

THE LUR'E EQUATION IN A BEHAVIORAL CONTEXT *

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Abstract. The well-known Kalman-Yakubovich-Popov Lemma establishes an equivalence between dissipativity and the solvability of a linear matrix inequality, see [25]. In this paper we strengthen this result by showing the equivalence of dissipativity to the solvability of a so-called *Lur'e equation*, which mainly is a linear matrix inequality with a rank minimizing condition. Finally, we apply the result to standard systems to obtain the well-known result about the solvability of the algebraic Riccati equation.

1. Introduction. The importance of the algebraic Riccati equation in control theory is widely accepted [26]. However, in its original form, the algebraic Riccati equation is only applicable to standard system, i.e., to systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (1.1)$$

where $E = I$ is the identity matrix, $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $x(t) \in \mathbb{C}^n$ is the state, and $u(t) \in \mathbb{C}^m$ is the input. A generalization of the algebraic Riccati equation to descriptor systems, i.e., to systems of the form (1.1), where $E \neq I$ is not necessarily the identity, is considered in [16]. A drawback of the results in [16, 26] is that one has to assume regular cost/supply functions, i.e., one has to assume that the cost/supply function

$$s(u(t), x(t)) := \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

(with $Q = Q^* \in \mathbb{C}^{n,n}$, $S \in \mathbb{C}^{n,m}$, and $R = R^* \in \mathbb{C}^{m,m}$) satisfies the property that R is invertible. Avoiding this assumption leads to the Lur'e equation, for the case of standard systems, see [21].

In this paper we will generalize these results to behavioral system, cf. [29, 30, 31], which will be introduced in Definition 1.2. The advantage of a behavioral approach to this topic is that it incorporates systems of the form (1.1) as a special case, while at the same time the results get simpler and one does not have to determine a-priori which of the variables in (1.1) are inputs u or states x . Also, many systems seem to be described in a behavioral way most naturally [32], especially electrical RLC circuits [33].

EXAMPLE 1.1. Consider the following diagram of an electrical RLC circuit, which is taken from [8].

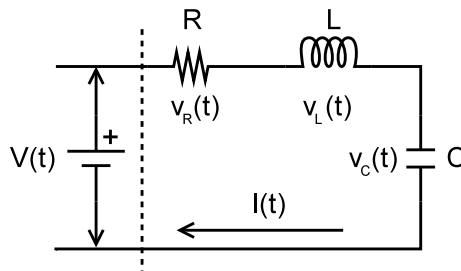


FIG. 1.1. The simple RLC circuit from [8].

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By $V(t)$ and $I(t)$ we denote the voltage and current, resp., at time t through the source on the left. R denotes the resistance, L the inductance, and C denotes the capacity. Since the whole circuit consists of only one loop, i.e., since all electrical elements in the circuit are in series, the current through each element is equal to $I(t)$ because of Kirchoff's current law. However, the voltages across the individual elements may vary. Let $v_R(t)$, $v_L(t)$, and $v_C(t)$ denote the voltages across the corresponding elements at time t . Then with the definitions

$$F := \begin{bmatrix} L & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad G := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ R & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix}, \quad z := \begin{bmatrix} I \\ v_L \\ v_C \\ v_R \\ V \end{bmatrix}$$

the equations which describe the electrical behavior of the circuit in Fig. 1.1 can be written as

$$F\dot{z}(t) + Gz(t) = 0, \quad (1.2)$$

see [8]. With this notation and the definition

$$H := \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the power supplied to the system (1.2) via the port indicated by the dashed line is measured via the quadratic function s , which is called the supply function, given by

$$s(z(t)) := z^*(t)Hz(t) = V(t)I(t). \quad (1.3)$$

We use the following notation. By \mathcal{C}_∞^q we denote the infinitely often differentiable functions, mapping from \mathbb{R} to \mathbb{C}^q . By \mathcal{C}_+^q we denote the decaying elements of \mathcal{C}_∞^q , i.e., all $z \in \mathcal{C}_\infty^q$ such that for every $i \in \mathbb{N}_0$ there exist $a_i, b_i > 0$ such that the inequality $\|z(t)\| \leq b_i e^{-a_i t}$ holds for all $t \in [0, \infty)$. By \mathcal{C}_c^q we denote all elements of \mathcal{C}_∞^q , which have compact support, i.e., which vanish outside of a bounded interval. We use \mathcal{C}_∞ , \mathcal{C}_+ , and \mathcal{C}_c as abbreviations for \mathcal{C}_∞^1 , \mathcal{C}_+^1 , and \mathcal{C}_c^1 , resp. Furthermore, by $\mathbb{C}[\lambda]$ we denote the polynomials with coefficients in \mathbb{C} , and by $\mathbb{C}(\lambda)$ we denote the associated quotient field, i.e., the rational functions. We use the symbol $\mathbb{C}[\lambda]^{p,q}$ to denote the p -by- q polynomial matrices and the symbol $\mathbb{C}(\lambda)^{p,q}$ to denote the p -by- q rational matrices. For $P \in \mathbb{C}[\lambda]^{p,q}$ of the form $P(\lambda) = \sum_{i=0}^K \lambda^i P_i$, with $P_i \in \mathbb{C}^{p,q}$ for $i = 0, \dots, K$ and $P_K \neq 0$ we call K the *degree* or *order* of P and we will use the symbol $\text{rank}_{\mathbb{C}(\lambda)}(P)$ to denote the so-called *normal rank*, i.e., the rank of P over the field of the rational functions. In contrast to this, for $\lambda_0 \in \mathbb{C}$ the expression $\text{rank}(P(\lambda_0))$ denotes the rank of the matrix $P(\lambda_0) \in \mathbb{C}^{p,q}$ in the usual sense. Using the Smith canonical form [9, p. 141, Theorem 3] one can show that

$$\text{rank}_{\mathbb{C}(\lambda)}(P) = \max_{\lambda_0 \in \mathbb{C}} \text{rank}(P(\lambda_0)). \quad (1.4)$$

We will call $\lambda_0 \in \mathbb{C}$ a *zero* of P , if $\text{rank}(P(\lambda_0)) < \text{rank}_{\mathbb{C}(\lambda)}(P)$ and the set of all zeros of P will be denoted by $\mathfrak{Z}(P)$. For a polynomial matrix $P \in \mathbb{C}[\lambda]^{p,q}$ we define the *para-conjugate*

transposed $P^\sim \in \mathbb{C}[\lambda]^{q,p}$ via $P^\sim(\lambda) := P^*(-\bar{\lambda})$ and a matrix $\mathcal{N} \in \mathbb{C}[\lambda]^{k,k}$ is called *para-Hermitian* if $\mathcal{N} = \mathcal{N}^\sim$, cf. [35]. By \mathbb{C}_- we denote the strict left half plane of the complex numbers and by $\overline{\mathbb{C}_-}$ the closed left half plane. We use i to denote the imaginary unit.

In behavioral systems theory a system is usually viewed as a subset \mathfrak{B} , called the behavior, of some superset \mathcal{U} , called the universe, see [31]. Since in this paper we will only deal with linear, time-invariant, first-order system we only consider universes of the form $\mathcal{U} = \mathcal{C}_\infty^q$ and make the following definition.

DEFINITION 1.2. [18] Let $P(\lambda) := \lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order polynomial matrix. Then the set

$$\mathfrak{B}(\lambda F + G) := \left\{ z \in \mathcal{C}_\infty^q \mid P\left(\frac{d}{dt}\right)z = 0 \right\} = \text{kernel}_{\mathcal{C}_\infty}\left(P\left(\frac{d}{dt}\right)\right)$$

is called the behavior of $\lambda F + G$,

$$\mathfrak{B}_+(\lambda F + G) := \left\{ z \in \mathcal{C}_+^q \mid P\left(\frac{d}{dt}\right)z = 0 \right\} = \text{kernel}_{\mathcal{C}_+}\left(P\left(\frac{d}{dt}\right)\right)$$

is called the decaying behavior of $\lambda F + G$, and

$$\mathfrak{B}_c(\lambda F + G) := \left\{ z \in \mathcal{C}_c^q \mid P\left(\frac{d}{dt}\right)z = 0 \right\} = \text{kernel}_{\mathcal{C}_c}\left(P\left(\frac{d}{dt}\right)\right)$$

is called the compact behavior of $\lambda F + G$. The elements of $\mathfrak{B}(\lambda F + G)$, $\mathfrak{B}_+(\lambda F + G)$, and $\mathfrak{B}_c(\lambda F + G)$ are called (decaying/compact) trajectories of the system.

Let $H = H^* \in \mathbb{C}^{q,q}$ be a Hermitian matrix. Then we measure the power supplied to the system along the trajectory $z \in \mathfrak{B}(\lambda F + G)$ at the time point $t \in \mathbb{R}$ via the expression $z^*(t)Hz(t)$. Consequently, the amount of energy supplied to a system over a time interval $[t_0, t_1]$ is then measured by

$$\int_{t_0}^{t_1} z^*(t)Hz(t)dt. \quad (1.5)$$

If the expression in (1.5) becomes negative, then we say that energy is extracted from the system, otherwise we say that energy is supplied. Of special physical interest are systems which do not generate energy, i.e., systems from which one can never extract more energy than the amount that has been supplied to it. This is formalized in the following definition.

DEFINITION 1.3. [27, 28] Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order polynomial matrix and let $H = H^* \in \mathbb{C}^{q,q}$ be Hermitian. Then we say that $(\lambda F + G, H)$ is dissipative if there exists a so-called storage function, i.e., a continuous function $\Theta : \mathbb{C}^q \rightarrow \mathbb{R}$ with $\Theta(0) = 0$, which satisfies the so-called dissipation inequality

$$\Theta(z(t_1)) - \Theta(z(t_0)) \leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt. \quad (1.6)$$

The storage function Θ measures the amount of energy which is stored internally in the system. Thus, the right hand side of the dissipation inequality (1.6) measures the gain in internally stored energy over the time interval $[t_0, t_1]$.

EXAMPLE 1.4. (Example 1.1 revisited) To show that the circuit in Fig. 1.1 is dissipative, we can assume that the energy, which is stored in the capacitor with capacity C , is given by $\frac{1}{2}Cv_C^2$ and the energy, which is stored in the inductor with inductance L , is given by $\frac{1}{2}LI^2$. Thus, for (1.2) and (1.3) we define the storage function

$$\Theta(z(t)) := \frac{1}{2}Cv_C(t)^2 + \frac{1}{2}LI(t)^2. \quad (1.7)$$

Using (1.3) and the individual equations which make up (1.2) we find

$$\begin{aligned} \frac{d}{dt}\Theta(z(t)) &= C\dot{v}_C(t)v_C(t) + L\dot{I}(t)I(t) \\ &= I(t)v_C(t) + I(t)v_L(t) = I(t)(v_C(t) + v_L(t)) \\ &= I(t)(V(t) - v_R(t)) = I(t)V(t) - I(t)v_R(t) = s(z(t)) - RI^2(t) \leq z^*(t)Hz(t). \end{aligned} \quad (1.8)$$

Integrating equation (1.8) from t_0 to t_1 and using the fundamental theorem of calculus we obtain that (1.7) is indeed a storage function of the system (1.2) which implies dissipativity.

DEFINITION 1.5. Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$. Then we call

$$\mathfrak{R}(\lambda F + G) := \{\hat{z} \in \mathbb{C}^q \mid \exists z \in \mathfrak{B}(\lambda F + G) \text{ such that } \hat{z} = z(0)\}$$

the reachable set of $\lambda F + G$,

$$\mathfrak{R}_+(\lambda F + G) := \{\hat{z} \in \mathbb{C}^q \mid \exists z \in \mathfrak{B}_+(\lambda F + G) \text{ such that } \hat{z} = z(0)\}$$

the decaying reachable set of $\lambda F + G$, and

$$\mathfrak{R}_c(\lambda F + G) := \{\hat{z} \in \mathbb{C}^q \mid \exists z \in \mathfrak{B}_c(\lambda F + G) \text{ such that } \hat{z} = z(0)\}$$

the compact reachable set of $\lambda F + G$.

The main result of this paper states that (under the assumptions of controllability, which is given in Definition 2.6, and feasibility, see Section 5) dissipativity is equivalent to the solvability of a Lur'e equation. The Lur'e equation is introduced in the following definition.

DEFINITION 1.6. Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order matrix polynomial and let $H = H^* \in \mathbb{C}[\lambda]^{q,q}$ be Hermitian. Form the para-Hermitian polynomial

$$\mathcal{N}(\lambda) := \lambda N_1 + N_0 := \lambda \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix}. \quad (1.9)$$

Then we call $Z \in \mathbb{C}^{p,q}$ a solution of the Lur'e equation (associated with $(\lambda F + G, H)$), if the following conditions are fulfilled:

- 1.) $F^*Z = Z^*F$
- 2.) $0 \leq G^*Z + Z^*G + H$
- 3.) $\text{rank}(G^*Z + Z^*G + H) = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) =: s$
- 4.) for every Cholesky factor $\mathcal{C} \in \mathbb{C}^{s,q}$ of the matrix $G^*Z + Z^*G + H = \mathcal{C}^*\mathcal{C}$, we have that

$$\mathfrak{B}_+(\mathcal{N}) = \begin{bmatrix} Z \\ I \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) \oplus \begin{bmatrix} I \\ 0 \end{bmatrix} \mathfrak{B}_+(-\lambda F^* + G^*), \quad (1.10)$$

and we have

$$\mathfrak{Z} \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) = \mathfrak{Z}(\mathcal{N}) \cap \overline{\mathbb{C}_-}. \quad (1.11)$$

Furthermore, Z is called a strong solution of the Lur'e equation, if in addition to the above, for every Cholesky factor $\mathcal{C} \in \mathbb{C}^{s,q}$ the following condition is fulfilled:

- 5.) for every $\hat{z} \in \mathfrak{R}_+(\lambda F + G)$ there exists a $z_0 \in \mathfrak{R}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right)$ such that $F\hat{z} = Fz_0$.

It is obvious that the properties 2.) and 3.) of Definition 1.6 are equivalent to the existence of a Cholesky factor $\mathcal{C} \in \mathbb{C}^{s,q}$ which fulfills $G^*Z + Z^*G + H = \mathcal{C}^*\mathcal{C}$. Thus properties 1.) - 3.) of Definition 1.6 can be rewritten as

$$F^*Z = Z^*F, \quad G^*Z + Z^*G + H = \mathcal{C}^*\mathcal{C}, \quad (1.12)$$

which is an equation in the indeterminates Z and \mathcal{C} . The identity (1.12) constitutes the actual Lur'e equation, cf. [21].

In the following example we show that the Lur'e equation for the circuit in Fig. 1.1 is solvable. We do this by an educated guess of the solution. In the remainder of the paper we will develop a constructive method to compute the solution. Since this method is quite involved it is not convenient for manual calculations, but it gives a hint about how a solution of the Lur'e equation can be computed numerically. Further notes concerning the numerical aspects can be found in the Conclusion, Section 9.

EXAMPLE 1.7. (*Example 1.1 revisited*) We show that the Lur'e equation associated with (1.2) and (1.3) has the solution

$$Z = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (1.13)$$

To prove point 1.) of Definition 1.6 note that

$$Z^*F = \text{diag} \left(0, 0, \frac{C}{2}, 0, \frac{L}{2} \right), \quad (1.14)$$

is symmetric and point 2.) is fulfilled since

$$Z^*G + G^*Z + H = \text{diag} (R, 0, 0, 0, 0) \geq 0. \quad (1.15)$$

Let the matrices \mathcal{N} , N_1 , and N_2 be according to (1.9) with F , G , and H as in (1.2) and (1.3). Then, using the identity (1.4), we find

$$9 = \text{rank} (1 \cdot N_1 + N_0) \leq \max_{\lambda_0 \in \mathbb{C}} \text{rank} (\mathcal{N}(\lambda_0)) = \text{rank}_{\mathbb{C}(\lambda)} (\mathcal{N}) \leq 9,$$

and

$$4 = \text{rank} (1 \cdot F + G) \leq \max_{\lambda_0 \in \mathbb{C}} \text{rank} (\lambda_0 F + G) = \text{rank}_{\mathbb{C}(\lambda)} (\lambda F + G) \leq 4.$$

This means that in Definition 1.6 we have $s = 1$, which with (1.15) implies that point 3.) is fulfilled. Let $\mathcal{C} \in \mathbb{C}^{1,5}$ be a Cholesky factor of $H + G^*Z + Z^*G = \mathcal{C}^*\mathcal{C}$, i.e. let

$$\mathcal{C} = [\sqrt{R} \quad 0 \quad 0 \quad 0 \quad 0].$$

Then, by basic manipulations we find that

$$\mathfrak{B}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) = \mathfrak{B}_+ \left(\begin{bmatrix} \lambda L & -1 & 0 & 0 & 0 \\ -1 & 0 & \lambda C & 0 & 0 \\ R & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 \\ \sqrt{R} & 0 & 0 & 0 & 0 \end{bmatrix} \right) = \{0\}, \quad (1.16)$$

and also that $\mathfrak{B}_+(-\lambda F^* + G^*) = \{0\}$. Also, we have that $\mathfrak{B}_+(\mathcal{N}) = \{0\}$, if and only if

$$\begin{aligned}
\{0\} &= \mathfrak{B}_+ \left(\begin{bmatrix} I & 0 \\ Z^* & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \right) \\
&= \mathfrak{B}_+ \left(\begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & C^* C \end{bmatrix} \right) \\
&= \mathfrak{B}_+ \left(\begin{bmatrix} 0 & 0 & 0 & 0 & \lambda L & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & \lambda C & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \\ -\lambda L & -1 & R & 0 & R & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda C & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right). \quad (1.17)
\end{aligned}$$

The correctness of (1.17), however, can again be determined by basic manipulations. Since one can see by a similar argument that the identity (1.11) holds, we conclude that point 4.) of Definition 1.6 is fulfilled, which means that (1.13) is indeed a solution of the Lur'e equation associated with the circuit in Fig. 1.1. As a final remark, note that with (1.14) we can write the storage function Θ in (1.7) as

$$\Theta(z(t)) = z^*(t)Z^*Fz(t).$$

Similar to the derivation in Example 1.4 one can easily show that the solvability of the Lur'e equation implies dissipativity. This is done in the following way. Let $Z \in \mathbb{C}^{p,q}$ be a solution of the Lur'e equation and define $\Theta : \mathbb{C}^q \rightarrow \mathbb{R}$ through

$$\Theta(\hat{z}) := \hat{z}^* F^* Z \hat{z}.$$

Then Θ is continuous and fulfills $\Theta(0) = 0$. Also, we find that for all $z \in \mathfrak{B}(\lambda F + G)$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{d}{dt}\Theta(z(t)) &= \dot{z}^*(t)F^*Zz(t) + z^*(t)Z^*F\dot{z}(t) \\
&= -z^*(t)G^*Zz(t) - z^*(t)Z^*Gz(t) \leq z^*(t)Hz(t).
\end{aligned}$$

Integration over $[t_0, t_1]$ results in

$$\Theta(z(t_1)) - \Theta(z(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt}\Theta(z(t))dt \leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt,$$

which shows that Θ is a storage function. The other direction is harder to show, and the proof is the content of the remainder of the paper.

