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Thermodynamics of multiphase problems in viscoelasticity

Laetitia Paoli[1], Adrien Petrov[2]

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[1] Université de Lyon, LaMUSE
23 rue Paul Michelon
42023 Saint-Etienne Cedex 02
France
E-Mail: laetitia.paoli@univ-st-etienne.fr

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Adrien.Petrov@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper deals with a three-dimensional mixture model describing materials undergoing phase transition with thermal expansion. The problem is formulated within the framework of generalized standard solids by the coupling of the momentum equilibrium equation and the flow rule with the heat transfer equation. A global solution for this thermodynamically consistent problem is obtained by using a fixed-point argument combined with global energy estimates.

1 Introduction

Shape-memory alloys (SMA) are employed nowadays in a large number of applications in different fields like biomedical or structural engineering. The increasing interest in SMA materials is deeply simulating the research on constitutive laws. Many models were developed during the last two decades, see for instance [Fré90, SMZ98, MiT99, Paw00, HaG02, GMH02, AuP04, GHH07]. These models are able to reproduce one or both of the well-known SMA behaviors; the *pseudo-elasticity* and the *shape-memory effect*. These two peculiar behaviors allow to include SMA into the smart material category. It is well known that the temperature plays a crucial role on the mechanical behavior of SMA allowing an austenite-martensite phase transition (see [Bha03] and the references therein). From a mechanical viewpoint, these phase transitions can give rise to stress-strain hysteresis loops. Consequently the temperature should be taken into account in the modelization. Some three-dimensional models for SMA suppose that the temperature is given a priori as a data (see [Mie07, MiP07]). This assumption is commonly used in engineering if the characteristic dimension of the material is small in at least one direction. Then the excessive or missing heat can be balanced through the environment. However many industrial applications do not fit this dimension property and the description of the mechanical behavior has to be coupled with the heat transfer equation.

Existence results have already been obtained for some of the previously mentioned models (see for instance [CoS92, PaZ05, Rou10, PaP11]). In this work, we are interested in a three-dimensional modelization describing austenite-martensite phase transition by using phase fractions. This mixture model is written in accordance with the formalism of generalized standard materials due to Halphen and Nguyen (see [HaN75]) and it is composed of the *momentum equilibrium equation* (1.1a) and the *flow rule* (1.1b), coupled with the heat-transfer equation (1.1c).

More precisely we denote by $W(\mathrm{e}(u), z, \nabla z, \theta)$ the Helmholtz free energy, depending on the *infinitesimal strain tensor* $\mathrm{e}(u) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}}) \in \mathbb{R}^{3\times 3}_{\text{sym}}$ for the *displacement* $u : \Omega \times (0,T) \to \mathbb{R}^3$, the *internal variable* $z : \Omega \times (0,T) \to \mathbb{R}^{N-1}$, where N is the total number of phases, i.e. the austenite and all the variants of martensite, and the *temperature* $\theta : \Omega \times (0,T) \to \mathbb{R}$. Here $\Omega \subset \mathbb{R}^3$ denotes a reference configuration. We assume that W can be decomposed as $W(\mathrm{e}(u), z, \nabla z, \theta) \stackrel{\text{def}}{=} W_1(\mathrm{e}(u), z, \nabla z) - W_0(\theta) + \theta W_2(\mathrm{e}(u), z)$. This decomposition ensures that entropy separates the thermal and mechanical variables. Note that $\theta W_2(\mathrm{e}(u), z)$ allows for coupling effects between the temperature and the internal variable. Under the assumptions of small deformations, the problem is formulated as follows

$$
\begin{align}
&-\operatorname{div}(\sigma_{\text{el}}+\mathbb{L}\mathrm{e}(\dot u))=\ell, \tag{1.1a}\\
&\partial\Psi(\dot z)+\mathbb{M}\dot z+\sigma_{\text{in}}\ni 0, \tag{1.1b}\\
&c(\theta)\dot\theta-\operatorname{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)=\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)+\theta\partial_t W_2(\mathrm{e}(u),z)+\Psi(\dot z)+\mathbb{M}\dot z.\dot z, \tag{1.1c}
\end{align}
$$

