For each face  $F$  one can always do a slight change to the supporting sphere of a  $(d-1)$ -face that contains  $F$  to derive a supporting sphere of  $F$ . The second statement implies the third and this in turn the last one. For more details and also for a proof that the last statement implies the first, we refer to Edelsbrunner [7, pp. 7 and 99].  $\square$

**Definition 7** (stellar subdivision). Let  $p$  be a point inside a full-dimensional simplex  $\sigma$  of a triangulation  $\mathcal{T}$ . A *single stellar subdivision* of  $\mathcal{T}$  at  $\sigma$  (by  $p$ ) is the triangulation that replaces  $\sigma$  by the simplices spanned by  $p$  and a proper face of  $\sigma$ . We call a triangulation a *multiple stellar subdivision* of  $\mathcal{T}$  at  $\sigma$  if one or more single stellar subdivisions have been applied.

In the following, we will discuss which Delaunay triangulations can be generated by multiple stellar subdivisions of a triangulation that has just one full-dimensional simplex.

**Lemma 8.** *A single stellar subdivision of the triangulation that has just one full-dimensional simplex is always a Delaunay triangulation.*

*Proof.* If the circumsphere of a new full-dimensional simplex  $\sigma$  would contain the vertex  $v$  that does not lie in  $\sigma$ , then it would contain all points of the original simplex, and hence it would contain the new vertex  $c$  in its interior. See Figure 3.  $\square$

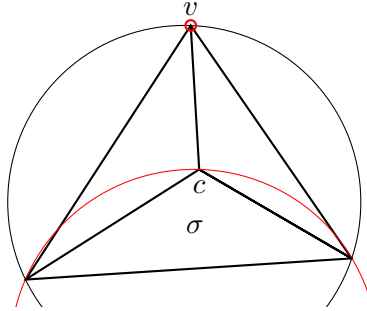


Figure 3: The circumcircle of  $\sigma$  cannot contain  $v$ .

**Definition 9** (dual tree of a stellar subdivision). The *rooted tree*  $T_{\mathcal{T}}$  of a multiple stellar subdivision  $\mathcal{T}$  of a  $(d-1)$ -simplex  $\sigma$ , for  $d \geq 3$ , has one node for every vertex that is inserted by a single stellar subdivision, or equivalently for every  $(d-1)$ -face that it destroys. The root node  $r$  corresponds to the (first) single stellar subdivision of  $\sigma$ . The node  $v'$  is a child of the node  $v$  if it corresponds to a single stellar subdivision of a  $(d-1)$ -face that was created in the single stellar subdivision corresponding to  $v$ .

Figure 2 shows a multiple stellar subdivision of a 2-simplex and the corresponding dual tree.

**Example 10.** Figure 4 illustrates the construction of a multiple stellar subdivision of a  $(d-1)$ -simplex (generalizing Lemma 8) where the dual tree is a path.

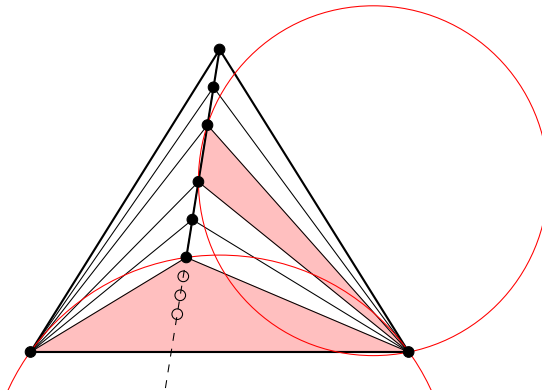


Figure 4: Take a full dimensional simplex and a ray from a vertex to the interior of the simplex. Apply  $n$  single stellar subdivisions to the complex by introducing new vertices on the ray. The result is a Delaunay triangulation.

**Corollary 11** (a stellar subdivision can be undone). *Let the triangulation  $\mathcal{T}'$  be obtained from a triangulation  $\mathcal{T}$  of an affinely spanning point set  $V \subset \mathbb{R}^{d-1}$  by a single stellar subdivision. If  $\mathcal{T}'$  is a Delaunay triangulation, then so is  $\mathcal{T}$ .*

*Proof.* A stellar subdivision does not destroy a  $(d-2)$ -face, thus among the supporting spheres for  $(d-2)$ -faces in  $\mathcal{T}'$  we have the supporting spheres for all  $(d-2)$ -faces in  $\mathcal{T}$ .  $\square$

We end this section with a first example of a triangulation that *cannot* be realized as a Delaunay triangulation. The last sentence of the following lemma yields a more precise statement that will turn out to be crucially important later.

**Lemma 12.** *Apply a single stellar subdivision by a point  $x$  to a triangle  $\text{conv}\{A, B, C\}$  and then single stellar subdivisions by points  $a, b, c$  to the three new triangles.*

*The resulting triangulation is not a Delaunay triangulation: At least one of the three edges  $Ax$ ,  $Bx$  and  $Cx$  violates the locally Delaunay criterion.*

*Proof.* The nine angles at  $a, b$  and  $c$  sum up to  $6\pi$ . The three angles that lie in triangles that contain a boundary edge are each smaller than  $\pi$ , so the remaining six angles must sum to more than  $3\pi$ . However, each of the three edges  $Ax$ ,  $Bx$  and  $Cx$  is locally Delaunay if and only if its two opposite angles sum to less than  $\pi$ . Hence not all three edges can be locally Delaunay. See Figure 5.  $\square$

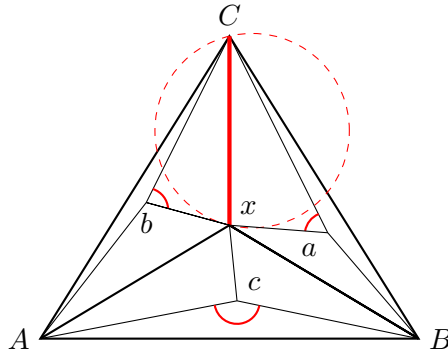


Figure 5: In any triangulation of this combinatorial type at least one of the three edges  $Ax$ ,  $Bx$  and  $Cx$  is not locally Delaunay.