2. Preliminaries. In this section we first introduce the Kronecker canonical form, a very helpful tool for the analysis of behaviors, as defined in Definition 1.2. With this we will introduce the reachable sets and the notion of a strong solution of the Lur'e equation. Afterwards, we will derive some basic properties which solutions of the Lur'e equation fulfill and then, finally, we will introduce the Thompson canonical form, which is the equivalent to the Kronecker canonical form for para-Hermitian pencils.

THEOREM 2.1 (Kronecker canonical form). [10, 21] Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order matrix polynomial. Then there exist invertible matrices $S \in \mathbb{C}^{p,p}$ and $T \in \mathbb{C}^{q,q}$ such that

$$S(\lambda F + G)T = \text{diag}(\mathcal{K}_1(\lambda), \dots, \mathcal{K}_d(\lambda)), \quad (2.1)$$

where each of the blocks $\mathcal{K}_j(\lambda)$ takes one of the forms (K1) - (K4) below.

1. Right singular block with k_j rows

$$\lambda \begin{bmatrix} -1 & 0 & & \\ & \ddots & \ddots & \\ & & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \quad (K1)$$

2. Finite eigenvalue $\lambda_j \in \mathbb{C}$ of size k_j

$$\lambda \begin{bmatrix} -1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} + \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix} \quad (K2)$$

3. Infinite eigenvalue of size k_j

$$\lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad (K3)$$

4. Left singular block with k_j columns

$$\lambda \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & \ddots & 1 & \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & & 1 \end{bmatrix} \quad (K4)$$

EXAMPLE 2.2. (Example 1.1 revisited) Defining the transformation matrices $T \in \mathbb{C}^{5,5}$ and $S \in \mathbb{C}^{4,4}$ via

$$S := \begin{bmatrix} 0 & -\frac{1}{C} & 0 & 0 \\ -\frac{1}{LC} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } T := \begin{bmatrix} 0 & C & 0 & 0 & 0 \\ 0 & 0 & LC & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & RC & 0 & -1 & 0 \\ 1 & RC & LC & -1 & 1 \end{bmatrix}, \quad (2.2)$$

we have (with F , G , and H defined as in (1.2) and (1.3)) that

$$S(\lambda F + G)T = \left[\begin{array}{ccc|cc} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right], \quad (2.3)$$

which means that the Kronecker canonical form for the system depicted in Fig. 1.1 has one block of type (K1) with size $k_j = 2$ and two blocks of type (K3) with size $k_j = 1$.

The Kronecker canonical form can be used to compute the reachable sets of Definition 1.5, by considering the reachable sets of each block (K1)-(K4) separately. From elementary considerations we obtain that for blocks \mathcal{K}_j of type (K1) we have

$$\mathfrak{R}(\mathcal{K}_j) = \mathfrak{R}_+(\mathcal{K}_j) = \mathfrak{R}_c(\mathcal{K}_j) = \mathbb{C}^{k_j}, \quad (2.4)$$

for blocks \mathcal{K}_j of type (K2) we have

$$\mathfrak{R}(\mathcal{K}_j) = \mathbb{C}^{k_j}, \quad \mathfrak{R}_+(\mathcal{K}_j) = \begin{cases} \mathbb{C}^{k_j} & \text{for } \operatorname{Re}(\lambda_j) < 0 \\ \{0\} & \text{for } \operatorname{Re}(\lambda_j) \geq 0 \end{cases}, \quad \mathfrak{R}_c(\mathcal{K}_j) = \{0\}, \quad (2.5)$$

and for blocks \mathcal{K}_j of type (K3) and (K4) we have

$$\mathfrak{R}(\mathcal{K}_j) = \mathfrak{R}_+(\mathcal{K}_j) = \mathfrak{R}_c(\mathcal{K}_j) = \{0\}. \quad (2.6)$$

With these identities and with the columns of the transformation matrix T in (2.1) one can specify the reachable sets of arbitrary behaviors. A more detailed exposition of this procedure can be found in [2, Lemma 2.19].

Also, the Kronecker canonical form reveals the set of zeros of a matrix pencil. Again, looking at the zeros of each block (K1)-(K4) separately we find that $\mathfrak{Z}(\mathcal{K}_j) = \emptyset$ for blocks \mathcal{K}_j of type (K1), (K3), and (K4). Only blocks \mathcal{K}_j of type (K2) have zeros, namely $\mathfrak{Z}(\mathcal{K}_j) = \{\lambda_j\}$. Since with (2.1) we have

$$\mathfrak{Z}(\lambda F + G) = \mathfrak{Z}(S(\lambda F + G)T) \quad (2.7)$$

we conclude that the set of zeros of a pencil $\lambda F + G$ is given by the union of all λ_j , corresponding to blocks of type (K2).

EXAMPLE 2.3. (*Example 1.1 revisited*) From the Kronecker canonical form (2.3) of the circuit in Fig. 1.1 we conclude with (2.4)-(2.6) that all three reachable sets $\mathfrak{R}(\lambda F + G)$, $\mathfrak{R}_+(\lambda F + G)$, and $\mathfrak{R}_c(\lambda F + G)$ are given by the span of the first three columns of the matrix T from (2.2). However, from (1.16) we conclude that

$$\mathfrak{R}_+ \left(\begin{bmatrix} \lambda F + G \\ C \end{bmatrix} \right) = \{0\},$$

which implies that the solution of the Lur'e equation Z given in (1.13) is not a strong solution.

The advantage of a strong solution of a Lur'e equation is that we can use it to make a statement about its uniqueness, as shown in the following Lemma.

LEMMA 2.4. Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$ and let $Z_1, Z_2 \in \mathbb{C}^{p,q}$ be two strong solutions of the Lur'e equation. Then

$$\hat{z}^* F^* Z_1 \hat{z} = \hat{z}^* F^* Z_2 \hat{z}$$

for all $\hat{z} \in \mathfrak{R}_+(\lambda F + G)$.

Proof. For the proof we introduce the so-called *available storage* [2, 26, 34] associated with $(\lambda F + G, H)$ as the function $\Theta_+ : \mathbb{C}^q \rightarrow \mathbb{R} \cup \{\pm\infty\}$, which is defined through

$$\Theta_+(\hat{z}) := - \inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ z(0) = \hat{z}}} \int_0^\infty z^*(t) H z(t) dt.$$

Let Z be any strong solution of the Lur'e equation as in Definition 1.6. Let $\hat{z} \in \mathfrak{R}_+(\lambda F + G)$ be arbitrary. Using property 5.) of Definition 1.6, let

$$z_0 \in \mathfrak{R}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right)$$

with $F\hat{z} = Fz_0$. Furthermore, let

$$z \in \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right)$$

with $z(0) = z_0$ and define $\mu := Zz$. Then from property 4.) of Definition 1.6 we conclude that $(\mu, z) \in \mathfrak{B}_+(\mathcal{N})$. Using [3, Theorem 5] we conclude that this implies

$$-\Theta_+(z_0) = \int_0^\infty z^*(t)Hz(t)dt,$$

from which we deduce that

$$\begin{aligned} \Theta_+(\hat{z}) &= \Theta_+(z_0) = - \int_0^\infty z^*(t)Hz(t)dt \\ &= - \int_0^\infty z^*(t)(\mathcal{C}^*\mathcal{C} - G^*Z - Z^*G)z(t)dt \\ &= \int_0^\infty z^*(t)G^*Zz(t) + z^*(t)Z^*Gz(t)dt \\ &= - \int_0^\infty z^*(t)F^*Zz(t) + z^*Z^*G\dot{z}(t)dt \\ &= - \int_0^\infty \frac{d}{dt} (z^*(t)F^*Zz(t)) dt \\ &= z^*(0)F^*Zz(0) = z_0^*F^*Zz_0 = \hat{z}^*F^*Z\hat{z}. \end{aligned}$$

Since the available storage Θ_+ does not depend on Z we obtain the assertion. \square

We make the following observation for every matrix Z which at least fulfills point 1.) of Definition 1.6.

LEMMA 2.5. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let $Z \in \mathbb{C}^{p,q}$ be such that $F^*Z = Z^*F$ and define*

$$s := \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \right) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)} (\lambda F + G)$$

as in Definition 1.6. Then we have

$$\text{rank} (H + G^*Z + Z^*G) \geq s.$$

Proof. Since

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H + G^*Z + Z^*G \end{bmatrix}$$

we obtain that

$$\begin{aligned} &\text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \right) = \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H + G^*Z + Z^*G \end{bmatrix} \right) \\ &\leq \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & 0 \end{bmatrix} \right) + \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & 0 \\ 0 & H + G^*Z + Z^*G \end{bmatrix} \right) \\ &= 2 \cdot \text{rank}_{\mathbb{C}(\lambda)} (\lambda F + G) + \text{rank} (H + G^*Z + Z^*G), \end{aligned}$$

and thus the claim follows. \square

Lemma 2.5 is the reason why a solution of the Lur'e equation can also be called a *rank minimizing solution*, cf. [26, Remark 10].

DEFINITION 2.6. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$. Then we call the system $\mathfrak{B}(\lambda F + G)$ trim if $\mathfrak{R}(\lambda F + G) = \mathbb{C}^q$, cf. [25]. Furthermore, we call $\mathfrak{B}(\lambda F + G)$ controllable if $\mathfrak{R}(\lambda F + G) = \mathfrak{R}_c(\lambda F + G)$ and we call $\mathfrak{B}(\lambda F + G)$ stabilizable if $\mathfrak{R}(\lambda F + G) = \mathfrak{R}_+(\lambda F + G)$.*

The following Lemma characterizes the concepts from Definition 2.6 through properties of the Kronecker canonical form.

LEMMA 2.7. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ have the Kronecker canonical form (2.1). Then we have the following assertions:*

1. $\mathfrak{B}(\lambda F + G)$ is trim if and only if there are no blocks of type (K3) and (K4).
2. $\mathfrak{B}(\lambda F + G)$ is controllable if and only if there are no blocks of type (K2).
3. $\mathfrak{B}(\lambda F + G)$ is stabilizable if and only if all blocks of type (K2) satisfy $\operatorname{Re}(\lambda_j) < 0$.

Proof. The proof follows immediately from the identities (2.4)-(2.6). \square

The following Corollary characterizes the concepts from Definition 2.6 algebraically.

COROLLARY 2.8. *Let $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$. Then we have the following:*

1. $\mathfrak{B}(\lambda F + G)$ is trim if and only if $\operatorname{rank}(F) = p$.
2. $\mathfrak{B}(\lambda F + G)$ is controllable if and only if $\mathfrak{Z}(\lambda F + G) = \emptyset$.
3. $\mathfrak{B}(\lambda F + G)$ is stabilizable if and only if $\mathfrak{Z}(\lambda F + G) \subset \mathbb{C}_-$.

Proof. The proof follows from Lemma 2.7, identity (2.7), and the Kronecker canonical form (2.1). A more detailed exposition can be found in [2, Lemma 2.21]. \square

Looking at property 4.) of Definition 1.6 we see that systems of the form $\mathfrak{B}_+(\mathcal{N})$, with $\mathcal{N} = \mathcal{N}^\sim$ being para-Hermitian, play a special role. Although such systems can in principle also be analyzed via the Kronecker canonical form, it would be preferable to respect the para-Hermitian structure of \mathcal{N} . This can be done with the Thompson canonical form, which is introduced in the following Theorem 2.9.

Note that we only use the Thompson canonical form for theoretical purposes. Since the Thompson canonical form involves the transformation with non-unitary matrices X , we conclude that the computation of the Thompson canonical form is not stable, and thus inappropriate from a numerical point of view. However, by unitary transformations it is possible to obtain a so-called *staircase form* in a numerically stable way, cf. [4, 5], from which important characteristics of the Thompson canonical form can be obtained, see also the Conclusion, Section 9. The computation of a solution of the Lur'e equation via unitary transformations will be discussed in a forthcoming paper.

THEOREM 2.9 (Thompson canonical form). [22, 23] *Let $\lambda N_1 + N_0 \in \mathbb{C}[\lambda]^{k,k}$ be a para-Hermitian first-order matrix polynomial. Then there exists an invertible matrix $X \in \mathbb{C}^{k,k}$ such that*

$$X^* (\lambda N_1 + N_0) X = \operatorname{diag}(\mathcal{T}_1(\lambda), \dots, \mathcal{T}_d(\lambda)), \quad (2.8)$$

where each of the blocks $\mathcal{T}_j(\lambda)$ takes one of the forms (T1) - (T6) below. By \hat{k}_j we denote the size of block j and by τ_1, \dots, τ_6 we denote the number of blocks of type (T1), ..., (T6).

We can use the Thompson canonical form to specify the zeros of a para-Hermitian pencil, by looking at the zeros of each block (T1) - (T6) separately. This shows that $\mathfrak{Z}(\mathcal{T}_j) = \emptyset$ for all blocks \mathcal{T}_j of type (T4), (T5), and (T6). For all blocks \mathcal{T}_j of type (T2) and (T3) we have $\mathfrak{Z}(\mathcal{T}_j) = \{\iota\omega_j\}$, and for all blocks \mathcal{T}_j of type (T1) we find $\mathfrak{Z}(\mathcal{T}_j) = \{\lambda_j, -\bar{\lambda}_j\}$.

DEFINITION 2.10. *Let $\mathcal{N} \in \mathbb{C}[\lambda]^{k,k}$ be a para-Hermitian first-order pencil with Thompson canonical form (2.8). Then we say that \mathcal{N} has index $\nu \in \mathbb{N} \setminus \{0\}$ if there is at least one block of type (T4) or (T5) with size $\hat{k}_j = \nu$ and all blocks of type (T4) or (T5) have size $\hat{k}_j \leq \nu$. We say that \mathcal{N} has index $\nu = 0$, if there are no blocks of type (T4) and (T5).*

3. Inertia results. In this section we first determine the inertia indices of the blocks in the Thompson canonical form (T1) - (T6) for points on the imaginary axis. This can already be found in [7, Section 4] and is repeated here, for the sake of completeness. Afterwards we will discuss how the results can be used to make statements about the Thompson canonical form of the pencil \mathcal{N} from (1.9) in the case of a dissipative system. These results are a simplification of the results in [6, 7].

DEFINITION 3.1. *Let the sign function*

$$\text{sign} : \mathbb{R} \cup \{\pm\infty\} \rightarrow \{-1, 1\},$$

be defined as $\text{sign}(\omega) = 1$ for $\omega \geq 0$ and as $\text{sign}(\omega) = -1$ for $\omega < 0$. Furthermore, for a Hermitian matrix $B = B^* \in \mathbb{C}^{k,k}$ with π positive, ζ zero, and ν negative eigenvalues we call the 1-by-3 vector $\text{In}(B) := [\pi \quad \zeta \quad \nu]$ the inertia index of B and with this we define the sign-sum function through

$$\eta(B) := \pi + \zeta - \nu.$$

For a first-order para-Hermitian matrix polynomial $\mathcal{N}(\lambda) = \lambda N_1 + N_0 \in \mathbb{C}[\lambda]^{k,k}$ we define

$$\mathcal{N}(\iota\infty) := \iota N_1 \quad \text{and} \quad \mathcal{N}(-\iota\infty) := -\iota N_1.$$

LEMMA 3.2. *For every $\omega \in \mathbb{R} \cup \{\pm\infty\}$ and every block of the form (T1)-(T6) the expression $\mathcal{T}_j(\iota\omega)$ is a Hermitian matrix. Thus, $\eta(\mathcal{T}_j(\iota\omega))$ is well-defined and its values can be specified as in Table 3.1.*

TABLE 3.1
Sign-sum values of the blocks (T1) - (T6) on the imaginary axis

Block	sign-sum for $\omega \in \mathbb{R} \cup \{\pm\infty\}$
(T1)	$\eta(\mathcal{T}_j(\iota\omega)) = 0$
(T2)	$\eta(\mathcal{T}_j(\iota\omega)) = \begin{cases} 0 & \omega \neq \omega_j \\ 1 + \text{sign}(\epsilon_j) & \omega = \omega_j \end{cases}$
(T3)	$\eta(\mathcal{T}_j(\iota\omega)) = \begin{cases} \text{sign}(\epsilon_j(\omega_j - \omega)) & \omega \neq \omega_j \\ 1 & \omega = \omega_j \end{cases}$
(T4)	$\eta(\mathcal{T}_j(\iota\omega)) = \begin{cases} 0 & \omega \neq \pm\infty \\ 1 \mp \text{sign}(\epsilon_j) & \omega = \pm\infty \end{cases}$
(T5)	$\eta(\mathcal{T}_j(\iota\omega)) = \begin{cases} \text{sign}(\epsilon_j) & \omega \neq \pm\infty \\ 1 & \omega = \pm\infty \end{cases}$
(T6)	$\eta(\mathcal{T}_j(\iota\omega)) = 1$

Proof. We compute the inertia index of each block of the form (T1)-(T6), from which the sign-sum function is then easily computed.