where we used the notations $\sigma_{\text{in}} \overset{\text{def}}{=} \mathrm{D}_z W(\mathrm{e}(u),z,\nabla z,\theta)-\operatorname{div}\mathrm{D}_{\nabla z}W(\mathrm{e}(u),z,\nabla z,\theta)$ and $\sigma_{\text{el}} \overset{\text{def}}{=} \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\nabla z,\theta)$. Here $\mathbb{L}$ and $\mathbb{M}$ are two viscosity tensors, ℓ is the applied mechanical loading, $c(\theta)$ is the *heat capacity*, $\kappa(\mathrm{e}(u),z,\theta)$ is the *conductivity* and Ψ is the dissipation potential, which is assumed to be positively homogeneous of degree 1, i.e., $\Psi(\gamma z)=\gamma\Psi(z)$ for all $\gamma\geq 0$. As usual, $(\dot{\ })$, D^i_z and ∂ denote the time derivative $\frac{\partial}{\partial t}$, the i-th derivative with respect to z and the subdifferential in the sense of convex analysis (for more details see [Bre73]), respectively. Moreover $\mathrm{e}_1{:}\mathrm{e}_2$ and $z_1.z_2$ denote the inner product of e_1 and e_2 in $\mathbb{R}^{3\times 3}_{\text{sym}}$ and z_1 and z_2 in $\mathbb{R}^{N-1}$.

We establish below a global existence result for such system by using fixed point argument. The paper is organized as follows. In Section 2, we check the thermodynamic consistency of this model and we present the mathematical formulation of the problem. Then we reformulate it by applying the enthalpy transformation. In Section 3, we consider first the system composed by the momentum equilibrium equation and flow rule for a given temperature θ and we obtain existence and regularity results. Therefore, we prove, with a fixed point argument, a local existence result in Section 4. Finally a global energy estimate is established in Section 5 leading to a global existence result for the system (1.1).

2 Mechanical model and mathematical formulation

We give here a rigorous justification of the thermodynamic consistency of the system (1.1). Let us define the specific *entropy* s via the Gibb's relation $s \overset{\text{def}}{=} -\mathrm{D}_\theta W(\mathrm{e}(u),z,\nabla z,\theta)$ and the *internal energy* $W_{\text{in}}(\mathrm{e}(u),z,\nabla z,\theta) \overset{\text{def}}{=} W(\mathrm{e}(u),z,\nabla z,\theta)+\theta s$ where we recall that W is the Helmholtz free energy. Then the *entropy equation* is given by $\theta\dot s-\operatorname{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)=\xi$ where $\xi=\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)+\mathbb{M}\dot z.\dot z+\Psi(\dot z)\geq 0$ is the dissipation rate. We can check that the second law of thermodynamics is satisfied if $\theta>0$. Indeed, we may divide the entropy equation by θ, and, assuming that the system is thermally isolated, we obtain

$$
\begin{aligned}
\int_\Omega \dot s\,\mathrm{d}x &= \int_\Omega \frac{\operatorname{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)}{\theta}\,\mathrm{d}x+\int_\Omega \frac{\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)+\mathbb{M}\dot z.\dot z+\Psi(\dot z)}{\theta}\,\mathrm{d}x\\
&= \int_\Omega \frac{\kappa(\mathrm{e}(u),z,\theta)\nabla\theta\cdot\nabla\theta}{\theta^2}\,\mathrm{d}x+\int_\Omega \frac{\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)+\mathbb{M}\dot z.\dot z+\Psi(\dot z)}{\theta}\,\mathrm{d}x\geq 0.
\end{aligned}
$$