## 2.4 Stereographic projection

The *stereographic projection*

$$\pi : S^{d-1} \setminus \{N\} \longrightarrow \mathbb{R}^{d-1} \times \{0\}$$

is the bijective map that projects every point  $x \neq N$  of the sphere  $S^{d-1}$  along the ray through  $x$  starting in the north pole  $N$  to the equator hyperplane of the sphere, which we identify with  $\mathbb{R}^{d-1}$ . The inversion  $x \mapsto N + 2\frac{x-N}{\|x-N\|^2}$  in the sphere with center  $N$  and radius  $\sqrt{2}$  extends this map to a bijection  $\hat{\pi} : \mathbb{R}^d \cup \{\infty\} \rightarrow \mathbb{R}^d \cup \{\infty\}$ . This sphere inversion is a Möbius transformation: It maps spheres to spheres, where spheres through  $N$  are mapped to hyperplanes (that is, spheres through  $\infty$ ).

The stereographic projection identifies (the vertex sets of) inscribed  $d$ -polytopes that have a vertex at the north pole with (the vertex sets of) Delaunay subdivisions in  $\mathbb{R}^{d-1}$ .

**Proposition 13** (inscribed polytopes and Delaunay subdivisions). *Let  $S^{d-1}$  denote the standard unit sphere in  $\mathbb{R}^d$  and let  $P$  be an inscribed  $d$ -polytope whose vertex set  $V \cup \{N\} \subset S^{d-1}$  includes the north pole  $N$  of  $S^{d-1}$ .*

*Then  $\pi(V)$  is the vertex set of a Delaunay subdivision in  $\mathbb{R}^{d-1}$  whose  $(d-1)$ -faces correspond to the facets of  $P$  that do not contain the vertex  $N$ .*

*Conversely, if  $W$  is a finite set that affinely spans  $\mathbb{R}^{d-1} \times \{0\}$ , then  $\pi^{-1}(W) \cup \{N\}$  is the vertex set of an inscribed  $d$ -polytope  $P$  whose facets that miss  $N$  are given by the  $(d-1)$ -faces of the Delaunay subdivision of  $W$  and the facets of  $P$  that contain the north pole  $N$  are exactly the convex hulls  $\text{conv}(\pi^{-1}(F \cap W) \cup \{N\})$  given by the facet  $F$  of  $\bar{P} := \text{conv}(W)$ .*

In the following, we will need this theorem specialized to the simplicial case.

**Proposition 14** (inscribed simplicial polytopes and Delaunay triangulations). *Let  $S^{d-1}$  denote the standard unit sphere in  $\mathbb{R}^d$  and let  $P$  be an inscribed simplicial  $d$ -polytope whose vertex set  $V \cup \{N\} \subset S^{d-1}$  includes the north pole  $N$  of  $S^{d-1}$ .*

*Then  $\pi(V)$  is the vertex set of a Delaunay triangulation in  $\mathbb{R}^{d-1}$  whose  $(d-1)$ -simplices correspond to the facets of  $P$  that do not contain the vertex  $N$ .*

*Conversely, if  $W \subset \mathbb{R}^{d-1} \times \{0\}$  is a finite set with a Delaunay triangulation  $\mathcal{T}$  whose convex hull  $\bar{P} := \text{conv}(W)$  is a simplicial  $(d-1)$ -polytope that has no points of  $W$  on the boundary except for the vertices, then  $\pi^{-1}(W) \cup \{N\}$  is the vertex set of a simplicial inscribed  $d$ -polytope  $P = \text{conv}(\pi^{-1}(W) \cup \{N\})$  whose facets that miss  $N$  are given by the  $(d-1)$ -simplices of the Delaunay triangulation  $\mathcal{T}$  and the facets of  $P$  that contain the north pole  $N$  are exactly the convex hulls  $\text{conv}(\pi^{-1}(F \cap W) \cup \{N\})$  given by the facet  $F$  of  $\bar{P}$ .*

As a corollary to this we obtain that the Main Theorems 1 (for polytopes) and 2 (for Delaunay triangulations) are equivalent. We will prove Theorem 2 below. We also obtain that the Lower Bound Theorem is tight for inscribable polytopes, by applying Proposition 14 to Example 10:

**Corollary 15.** *For all  $d \geq 2$ ,  $n \geq 0$ , there is an inscribed stacked  $d$ -polytope on  $d + 1 + n$  vertices.*

We end this section with a proof of a very special (but crucial) part and case of Theorem 2.