T1 Since $\lambda_j \in \mathbb{C} \setminus i\mathbb{R}$, we see that for every $\omega \in \mathbb{R} \cup \{\pm\infty\}$ there exists a non-singular matrix $A \in \mathbb{C}^{k_j, k_j}$ such that we have

$$\mathcal{T}_j(i\omega) = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}.$$

Let $A = U\Sigma V^*$ be a singular value decomposition of A , cf. [11]. Then we have

$$\begin{aligned} \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} &= \begin{bmatrix} U & \\ & V \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \begin{bmatrix} U^* & \\ & V^* \end{bmatrix} \sim \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} = \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & -\Sigma \end{bmatrix} = \begin{bmatrix} 2\Sigma & 0 \\ 0 & -2\Sigma \end{bmatrix} \end{aligned}$$

and, thus, since Σ is non-singular, the inertia is given by

$$\text{In}(\mathcal{T}_j(i\omega)) = [k_j \quad 0 \quad k_j].$$

T2 For every $\omega \in \mathbb{R} \cup \{\pm\infty\}$ with $\omega \neq \omega_j$ there exists an invertible matrix $A \in \mathbb{C}^{k_j, k_j}$ and a $B \in \mathbb{C}^{k_j, k_j}$ such that we have

$$\mathcal{T}_j(i\omega) = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix} \sim \begin{bmatrix} I & -\frac{1}{2}BA^{-*} \\ 0 & I \end{bmatrix} \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -\frac{1}{2}A^{-1}B & I \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix},$$

which means that the inertia is given by

$$\text{In}(\mathcal{T}_j(i\omega)) = [k_j \quad 0 \quad k_j].$$

Also, we see that

$$\mathcal{T}_j(i\omega_j) = \epsilon_j \begin{bmatrix} & & & & & & & & 1 & 0 \\ & & & & & & & & \cdot & \cdot \\ & & & & & & & & \cdot & \cdot \\ & & & & 1 & \cdot & \cdot & & \cdot & \cdot \\ & & & & 1 & \cdot & \cdot & & \cdot & \cdot \\ & & & & 1 & 0 & & & & \\ & & & & 1 & 0 & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & 1 & \cdot & \cdot & & & \\ & & & & 1 & \cdot & \cdot & & & \\ & & & & & & & & & 1 \\ & & & & & & & & & 0 \end{bmatrix} \sim \epsilon_j \begin{bmatrix} & & & & & & & & 1 & \\ & & & & & & & & \cdot & \\ & & & & & & & & \cdot & \\ & & & & & & & & \cdot & \\ & & & & 1 & \cdot & \cdot & & \cdot & \\ & & & & 1 & \cdot & \cdot & & \cdot & \\ & & & & 1 & 0 & & & & \\ & & & & 1 & 0 & & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & & \\ & & & & 1 & \cdot & \cdot & & & \\ & & & & 1 & \cdot & \cdot & & & \\ & & & & & & & & & 1 \\ & & & & & & & & & 0 \end{bmatrix},$$

from which we deduce that

$$\text{In}(\mathcal{T}_j(i\omega_j)) = [k_j - 1 \quad 1 \quad k_j - 1] + \text{In}(\epsilon_j),$$

where $\text{In}(\epsilon_j)$ is the inertia of the 1-by-1 matrix with the entry $\epsilon_j = \pm 1$, i.e., $\text{In}(1) = [1 \quad 0 \quad 0]$ and $\text{In}(-1) = [0 \quad 0 \quad -1]$. Later, we will also need that $\text{In}(0) = [0 \quad 1 \quad 0]$.

T3 For every $\omega \in \mathbb{R} \cup \{\pm\infty\}$ with $\omega \neq \omega_j$ there exists an invertible matrix $A \in \mathbb{C}^{k_j, k_j}$ and $b \in \mathbb{C}^{k_j, 1}$ such that we have

$$\begin{aligned} \mathcal{T}_j(i\omega) &= \begin{bmatrix} 0 & b & A \\ b^* & \epsilon_j(\omega_j - \omega) & \\ A^* & 0 & \end{bmatrix} \\ &\sim \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & -b^*A^{-*} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & b & A \\ b^* & \epsilon_j(\omega_j - \omega) & \\ A^* & 0 & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -A^{-1}b & I \end{bmatrix} \\ &\sim \begin{bmatrix} 0 & 0 & A \\ 0 & \epsilon_j(\omega_j - \omega) & \\ A^* & 0 & \end{bmatrix} \sim \begin{bmatrix} I & & \\ & -I & \\ & & \epsilon_j(\omega_j - \omega) \end{bmatrix}, \end{aligned}$$

which implies that

$$\text{In}(\mathcal{T}_j(\iota\omega)) = \begin{bmatrix} k_j & 0 & k_j \end{bmatrix} + \text{In}(\epsilon_j(\omega_j - \omega)).$$

Also, we see that

$$\mathcal{T}_j(\iota\omega_j) = \epsilon_j \left[\begin{array}{c|c|c} & & 1 \\ & \ddots & \\ & 1 & \\ \hline & & \\ & \ddots & \\ & 1 & \\ \hline 1 & & \\ & & \\ & & 0 \end{array} \right],$$

from which we deduce

$$\text{In}(\mathcal{T}_j(\iota\omega_j)) = \begin{bmatrix} k_j & 1 & k_j \end{bmatrix}.$$

T4 For every $\omega \in \iota\mathbb{R}$ there exists an invertible matrix $A \in \mathbb{C}^{k_j, k_j}$ and a $B = B^* \in \mathbb{C}^{k_j, k_j}$ such that we have

$$\mathcal{T}_j(\iota\omega) = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix},$$

which as above implies that

$$\text{In}(\mathcal{T}_j(\iota\omega)) = \begin{bmatrix} k_j & 0 & k_j \end{bmatrix}.$$

Also, we see that

$$\mathcal{T}_j(\pm\iota\infty) = \pm\epsilon_j \left[\begin{array}{c|c|c} & & -1 & 0 \\ & & \ddots & \\ & -1 & \ddots & \\ \hline & -1 & 0 & \\ & \ddots & \ddots & \\ & -1 & 0 & \\ \hline -1 & \ddots & \ddots & \\ & \ddots & \ddots & \\ & & & 0 \end{array} \right] \sim \mp\epsilon_j \left[\begin{array}{c|c|c} & & 1 \\ & \ddots & \\ & 1 & \\ \hline & & \\ & \ddots & \\ & 1 & \\ \hline 1 & & \\ & & \\ & & 0 \end{array} \right],$$

from which we deduce that

$$\text{In}(\mathcal{T}_j(\pm\iota\infty)) = \begin{bmatrix} k_j - 1 & 1 & k_j - 1 \end{bmatrix} + \text{In}(\mp\epsilon_j).$$

T5 For every $\omega \in \iota\mathbb{R}$ there exists an invertible matrix $A \in \mathbb{C}^{k_j, k_j}$ and $b \in \mathbb{C}^{k_j, 1}$ such that we have

$$\mathcal{T}_j(\iota\omega) = \begin{bmatrix} 0 & b & A \\ b^* & \epsilon_j & \\ A^* & 0 & \end{bmatrix},$$

which, similar to the argument for the blocks of type (T3), implies that

$$\text{In}(\mathcal{T}_j(\iota\omega)) = \begin{bmatrix} k_j & 0 & k_j \end{bmatrix} + \text{In}(\epsilon_j).$$

Proof. Using the assumption, Corollary 2.8, and Theorem 3.3 we obtain that

$$\eta(\mathcal{N}(i\omega)) = p + q - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) = q - p, \quad (3.1)$$

for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$. Assume to the contrary that there is a block of type (T3). Choose $\epsilon > 0$ such that the interval $[\omega_j - \epsilon, \omega_j + \epsilon]$ contains no other zero. Since from (3.1) we obtain that $\eta(\mathcal{N}(i\omega))$ is constant for all $\omega \in [\omega_j - \epsilon, \omega_j + \epsilon] \setminus \{\omega_j\}$ we deduce that there are as many blocks of type (T3) with negative sign, as there are with positive sign, let us say ℓ -many. In this case, however, we have that

$$\eta(\mathcal{N}(i\omega)) + 2\ell \leq \eta(\mathcal{N}(i\omega_j)),$$

for all $\omega \in [\omega_j - \epsilon, \omega_j + \epsilon] \setminus \{\omega_j\}$, since there might also be blocks of type (T2) with $\epsilon_j = 1$. This contradicts (3.1). Thus, there can be no blocks of type (T3). Blocks of type (T2) with $\epsilon_j = 1$ are then not possible, since they would also increase the rank at one point.

In the same way, blocks of type (T4) and of type (T5) with $\epsilon_j = -1$ are not possible, since they would increase the rank at $\pm\infty$. \square

We construct the following canonical form, for para-Hermitian polynomials, which fulfill the properties 1. - 4. of Corollary 3.4. From this canonical form it is then in principle possible to obtain, what in [21] is called *Lagrangian subspaces*.

LEMMA 3.5. *Consider the first-order para-Hermitian matrix polynomial $\lambda N_1 + N_0 \in \mathbb{C}[\lambda]_1^{k,k}$. Let the Thompson canonical form of $\lambda N_1 + N_0$ be given by (2.8) together with the other notations, which were introduced in Theorem 2.9. Assume that (2.8) satisfies the properties 1. - 4. of Corollary 3.4. Then the rank of N_1 is even, let us say $2m$, and there exist matrices $\tilde{Y}_1 \in \mathbb{C}^{k,m}$, $\tilde{Y}_2 \in \mathbb{C}^{k,\tau_5}$, and $\tilde{Y}_3 \in \mathbb{C}^{k,m+\tau_6}$ such that $\tilde{Y} := [\tilde{Y}_1 \quad \tilde{Y}_2 \quad \tilde{Y}_3] \in \mathbb{C}^{k,k}$ is invertible and we have*

$$\tilde{Y}^* \mathcal{N} \tilde{Y} = \tilde{Y}^* (\lambda N_1 + N_0) \tilde{Y} = \begin{bmatrix} T & \lambda S & \lambda R_1 + R_0 \\ -\lambda S^* & I & 0 \\ -\lambda R_1^* + R_0^* & 0 & 0 \end{bmatrix},$$

where $\lambda R_1 + R_0 \in \mathbb{C}[\lambda]^{m,m+\tau_6}$ has full row rank over $\mathbb{C}(\lambda)$, $S \in \mathbb{C}^{m,\tau_5}$ such that $[S \quad R_1]$ has full row rank over \mathbb{C} , $T \in \mathbb{C}^{m,m}$ fulfills $T = T^* \leq 0$, $\mathfrak{B}([\lambda S \quad \lambda R_1 + R_0])$ is trim, and we have that

$$\mathfrak{B}_+(\mathcal{N}) = [\tilde{Y}_2 \quad \tilde{Y}_3] \mathfrak{B}_+ \left(\begin{bmatrix} \lambda S & \lambda R_1 + R_0 \\ I & 0 \end{bmatrix} \right)$$

and

$$\mathfrak{z}(\mathcal{N}) \cap \overline{\mathbb{C}_-} = \mathfrak{z} \left(\begin{bmatrix} \lambda S & \lambda R_1 + R_0 \\ I & 0 \end{bmatrix} \right).$$

Furthermore, if the index of \mathcal{N} is less or equal to 1, then $S = 0$. Also, if \mathcal{N} has no purely imaginary zeros (i.e., the Thompson canonical form has no blocks of type (T2)), then $T = 0$ and $\mathfrak{B}([\lambda S \quad \lambda R_1 + R_0])$ is stabilizable.

Proof. Partition the transformation matrix in (2.8) as

$$X =: [X_1 \quad \dots \quad X_d],$$

according to the diagonal structure in (2.8), i.e., such that $X_j \in \mathbb{C}^{k,\hat{k}_j}$. This means that we have $X_j^* (\lambda N_1 + N_0) X_j = \mathcal{T}_j(\lambda)$ and $X_j^* (\lambda N_1 + N_0) X_i = 0$ for all $i \neq j$. With 0_{k_j} denoting

the vector of length k_j which only contains zeros, set

$$C_j := \begin{bmatrix} I_{k_j} \\ 0 \end{bmatrix},$$

$$D_j := \begin{cases} \begin{bmatrix} 0_{k_j} \\ 1 \end{bmatrix} & \text{if } \mathcal{T}_j \text{ has type (T5),} \\ \in \mathbb{C}^{\hat{k}_j, 0} & \text{if } \mathcal{T}_j \text{ has type (T1), (T2), (T6),} \end{cases}$$

$$E_j := \begin{cases} \begin{bmatrix} 0 \\ I_{k_j} \\ 0 \end{bmatrix} & \text{if } \mathcal{T}_j \text{ has type (T1), (T2), (T5),} \\ \begin{bmatrix} 0 \\ I_{k_j+1} \end{bmatrix} & \text{if } \mathcal{T}_j \text{ has type (T6),} \end{cases}$$

for $j = 1, \dots, d$. With this define the block matrices,

$$\begin{aligned} \mathcal{T} &:= \text{diag}(\mathcal{T}_1(\lambda), \dots, \mathcal{T}_d(\lambda)) \in \mathbb{C}[\lambda]^{k, k}, \\ C &:= \text{diag}(C_1, \dots, C_d) \in \mathbb{C}^{k, m}, \\ D &:= \text{diag}(D_1, \dots, D_d) \in \mathbb{C}^{k, \tau_5}, \\ E &:= \text{diag}(E_1, \dots, E_d) \in \mathbb{C}^{k, m + \tau_6}. \end{aligned}$$

We see that $k = 2m + \tau_5 + \tau_6$ and that

$$\begin{bmatrix} C & D & E \end{bmatrix},$$

is a permutation matrix, and thus invertible. Thus, also the matrix

$$\tilde{Y} := X \begin{bmatrix} C & D & E \end{bmatrix}$$

is invertible and we have

$$\tilde{Y}^* (\lambda N_1 + N_0) \tilde{Y} = \begin{bmatrix} C^* \\ D^* \\ E^* \end{bmatrix} \mathcal{T} \begin{bmatrix} C & D & E \end{bmatrix} = \begin{bmatrix} C^* \mathcal{T} C & C^* \mathcal{T} D & C^* \mathcal{T} E \\ (C^* \mathcal{T} D)^\sim & D^* \mathcal{T} D & D^* \mathcal{T} E \\ (C^* \mathcal{T} E)^\sim & (D^* \mathcal{T} E)^\sim & E^* \mathcal{T} E \end{bmatrix}.$$

Looking at the blocks in this matrix, we find that

$$\begin{aligned} E^* \mathcal{T} E &= \text{diag}(E_1^* \mathcal{T}_1 E_1, \dots, E_d^* \mathcal{T}_d E_d) = \text{diag}(0, \dots, 0) = 0, \\ D^* \mathcal{T} E &= \text{diag}(D_1^* \mathcal{T}_1 E_1, \dots, D_d^* \mathcal{T}_d E_d) = \text{diag}(0, \dots, 0) = 0, \\ D^* \mathcal{T} D &= \text{diag}(D_1^* \mathcal{T}_1 D_1, \dots, D_d^* \mathcal{T}_d D_d) = I. \end{aligned}$$

Furthermore, we see that there exists a matrix polynomial $\lambda R_1 + R_0 \in \mathbb{C}[\lambda]^{m, m + \tau_6}$ and matrices $S \in \mathbb{C}^{m, \tau_5}$ and $T \in \mathbb{C}^{m, m}$ with $T = T^* \leq 0$ such that

$$\begin{aligned} C^* \mathcal{T} E &= \lambda R_1 + R_0, \\ C^* \mathcal{T} D &= \lambda S, \end{aligned} \tag{3.2}$$

$$C^* \mathcal{T} C = -T. \tag{3.3}$$

From (3.2) we see that S vanishes, if all blocks of type (T5) have size $\hat{k}_j = 1$, which by definition is the case if the index of \mathcal{N} is less or equal to 1. From (3.3) we see that $T = 0$,

if there are no blocks of type (T2). Also, we observe that $C^*\mathcal{T}(\infty) \begin{bmatrix} D & E \end{bmatrix} = \begin{bmatrix} S & R_1 \end{bmatrix}$ has full row rank. With Corollary 2.8 this implies that

$$\mathfrak{B}(C^*\mathcal{T} \begin{bmatrix} D & E \end{bmatrix}) = \mathfrak{B}(\begin{bmatrix} \lambda S & \lambda R_1 + R_0 \end{bmatrix})$$

is trim and, furthermore, stabilizable if \mathcal{N} has no blocks of type (T2). From the construction of C, D, E we obtain that

$$\mathfrak{Z}(\mathcal{N}) \cap \overline{\mathfrak{C}}_- = \mathfrak{Z}(\lambda R_1 + R_0) = \mathfrak{Z}\left(\begin{bmatrix} \lambda S & \lambda R_1 + R_0 \\ I & 0 \end{bmatrix}\right)$$

and also that $\mathfrak{B}_+(-\lambda R_1^* + R_0^*) = \{0\}$, from which we conclude that

$$\begin{aligned} \mathfrak{B}_+(\mathcal{N}) &= \mathfrak{B}_+\left(\tilde{Y}^{-*} \begin{bmatrix} T & \lambda S & \lambda R_1 + R_0 \\ -\lambda S^* & I & 0 \\ \lambda R_1 + R_0^\sim & 0 & 0 \end{bmatrix} \tilde{Y}^{-1}\right) \\ &= \mathfrak{B}_+\left(\begin{bmatrix} T & \lambda S & \lambda R_1 + R_0 \\ -\lambda S^* & I & 0 \\ \lambda R_1 + R_0^\sim & 0 & 0 \end{bmatrix} \tilde{Y}^{-1}\right) \\ &= \tilde{Y} \mathfrak{B}_+\left(\begin{bmatrix} T & \lambda S & \lambda R_1 + R_0 \\ -\lambda S^* & I & 0 \\ \lambda R_1 + R_0^\sim & 0 & 0 \end{bmatrix}\right) = \tilde{Y} \left[\mathfrak{B}_+\left(\begin{bmatrix} \lambda S & \lambda R_1 + R_0 \\ -\lambda I & 0 \\ 0 & 0 \end{bmatrix}\right) \right] \\ &= [Y_2 \ Y_3] \mathfrak{B}_+\left(\begin{bmatrix} \lambda S & \lambda R_1 + R_0 \\ -\lambda I & 0 \end{bmatrix}\right), \end{aligned}$$

and thus the claim follows. \square

The main result of this section, Lemma 3.5, states that a para-Hermitian pencil \mathcal{N} which fulfills the properties 1. - 4. of Corollary 3.4 has a certain structure. In Corollary 3.4, however, we saw that for dissipative systems these properties are fulfilled, for the special para-Hermitian matrix \mathcal{N} which is given by (1.9). In the next section we will use these two results to construct a solution of the Lur'e equation from the matrix \tilde{Y} in Lemma 3.5.