Next we differentiate $W_{\text{in}}(\mathrm{e}(u),z,\nabla z,\theta)$ with respect to time, and we integrate over Ω. By using the entropy equation, we find

$$
\begin{aligned}
\int_\Omega \dot W_{\text{in}}(\mathrm{e}(u),z,\nabla z,\theta)\,\mathrm{d}x &= \int_\Omega \mathrm{D}_{\mathrm{e}(u)}W(\mathrm{e}(u),z,\nabla z,\theta):\mathrm{e}(\dot u)\,\mathrm{d}x\\
&+\int_\Omega \mathrm{D}_z W(\mathrm{e}(u),z,\nabla z,\theta).\dot z\,\mathrm{d}x+\int_\Omega \mathrm{D}_{\nabla z}W(\mathrm{e}(u),z,\nabla z,\theta)\cdot\nabla\dot z\,\mathrm{d}x\\
&+\int_\Omega \big(\operatorname{div}(\kappa(\mathrm{e}(u),z,\theta)\nabla\theta)+\mathbb{L}\mathrm{e}(\dot u):\mathrm{e}(\dot u)+\mathbb{M}\dot z.\dot z+\Psi(\dot z)\big)\,\mathrm{d}x.
\end{aligned} \tag{2.1}
$$

We recalculate the left hand side of (2.1) by using (1.1a) and (1.1b). More precisely, we test (1.1a) with $\dot{u}$ and (1.1b) with $\dot{z}$. Reminding that Ψ is positively homogeneous of degree 1, we get

$$\int_\Omega \mathrm{D}_{\mathrm{e}(u)} W(\mathrm{e}(u), z, \nabla z, \theta) : \mathrm{e}(\dot{u}) \,\mathrm{d}x + \int_\Omega \mathbb{L}\mathrm{e}(\dot{u}) : \mathrm{e}(\dot{u}) \,\mathrm{d}x = \int_\Omega \ell \cdot \dot{u} \,\mathrm{d}x \tag{2.2}$$

and

$$\begin{aligned} &\int_\Omega \mathrm{D}_z W(\mathrm{e}(u), z, \nabla z, \theta) . \dot{z} \,\mathrm{d}x + \int_\Omega \mathrm{D}_{\nabla z} W(\mathrm{e}(u), z, \nabla z, \theta) \cdot \nabla \dot{z} \,\mathrm{d}x \\ &+ \int_\Omega \mathbb{M}\dot{z}.\dot{z} \,\mathrm{d}x + \int_\Omega \Psi(\dot{z}) \,\mathrm{d}x = 0. \end{aligned} \tag{2.3}$$

Then we insert (2.2) and (2.3) into (2.1), and we obtain

$$\int_\Omega \dot{W}_{\mathrm{in}}(\mathrm{e}(u), z, \nabla z, \theta) \,\mathrm{d}x = \int_\Omega \ell \cdot \dot{u} \,\mathrm{d}x + \int_{\partial\Omega} \kappa(\mathrm{e}(u), z, \theta) \nabla\theta \cdot \eta \,\mathrm{d}x,$$

which means that the total energy balance can be expressed in terms of the internal energy, as the sum of power of external load and heat. Finally, we have $s \stackrel{\text{def}}{=} \mathrm{D}_\theta W_0(\theta) - W_2(\mathrm{e}(u), z)$ and we may deduce from the entropy equation that the heat-transfer equation (1.1c) holds with the heat capacity given by $c(\theta) = \theta \mathrm{D}_\theta^2 W_0(\theta)$.