*Proof of the “only if” part of Theorem 2 for  $d - 1 = 2$ .* By Lemma 6 and Corollary 11 it suffices to show that the configuration of Figure 5 cannot be realized as a Delaunay triangulation. (This was proved in Lemma 12. Via Proposition 14 it is equivalent to the fact that the polytope obtained by stacking onto all four facets of a tetrahedron is not inscribable, which was first proved by Steinitz [16] [9, Sect. 13.5].)  $\square$

## 2.5 Some inscribable polytopes

### 2.5.1 3-dimensional polytopes

**Proposition 16.** *All  $f$ -vectors of 3-polytopes occur for inscribable 3-polytopes.*

*Proof.* According to Steinitz [16] [9, Sect. 10.3], the set of all  $f$ -vectors of convex 3-polytopes is

$$\{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_2 \leq 2f_0 - 4, f_0 \leq 2f_2 - 4, f_1 = f_0 + f_2 - 2\}.$$

In Figure 6 one can see three types of inscribed 3-polytopes. These can be described as wedges over an  $n$ -gon that have been stacked  $k$  times. By performing these constructions for arbitrary  $n \geq 3$  and  $k \geq 0$  we get three families of inscribable 3-polytopes. Their  $f$ -vectors are

$$\begin{aligned} f_{\text{left}} &= (2n - 2 + k, 3n - 3 + 3k, n + 1 + 2k) = (f_0, f_1, \frac{f_0}{2} + 2 + \frac{3}{2}k) \\ f_{\text{middle}} &= (2n - 1 + k, 3n - 1 + 3k, n + 2 + 2k) = (f_0, f_1, \frac{f_0}{2} + 2 + \frac{3}{2}k + \frac{1}{2}) \\ f_{\text{right}} &= (2n + k, 3n + 1 + 3k, n + 3 + 2k) = (f_0, f_1, \frac{f_0}{2} + 2 + \frac{3}{2}k + 1) \end{aligned}$$

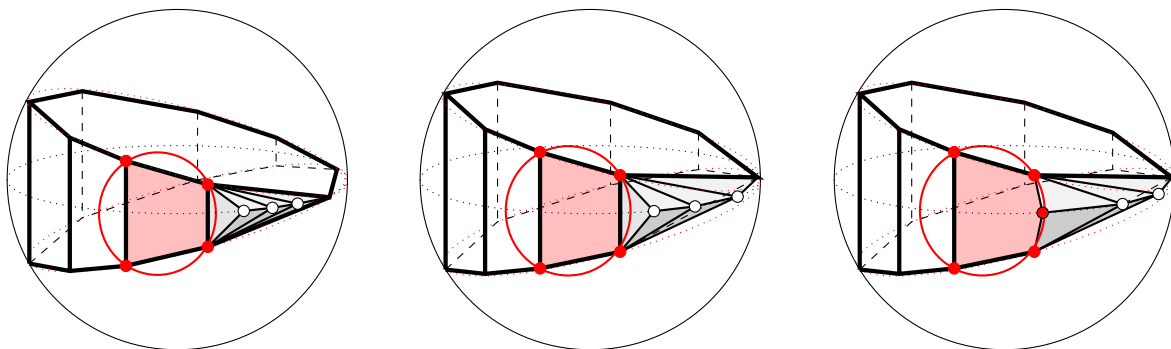


Figure 6: Three constructions for inscribed 3-polytopes that produce all possible  $f$ -vectors.

For  $k = 0$  the first type produces the  $f$ -vectors of simple polytopes; the first two types provide all  $f$ -vectors with the minimal number of facets for any given number of vertices. For  $n = 3$ , the first type produces inscribable stacked 3-polytopes with arbitrary number of vertices. These are simplicial and hence give the maximal number of facets for any given number of vertices. It is easy to see that for any number of vertices all permissible numbers of facets can be obtained by choosing the right  $n$ ,  $k$ , and type.  $\square$

## 2.5.2 Neighborly polytopes

**Proposition 17.** *The  $d$ -dimensional cyclic polytope  $C_d(n)$  with  $n$  vertices is inscribable for all  $d \geq 2$  and  $n \geq d + 1$ .*

*Thus all  $f$ -vectors of neighborly polytopes occur for inscribable polytopes.*

We will sketch three simple proofs for this.

*Proof 1.* The standard moment curve in  $\mathbb{R}^{d-1}$  is given by

$$\gamma(t) := (t, t^2, \dots, t^{d-1}).$$

This is a curve of order  $d$  by Vandermonde's determinant formula. If the sequence of parameters  $t_1 < t_2 < \dots < t_{n-1}$  grows fast enough, then for each  $i > d$  the point  $\gamma(t_i)$  lies outside all circumspheres of the Delaunay triangulation of  $\{\gamma(t_1), \dots, \gamma(t_{i-1})\}$ . Thus the Delaunay triangulation of  $\{\gamma(t_1), \dots, \gamma(t_{i-1})\}$  is obtained by induction on  $i$ , where for suitably large  $t_i$  the new facets are given by the "upper" facets of  $\text{conv}\{\gamma(t_1), \dots, \gamma(t_{i-1})\}$  joined to the new vertex  $\gamma(t_i)$ .



One checks, using Gale’s evenness criterion, that thus the facets of the Delaunay triangulation of the point set  $\{\gamma(t_1), \gamma(t_2), \dots, \gamma(t_{n-1})\}$  correspond exactly to the facets of  $C_d(n)$  that do not contain the last vertex.

Finish the proof via Proposition 14. □

*Proof 2* (Seidel [13, p. 521]). The *spherical moment curve* is given by

$$C : \mathbb{R}^+ \rightarrow \mathbb{R}^d, \quad c(t) := \frac{1}{1 + t^2 + t^4 + \dots + t^{2(d-1)}}(1, t, t^2, \dots, t^{d-1}).$$

This curve lies on the image of the hyperplane  $x_1 = 1$  under inversion in the origin, that is, on the sphere with center  $\frac{1}{2}e_1$  and radius  $\frac{1}{2}$ . Using Descartes’ rule of signs one gets that this curve (restricted to the domain  $t > 0!$ ) is of order  $d$ , and thus the convex hull of any  $n$  distinct points on this curve is an inscribed realization of  $C_d(n)$ . □

*Proof 3* (Grünbaum [9, p. 67]). For even  $d \geq 2$ , we consider the *trigonometric moment curve*

$$c : (-\pi, \pi] \rightarrow \mathbb{R}^d, \quad c(t) := (\sin(t), \cos(t), \sin(2t), \cos(2t), \dots, \sin(\frac{d}{2}t), \cos(\frac{d}{2}t))$$

Obviously its image lies on a sphere. We verify that this is a curve of order  $d$  using the fact that any nonzero trigonometric polynomial of degree  $\frac{d}{2}$  has at most  $d$  zeros per period (see e.g. Polya & Szegő [12, pp. 72-73]). Thus we get that the convex hull of any  $n$  points on this curve yields an inscribed realization of  $C_d(n)$ . (Compare [17, pp. 75-76].)