4. Controllable and trim systems. In this section we work out the main result for the special case where $\mathfrak{B}(\lambda F + G)$ is controllable and trim, see Lemma 4.3. For this, we first formulate the following lemma, wherein the main part is inspired by [14].

LEMMA 4.1. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and let $H = H^* \in \mathbb{C}[\lambda]^{q,q}$. Form the para-Hermitian polynomial $\mathcal{N} = \mathcal{N}^\sim \in \mathbb{C}[\lambda]^{p+q,p+q}$ via*

$$\mathcal{N}(\lambda) := \lambda N_1 + N_0 := \lambda \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix}.$$

Assume that $(\lambda F + G, H)$ is dissipative and that $\mathfrak{B}(\lambda F + G)$ is controllable and trim. Then there exist matrices $Y_1 \in \mathbb{C}^{p,p}$, $Y_2 \in \mathbb{C}^{p,q}$, $Y_3 \in \mathbb{C}^{q,p}$, and $Y_4 \in \mathbb{C}^{q,q}$ such that the matrices

$$Y := \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \in \mathbb{C}^{p+q,p+q},$$

and Y_4 are invertible, $Y_4 = Y_4^$, and we have*

$$Y^* (\lambda N_1 + N_0) Y = \begin{bmatrix} -Q_1 & \lambda \tilde{F} + \tilde{G} \\ -\lambda \tilde{F}^* + \tilde{G}^* & Q_2 \end{bmatrix},$$

where $\lambda\tilde{F} + \tilde{G} \in \mathbb{C}[\lambda]^{p,q}$ has full column rank, \tilde{F} has full column rank, $Q_1 = Q_1^* \geq 0$, $Q_2 = Q_2^* \geq 0$, $\text{rank}(Q_2) = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$,

$$\mathfrak{B}_+(\mathcal{N}) = \begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda\tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right),$$

and

$$\mathfrak{Z}(\mathcal{N}) \cap \overline{\mathbb{C}_-} = \mathfrak{Z} \left(\begin{bmatrix} \lambda\tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right).$$

Furthermore, if \mathcal{N} has index less or equal to 1 and no purely imaginary zeros, then for every $\hat{y} \in \mathbb{C}^q$ there exists a

$$y_0 \in \mathfrak{R}_+ \left(\begin{bmatrix} \lambda\tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right)$$

with $\tilde{F}\hat{y} = \tilde{F}y_0$ and we have $Q_1 = 0$.

Proof. Using part 1. of Corollary 2.8 we see that the dimension of the kernel of N_1 is given by

$$p + q - 2 \cdot \text{rank}(F) = q - p,$$

which means that

$$\tau_5 + \tau_6 = q - p, \tag{4.1}$$

since (in the absence of blocks of type (T4)) only blocks of type (T5) and (T6) contribute to the dimension of the kernel of the leading matrix; each block of type (T5) and (T6) by 1. Also, the dimension of the kernel of \mathcal{N} (over $\mathbb{C}(\lambda)$), and thus the number of blocks of type (T6), is given by

$$\tau_6 = p + q - \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) = q - p - (\text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)),$$

which together with (4.1) gives that

$$\tau_5 = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G).$$

Thus, from Lemma 3.5 (with $m = \text{rank} F = p$, $m + \tau_5 + \tau_6 = q$) we deduce the existence of $Y_1 \in \mathbb{C}^{p,p}$, $\tilde{Y}_2 \in \mathbb{C}^{p,q}$, $Y_3 \in \mathbb{C}^{q,p}$, and $\tilde{Y}_4 \in \mathbb{C}^{q,q}$, such that

$$\tilde{Y} := \begin{bmatrix} Y_1 & \tilde{Y}_2 \\ Y_3 & \tilde{Y}_4 \end{bmatrix}$$

is invertible and

$$\tilde{Y}^* \mathcal{N} \tilde{Y} = \begin{bmatrix} -Q_1 & \lambda\tilde{R}_1 + \tilde{R}_0 \\ -\lambda\tilde{R}_1^* + \tilde{R}_0^* & \tilde{Q}_2 \end{bmatrix}, \tag{4.2}$$

where $\text{rank}(\tilde{Q}_2) = \tau_5 = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$. If, additionally, \mathcal{N} has index less or equal to 1 and no purely imaginary zeros we obtain from Lemma 3.5 (since then $S = 0$ and $\mathfrak{B}(\lambda\tilde{R}_1 + \tilde{R}_0)$ is stabilizable and trim) that $\lambda\tilde{R}_1 + \tilde{R}_0$ and \tilde{Q}_2 are such that for every $\hat{x} \in \mathbb{C}^q = \mathfrak{R}_+(\lambda\tilde{R}_1 + \tilde{R}_0)$ there exists an

$$x_0 \in \mathfrak{R}_+ \left(\begin{bmatrix} \lambda\tilde{R}_1 + \tilde{R}_0 \\ \tilde{Q}_2 \end{bmatrix} \right) \tag{4.3}$$

with $\tilde{R}_1 \hat{x} = \tilde{R}_1 x_0$. Using the singular value decomposition [11] of \tilde{Y}_4 we obtain the existence of invertible matrices U and V such that

$$\tilde{Y}_4 = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V.$$

Transform (4.2) with $\text{diag}(I, W)$ where $W := V^{-1}U^*$, i.e., define $Y_4 := \tilde{Y}_4 W$, $Y_2 := \tilde{Y}_2 W$, $\tilde{F} := \tilde{R}_1 W$, $\tilde{G} := \tilde{R}_0 W$, $Q_2 = W^* \tilde{Q}_2 W$, and

$$Y := \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix},$$

to obtain

$$Y^* \mathcal{N} Y = \begin{bmatrix} -Q_1 & \lambda \tilde{F} + \tilde{G} \\ -\lambda \tilde{F}^* + \tilde{G}^* & Q_2 \end{bmatrix}, \quad (4.4)$$

with

$$Y_4 = \tilde{Y}_4 W = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U^* = Y_4^*$$

being a Hermitian matrix. Using Lemma 3.5 we find

$$\mathfrak{B}_+(\mathcal{N}) = \begin{bmatrix} \tilde{Y}_2 \\ \tilde{Y}_4 \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda \tilde{R}_1 + \tilde{R}_0 \\ \tilde{Q}_2 \end{bmatrix} \right) = \begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right)$$

and also that

$$\mathfrak{Z}(\mathcal{N}) \cap \overline{\mathfrak{C}_-} = \mathfrak{Z} \left(\begin{bmatrix} \lambda \tilde{R}_1 + \tilde{R}_0 \\ \tilde{Q}_2 \end{bmatrix} \right) = \mathfrak{Z} \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right).$$

In the case that \mathcal{N} has index less or equal to 1 and no purely imaginary zeros, let $\hat{y} \in \mathbb{C}^q$. Then $\hat{x} := W \hat{y} \in \mathbb{C}^q$ and thus we obtain the existence of an x_0 such that (4.3) holds. Defining $y_0 := W^{-1} x_0$ then shows that

$$y_0 = W^{-1} x_0 \in W^{-1} \mathfrak{R}_+ \left(\begin{bmatrix} \lambda \tilde{R}_1 + \tilde{R}_0 \\ \tilde{Q}_2 \end{bmatrix} \right) = \mathfrak{R}_+ \left(\begin{bmatrix} \lambda \tilde{R}_1 + \tilde{R}_0 \\ \tilde{Q}_2 \end{bmatrix} W \right) = \mathfrak{R}_+ \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right),$$

while $\tilde{F} y_0 = \tilde{F} W^{-1} x_0 = \tilde{R}_1 x_0 = \tilde{R}_1 \hat{x} = \tilde{R}_1 W \hat{y} = \tilde{F} \hat{y}$.

Finally, to show that Y_4 is invertible, let $M \in \mathbb{C}^{q, \ell}$ be an orthonormal basis of kernel (Y_4), i.e., let kernel (Y_4) = image (M) with $M^* M = I$. We will show that $\ell = 0$. To do so, note that from (4.4) we obtain that

$$\begin{aligned} \begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} M &= Y^* \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} M \\ &= \begin{bmatrix} Y_1^* & Y_3^* \\ Y_2^* & Y_4^* \end{bmatrix} \begin{bmatrix} 0 \\ (-\lambda F^* + G^*) Y_2 M \end{bmatrix} \\ &= \begin{bmatrix} Y_3^* \\ Y_4^* \end{bmatrix} (-\lambda F^* + G^*) Y_2 M \end{aligned} \quad (4.5)$$

and from this we obtain

$$M^* Q_2 M = M^* Y_4^* (-\lambda F^* + G^*) Y_2 M = (Y_4 M)^* (-\lambda F^* + G^*) Y_2 M = 0.$$

Since Q_2 is positive semi-definite, there exists an $\mathcal{C} \in \mathbb{C}^{s,q}$ such that $Q_2 = \mathcal{C}^* \mathcal{C}$, which implies that for every $y \in \mathbb{C}^\ell$ we have

$$0 = y^* M^* Q_2 M y = y^* M^* \mathcal{C}^* \mathcal{C} M y = \|\mathcal{C} M y\|_2^2,$$

and thus $0 = L M y$ and also $0 = \mathcal{C}^* \mathcal{C} M y = Q_2 M y$. From this we conclude that

$$Q_2 M = 0,$$

and thus (4.5) implies that

$$0 = Y_4^* (-\lambda F^* + G^*) Y_2 M = Y_4 (-\lambda F^* + G^*) Y_2 M,$$

from which we conclude that $\text{image}(-F^* Y_2 M) \subset \text{image}(M)$ and, on the other hand, also that $\text{image}(G^* Y_2 M) \subset \text{image}(M)$. This ensures the existence of matrices $\tilde{E}, \tilde{A} \in \mathbb{C}^{\ell,\ell}$ such that

$$(-\lambda F^* + G^*) Y_2 M = M(\lambda \tilde{E} + \tilde{A}).$$

Furthermore, we find that the matrix

$$\begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} M = \begin{bmatrix} Y_2 M \\ 0 \end{bmatrix},$$

has full column rank ℓ and thus also the matrix $Y_2 M$. Since by assumption the matrix F^* and for all $\mu \in \mathbb{C}$ the matrices $-\mu F^* + G^*$ have full column rank we obtain that also the matrices $F^* Y_2 M$ and $(-\mu F^* + G^*) Y_2 M$ have full column rank ℓ , for all $\mu \in \mathbb{C}$. Using the following Lemma 4.2 with $E = -F^* Y_2$ and $A = G^* Y_2$, this proves that $\ell = 0$, and the proof is finished. \square

LEMMA 4.2. *Let $\lambda E + A \in \mathbb{C}[\lambda]^{q,q}$, $\lambda \tilde{E} + \tilde{A} \in \mathbb{C}[\lambda]^{\ell,\ell}$, and $M \in \mathbb{C}^{q,\ell}$ be such that $M^* M = I$,*

$$(\lambda E + A) M = M(\lambda \tilde{E} + \tilde{A}),$$

and $\text{rank}(EM) = \ell$ and $\text{rank}((\mu E + A)M) = \ell$ for all $\mu \in \mathbb{C}$. Then $\ell = 0$.

Proof. Assume to the contrary that $\ell > 0$. Then, from the assumptions we see that \tilde{E} is invertible. Thus, there exists a $\tilde{\mu} \in \mathbb{C}$ and a $\tilde{v} \in \mathbb{C}^\ell \setminus \{0\}$ such that $(\tilde{\mu} \tilde{E} + \tilde{A}) \tilde{v} = 0$ which implies that

$$0 = M(\tilde{\mu} \tilde{E} + \tilde{A}) \tilde{v} = (\tilde{\mu} E + A) M \tilde{v},$$

in contrast to the assumption that $\text{rank}((\mu E + A)M) = \ell$ for all $\mu \in \mathbb{C}$. This proves the assertion. \square

The following Lemma corresponds to the main result (cf. Theorem 6.2), in the case of controllable and trim systems.

LEMMA 4.3. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order matrix polynomial and let $H = H^* \in \mathbb{C}[\lambda]^{q,q}$ be Hermitian. Assume that $(\lambda F + G, H)$ is dissipative and that $\mathfrak{B}(\lambda F + G)$ is controllable and trim. Then there exists a solution $Z \in \mathbb{C}^{p,q}$ of the Lur'e equation.*

Furthermore, if the pencil \mathcal{N} from (1.9) has index less or equal to 1 and no purely imaginary zeros, then there exists a strong solution of the Lur'e equation.

Proof. Using Lemma 4.1 we verify the existence of invertible matrices $Y \in \mathbb{C}^{p+q,p+q}$ and $Y_4 = Y_4^* \in \mathbb{C}^{q,q}$ such that

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$$

and

$$Y^* \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} Y = \begin{bmatrix} -Q_1 & \lambda \tilde{F} + \tilde{G} \\ -\lambda \tilde{F}^* + \tilde{G}^* & Q_2 \end{bmatrix}. \quad (4.6)$$

Transforming this equation further we can bring Y into block-upper-triangular form with identity matrices in the two block-diagonal positions. To be more specific, define $Z := Y_2 Y_4^{-1}$ and observe that

$$\begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} = \begin{bmatrix} Y_1 - ZY_3 & 0 \\ Y_3 & Y_4 \end{bmatrix} =: \hat{Y}$$

and thus that $(Y_1 - ZY_3)$ is invertible. Define

$$U := \begin{bmatrix} (Y_1 - ZY_3)^{-1} & 0 \\ -Y_4^{-1}Y_3(Y_1 - ZY_3)^{-1} & Y_4^{-1} \end{bmatrix} = \hat{Y}^{-1}.$$

Then, using (4.6), we have

$$U^* Y^* \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} Y U = \begin{bmatrix} * & * \\ * & Y_4^{-*} Q_2 Y_4^{-1} \end{bmatrix}, \quad (4.7)$$

and

$$Y U = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}.$$

This means that $\tilde{Q} := Y_4^{-*} Q_2 Y_4^{-1}$ still is Hermitian, positive semi-definite, and has rank $\text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$ and that

$$\begin{aligned} & \begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H + G^* Z - \lambda F^* Z \end{bmatrix} \\ & = \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H + G^* Z + Z^* G + \lambda Z^* F - \lambda F^* Z \end{bmatrix}, \end{aligned} \quad (4.8)$$

which together with (4.7) shows that $H + G^* Z + Z^* G = \tilde{Q} \geq 0$ and $F^* Z = Z^* F$. The identities (1.11) and (1.10) hold, again, with Lemma 4.1, since

$$\begin{aligned} \mathfrak{B}_+(\mathcal{N}) &= \begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right) = \begin{bmatrix} Z \\ I \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} Y_4^{-1} \right) \\ &= \begin{bmatrix} Z \\ I \end{bmatrix} \mathfrak{B}_+ \left(U^* \begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} Y_4^{-1} \right) = \begin{bmatrix} Z \\ I \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F + G \\ \tilde{Q} \end{bmatrix} \right) \end{aligned}$$

and

$$\mathfrak{z}(\mathcal{N}) \cap \overline{\mathbb{C}_-} = \mathfrak{z} \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right) = \mathfrak{z} \left(U^* \begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} Y_4^{-1} \right) = \mathfrak{z} \left(\begin{bmatrix} \lambda F + G \\ \tilde{Q} \end{bmatrix} \right).$$

Finally, assume that \mathcal{N} has index less or equal to 1 and no purely imaginary zeros. Let $\hat{z} \in \mathfrak{R}_+(\lambda F + G) = \mathbb{C}^q$. Then Lemma 4.1 shows that for $\hat{y} := Y_4^{-1} \hat{z} \in \mathbb{C}^q$ there exists a $y_0 \in \mathfrak{R}_+ \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right)$ with $\tilde{F} \hat{y} = \tilde{F} y_0$. Define $z_0 := Y_4 y_0$. Lemma 4.1 shows that $Q_1 = 0$

and, thus, from (4.8) and (4.6) we obtain

$$\begin{aligned} \begin{bmatrix} \lambda F + G \\ \tilde{Q} \end{bmatrix} &= \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & \tilde{Q} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} = U^* Y^* \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} Y U \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= U^* \begin{bmatrix} 0 & \lambda \tilde{F} + \tilde{G} \\ -\lambda \tilde{F}^* + \tilde{G}^* & Q_2 \end{bmatrix} U \begin{bmatrix} 0 \\ I \end{bmatrix} = U^* \begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} Y_4^{-1}, \end{aligned} \quad (4.9)$$

which implies that

$$z_0 = Y_4 y_0 \in Y_4 \mathfrak{R}_+ \left(\begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} \right) = \mathfrak{R}_+ \left(U^* \begin{bmatrix} \lambda \tilde{F} + \tilde{G} \\ Q_2 \end{bmatrix} Y_4^{-1} \right) = \mathfrak{R}_+ \left(\begin{bmatrix} \lambda F + G \\ \tilde{Q} \end{bmatrix} \right).$$

Since from (4.9) we obtain that

$$\begin{bmatrix} F \\ 0 \end{bmatrix} = U^* \begin{bmatrix} \tilde{F} \\ 0 \end{bmatrix} Y_4^{-1} = \begin{bmatrix} (Y_1 - ZY_3)^{-*} \tilde{F} Y_4^{-1} \\ 0 \end{bmatrix}$$

and we also have $(Y_1 - ZY_3)^{-*} \tilde{F} \hat{y} = (Y_1 - ZY_3)^{-*} \tilde{F} y_0$, we find that

$$F \hat{z} = (Y_1 - ZY_3)^{-*} \tilde{F} Y_4^{-1} Y_4 \hat{y} = (Y_1 - ZY_3)^{-*} \tilde{F} Y_4^{-1} Y_4 y_0 = F z_0,$$

which proves the claim. \square

The following example shows that there exist situations where a solution of the Lur'e equation exists, but no strong solution.