We will focus on the case where

$$\begin{aligned} &W_1(\mathrm{e}(u), z, \nabla z) \stackrel{\text{def}}{=} \tfrac{1}{2}\mathbb{E}(\mathrm{e}(u) - E(z)) : (\mathrm{e}(u) - E(z)) + \tfrac{\nu}{2}|\nabla z|^2 + H_1(z), \\ &W_2(\mathrm{e}(u), z) \stackrel{\text{def}}{=} \alpha \mathrm{tr}(\mathrm{e}(u)) + H_2(z). \end{aligned}$$

Here $\alpha \geq 0$ is the isotropic thermal expansion coefficient, $\mathbb{E}$ is the elastic tensor, H_i, $i = 1, 2$, are two hardening functionals, $\nu > 0$ is a coefficient that measures some non local interaction effect for the internal variable z and $E(z)$ is the effective transformation strain of the mixture given by

$$E(z) \stackrel{\text{def}}{=} \sum_{k=1}^{N-1} z^k E^k + \Big(1 - \sum_{k=1}^{N-1} z^k\Big) E^N \tag{2.4}$$

where E^k is the transformation strain of the phase k. In the systems described in [MiT99, Mie00, HaG02, GMH02, MTL02, GHH07], the temperature-dependent hardening functional $H(z, \theta)$ is the sum of a smooth part $w(z, \theta)$ and the indicator function of the set $[0, 1]^{N-1}$. Following the ideas proposed in [MiP07], we will consider here a regularization given by

$$H^\delta(z, \theta) \stackrel{\text{def}}{=} w(z, \theta) + \sum_{k=1}^{N-1} \tfrac{((-z_k)_+)^4 + ((z_k - 1)_+)^4}{\delta(1 + |z_k|^2)}, \quad 0 < \delta \ll 1$$

and we define H_1 and H_2 in order that $H_1(z) + \theta H_2(z)$ is an affine approximation of $H^\delta(z, \theta)$. If we define $E_0 \in \mathcal{L}(\mathbb{R}^{N-1}; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ by $E_0(z) \stackrel{\text{def}}{=} \sum_{k=1}^{N-1} z^k (E^k - E^N)$, the system (1.1) is rewritten as follows:

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u) - E(z)) + \alpha\theta \mathrm{I} + \mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{2.5a}$$

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - E_0^{\mathsf{T}} \mathbb{E}(\mathrm{e}(u) - E(z)) + \mathrm{D}_z H_1(z) + \theta \mathrm{D}_z H_2(z) - \nu \Delta z \ni 0, \tag{2.5b}$$

$$c(\theta)\dot{\theta} - \operatorname{div}(\kappa(\mathrm{e}(u), z, \theta) \nabla\theta) = \mathbb{L}\mathrm{e}(\dot{u}) : \mathrm{e}(\dot{u}) + \theta(\alpha \mathrm{tr}(\mathrm{e}(\dot{u})) + \mathrm{D}_z H_2(z).\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}.\dot{z}, \tag{2.5c}$$

where I is the identity matrix. We have naturally to prescribe initial and boundary conditions for the displacement, the internal variables, and the temperature, namely

$$u(\cdot,0)=u^0,\quad z(\cdot,0)=z^0,\quad \theta(\cdot,0)=\theta^0, \tag{2.6a}$$

$$u_{|\partial\Omega}=0,\quad \nabla z\cdot\eta_{|\partial\Omega}=0,\quad \kappa\nabla\theta\cdot\eta_{|\partial\Omega}=0, \tag{2.6b}$$

where η denotes the outward normal to the boundary $\partial\Omega$ of Ω.

In order to get another formulation of this problem, we define the enthalpy transformation $g(\theta)=\vartheta \stackrel{\text{def}}{=} \int_0^\theta c(s)\,\mathrm{d}s$. We will assume that c is continuous and bounded from below by a positive constant c^c. Hence we deduce that g is a bijection from $[0,\infty)$ into $[0,\infty)$ which allows us to define the mapping ζ by $\zeta(\vartheta)\stackrel{\text{def}}{=} g^{-1}(\vartheta)$ if $\vartheta\geq 0$ and $\zeta(\vartheta)\stackrel{\text{def}}{=}0$ otherwise where g^{-1} is the inverse of g (see also [Rou09] for more details on the enthalpy transformation). Let $\kappa^c(\mathrm{e}(u),z,\vartheta)\stackrel{\text{def}}{=}\frac{\kappa(\mathrm{e}(u),z,\zeta(\vartheta))}{c(\zeta(\vartheta))}$. The system (2.5) is transformed into the following form