For odd  $d \geq 3$ , we check using Gale’s evenness criterion that any “vertex splitting” on  $C_{d-1}(n-1)$  results in a realization of  $C_d(n)$ ; this yields inscribed realizations of  $C_d(n)$  where all vertices except for those labeled 1 and  $n$  lie on a hyperplane. (See e.g. Seidel [13, p. 528], where the “vertex splitting” is called “pseudo-bipyramid”.) □

### 3 Stacked polytopes of dual degree at most 3 are inscribable

The following proposition establishes the “if” part of Main Theorem 2 (and thus also of Main Theorem 1).

**Proposition 18.** *Let  $\mathcal{T}$  be a Delaunay triangulation in  $\mathbb{R}^{d-1}$ , let  $c$  be an interior vertex of degree  $d$ . Then one can perform single stellar subdivisions on two arbitrary  $(d-1)$ -faces  $F_1$  and  $F_2$  of  $\mathcal{T}$  that contain  $c$  such that the resulting triangulation is again Delaunay.*

*Proof.* Let  $F_1, \dots, F_d$  be the  $(d-1)$ -faces of  $\mathcal{T}$  that contain  $c$ , and let  $\mathcal{R}$  be the set of all other  $(d-1)$ -faces of  $\mathcal{T}$ . Let  $v_1, \dots, v_d$  be the vertices of  $F_1, \dots, F_d$  such that  $v_i$  is not contained in  $F_i$ . The circumspheres of  $F_3, \dots, F_d$  contain  $c$ . The intersection of the tangent hyperplanes to these  $d-2$  spheres in the point  $c$  contains a line  $t$  through  $c$  that lies tangent to all those spheres.

Let  $U$  be a small open ball around  $c$  that, like  $c$ , lies outside the circumspheres of all cells in  $\mathcal{R}$ . Then  $U \cap t \setminus \{c\}$  consists of two disjoint, open line segments. Choose two points,  $x_1, x_2$ , one in each line segment, and use them for single stellar subdivisions of  $F_1$  resp.  $F_2$  (cf. Figure 7).

We claim that the resulting triangulation  $\mathcal{T}'$  is again Delaunay. First we check that  $x_1$  and  $x_2$  lie inside  $F_1$  resp.  $F_2$ : They lie outside all facets  $F_3, \dots, F_d$  but inside  $\text{conv}\{v_1, \dots, v_d\}$ , hence they lie in  $F_1 \cup F_2$ . Because  $t$  contains  $c$ , which is a vertex of  $F_1$  and  $F_2$ , only one

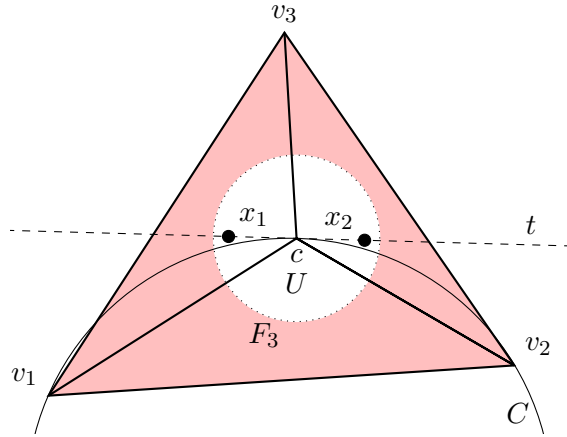


Figure 7: Choice of the subdivision points.

component of  $t \setminus \{c\}$  can be contained in  $F_1$  and only one can be contained in  $F_2$ . Hence we can assume that  $x_1$  lies in the relative interior of  $F_1$  and  $x_2$  lie in the relative interior of  $F_2$ .

Now we need to show that all interior  $(d-2)$ -faces of  $\mathcal{T}'$  are locally Delaunay. The cells in  $\mathcal{R} \cup \{F_3, \dots, F_d\}$  lie in both triangulations  $\mathcal{T}$  and in  $\mathcal{T}'$ . They have empty circumpheres in  $\mathcal{T}$  by assumption, and in  $\mathcal{T}'$  by construction.

Let  $\mathcal{I}$  be the faces of  $\mathcal{T}'$  that are not faces of  $\mathcal{T}$ . It remains to show that all  $(d-2)$ -faces in  $\mathcal{T}'$  that are contained in two facets of  $\mathcal{I}$  are locally Delaunay. The first type lies in  $(d-1)$ -faces that both contain  $x_1$  or both contain  $x_2$ . In this case the locally Delaunay condition is given by Lemma 8. The second type lies in a  $(d-1)$ -face that contains  $x_1$  and an other  $(d-1)$ -face that contains  $x_2$ . There is only one such  $(d-2)$ -face, namely the intersection of  $F_1$  and  $F_2$ . Let's call this face  $K$ . The circumphere of  $\text{conv}(K \cup \{x_2\})$  does not contain  $x_1$ , because  $x_1, c, x_2$  are collinear and  $c$  lies between  $x_1$  and  $x_2$  (see Figure 8). Hence also  $K$  is locally Delaunay and thus all interior  $(d-2)$ -faces of  $\mathcal{T}'$  are locally Delaunay. Hence  $\mathcal{T}'$  is a Delaunay triangulation.  $\square$