EXAMPLE 4.4. Consider the system $\lambda F + G := [\lambda \quad 1] \in \mathbb{C}[\lambda]^{1,2}$ together with the supply

$$H := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\lambda F + G$ is already in Kronecker canonical form (2.1) we can use Lemma 2.7 to obtain that $\mathfrak{B}(\lambda F + G)$ is controllable and trim. Thus, Lemma 4.3 guarantees the existence of a solution of the Lur'e equation. Setting

$$\begin{aligned} \mathcal{N}(\lambda) &:= \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 1 \\ -\lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \lambda & 1 \\ -\lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 1 \\ \lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \end{aligned}$$

shows that \mathcal{N} does not have index less or equal 1. Let $Z = [z_1 \quad z_2]$ be a solution of the Lur'e equation. Then

$$\begin{bmatrix} z_1 & z_2 \\ 0 & 0 \end{bmatrix} = F^* Z = Z^* F = \begin{bmatrix} \bar{z}_1 & 0 \\ \bar{z}_2 & 0 \end{bmatrix},$$

which means that $z_1 \in \mathbb{R}$ and $z_2 = 0$. Furthermore, we have

$$0 \leq H + G^* Z + Z^* G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [z_1 \quad 0] + \begin{bmatrix} z_1 \\ 0 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 1 & z_1 \\ z_1 & 0 \end{bmatrix},$$

which implies that $z_1 = 0$ and thus $Z = 0$. Thus, there is only one full rank Cholesky factor \mathcal{C} of $H + G^*Z + Z^*G = H$, namely, $\mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. However, since $\mathfrak{R}_+(\lambda F + G) = \mathfrak{R}_+(\begin{bmatrix} \lambda & 1 \\ 0 & 0 \end{bmatrix}) = \mathbb{C}^2$ and

$$\mathfrak{R}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) = \mathfrak{R}_+ \left(\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} \right) = \{0\},$$

we find that, e.g., for $\hat{z} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \in \mathfrak{R}_+(\lambda F + G)$ there exists no $z_0 \in \mathfrak{R}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right)$ with $Fz_0 = F\hat{z}$.

In this section we have seen that for controllable and trim systems the solvability of the Lur'e equation is equivalent to dissipativity. Also, we have seen an example of a system for which a solution of the Lur'e equation exists but no strong solution of the Lur'e equation exists.

5. Feasibility of the supply function. In this section we will introduce the notion of feasibility in Definition 5.1. This property will be needed as an assumption in the main result, Theorem 6.2. However, in this section we will show that every supply function can be made feasible without changing how power is measured, see Theorem 5.3. Also, we will see that a wide class of problems is already feasible by itself, cf. Theorem 5.4.

DEFINITION 5.1. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and let $H = H^* \in \mathbb{C}^{q,q}$. Then we call $(\lambda F + G, H)$ feasible, if there exist invertible matrices $T \in \mathbb{C}^{q,q}$ and $S \in \mathbb{C}^{p,p}$ and a matrix $\tilde{Z} \in \mathbb{C}^{p,q}$ such that*

$$S(\lambda F + G)T =: \lambda \tilde{F} + \tilde{G} = \begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \end{bmatrix}, \quad (5.1)$$

where $\lambda F_1 + G_1 \in \mathbb{C}[\lambda]^{p_1, q_1}$, $\lambda F_2 + G_2 \in \mathbb{C}[\lambda]^{p_2, q_2}$, $\mathfrak{B}(\lambda F_1 + G_1)$ is trim, $\mathfrak{R}(\lambda F_2 + G_2) = \{0\}$, $\tilde{F}^* \tilde{Z} = \tilde{Z}^* \tilde{F}$, and

$$\left(T^* H T + \tilde{G}^* \tilde{Z} + \tilde{Z}^* \tilde{G} \right) \begin{bmatrix} 0 \\ I_{q_2} \end{bmatrix} = 0.$$

The following example shows that not all supply functions are feasible.

EXAMPLE 5.2. *Consider the system which consists of one block of type (K3), i.e., define*

$$\lambda F + G := \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix},$$

and

$$H := \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\mathfrak{R}(\lambda F + G) = \{0\}$, it is sufficient to show that there exists no $Z = \tilde{Z} \in \mathbb{C}^{2,2}$ such that $F^*Z = Z^*F$ and $0 = H + G^*Z + Z^*G$. Denote the entries of Z as

$$Z =: \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix},$$

and observe that then $F^*Z = Z^*F$ reads

$$\begin{bmatrix} 0 & 0 \\ z_{11} & z_{12} \end{bmatrix} = \begin{bmatrix} 0 & \overline{z_{11}} \\ 0 & \overline{z_{12}} \end{bmatrix},$$

which means that Z is supposed to have the form

$$Z =: \begin{bmatrix} 0 & r \\ z_{21} & z_{22} \end{bmatrix},$$

where $r \in \mathbb{R}$ is a real number. This implies the equation

$$0 = H + G^*Z + Z^*G = \begin{bmatrix} -1 & r + \overline{z_{21}} \\ r + z_{21} & z_{22} + \overline{z_{22}} \end{bmatrix},$$

which is a contradiction.

The following theorem shows that every problem can be made feasible by changing the supply function, without changing how power is measured.

THEOREM 5.3. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and let $\hat{H} = \hat{H}^* \in \mathbb{C}^{q,q}$. Then there exists $H = H^* \in \mathbb{C}^{q,q}$ such that*

$$z^*(t)Hz(t) = z^*(t)\hat{H}z(t),$$

for all $z \in \mathfrak{B}(\lambda F + G)$ and all $t \in \mathbb{R}$ and such that $(\lambda F + G, H)$ is feasible.

Proof. Consider the Kronecker canonical form of $\lambda F + G$ given by (2.1) and assume that the blocks \mathcal{K}_j are ordered such that on the diagonal first all blocks of type (K1) and (K2) appear and then all blocks of type (K3) and (K4). This means that

$$S(\lambda F + G)T = \begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \end{bmatrix}. \quad (5.2)$$

Since $\lambda F_1 + G_1$ only consists of blocks of type (K1) and (K2) we see that $\mathfrak{B}(\lambda F_1 + G_1)$ is trim. Since $\lambda F_2 + G_2$ only consists of blocks of type (K3) and (K4) we see that $\mathfrak{B}(\lambda F_2 + G_2) = \{0\}$. Partition

$$T^* \hat{H} T =: \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix},$$

according to the block structure in (5.2) and define $\tilde{Z} := 0$ and

$$H := T^{-*} \begin{bmatrix} H_{11} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Then obviously we have $\tilde{F}^* \tilde{Z} = 0 = \tilde{Z}^* \tilde{F}$ and

$$\left(T^* H T + \tilde{G}^* \tilde{Z} + \tilde{Z}^* \tilde{G} \right) \begin{bmatrix} 0 \\ I_{q_2} \end{bmatrix} = \begin{bmatrix} H_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{q_2} \end{bmatrix} = 0,$$

which means that $(\lambda F + G, H)$ is feasible.

Let $z \in \mathfrak{B}(\lambda F + G)$. Then we have that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := y := T^{-1} z \in \mathfrak{B} \left(\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \end{bmatrix} \right),$$

which implies that $y_2 \in \mathfrak{B}(\lambda F_2 + G_2)$ and thus $y_2 = 0$. From this we conclude that

$$\begin{aligned} z^*(t)Hz(t) &= y^*(t)T^*HTy(t) = \begin{bmatrix} y_1(t) \\ 0 \end{bmatrix}^* \begin{bmatrix} H_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} y_1(t) \\ 0 \end{bmatrix}^* \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ 0 \end{bmatrix} = z^*(t)T^{-*} \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} T^{-1}z(t) = z^*(t)\hat{H}z(t), \end{aligned}$$

and thus the claim follows. \square

The following Theorem shows that a wide class of problems is already feasible by itself.

THEOREM 5.4. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and let (2.1) be its Kronecker canonical form. Let $H = H^* \in \mathbb{C}^{q,q}$. Assume that $\mathfrak{B}(\lambda F + G)$ is controllable, that all blocks of type (K3) have size $k_j \leq 2$ and all blocks of type (K4) have size $k_j \leq 2$. Then $(\lambda F + G, H)$ is feasible.*

Proof. The proof can be found in the appendix. \square

6. Non-trim systems. In this section we generalize Lemma 4.3 to non-trim systems to obtain the main result. In this course the definition of feasibility, which was introduced in the previous section, is essential. We begin with the following Lemma.

LEMMA 6.1. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let $Z \in \mathbb{C}^{p,q}$ fulfill $F^*Z = Z^*F$. Then $(\lambda F + G, H)$ is dissipative if and only if $(\lambda F + G, H + G^*Z + Z^*G)$ is dissipative.*

Proof. It is sufficient to show that dissipativity of $(\lambda F + G, H)$ implies dissipativity of $(\lambda F + G, H + G^*Z + Z^*G)$, since then the other direction follows by replacing Z with $-Z$. Thus, assume that $(\lambda F + G, H)$ is dissipative, i.e., let $\Theta : \mathbb{C}^q \rightarrow \mathbb{R}$ be a continuous function which fulfills

$$\Theta(z(t_1)) - \Theta(z(t_0)) \leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt.$$

Then we have

$$\begin{aligned} & \int_{t_0}^{t_1} z^*(t) (H + G^*Z + Z^*G) z(t) dt \\ &= \int_{t_0}^{t_1} z^*(t)Hz(t) - \dot{z}^*(t)F^*Zz(t) - z^*(t)Z^*F\dot{z}(t) dt \\ &= \int_{t_0}^{t_1} z^*(t)Hz(t) - \frac{d}{dt} (z^*(t)F^*Zz(t)) dt \\ &= \int_{t_0}^{t_1} z^*(t)Hz(t) dt - z^*(t_1)F^*Zz(t_1) - z^*(t_0)F^*Zz(t_0) \\ &\geq \Theta(z(t_1)) - \Theta(z(t_0)) - z^*(t_1)F^*Zz(t_1) + z^*(t_0)F^*Zz(t_0), \end{aligned}$$

which implies that $\tilde{\Theta}(\hat{z}) := \Theta(\hat{z}) - \hat{z}^*F^*Z\hat{z}$ is a storage function for $(\lambda F + G, H + G^*Z + Z^*G)$. \square

The following Theorem can be considered as a strengthening of the Kalman-Yakubovich-Popov Lemma, cf. [25], and it constitutes the main result of this paper.

THEOREM 6.2. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order matrix polynomial and let $H = H^* \in \mathbb{C}[\lambda]^{q,q}$ be Hermitian. Assume that $\mathfrak{B}(\lambda F + G)$ is controllable and that $(\lambda F + G, H)$ is dissipative and feasible. Then there exists a solution $Z \in \mathbb{C}^{p,q}$ of the Lur'e equation.*

Furthermore, if the pencil \mathcal{N} from (1.9) has index less or equal to 1 and no purely imaginary zeros, then there exists a strong solution of the Lur'e equation.

Proof. Let $T \in \mathbb{C}^{q,q}$, $S \in \mathbb{C}^{p,p}$, and $\tilde{Z} \in \mathbb{C}^{p,q}$ be according to Definition 5.1. Partition

$$\tilde{Z} =: \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} \text{ and } T^*HT =: \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix},$$

according to the block structure in (5.1). Then from $\tilde{F}^*\tilde{Z} = \tilde{Z}^*\tilde{F}$ we obtain that

$$\begin{bmatrix} F_1^*\tilde{Z}_{11} & F_1^*\tilde{Z}_{12} \\ F_2^*\tilde{Z}_{21} & F_2^*\tilde{Z}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{Z}_{11}^*F_1 & \tilde{Z}_{21}^*F_2 \\ \tilde{Z}_{12}^*F_1 & \tilde{Z}_{22}^*F_2 \end{bmatrix},$$

and from $(T^*HT + \tilde{G}^*\tilde{Z} + \tilde{Z}^*\tilde{G}) \begin{bmatrix} 0 & I_{q_2} \end{bmatrix}^* = 0$ we obtain that

$$\begin{bmatrix} H_{12} + G_1^*\tilde{Z}_{12} + \tilde{Z}_{21}^*G_2 \\ H_{22} + G_2^*\tilde{Z}_{22} + \tilde{Z}_{22}^*G_2 \end{bmatrix} = 0. \quad (6.1)$$

Since $\lambda F + G$ is controllable we see from Definition 5.1 that also $\lambda F_1 + G_1$ is controllable and trim. Define

$$\mathcal{N}_{11} := \begin{bmatrix} 0 & \lambda F_1 + G_1 \\ -\lambda F_1^* + G_1^* & H_{11} + G_1^*\tilde{Z}_{11} + \tilde{Z}_{11}^*G_1 \end{bmatrix} \in \mathbb{C}^{p_1+q_1, p_1+q_1}.$$

With Lemma 6.1 we obtain that $(\lambda F_1 + G_1, H_{11} + G_1^*\tilde{Z}_{11} + \tilde{Z}_{11}^*G_1)$ is dissipative. Thus, using Lemma 4.3, we obtain the existence of a Z_{11} , which solves the Lur'e equation associated with $(\lambda F_1 + G_1, H_{11} + G_1^*\tilde{Z}_{11} + \tilde{Z}_{11}^*G_1)$, i.e., such that $F_1^*Z_{11} = Z_{11}^*F_1$, $0 \leq G_1^*(Z_{11} + \tilde{Z}_{11}) + (Z_{11} + \tilde{Z}_{11})^*G_1 + H_{11}$, and

$$\text{rank} \left(G_1^*(Z_{11} + \tilde{Z}_{11}) + (Z_{11} + \tilde{Z}_{11})^*G_1 + H_{11} \right) = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}_{11}) - 2p_1 =: s_1,$$

and for every Cholesky factor $\mathcal{C}_1 \in \mathbb{C}^{s_1, q_1}$ of

$$\tilde{G}^*(Z_{11} + \tilde{Z}_{11}) + (Z_{11} + \tilde{Z}_{11})^*\tilde{G} + H_{11} = \mathcal{C}_1^*\mathcal{C}_1$$

we have that

$$\mathfrak{B}_+(\mathcal{N}_{11}) = \begin{bmatrix} Z_{11} \\ I \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F_1 + G_1 \\ \mathcal{C}_1 \end{bmatrix} \right) \text{ and } \mathfrak{Z} \left(\begin{bmatrix} \lambda F_1 + G_1 \\ \mathcal{C}_1 \end{bmatrix} \right) = \mathfrak{Z}(\mathcal{N}_{11}) \cap \overline{\mathbb{C}_-}. \quad (6.2)$$

We now show that $Z \in \mathbb{C}^{p, q}$ defined via

$$Z := S^* \begin{bmatrix} Z_{11} + \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} T^{-1}$$

is a solution of the Lur'e equation. Therefore, note that we have

$$\begin{aligned} F^*Z &= T^{-*} \begin{bmatrix} F_1^* & \\ & F_2^* \end{bmatrix} S^{-*} S^* \begin{bmatrix} Z_{11} + \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} T^{-1} \\ &= T^{-*} \begin{bmatrix} F_1^*Z_{11} + F_1^*\tilde{Z}_{11} & F_1^*\tilde{Z}_{12} \\ F_2^*\tilde{Z}_{21} & F_2^*\tilde{Z}_{22} \end{bmatrix} T^{-1} = T^{-*} \begin{bmatrix} Z_{11}^*F_1 + \tilde{Z}_{11}^*F_1 & \tilde{Z}_{21}^*F_2 \\ \tilde{Z}_{12}^*F_1 & \tilde{Z}_{22}^*F_2 \end{bmatrix} T^{-1} \\ &= T^{-*} \begin{bmatrix} Z_{11}^* + \tilde{Z}_{11}^* & \tilde{Z}_{21}^* \\ \tilde{Z}_{12}^* & \tilde{Z}_{22}^* \end{bmatrix} S S^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} T^{-1} = Z^*F \end{aligned}$$

and similarly we obtain with (6.1) that

$$\begin{aligned} 0 &\leq \begin{bmatrix} H_{11} + G_1^*(Z_{11} + \tilde{Z}_{11}) + (Z_{11} + \tilde{Z}_{11})^*G_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} H_{11} + G_1^*(Z_{11} + \tilde{Z}_{11}) + (Z_{11} + \tilde{Z}_{11})^*G_1 & H_{12} + G_1^*\tilde{Z}_{12} + \tilde{Z}_{21}^*G_2 \\ H_{12}^* + \tilde{Z}_{12}^*G_1 + G_2^*\tilde{Z}_{21} & H_{22} + G_2^*\tilde{Z}_{22} + \tilde{Z}_{22}^*G_2 \end{bmatrix} \\ &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} + \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix} \begin{bmatrix} Z_{11} + \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} + \begin{bmatrix} Z_{11}^* + \tilde{Z}_{11}^* & \tilde{Z}_{21}^* \\ \tilde{Z}_{12}^* & \tilde{Z}_{22}^* \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \quad (6.3) \end{aligned}$$

has rank s_1 . Multiplying this inequality from the left with T^{-*} and from the right with T^{-1} shows that also

$$\begin{aligned}
0 &\leq T^{-*} \left(\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} + \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix} \begin{bmatrix} Z_{11} + \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} Z_{11}^* + \tilde{Z}_{11}^* & \tilde{Z}_{21}^* \\ \tilde{Z}_{12}^* & \tilde{Z}_{22}^* \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \right) T^{-1} \\
&= H + T^{-*} \begin{bmatrix} G_1^* & 0 \\ 0 & G_2^* \end{bmatrix} S^{-*} S^* \begin{bmatrix} Z_{11} + \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \end{bmatrix} T^{-1} \\
&\quad + T^{-*} \begin{bmatrix} Z_{11}^* + \tilde{Z}_{11}^* & \tilde{Z}_{21}^* \\ \tilde{Z}_{12}^* & \tilde{Z}_{22}^* \end{bmatrix} S S^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} T^{-1} \\
&= H + G^* Z + Z^* G
\end{aligned}$$

has rank

$$s_1 = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}_{11}) - 2p_1 = \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}_{11}) + 2q_2 - 2(p_1 + q_2).$$

Since $\lambda F_2 + G_2$ has full column rank over $\mathbb{C}(\lambda)$ and $\lambda F_1 + G_1$ has full row rank over $\mathbb{C}(\lambda)$, this shows that the rank is given by

$$\begin{aligned}
s_1 &= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & \lambda F_1 + G_1 & 0 & 0 \\ -\lambda F_1^* + G_1^* & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ 0 & 0 & -\lambda F_2^* + G_2^* & 0 \end{bmatrix} \right) \\
&\quad - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \end{bmatrix} \right) \\
&= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & 0 \\ 0 & -\lambda F_2^* + G_2^* & 0 & 0 \end{bmatrix} \right) \\
&\quad - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)} \left(S^{-1} \begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \end{bmatrix} T^{-1} \right) \\
&= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & H_{12} + G_1^* \tilde{Z}_{12} + \tilde{Z}_{21}^* G_2 \\ 0 & -\lambda F_2^* + G_2^* & H_{21} + G_2^* \tilde{Z}_{21} + \tilde{Z}_{12}^* G_1 & H_{22} + G_2^* \tilde{Z}_{22} + \tilde{Z}_{22}^* G_2 \end{bmatrix} \right) \\
&\quad - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) \\
&= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} I & 0 \\ \tilde{Z}^* & I \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} & H_{12} \\ 0 & -\lambda F_2^* + G_2^* & H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I & \tilde{Z} \\ 0 & I \end{bmatrix} \right) \\
&\quad - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) \\
&= \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) =: s.
\end{aligned}$$

Let $\mathcal{C} \in \mathbb{C}^{s,q}$ be a Cholesky factor of $H + G^* Z + Z^* G$. Using the formulas at (6.3) we see that in this case with $[\mathcal{C}_1 \ \mathcal{C}_2] := \mathcal{C} T$ we have

$$\begin{bmatrix} H_{11} + G_1^*(Z_{11} + \tilde{Z}_{11}) + (Z_{11} + \tilde{Z}_{11})^* G_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{C}_1^* \\ \mathcal{C}_2^* \\ 0 \end{bmatrix} [\mathcal{C}_1 \ \mathcal{C}_2]$$

Since $[\mathcal{C}_1 \ \mathcal{C}_2]$ has full row rank and $\mathcal{C}_2^* [\mathcal{C}_1 \ \mathcal{C}_2] = 0$, this implies that $\mathcal{C}_2 = 0$. Thus we obtain that

$$\begin{aligned}
& \mathfrak{B}_+(\mathcal{N}) \\
&= \begin{bmatrix} S^* & 0 \\ 0 & T \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} & H_{12} \\ 0 & -\lambda F_2^* + G_2^* & H_{12}^* & H_{22} \end{bmatrix} \right) \\
&= \begin{bmatrix} S^* & S^* \tilde{Z} \\ 0 & T \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & 0 \\ 0 & -\lambda F_2^* + G_2^* & 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} S^* & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I & \tilde{Z} \\ 0 & I \end{bmatrix} \left\{ \begin{bmatrix} \mu_1 \\ y \\ z_1 \\ 0 \end{bmatrix} \mid (\mu_1, z_1) \in \mathfrak{B}_+(\mathcal{N}_{11}), y \in \mathfrak{B}_+(-\lambda F_2^* + G_2^*) \right\},
\end{aligned} \tag{6.4}$$

which together with (6.2) implies that

$$\begin{aligned}
\mathfrak{B}_+(\mathcal{N}) &= \begin{bmatrix} S^* & S^* \tilde{Z} \\ 0 & T \end{bmatrix} \left(\left\{ \begin{bmatrix} Z_{11} z_1 \\ 0 \\ z_1 \\ 0 \end{bmatrix} \mid z_1 \in \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F_1 + G_1 \\ \mathcal{C}_1 \end{bmatrix} \right) \right\} \right. \\
&\quad \left. \oplus \left\{ \begin{bmatrix} 0 \\ y \\ 0 \\ 0 \end{bmatrix} \mid y \in \mathfrak{B}_+(-\lambda F_2^* + G_2^*) \right\} \right)
\end{aligned}$$

since $\mathfrak{B}_+(\lambda F_2 + G_2) = \{0\}$ and also $\mathfrak{B}_+(-\lambda F_1^* + G_1^*) = \{0\}$. We conclude that

$$\begin{aligned}
& \mathfrak{B}_+(\mathcal{N}) \\
&= \begin{bmatrix} S^* & S^* \tilde{Z} \\ 0 & T \end{bmatrix} \left(\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \\ I & I \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \\ \mathcal{C}_1 & 0 \end{bmatrix} \right) \oplus \right. \\
&\quad \left. \begin{bmatrix} I \\ 0 \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} -\lambda F_1^* + G_1^* & 0 \\ 0 & -\lambda F_2^* + G_2^* \end{bmatrix} \right) \right) \\
&= \begin{bmatrix} S^* & S^* \tilde{Z} \\ 0 & T \end{bmatrix} \left(\begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \\ I & I \end{bmatrix} T^{-1} \mathfrak{B}_+ \left(\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \\ \mathcal{C}_1 & 0 \end{bmatrix} T^{-1} \right) \oplus \right. \\
&\quad \left. \begin{bmatrix} I \\ 0 \end{bmatrix} S^{-*} \mathfrak{B}_+ \left(T^{-*} \begin{bmatrix} -\lambda F_1^* + G_1^* & 0 \\ 0 & -\lambda F_2^* + G_2^* \end{bmatrix} S^{-*} \right) \right) \\
&= \begin{bmatrix} S^* \left(\tilde{Z} + \begin{bmatrix} Z_{11} & 0 \\ 0 & 0 \\ I & I \end{bmatrix} T^{-1} \right) \\ 0 & T \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) \oplus \begin{bmatrix} I \\ 0 \end{bmatrix} \mathfrak{B}_+(-\lambda F^* + G) \\
&= \begin{bmatrix} Z \\ I \end{bmatrix} \mathfrak{B}_+ \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) \oplus \begin{bmatrix} I \\ 0 \end{bmatrix} \mathfrak{B}_+(-\lambda F^* + G)
\end{aligned}$$

Also, observe that with the transformation used already in (6.4) we have

$$\begin{aligned}
& \mathfrak{Z}(\mathcal{N}) \cap \overline{\mathcal{C}}_- \\
&= \mathfrak{Z} \left(\begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_2^* & 0 & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & 0 \\ 0 & -\lambda F_2^* + G_2^* & 0 & 0 \end{bmatrix} \right) \cap \overline{\mathcal{C}}_- \\
&= \mathfrak{Z}(\mathcal{N}_{11}) \cap \overline{\mathcal{C}}_-,
\end{aligned}$$

since $\lambda F_2 + G_2$ has no zeros. Together with (6.2) this implies that

$$\begin{aligned}
\mathfrak{Z}(\mathcal{N}) \cap \overline{\mathcal{C}}_- &= \mathfrak{Z} \left(\begin{bmatrix} \lambda F_1 + G_1 \\ \mathcal{C}_1 \end{bmatrix} \right) \\
&= \mathfrak{Z} \left(\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \\ \mathcal{C}_1 & 0 \end{bmatrix} \right) \\
&= \mathfrak{Z} \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right).
\end{aligned}$$

Finally, assume that \mathcal{N} has index less or equal to 1 and no purely imaginary zeros. We have

$$\begin{aligned}
\mathcal{N} &= \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \sim \begin{bmatrix} S & 0 \\ 0 & T^* \end{bmatrix} \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} \begin{bmatrix} S^* & 0 \\ 0 & T \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} & H_{12} \\ 0 & -\lambda F_2^* + G_2^* & H_{12}^* & H_{22} \end{bmatrix} \\
&\sim \begin{bmatrix} I & 0 \\ \tilde{Z}^* & I \end{bmatrix} \begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} & H_{12} \\ 0 & -\lambda F_2^* + G_2^* & H_{12}^* & H_{22} \end{bmatrix} \begin{bmatrix} I & \tilde{Z} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \lambda F_1 + G_1 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ -\lambda F_1^* + G_1^* & 0 & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & 0 \\ 0 & -\lambda F_2^* + G_2^* & 0 & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} 0 & \lambda F_1 + G_1 & 0 & 0 \\ -\lambda F_1^* + G_1^* & H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda F_2 + G_2 \\ 0 & 0 & -\lambda F_2^* + G_2^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{N}_{11} & 0 & 0 \\ 0 & 0 & \lambda F_2 + G_2 \\ 0 & -\lambda F_2^* + G_2^* & 0 \end{bmatrix},
\end{aligned}$$

which means that if the Thompson canonical form (2.8) of \mathcal{N} has no blocks of a certain type, so does the Thompson canonical form of \mathcal{N}_{11} . This means that also \mathcal{N}_{11} has index less or equal to 1 and no purely imaginary zeros. With Lemma 4.3 this implies that we can assume that Z_{11} is a strong solution of the Lur'e equation associated with $(\lambda F_1 + G_1, H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1)$. Let $\hat{z} \in \mathfrak{R}_+(\lambda F + G)$. Set $\hat{y} := T^{-1} \hat{z}$. Then from Definition 5.1 and the

assumption of controllability we obtain that

$$\begin{aligned}\hat{y} &= T^{-1}\hat{z} \in T^{-1}\mathfrak{R}_+(\lambda F + G) = T^{-1}\mathfrak{R}(\lambda F + G) = \mathfrak{R}(S(\lambda F + G)T) \\ &= \mathfrak{R}\left(\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \end{bmatrix}\right) = \text{image}\left(\begin{bmatrix} I \\ 0 \end{bmatrix}\right),\end{aligned}$$

which means that there exists a $\hat{x} \in \mathbb{C}^{q_1} = \mathfrak{R}_+(\lambda F_1 + G_1)$ such that $\hat{y} = (\hat{x}, 0)$. Using that Z_{11} is a strong solution of the Lur'e equation associated with $(\lambda F_1 + G_1, H_{11} + G_1^* \tilde{Z}_{11} + \tilde{Z}_{11}^* G_1)$ we deduce that there exists an

$$x_0 \in \mathfrak{R}_+\left(\begin{bmatrix} \lambda F_1 + G_1 \\ \mathcal{C}_1 \end{bmatrix}\right),$$

such that $F_1 x_0 = F_1 \hat{x}$. Set $z_0 := T \begin{bmatrix} x_0^T & 0 \end{bmatrix}^T$. Then we have

$$\begin{aligned}Fz_0 &= FTT^{-1}z_0 = S^{-1}\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}\begin{bmatrix} x_0 \\ 0 \end{bmatrix} \\ &= S^{-1}\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}\begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} FT\hat{y} = F\hat{z},\end{aligned}$$

and, since $\mathfrak{R}_+(\lambda F_2 + G_2) = \{0\}$, we have

$$\begin{aligned}z_0 &= T \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in T\mathfrak{R}_+\left(\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \\ \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix}\right) \\ &= \mathfrak{R}_+\left(\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \lambda F_1 + G_1 & 0 \\ 0 & \lambda F_2 + G_2 \\ \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix}T^{-1}\right) = \mathfrak{R}_+\left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix}\right)\end{aligned}$$

which proves that in this case Z is also a strong solution. \square

7. Connection to the Popov function. In this section we will introduce the Popov function, cf. Definition 7.2. The relevance of the Popov function for systems theory is well understood, see [12]. We will show in Lemma 7.3 that the quantity s in Definition 1.6 is nothing else than the rank (over $\mathbb{C}(\lambda)$) of the Popov function. Also note that in [1] it was shown that for controllable systems the zeros of the matrix \mathcal{N} from (1.9) are the zeros of the Popov function.

DEFINITION 7.1. Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and let $r := \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$ denote its rank. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}(\lambda)^{q,r}$ be such that

- $(\lambda F + G)U(\lambda) = 0$,
- $\text{rank}_{\mathbb{C}(\lambda)}((\lambda F + G)V(\lambda)) = r$, and
- $\begin{bmatrix} U & V \end{bmatrix}$ is invertible (over $\mathbb{C}(\lambda)$).

Then we call U and V a kernel-spanning matrix and a co-kernel-spanning matrix of $\lambda F + G$.

With this we can define the notion of the Popov function as follows.

DEFINITION 7.2. Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ with $r := \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$ and let $H = H^* \in \mathbb{C}^{q,q}$. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel-spanning matrix of $\lambda F + G$. Then we call

$$\Pi := U^* H U,$$

a Popov function of $(\lambda F + G, H)$ or the Popov function associated with the kernel-spanning matrix U .

It is well-known that a controllable system is dissipative, if and only if one of its Popov functions is positive semi-definite along the imaginary axis, see [34]. In this paper we, however, only want to concentrate on the following fact.

LEMMA 7.3. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ be a first-order matrix polynomial and let $H = H^* \in \mathbb{C}[\lambda]^{q,q}$ be Hermitian. As in Definition 1.6, form the para-Hermitian polynomial $\mathcal{N} = \mathcal{N}^\sim \in \mathbb{C}[\lambda]^{p+q,p+q}$ via*

$$\mathcal{N}(\lambda) := \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix},$$

and let the quantity s be given by

$$s := \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) - 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(\lambda F + G).$$

Let Π be a Popov function of $(\lambda F + G, H)$. Then we have

$$s = \text{rank}_{\mathbb{C}(\lambda)}(\Pi).$$

Proof. We have

$$\begin{aligned} \text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) &= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} I & 0 \\ 0 & [U \ V] \end{bmatrix}^\sim \begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & [U \ V] \end{bmatrix} \right) \\ &= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & U^\sim HV \\ V^\sim P^\sim & V^\sim HU & V^\sim HV \end{bmatrix} \right), \end{aligned}$$

and thus, since PV has full column rank also

$$\text{rank}_{\mathbb{C}(\lambda)}(\mathcal{N}) = \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & 0 \\ V^\sim P^\sim & 0 & 0 \end{bmatrix} \right) = \text{rank}_{\mathbb{C}(\lambda)}(\Pi) + 2 \cdot \text{rank}_{\mathbb{C}(\lambda)}(PV),$$

which proves the claim. \square

In this section we have shown that the quantity s from Definition 1.6 is the rank of the Popov function. We will use this property in the next section to obtain the uniqueness of the solution of the algebraic Riccati equation in the case were the Popov function does not have any zeros one the imaginary axis.

8. Specialization to standard state-space systems. In this section we will deduce the well-known results about the solvability of the algebraic Riccati equation, see [26], from the results which we have obtained above. Consider the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{8.1}$$

where $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $x \in \mathcal{C}_\infty^n$ is called the state, and $u \in \mathcal{C}_\infty^m$ is called the input. With this, build the matrix polynomial

$$\lambda F + G := \lambda \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} -A & -B \end{bmatrix}. \tag{8.2}$$

We call system (8.1) *controllable*, if with (8.2) we have that $\mathfrak{B}(\lambda F + G)$ is controllable in the sense of Definition 2.6.

In the literature, e.g., [16], for systems of the form (8.1) the supply is frequently measured by a quadratic function $s : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ of the form

$$s(u, x) := \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad (8.3)$$

where $Q = Q^T \in \mathbb{R}^{n,n}$, $S \in \mathbb{R}^{n,m}$, and $R = R^T \in \mathbb{R}^{m,m}$. Under these circumstances system (8.1) is called dissipative with respect to s , if there exists a storage function $\Theta : \mathbb{C}^n \rightarrow \mathbb{R}$, which fulfills

$$\Theta(x(t_1)) - \Theta(x(t_0)) \leq \int_{-\infty}^{\infty} s(u(t), x(t)) dt, \quad (8.4)$$

for all trajectories $(u, x) \in \mathcal{C}_c^m \times \mathcal{C}_c^n$ which satisfy (8.1) for all $t_0 \leq t_1$. Note the following difference to Definition 1.3. In Definition 1.3 the storage function may depend on z (and its derivatives up to some degree) just as the term on the right hand side of the dissipation inequality in Definition 1.3 may depend on all components of z . In (8.4), however, the right hand side still depends on all components of u and x , while the storage function Θ only depends on the state x . Nevertheless, one can show that Definition 1.3 and inequality (8.4) are equivalent. This can be done in the following way. In [24] it was shown that every storage function Θ is a function of a so-called state-map. Furthermore, in [20] it was shown, that for systems of the form (8.1) a state-map is given by $(x, u) \mapsto x$.