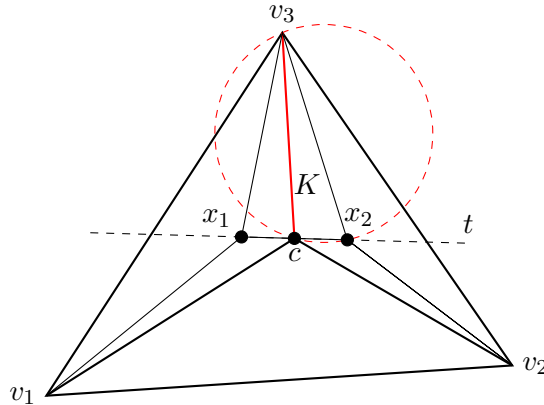


Figure 8: The circumphere does not contain  $x_1$  because  $x_1, c$  and  $x_2$  are collinear.

Using this result, we also obtain examples of stacked polytopes that go beyond the rather special construction given by Corollary 15.

**Corollary 19** (inscribable stacked polytopes with bounded vertex degree). *For all  $d \geq 2$  and  $n \geq 0$  there exists a stacked inscribed polytope of dimension  $d$  that has  $d + 1 + n$  vertices such that no vertex has degree more than  $2d$ .*

*Proof.* We may assume  $d > 2$  (where the inductive steps discussed in the following do not destroy edges).

We start with an arbitrary  $d$ -simplex, which is inscribed. All its vertices are simple; we label them  $1, \dots, d+1$ . Now for  $k = 1, \dots, n$  we refer to Proposition 18 in order to stack a new vertex  $d+1+k$  onto the facet  $\{k+1, k+2, \dots, k+d\}$  at the simple vertex  $d+1+(k-1) = d+k$ . This in particular destroys the facet  $\{k+1, k+2, \dots, k+d\}$  (which contains the vertex labeled  $k+1$ , which will not be touched again) and creates the new facet  $\{k+2, \dots, k+1+d\}$ , adjacent to the new simple vertex  $d+1+k$ .

In the stacked inscribed polytope created this way, vertices  $i$  and  $j$  are adjacent exactly if  $|i - j| \leq d$ .  $\square$

## 4 Three stellar subdivisions are impossible

The following establishes the “only if” part of Main Theorem 2 (and thus also of Main Theorem 1): If multiple stellar subdivisions are performed on three facets  $F_1, F_2$  and  $F_3$  at a simple interior vertex of an arbitrary triangulation, then the resulting triangulation is not a Delaunay triangulation.

For this, it suffices to consider the complex  $\Delta$  that arises by a single stellar subdivision of a  $(d-1)$ -simplex  $\sigma \subset \mathbb{R}^{d-1}$  using an arbitrary interior point  $c \in \sigma$ . This complex  $\Delta$  with  $(d-1)$ -faces  $F_1, \dots, F_d$  is Delaunay by Lemma 8. Now for  $d \geq 3$  we apply single stellar subdivisions to the cells  $F_1, F_2, F_3$  by arbitrary interior points  $r_1, r_2, r_3$ . Our claim is that the resulting triangulation  $\mathcal{T}$  cannot be Delaunay.

In order to prove this claim, we first construct a point  $x$  that depends only on  $\Delta$ . Its position with respect to  $\Delta$  is established in Lemma 20. Then Lemma 21 records the properties of  $x$  with respect to the subdivision  $\mathcal{T}$ . Finally, we establish in Proposition 22 that  $\mathcal{T}$  cannot be Delaunay: For that we use an inversion in a sphere centered at  $x$  in order to simplify the situation so that a projection argument reduces the claim to the case  $d = 3$ , which was established in Lemma 12.

Let  $\sigma = \text{conv}\{v_1, \dots, v_d\}$  be a  $(d-1)$ -simplex in  $\mathbb{R}^{d-1}$ , let  $c \in \sigma$  be an interior point, and let  $\Delta$  be the single stellar subdivision of  $\sigma$  by  $c$ , with  $(d-1)$ -faces  $F_1, \dots, F_d$ , labeled such that  $v_i \notin F_i$ .

For some  $k$  ( $1 \leq k < d$ ) let  $\mathcal{F} := \{F_{k+1}, \dots, F_d\}$  and  $\mathcal{G} := \{F_1, \dots, F_k\}$ . Then  $V_{\mathcal{F}} := \{v_1, \dots, v_k\}$  is the set of vertices of  $\sigma$  that lie in all cells of  $\mathcal{F}$ , while  $V_{\mathcal{G}} := \{v_{k+1}, \dots, v_d\}$  is the set of vertices of  $\sigma$  that lie in all cells of  $\mathcal{G}$ .

Now  $E_{\mathcal{F}} := \text{aff}(V_{\mathcal{F}} \cup \{c\})$  is an affine subspace of dimension  $k$ , while  $E_{\mathcal{G}} := \text{aff}(V_{\mathcal{G}} \cup \{c\})$  has dimension  $d - k$ . The two spaces together affinely span  $\mathbb{R}^{d-1}$ , so by dimension reasons they intersect in a line  $\ell$ . This line intersects the two complementary faces  $\text{conv}(V_{\mathcal{F}})$  and  $\text{conv}(V_{\mathcal{G}})$  of  $\sigma$  in relatively interior points  $\bar{x}$  resp.  $\bar{y}$ .