Consider the rational matrices $U \in \mathbb{C}(\lambda)^{n+m,m}$ and $V \in \mathbb{C}(\lambda)^{n+m,n}$ given by

$$U(\lambda) := \begin{bmatrix} (\lambda I - A)^{-1} B \\ I \end{bmatrix} \quad \text{and} \quad V(\lambda) := \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Then we immediately see that U and V are kernel-spanning and co-kernel-spanning matrices of $\lambda F + G$ given by (8.2), in the sense of Definition 7.1. Thus, with

$$H := \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \quad (8.5)$$

we have that a Popov function of $(\lambda F + G, H)$ is given by

$$\begin{aligned} & U^\sim(\lambda) H U(\lambda) \\ &= \begin{bmatrix} (\lambda I - A)^{-1} B \\ I \end{bmatrix}^\sim \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (\lambda I - A)^{-1} B \\ I \end{bmatrix} \\ &= B^* (-\lambda I^* - A^*)^{-1} Q (\lambda I - A) B + B^* (-\lambda I^* - A^*)^{-1} S + S^* (\lambda I - A) B + R. \end{aligned} \quad (8.6)$$

COROLLARY 8.1. *Assume that system (8.1) is controllable and that in (8.3) the matrix R is invertible. Then (8.1) is dissipative with respect to the supply function (8.3) if and only if there exists a Hermitian matrix $X = X^* \in \mathbb{C}^{n,n}$ which solves the algebraic Riccati equation*

$$Q - A^* X - X^* A - (S - X^* B) R^{-1} (S^* - B^* X) = 0 \quad (8.7)$$

and all eigenvalues of the matrix

$$A - B R^{-1} (S^* - B^* X)$$

are in the closed left half plane.

Furthermore, if A has no eigenvalues on the imaginary axis and the Popov function (8.6) has no zeros on the imaginary axis, then X is unique.

Proof. To show that dissipativity implies the existence of a solution of the algebraic Riccati equation, let $\lambda F + G$ be defined through (8.2) and let H be defined through (8.5). Then the assumption implies that $(\lambda F + G, H)$ is dissipative, controllable, and (because of Corollary 2.8) trim. Thus, using Lemma 4.3, we obtain the existence of a $Z \in \mathbb{C}^{n, n+m}$ which solves the Lur'e equation. Partition

$$Z =: \begin{bmatrix} X & Y \end{bmatrix},$$

with $X \in \mathbb{C}^{n, n}$ and $Y \in \mathbb{C}^{n, m}$. Then we obtain from Definition 1.6 that

$$\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} = F^* Z = Z^* F = \begin{bmatrix} X^* \\ Y^* \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} = \begin{bmatrix} X^* & 0 \\ Y^* & 0 \end{bmatrix},$$

which implies that $X = X^*$ and $Y = 0$. Since R is assumed to be invertible we see that the Popov function in (8.6) has rank m over $\mathbb{C}(\lambda)$. Thus, from Definition 1.6 and Lemma 7.3 we obtain that

$$\begin{aligned} 0 \leq H + G^* Z + Z^* G &= \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} + \begin{bmatrix} -A^* \\ -B^* \end{bmatrix} \begin{bmatrix} X & 0 \end{bmatrix} + \begin{bmatrix} X^* \\ 0 \end{bmatrix} \begin{bmatrix} -A & -B \end{bmatrix} \\ &= \begin{bmatrix} Q - A^* X - X^* A & S - X^* B \\ S^* - B^* X & R \end{bmatrix}, \end{aligned}$$

has rank m and we see that R is positive definite. Building the Schur complement we find that also

$$\begin{aligned} 0 \leq & \begin{bmatrix} I & -(S - X^* B)R^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - A^* X - X^* A & S - X^* B \\ S^* - B^* X & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}(S^* - B^* X) & I \end{bmatrix} \\ &= \begin{bmatrix} I & -(S - X^* B)R^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - A^* X - X^* A - (S - X^* B)R^{-1}(S^* - B^* X) & S - X^* B \\ 0 & R \end{bmatrix} \\ &= \begin{bmatrix} Q - A^* X - X^* A - (S - X^* B)R^{-1}(S^* - B^* X) & 0 \\ 0 & R \end{bmatrix}, \end{aligned}$$

has rank m , which implies that

$$Q - A^* X - X^* A - (S - X^* B)R^{-1}(S^* - B^* X) = 0.$$

Let $R = T^* T$ be a Cholesky factorization of R . Then we can factor

$$\begin{aligned} 0 \leq & \begin{bmatrix} Q - A^* X - X^* A & S - X^* B \\ S^* - B^* X & R \end{bmatrix} = \begin{bmatrix} (S - X^* B)R^{-1}(S^* - B^* X) & S - X^* B \\ S^* - B^* X & R \end{bmatrix} \\ &= \begin{bmatrix} (S - X^* B)T^{-1} \\ T^* \end{bmatrix} \begin{bmatrix} T^{-*}(S^* - B^* X) & T \end{bmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} & 3 \left(\begin{bmatrix} \lambda I - A & -B \\ T^{-*}(S^* - B^* X) & T \end{bmatrix} \right) \\ &= 3 \left(\begin{bmatrix} \lambda I - A & -B \\ T^{-*}(S^* - B^* X) & T \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}(S^* - B^* X) & I \end{bmatrix} \right) \\ &= 3 \left(\begin{bmatrix} I & BT^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - A + BR^{-1}(S^* - B^* X) & -B \\ 0 & T \end{bmatrix} \right) \\ &= 3 \left(\begin{bmatrix} \lambda I - A + BR^{-1}(S^* - B^* X) & 0 \\ 0 & T \end{bmatrix} \right) = 3 (\lambda I - (A - BR^{-1}(S^* - B^* X))). \end{aligned}$$

Finally, assume that A is stable and that the Popov function (8.6) has no purely imaginary zeros. Define

$$\mathcal{N}(\lambda) := \begin{bmatrix} 0 & \lambda F + G \\ -\lambda F^* + G^* & H \end{bmatrix} = \begin{bmatrix} 0 & \lambda I - A & -B \\ -\lambda I - A^* & Q & S \\ -B^* & S^* & R \end{bmatrix}.$$

Since R is invertible, we see that \mathcal{N} is regular (i.e., in the Thompson canonical form (2.8) there are no blocks of type (T6)) and has index 1. Define

$$X(\lambda) := \begin{bmatrix} I & 0 & (\lambda I - A)^{-1}(Q(\lambda I - A)^{-1}B + S) \\ 0 & I & (\lambda I - A)^{-1}B \\ 0 & 0 & I \end{bmatrix}.$$

Since A has no purely imaginary eigenvalues we conclude that X has no poles on the imaginary axis. Thus, using (1.4), we find that the purely imaginary zeros of \mathcal{N} coincide with those of

$$X \sim \mathcal{N} X = \begin{bmatrix} 0 & \lambda I - A & 0 \\ -\lambda I - A^* & Q & 0 \\ 0 & 0 & \Pi \end{bmatrix},$$

where Π denotes the Popov function from (8.6). Since we assume that Π has no purely imaginary zeros, we conclude that also \mathcal{N} has no purely imaginary zeros. Thus, Lemma 4.3 implies that Z is a strong solution. Using Lemma 2.4 and the fact that with (8.2) we have $\mathfrak{R}_+(\lambda F + G) = \mathbb{C}^{n+m}$, we conclude the uniqueness of X . \square

In this section we have derived from the material in the previous sections that for controllable standard systems with regular R the solvability of the algebraic Riccati equation (8.7) is equivalent to dissipativity, a well-known result [26]. Also we proved that the solution is unique, if A is stable and the Popov function (8.6) has no zeros on the imaginary axis.

9. Conclusion. In this paper we have proven that, under the assumptions that $\mathfrak{B}(\lambda F + G)$ is controllable and $(\lambda F + G, H)$ is feasible, dissipativity is equivalent to the solvability of the Lur'e equation (Theorem 6.2). Furthermore, we saw that every problem $(\lambda F + G, \tilde{H})$ can be reformulated into an equivalent problem $(\lambda F + G, H)$ which is feasible (Theorem 5.3). This means that we essentially assume controllability of the system. It remains unclear, if the assumption of controllability is really necessary, and we conjecture that it can be weakened to stabilizability or that it can be dropped entirely. It is possible to derive the standard results about the algebraic Riccati equation from our results (Corollary 8.1).

All results have been formulated on the positive time-axis, although it would be no problem to formulate them analogously on the negative time-axis, i.e., one can show that for controllable systems with feasible supply there exists a $Z \in \mathbb{C}^{p \times q}$ such that 1.)-3.) of Definition 1.6 are fulfilled, while for every Cholesky factor $\mathcal{C} \in \mathbb{C}^{s \times q}$ of the matrix $G^*Z + Z^*G + \hat{H} = \mathcal{C}^*\mathcal{C}$, we have that

$$\mathfrak{B}_-(\mathcal{N}) = \begin{bmatrix} Z \\ I \end{bmatrix} \mathfrak{B}_- \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) \oplus \begin{bmatrix} I \\ 0 \end{bmatrix} \mathfrak{B}_-(-\lambda F^* + G^*),$$

(with \mathfrak{B}_- defined analogously to Definition 1.2 in the obvious way) and we have

$$\mathfrak{Z} \left(\begin{bmatrix} \lambda F + G \\ \mathcal{C} \end{bmatrix} \right) = \mathfrak{Z}(\mathcal{N}) \cap \overline{\mathbb{C}_+}.$$

Finally, the question arises, how the matrices Z and C from Definition 1.6 can be computed in a numerical reliable and efficient way. The results, especially Lemma 3.5 and the proof of Theorem 5.3, give a first hint at how this might be possible. However, the Thompson canonical form (2.8) cannot be used for actual computations, since it involves transformations with arbitrary invertible matrices X . From a numerical point-of-view it is desirable to only use unitary transformations. The characteristics of the singular blocks (T6) and the blocks of index higher than 1 (i.e., the blocks of type (T4) and (T5) with $\hat{k}_j > 1$) can be obtained from a so-called *staircase form*, which can be computed via unitary transformations only [4, 5]. This staircase form could be a helpful tool for the computation of solutions of the Lur'e equation.

It is well-known that Hamiltonian eigenvalue problems can be used to compute the stabilizing solution of the algebraic Riccati equation, compare [15]. Thus, we conjecture that this approach can be generalized to the setting employed in this paper via para-Hermitian eigenvalue problems. Methods for para-Hermitian eigenvalue problems are an active topic of research, compare [13, 17, 19].

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Appendix A. Proof of Theorem 5.4.

In this section we give the proof of Theorem 5.4. The basic idea is to use the Kronecker canonical form (2.1) to transform the equations of Definition 5.1 into a set of equations, which can be solved in a purely algebraic fashion. To do so, we have to consider the corresponding equations, which occur by looking at individual blocks of the Kronecker canonical form. This is done in the following Lemmata, which are needed for the proof of Theorem 5.4. The whole construction is very similar to the one in [2, Appendix A.5].

LEMMA A.1. *Consider a single block of type (K4) in the Kronecker canonical form, i.e., let $F, G \in \mathbb{C}^{\eta+1, \eta}$ be defined as*

$$F = \begin{bmatrix} 1 & & & \\ 0 & \ddots & & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix},$$

and consider the pencil $\lambda F + G \in \mathbb{C}[\lambda]^{\eta+1, \eta}$. Let $H = H^* \in \mathbb{C}^{\eta, \eta}$ be a Hermitian matrix. Then there exists a $X = X^* \in \mathbb{C}^{\eta+1, \eta+1}$ such that $0 = G^* X F + F^* X G + H$.

Proof. Denote the entries of the Hermitian matrix X in the form

$$X = \begin{bmatrix} x_{1,1} & \dots & x_{1, \eta+1} \\ \vdots & & \vdots \\ x_{\eta+1,1} & \dots & x_{\eta+1, \eta+1} \end{bmatrix} = [x_{i,j}]_{i,j=1, \dots, \eta+1} = [\overline{x_{j,i}}]_{i,j=1, \dots, \eta+1}$$

and the Hermitian matrix H in the form

$$H = \begin{bmatrix} h_{1,1} & \dots & h_{1, \eta} \\ \vdots & & \vdots \\ h_{\eta,1} & \dots & h_{\eta, \eta} \end{bmatrix} = [h_{i,j}]_{i,j=1, \dots, \eta} = [\overline{h_{j,i}}]_{i,j=1, \dots, \eta}.$$

Then we see that we are looking for an Hermitian X such that

$$\begin{aligned}
0 &= F^* X G + G^* X F + H \\
&= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,1} & \cdots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta+1,1} & \cdots & x_{\eta+1,\eta+1} \end{bmatrix} \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & 0 \\ & & & 1 \end{bmatrix} + (F^* X G)^* + H \\
&= \begin{bmatrix} x_{1,2} & \cdots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta,2} & \cdots & x_{\eta,\eta+1} \end{bmatrix} + \begin{bmatrix} \overline{x_{1,2}} & \cdots & \overline{x_{\eta,2}} \\ \vdots & & \vdots \\ \overline{x_{1,\eta+1}} & \cdots & \overline{x_{\eta,\eta+1}} \end{bmatrix} + \begin{bmatrix} h_{1,1} & \cdots & h_{1,\eta} \\ \vdots & & \vdots \\ h_{\eta,1} & \cdots & h_{\eta,\eta} \end{bmatrix} \\
&= [x_{i,j+1}]_{i,j=1,\dots,\eta} + [\overline{x_{j,i+1}}]_{i,j=1,\dots,\eta} + [h_{i,j}]_{i,j=1,\dots,\eta} \\
&= [x_{i,j+1} + \overline{x_{j,i+1}} + h_{i,j}]_{i,j=1,\dots,\eta} \\
&= [x_{i,j+1} + x_{i+1,j} + h_{i,j}]_{i,j=1,\dots,\eta}. \tag{A.1}
\end{aligned}$$

We construct such an X in the following recursive way. First, choose all $x_{i,i} = 0$ for $i = 1, \dots, \eta + 1$ and choose $x_{i,i+1} := x_{i+1,i} := -\frac{h_{i,i}}{2} \in \mathbb{R}$ for all $i = 1, \dots, \eta$. With this choice all $x_{i,j}$ with $|i - j| \leq 1$ are fixed and all equations in (A.1) with $|i - j| \leq 0$ are fulfilled.

As induction hypothesis, assume that for some $k \in \{1, \dots, \eta - 1\}$ we have that all $x_{i,j}$ with $|i - j| \leq k$ are fixed and all equations in (A.1) with $|i - j| \leq k - 1$ are fulfilled.

For the inductive step, note that all equations in (A.1) with $|i - j| = k$ are given by

$$0 = x_{j+k,j+1} + x_{j+k+1,j} + h_{j+k,j}$$

for $j = 0, \dots, \eta - k$ and their complex conjugate equations, which are not really additional equations. Since $|(j+k) - (j+1)| = k - 1 \leq k$, we know that all $x_{j+k,j+1}$ are already fixed but not the $x_{j+k+1,j}$, since $|(j+k+1) - j| = k + 1 > k$. Thus we define

$$\overline{x_{j,j+k+1}} := x_{j+k+1,j} := -x_{j+k,j+1} - h_{j+k,j},$$

for $j = 0, \dots, \eta - k$ and thus have fixed all $x_{i,j}$ with $|i - j| \leq k + 1$ while at the same time all equations in (A.1) with $|i - j| \leq k$ are fulfilled. Thus the inductive argument is finished and the claim is proved. \square

LEMMA A.2. *Let the pencils $\lambda F_1 + G_1 \in \mathbb{C}[\lambda]_1^{\eta_1+1,\eta}$ and $\lambda F_2 + G_2 \in \mathbb{C}[\lambda]_1^{\eta_2+1,\eta_2}$ both be of the form (K4), as defined in the Kronecker canonical form (2.1). Let $H_{12} \in \mathbb{C}^{\eta_1,\eta_2}$. Then there exists a matrix $X_{12} \in \mathbb{C}^{\eta_1+1,\eta_2+1}$ such that*

$$0 = F_1^* X_{12} G_2 + G_1^* X_{12} F_2 + H_{12}.$$

Proof. For the matrix X_{12} we introduce the notation

$$X_{12} = \begin{bmatrix} x_{1,1} & \cdots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \cdots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} = [x_{i,j}]_{\substack{i=1,\dots,\eta_1+1 \\ j=1,\dots,\eta_2+1}} \in \mathbb{C}^{\eta_1+1,\eta_2+1},$$

and for the matrix H_{12} analogously

$$H_{12} = \begin{bmatrix} h_{1,1} & \cdots & h_{1,\eta_2} \\ \vdots & & \vdots \\ h_{\eta_1,1} & \cdots & h_{\eta_1,\eta_2} \end{bmatrix} = [h_{i,j}]_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \in \mathbb{C}^{\eta_1,\eta_2}.$$

Then we see that we are looking for an X_{12} such that

$$\begin{aligned}
0 &= F_1^* X_{12} G_2 + G_1^* X_{12} F_2 + H_{12} \\
&= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,1} & \cdots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \cdots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \cdots & \\ & \ddots & 0 \\ & & & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,1} & \cdots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \cdots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & & 0 \end{bmatrix} + H_{12} \\
&= \begin{bmatrix} x_{1,2} & \cdots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1,2} & \cdots & x_{\eta_1,\eta_2+1} \end{bmatrix} + \begin{bmatrix} x_{2,1} & \cdots & x_{2,\eta_2} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \cdots & x_{\eta_1+1,\eta_2} \end{bmatrix} + \begin{bmatrix} h_{1,1} & \cdots & h_{1,\eta_2} \\ \vdots & & \vdots \\ h_{\eta_1,1} & \cdots & h_{\eta_1,\eta_2} \end{bmatrix} \\
&= [x_{i,j+1}]_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} + [x_{i+1,j}]_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} + [h_{i,j}]_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \\
&= [x_{i,j+1} + x_{i+1,j} + h_{i,j}]_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}}. \tag{A.2}
\end{aligned}$$

We construct such an X_{12} in the following recursive way. First, choose $x_{i,1} = 0$ for $i = 1, \dots, \eta_1 + 1$, choose $x_{i,2} := h_{i,1}$ for $i = 1, \dots, \eta_1$, and choose $x_{\eta_1+1,2}$ arbitrary. Then all $x_{i,j}$ with $j \leq 2$ are fixed and all equations in (A.2) with $j \leq 1$ are fulfilled.

As induction hypothesis, assume that for some $k \in \{2, \dots, \eta_2\}$ we have that all $x_{i,j}$ with $j \leq k$ are fixed and all equations in (A.2) with $j \leq k - 1$ are fulfilled.

For the inductive step, note that all equations in (A.2) with $j = k$ are given by

$$x_{i,k+1} + x_{i+1,k} + h_{i,k} = 0,$$

with $i = 1, \dots, \eta_1$. Because of the induction hypothesis all $x_{i+1,k}$ are already fixed but not the $x_{i,k+1}$. Thus we define

$$x_{i,k+1} := -x_{i+1,k} - h_{i,k},$$

for $i = 1, \dots, \eta_1$ and choose $x_{\eta_1+1,k+1}$ arbitrary. Then all $x_{i,j}$ with $j \leq k + 1$ are fixed and all equations in (A.2) with $j \leq k$ are fulfilled. Thus the inductive argument is finished and the claim is proved. \square

LEMMA A.3. *Let $w \in \mathbb{N}$ and consider the pencil*

$$\lambda F + G = \lambda \cdot \text{diag}(F_1, \dots, F_w) + \text{diag}(G_1, \dots, G_w),$$

where the pencils on the block diagonal $\lambda F_i + G_i \in \mathbb{C}^{\eta_i+1, \eta_i}$ are of the form (K_4) for $i = 1, \dots, w$. Set $\eta := \eta_1 + \dots + \eta_w$ and observe that $F, G \in \mathbb{C}^{\eta+w, \eta}$. Let an arbitrary $H = H^* \in \mathbb{C}^{\eta, \eta}$ be given. Then there exists a matrix $Z \in \mathbb{C}^{\eta+w, \eta}$ such that

$$\begin{aligned}
F^* Z &= Z^* F, \\
0 &= G^* Z + Z^* G + H.
\end{aligned}$$

Proof. We construct an $X = X^* \in \mathbb{C}^{\eta+w, \eta+w}$ such that $F^*XG + G^*XF + H = 0$. Then we obtain the assertion by setting $Z := XF$. Partition the matrix X according to the partition of F and G as

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1w} \\ \vdots & & \vdots \\ X_{w1} & \cdots & X_{ww} \end{bmatrix} \begin{array}{l} \eta_1 + 1 \\ \vdots \\ \eta_w + 1 \end{array}$$

and observe that from $X = X^*$ we obtain that $X_{ij} = X_{ji}^*$ for all $i, j = 1, \dots, w$. We see that we are looking for an X such that

$$\begin{aligned} 0 &= F^*XG + G^*XF + H \\ &= \begin{bmatrix} F_1^* & & \\ & \ddots & \\ & & F_w^* \end{bmatrix} \begin{bmatrix} X_{11} & \cdots & X_{1w} \\ \vdots & & \vdots \\ X_{1w}^* & \cdots & X_{ww} \end{bmatrix} \begin{bmatrix} G_1 & & \\ & \ddots & \\ & & G_w \end{bmatrix} + (F^*XG)^* + H \\ &= \begin{bmatrix} F_1^*X_{11}G_1 & \cdots & F_1^*X_{1w}G_w \\ \vdots & & \vdots \\ F_w^*X_{1w}^*G_1 & \cdots & F_w^*X_{ww}G_w \end{bmatrix} + \begin{bmatrix} G_1^*X_{11}F_1 & \cdots & G_1^*X_{1w}F_w \\ \vdots & & \vdots \\ G_w^*X_{1w}^*F_1 & \cdots & G_w^*X_{ww}F_w \end{bmatrix} \\ &\quad + \begin{bmatrix} H_{11} & \cdots & H_{1w} \\ \vdots & & \vdots \\ H_{1w}^* & \cdots & H_{ww} \end{bmatrix} \\ &= \begin{bmatrix} F_1^*X_{11}G_1 + G_1^*X_{11}F_1 + H_{11} & \cdots & F_1^*X_{1w}G_w + G_1^*X_{1w}F_w + H_{1w} \\ \vdots & & \vdots \\ F_w^*X_{1w}G_1 + G_w^*X_{1w}^*F_1 + H_{1w}^* & \cdots & F_w^*X_{ww}G_w + G_w^*X_{ww}F_w + H_{ww} \end{bmatrix}. \quad (\text{A.3}) \end{aligned}$$

Using Lemma A.1 we obtain $X_{ii} = X_{ii}^* \in \mathbb{C}^{\eta_i+1, \eta_i+1}$ such that $F_i^*X_{ii}G_i + G_i^*X_{ii}F_i + H_{ii} = 0$ for all $i = 1, \dots, w$. Using Lemma A.2 we obtain $X_{ij} \in \mathbb{C}^{\eta_i+1, \eta_j+1}$ such that $F_i^*X_{ij}G_j + G_i^*X_{ij}F_j + H_{ij} = 0$ for all $i, j = 1, \dots, w$ with $i < j$. Since the equations in the strict block-lower-left part of (A.3) are the conjugate transpose of the equations in the strict block-upper-right part, they are also fulfilled. \square

LEMMA A.4. *Let $\lambda F_1 + G_1 \in \mathbb{C}[\lambda]_1^{\epsilon, \epsilon+1}$ be in the form (K1) and $\lambda F_4 + G_4 \in \mathbb{C}^{k_j+1, k_j}$ be in the form (K4) with $k_j \leq 2$. Let $H_{14} \in \mathbb{C}^{\epsilon+1, \eta}$ be arbitrary. Then there exist $Z_{14} \in \mathbb{C}^{\epsilon, \eta}$ and $Z_{41} \in \mathbb{C}^{\eta+1, \epsilon+1}$ such that*

$$\begin{aligned} F_1^*Z_{14} &= Z_{41}^*F_4 \\ 0 &= G_1^*Z_{14} + Z_{41}^*G_4 + H_{14}. \end{aligned}$$

Proof. Denote the entries of H by $[h_{i,j}]$. If $\eta = 1$ then the matrices Z_{14} and Z_{41} take the form

$$Z_{14} = \begin{bmatrix} z_1 \\ \vdots \\ z_\epsilon \end{bmatrix}, \quad Z_{41} = \begin{bmatrix} \tilde{z}_{1,1} & \cdots & \tilde{z}_{1, \epsilon+1} \\ \tilde{z}_{2,1} & \cdots & \tilde{z}_{2, \epsilon+1} \end{bmatrix}.$$

With this we obtain

$$\begin{bmatrix} z_1 \\ \vdots \\ z_\epsilon \\ 0 \end{bmatrix} = F_1^* Z_{14} = Z_{41}^* F_4 = \begin{bmatrix} \overline{\tilde{z}_{1,1}} \\ \vdots \\ \overline{\tilde{z}_{1,\epsilon+1}} \end{bmatrix},$$

which implies

$$Z_{41} = \begin{bmatrix} \overline{z_1} & \cdots & \overline{z_\epsilon} & 0 \\ \overline{\tilde{z}_{2,1}} & \cdots & \overline{\tilde{z}_{2,\epsilon}} & \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix}.$$

With this notation at hand we can verify that

$$G_1^* Z_{14} + Z_{41}^* G_4 = - \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & & 1 \end{bmatrix} Z_{14} + \begin{bmatrix} z_1 & \overline{\tilde{z}_{2,1}} \\ \vdots & \vdots \\ z_\epsilon & \overline{\tilde{z}_{2,\epsilon}} \\ 0 & \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ z_1 \\ \vdots \\ z_\epsilon \end{bmatrix} + \begin{bmatrix} \overline{\tilde{z}_{2,1}} \\ \vdots \\ \overline{\tilde{z}_{2,\epsilon}} \\ \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix},$$

which proves that one can choose $Z_{14} = 0$ and $\overline{\tilde{z}_{2,i}} := -\overline{h_{i,1}}$ for $i = 1, \dots, \epsilon + 1$.

If $\eta = 2$ then the matrices Z_{14} and Z_{41} take the form

$$Z_{14} = \begin{bmatrix} z_{1,1} & z_{1,2} \\ \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} \end{bmatrix}, \quad Z_{41} = \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \cdots & \overline{\tilde{z}_{1,\epsilon+1}} \\ \overline{\tilde{z}_{2,1}} & \cdots & \overline{\tilde{z}_{2,\epsilon+1}} \\ \overline{\tilde{z}_{3,1}} & \cdots & \overline{\tilde{z}_{3,\epsilon+1}} \end{bmatrix}.$$

With this we obtain

$$\begin{bmatrix} z_{1,1} & z_{1,2} \\ \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} \\ 0 & 0 \end{bmatrix} = F_1^* Z_{14} = Z_{41}^* F_4 = \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \overline{\tilde{z}_{2,1}} \\ \vdots & \vdots \\ \overline{\tilde{z}_{1,\epsilon+1}} & \overline{\tilde{z}_{2,\epsilon+1}} \end{bmatrix}$$

which implies

$$Z_{41} = \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \cdots & \overline{\tilde{z}_{\epsilon,1}} & 0 \\ \overline{\tilde{z}_{1,2}} & \cdots & \overline{\tilde{z}_{\epsilon,2}} & 0 \\ \overline{\tilde{z}_{3,1}} & \cdots & \overline{\tilde{z}_{3,\epsilon}} & \overline{\tilde{z}_{3,\epsilon+1}} \end{bmatrix}.$$

With this notation at hand we can verify that

$$\begin{aligned} & G_1^* Z_{14} + Z_{41}^* G_4 \\ &= - \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & & 1 \end{bmatrix} Z_{14} + \begin{bmatrix} z_{1,1} & z_{1,2} & \overline{\tilde{z}_{3,1}} \\ \vdots & \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} & \overline{\tilde{z}_{3,\epsilon}} \\ 0 & 0 & \overline{\tilde{z}_{3,\epsilon+1}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ z_{1,1} & z_{1,2} \\ \vdots & \vdots \\ z_{\epsilon,1} & z_{\epsilon,2} \end{bmatrix} + \begin{bmatrix} z_{1,2} & \overline{\tilde{z}_{3,1}} \\ \vdots & \vdots \\ z_{\epsilon,2} & \overline{\tilde{z}_{3,\epsilon}} \\ 0 & \overline{\tilde{z}_{3,\epsilon+1}} \end{bmatrix}. \end{aligned}$$

Thus, choosing $z_{i,1} := \overline{h_{i+1,1}}$ for $i = 1, \dots, \epsilon$, $z_{1,2} := -\overline{h_{1,1}}$, $z_{i,2} := 0$ for $i = 2, \dots, \epsilon$, $\overline{\tilde{z}_{3,1}} := -\overline{h_{1,2}}$, $\overline{\tilde{z}_{3,2}} := \overline{z_{1,2}} - \overline{h_{2,2}}$, and $\overline{\tilde{z}_{3,i}} := -\overline{h_{i,2}}$ for $i = 3, \dots, \epsilon + 1$ we obtain the assertion. \square

With this we can prove Theorem 5.4, the statement of which is repeated here for clarity.

THEOREM A.5. *Let $\lambda F + G \in \mathbb{C}[\lambda]^{p,q}$ and let (2.1) be its Kronecker canonical form. Let $H = H^* \in \mathbb{C}^{q,q}$. Assume that $\mathfrak{B}(\lambda F + G)$ is controllable, that all blocks of type (K3) have size $k_j \leq 1$ and all blocks of type (K4) have size $k_j \leq 2$. Then $(\lambda F + G, H)$ is feasible.*

Proof. Let the Kronecker canonical form of $\lambda F + G$ be given by (2.1). Using the assumption of controllability and Lemma 2.7 we find that there are no blocks of type (K2). W.l.o.g. we can assume that the blocks \mathcal{K}_j are ordered such that first all blocks of type (K1) appear, then all blocks of type (K3), and then all blocks of type (K4). This means that

$$S(\lambda F + G)T = \begin{bmatrix} \lambda F_1 + G_1 & & \\ & \lambda \hat{F}_3 + \hat{G}_3 & \\ & & \lambda \hat{F}_4 + \hat{G}_4 \end{bmatrix}. \quad (\text{A.4})$$

Since $\lambda F_1 + G_1$ only consists of blocks of type (K1) we see that $\mathfrak{B}(\lambda F_1 + G_1)$ is trim. Since

$$\lambda F_2 + G_2 := \begin{bmatrix} \lambda \hat{F}_3 + \hat{G}_3 & \\ & \lambda \hat{F}_4 + \hat{G}_4 \end{bmatrix}$$

only consists of blocks of type (K3) and (K4) we see that $\mathfrak{B}(\lambda F_2 + G_2) = \{0\}$. Partition the matrix

$$T^*HT =: \begin{bmatrix} H_{11} & \hat{H}_{13} & \hat{H}_{14} \\ \hat{H}_{13}^* & \hat{H}_{33} & \hat{H}_{34} \\ \hat{H}_{14}^* & \hat{H}_{34}^* & \hat{H}_{44} \end{bmatrix},$$

according to the block structure of (A.4). Also partition the matrix

$$\tilde{Z} = \begin{bmatrix} Z_{11} & \hat{Z}_{13} & \hat{Z}_{14} \\ \hat{Z}_{31} & \hat{Z}_{33} & \hat{Z}_{34} \\ \hat{Z}_{41} & \hat{Z}_{43} & \hat{Z}_{44} \end{bmatrix}, \quad (\text{A.5})$$

which still has to be determined, according to (A.4). From Definition 5.1 we conclude that the equations

$$\begin{bmatrix} F_1^* Z_{11} & F_1^* \hat{Z}_{13} & F_1^* \hat{Z}_{14} \\ \hat{F}_3^* \hat{Z}_{31} & \hat{F}_3^* \hat{Z}_{33} & \hat{F}_3^* \hat{Z}_{34} \\ \hat{F}_4^* \hat{Z}_{41} & \hat{F}_4^* \hat{Z}_{43} & \hat{F}_4^* \hat{Z}_{44} \end{bmatrix} = \tilde{F}^* \tilde{Z} = \tilde{Z}^* \tilde{F} = \begin{bmatrix} Z_{11}^* F_1 & \hat{Z}_{31}^* \hat{F}_3 & \hat{Z}_{41}^* \hat{F}_4 \\ \hat{Z}_{13}^* F_1 & \hat{Z}_{33}^* \hat{F}_3 & \hat{Z}_{43}^* \hat{F}_4 \\ \hat{Z}_{14}^* F_1 & \hat{Z}_{34}^* \hat{F}_3 & \hat{Z}_{44}^* \hat{F}_4 \end{bmatrix}$$

and

$$0 = \left(T^*HT + \tilde{G}^* \tilde{Z} + \tilde{Z}^* \tilde{G} \right) \begin{bmatrix} 0 \\ I_{q_2} \end{bmatrix} = \begin{bmatrix} \hat{H}_{13} + G_1^* \hat{Z}_{13} + \hat{Z}_{31}^* \hat{G}_3 & \hat{H}_{14} + G_1^* \hat{Z}_{14} + \hat{Z}_{41}^* \hat{G}_4 \\ \hat{H}_{33} + \hat{G}_3^* \hat{Z}_{33} + \hat{Z}_{33}^* \hat{G}_3 & \hat{H}_{34} + \hat{G}_3^* \hat{Z}_{34} + \hat{Z}_{43}^* \hat{G}_4 \\ \hat{H}_{34}^* + \hat{G}_4^* \hat{Z}_{43} + \hat{Z}_{34}^* \hat{G}_3 & \hat{H}_{44} + \hat{G}_4^* \hat{Z}_{44} + \hat{Z}_{44}^* \hat{G}_4 \end{bmatrix}$$

have to be fulfilled. Since we assume that all blocks of type (K3) have size $k_j \leq 1$ we conclude that $\lambda \hat{F}_3 + \hat{G}_3 = \lambda 0 + I$. Choosing $Z_{11} = 0$, $\hat{Z}_{43} := 0$, and $\hat{Z}_{13} := 0$ we find that it would be sufficient to determine \tilde{Z} as in (A.5) such that

$$\begin{bmatrix} 0 & 0 & F_1^* \hat{Z}_{14} \\ 0 & 0 & 0 \\ \hat{F}_4^* \hat{Z}_{41} & 0 & \hat{F}_4^* \hat{Z}_{44} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \hat{Z}_{41}^* \hat{F}_4 \\ 0 & 0 & 0 \\ \hat{Z}_{14}^* F_1 & 0 & \hat{Z}_{44}^* \hat{F}_4 \end{bmatrix}$$

and

$$0 = \begin{bmatrix} \hat{H}_{13} + \hat{Z}_{31}^* & \hat{H}_{14} + G_1^* \hat{Z}_{14} + \hat{Z}_{41}^* \hat{G}_4 \\ \hat{H}_{33} + \hat{Z}_{33} + \hat{Z}_{33}^* & \hat{H}_{34} + \hat{Z}_{34} \\ \hat{H}_{34}^* + \hat{Z}_{34}^* & \hat{H}_{44} + \hat{G}_4^* \hat{Z}_{44} + \hat{Z}_{44}^* \hat{G}_4 \end{bmatrix},$$

which can be achieved in the following way. Set $\hat{Z}_{33} := -\frac{1}{2}\hat{H}_{33}$, $\hat{Z}_{34} := -\hat{H}_{34}$, and $\hat{Z}_{31} := -\hat{H}_{13}^*$. We use Lemma A.3 to construct a matrix \hat{Z}_{44} such that $\hat{Z}_{44}^* \hat{F}_4 = \hat{F}_4^* \hat{Z}_{44}$ and $0 = \hat{H}_{44} + \hat{Z}_{44}^* \hat{G}_4 + \hat{G}_4^* \hat{Z}_{44}$. We use Lemma A.4 to construct matrices \hat{Z}_{14} and \hat{Z}_{41} such that $F_1^* \hat{Z}_{14} = \hat{Z}_{41}^* \hat{F}_4$ and $0 = \hat{H}_{14} + G_1^* \hat{Z}_{14} + \hat{Z}_{41}^* \hat{G}_4$. \square