Let  $C_{\mathcal{F}}$  denote the unique  $(k-1)$ -sphere that contains  $V_{\mathcal{F}} \cup \{c\}$ , that is, the circumsphere of the  $k$ -simplex  $\text{conv}(V_{\mathcal{F}} \cup \{c\})$ , which is also the intersection of the circumspheres of  $F_{k+1}, \dots, F_d$ . The point  $c$  lies in the intersection  $\ell \cap C_{\mathcal{F}}$ . The line  $\ell$  also contains the point  $\ell \cap \text{conv}(V_{\mathcal{F}}) = \{\bar{x}\}$ , which is a relative-interior point of  $\text{conv}(V_{\mathcal{F}})$  and thus for  $k > 1$  lies in

the interior of the circumspheres of  $F_{k+1}, \dots, F_d$  and thus in the interior of the sphere  $C_{\mathcal{F}}$  relative to the subspace  $E_{\mathcal{F}}$ . Thus  $C_{\mathcal{F}} \cap \ell = \{c, x\}$ , where the second intersection point  $x$  is distinct from  $c$ , and lies outside  $\sigma$  for  $k > 1$ .

As for  $C_{\mathcal{F}}$  and  $x, \bar{x}$ , we define  $C_{\mathcal{G}}$  and  $y, \bar{y}$  for  $\mathcal{G}$ : See Figure 9.

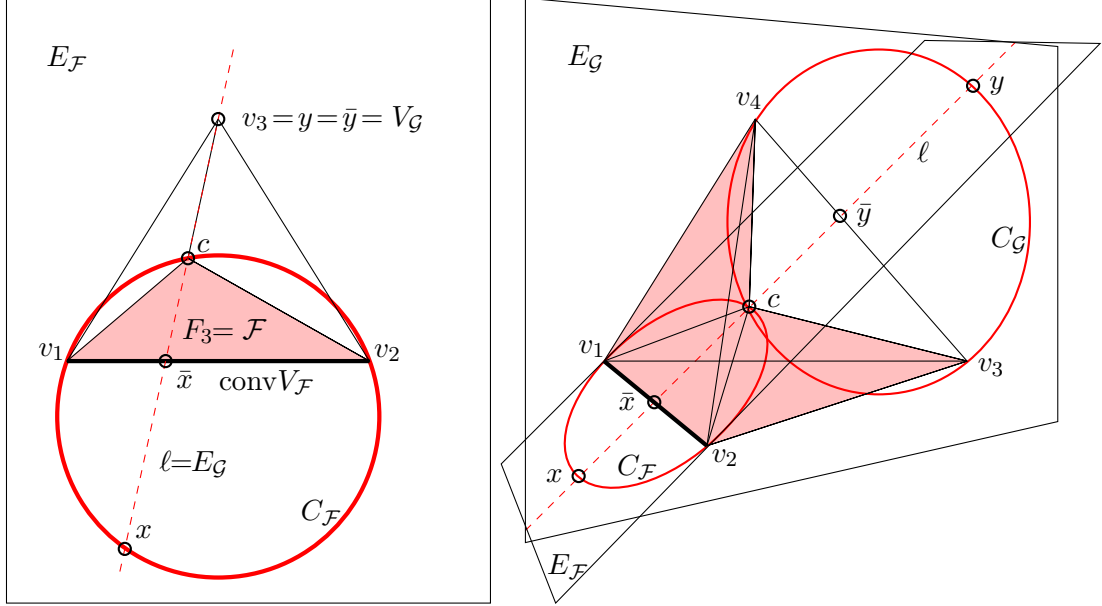


Figure 9: The situation of Lemma 20. The left figure illustrates  $d = 3, k = 2$ , the right one  $d = 4, k = 2$ .

**Lemma 20.** *In the situation just described, the point  $x$  lies outside the circumspheres of  $F_1, \dots, F_k$  and on the circumspheres of  $F_{k+1}, \dots, F_d$ .*

*Proof.* The five points  $x, \bar{x}, c, \bar{y}, y$  lie in this order along the line  $\ell$ , where the first two points coincide in the case  $k = 1$ , while the last two coincide for  $d - k = 1$ . The circumspheres of  $F_1, \dots, F_k$  intersect the line  $\ell$  in  $\{c, y\}$ , and thus the point  $x$  lies outside these spheres, while the circumspheres of  $F_{k+1}, \dots, F_d$  intersect the line  $\ell$  in  $\{c, x\}$ .  $\square$

**Lemma 21.** *If in the above situation the stellar subdivision of some or all of the facets  $F_1, \dots, F_k$  results in a Delaunay triangulation  $\mathcal{T}$ , then the point  $x$  lies outside all of the circumspheres of the newly created  $(d - 1)$ -faces.*

*Proof.* Without loss of generality, let us assume that  $\mathcal{T}$  is a single stellar subdivision of  $\Delta$  at  $F_1$  by a new vertex  $r$  inside  $F_1$ . This will result in  $d$  new facets  $F'_1, \dots, F'_d$ , whose vertex set consists of  $r$  together with all-but-one of the vertices of  $F_1$ , which are  $c, v_2, \dots, v_d$ .

We discuss them in three different cases (see Figure 10):

- (I) One new facet, say  $F'_1$ , does not contain  $c$ . Then  $c$  lies outside the circumsphere of  $F'_1$ , while all the vertices in  $V_{\mathcal{G}} := \{v_{k+1}, \dots, v_d\}$  are vertices of  $F'_1$ , so  $\bar{y}$  lies inside the circumsphere, or on its boundary (in the case  $d - k = 1$ ). In either case we conclude that  $x$  lies outside the circumsphere from the ordering on the line  $\ell$  described in the proof of Lemma 20.

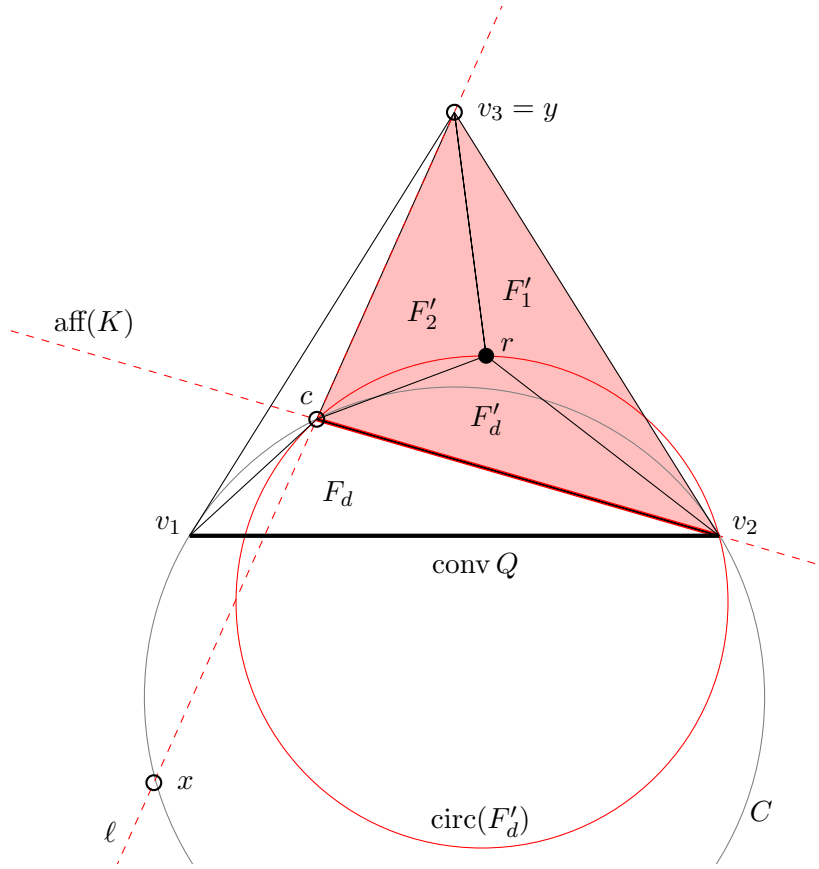


Figure 10: The three cases of Lemma 21, for  $d = 3$ ,  $k = 2$ .

- (II)  $k - 1$  new facets  $F'_2, \dots, F'_k$  do not contain a vertex  $v_j$ ,  $2 \leq j \leq k$ . In this case we argue as in Case (I).
- (III)  $d - k$  new facets  $F'_{k+1}, \dots, F'_d$  miss a vertex  $v_j$ ,  $(k + 1 \leq j \leq d)$  from  $V_G$ . Then  $F'_j$  is adjacent to the facet  $F_j$  because both share the  $(d - 2)$ -face  $K := \text{conv}(\{c, v_2, \dots, v_d\} \setminus v_j)$ . Their circumsheres intersect in  $\text{aff}(K)$ . The line  $\ell$  intersects  $\text{aff}(K)$  in  $c$ , hence  $x$  and  $\bar{x}$  lie on the same side of  $\text{aff}(K)$ , as well as  $v_1$ , because  $\bar{x}$  is a convex combination of  $v_1$  and  $\text{aff}(K)$ . So, the circumshere of  $F_j$  passes through  $x$  and  $v_1$  on the same side of  $\text{aff}(K)$ . Because  $\mathcal{T}$  is Delaunay, the circumshere of  $F'_j$  does not contain  $v_1$  and hence also not  $x$ .  $\square$

**Proposition 22.** *Let  $\Delta$  be a single stellar subdivision of a  $(d - 1)$ -simplex  $\sigma = \text{conv}\{v_1, \dots, v_d\}$  in  $\mathbb{R}^{d-1}$  by an interior point  $c \in \sigma$ , so the facets of  $\Delta$  are  $F_i = \text{conv}(\{c, v_1, \dots, v_d\} \setminus v_i)$ .*

*Let  $\mathcal{T}$  arise from this Delaunay triangulation  $\Delta$  by single stellar subdivisions of  $F_1, \dots, F_k$  by interior points  $r_i \in F_i$  ( $1 \leq i \leq k$ ).*

*If  $\mathcal{T}$  is a Delaunay triangulation, then  $k < 3$ .*

*Proof.* For  $d = 3$  this was established in Lemma 12, so we assume  $d > 3$ . As a single stellar subdivision can be undone without destroying the Delaunay property (Corollary 11), it is enough to show that  $\mathcal{T}$  cannot be a Delaunay triangulation if  $k = 3$ .

For the sake of contradiction we assume that such a  $\mathcal{T}$  is a Delaunay triangulation. Then we are in the situation discussed above, where we find the point  $x$  on the line  $\ell$ , which

by Lemmas 20 and 21 lies on the circumspheres of the facets  $F_4, \dots, F_d$ , but outside the circumspheres of all other facets of  $\mathcal{T}$ . Let  $\mathcal{R}$  denote the set of these other facets. The inversion of  $\mathbb{R}^{d-1}$  in the unit sphere centered at  $x$  sends all the vertices of  $\mathcal{T}$  to new points in  $\mathbb{R}^{d-1}$ . This inversion induces a simplicial map to a new triangulation  $\mathcal{T}'$ .

$$\Psi : \mathcal{R} \longrightarrow \mathcal{T}'.$$

As an abbreviation, we denote the images of  $\Psi$  by a prime ( $'$ ); for example,  $\Psi(v_1) = v'_1$ . Note that if we apply this to the images of simplices  $\sigma, F_1, \dots, F_3$ , then we refer to the simplices obtained by applying  $\Psi$  to the vertices.

The simplicial complex  $\mathcal{T}'$  is a part of the unique Delaunay subdivision of its vertex set, because for all cells in  $\mathcal{R}$ , an empty circumsphere is mapped to an empty circumsphere. This in particular shows that  $\mathcal{T}'$  is a simplicial complex. Let  $r'_1, r'_2, r'_3$  be the three images of the vertices that were used to perform single stellar subdivisions to  $F_1, F_2, F_3$ . Then these vertices are also interior vertices of  $\mathcal{T}'$  and hence  $\mathcal{T}'$  is the result of single stellar subdivisions of  $F'_1, F'_2$  and  $F'_3$  by  $r'_1, r'_2, r'_3$ . The inversion centered at  $x$  also implies that  $c', v'_1, v'_2$  and  $v'_3$  lie in a common 2-plane, as their preimages lie on a 2-sphere that passes through  $x$ . Note that no three of these four vertices lie in line. By checking some vertex incidences, we figure out that the structure of  $\mathcal{T}'$  can be described as follows: Take a  $(d-1)$ -simplex, split it into three simplices by inserting a vertex  $c'$  in the interior of a 2-face and then apply single stellar subdivisions to each of those three simplices by points  $r'_1, r'_2$  and  $r'_3$ . In particular, the support of  $\mathcal{T}'$  is convex, so  $T$  is the Delaunay triangulation of the set  $\{c', r'_1, r'_2, r'_3, v'_1, \dots, v'_d\}$  of  $d+4$  points in  $\mathbb{R}^{d-1}$ . (See the left part of Figure 11.)

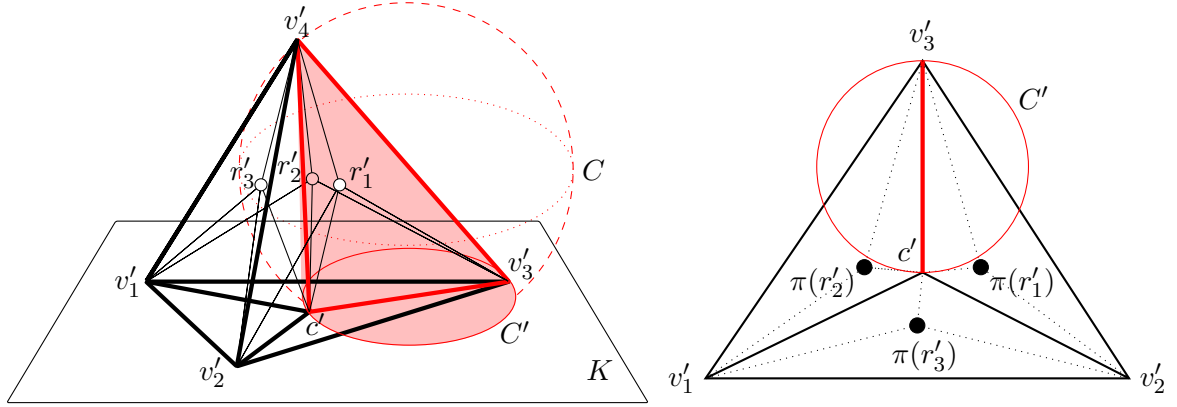


Figure 11: Left: An example for  $d = 4$ . Right: The projection image in the 2-plane  $K$ .

Let  $K$  be the 2-plane containing  $c', v'_1, v'_2$  and  $v'_3$  and let  $\mathcal{T}'_K$  denote the subcomplex of  $\mathcal{T}'$  that lies in  $K$ . We define barycentric coordinates by the points  $v'_1, \dots, v'_d$  and let  $\pi$  be the corresponding coordinate projection

$$\pi : \text{relint}(\mathcal{T}') \rightarrow \text{relint}(\mathcal{T}'_K) = \text{relint}(\text{conv}\{v_1, \dots, v_3\}).$$

As  $F_1$  and  $F_2$  share a  $(d-2)$ -face, we know that  $F'_1$  and  $F'_2$  also share the same  $(d-2)$ -face in  $\mathcal{T}'$ . It has vertex set  $\{c', v'_3, \dots, v'_d\}$  and it must have a supporting sphere. We pick one and call it  $C$ . The intersection of  $C$  with  $K$  is a supporting sphere for the edge  $(c', v'_3)$  in  $\mathcal{T}'_K$ . We call this 1-sphere  $C'$  and notice that the preimage  $\text{conv}(C')$  under  $\pi$ , which is contained in  $\text{conv}(C' \cup \{v'_4, \dots, v'_d\})$ , lies completely inside  $C$ .

(This crucial fact is illustrated in red in the right part of Figure 11: The sphere that contains  $C'$  as well as  $v'_4$  must enclose the whole truncated cone.)

This implies that the images of  $r'_1, r'_2$  and  $r'_3$  under the projection  $\pi$  lie outside  $C'$ , but in the interior of  $\mathcal{T}'_K$ . From this we derive that we can apply single stellar subdivisions to the three 2-faces of  $\mathcal{T}'_K$  by the vertices  $\pi(r'_1), \pi(r'_2)$  and  $\pi(r'_3)$  such that the edges  $(c', v'_1), (c', v'_2)$  and  $(c', v'_3)$  would still be locally Delaunay, as indicated in the right part of Figure 11. However, in Lemma 12 we have already proved that this is impossible.  $\square$

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