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u)-E(z))+\alpha\zeta(\vartheta)\mathrm{I}+\mathbb{L}\mathrm{e}(\dot u))=\ell, \tag{2.7a}$$

$$\partial\Psi(\dot z)+\mathbb{M}\dot z-E_0^{\top}\mathbb{E}(\mathrm{e}(u)-E(z))+\mathrm{D}_zH_1(z)+\zeta(\vartheta)\mathrm{D}_zH_2(z)-\nu\Delta z\ni 0, \tag{2.7b}$$

$$\dot\vartheta-\operatorname{div}(\kappa^c(\mathrm{e}(u),z,\vartheta)\nabla\vartheta)=\mathbb{L}\mathrm{e}(\dot u){:}\mathrm{e}(\dot u)+\zeta(\vartheta)(\alpha\mathrm{tr}(\mathrm{e}(\dot u))+\mathrm{D}_zH_2(z).\dot z)+\Psi(\dot z)+\mathbb{M}\dot z.\dot z, \tag{2.7c}$$

with initial and boundary conditions

$$u(\cdot,0)=u^0,\quad z(\cdot,0)=z^0,\quad \vartheta(\cdot,0)=\vartheta^0=g(\theta^0), \tag{2.8a}$$

$$u_{|\partial\Omega}=0,\quad \nabla z\cdot\eta_{|\partial\Omega}=0,\quad \kappa^c\nabla\vartheta\cdot\eta_{|\partial\Omega}=0. \tag{2.8b}$$

The identity (2.7c) will be called the enthalpy equation. As usual Korn's inequality plays a role in the mathematical analysis. We assume that Ω is a bounded domain such that $\partial\Omega$ is of class $\mathrm{C}^{2+\rho}$. Hence there exists $C^{\text{Korn}}>0$ such that for all $u\in\mathrm{H}^1_0(\Omega)$, we have $\|\mathrm{e}(u)\|^2_{\mathrm{L}^2(\Omega}\geq C^{\text{Korn}}\|u\|^2_{\mathrm{H}^1(\Omega)}$ (see [KoO88, DuL76]).

Let us introduce now the assumptions on the data.

(A1) The dissipation potential Ψ is positively homogeneous of degree 1, satisfies the triangle inequality, and there exists $C^\Psi>0$ such that for all $z,z_i\in\mathbb{R}^{N-1}$, $i=1,2$, and all $\gamma\geq 0$, we have

$$\Psi(\gamma z)=\gamma\Psi(z),\quad 0\leq\Psi(z)\leq C^\Psi|z|,\quad \Psi(z_1+z_2)\leq\Psi(z_1)+\Psi(z_2). \tag{2.9}$$

(A2) The hardening functionals H_i, $i=1,2$, belong to $\mathrm{C}^2(\mathbb{R}^{N-1};\mathbb{R})$ and that there exist $c^{H_1},\widetilde{c}^{H_1}>0$ and $C^{H_i}_{zz}>0$ such that for all $z\in\mathbb{R}^{N-1}$, we get

$$H_1(z)\geq c^{H_1}|z|^2-\widetilde{c}^{H_1}\quad\text{and}\quad|\mathrm{D}^2_zH_i(z)|\leq C^{H_i}_{zz}. \tag{2.10}$$

Then we may deduce that there exists $C^{H_i}_z>0$ such that for all $z\in\mathbb{R}^{N-1}$, we have

$$|\mathrm{D}_zH_i(z)|\leq C^{H_i}_z(1+|z|)\quad\text{and}\quad|H_i(z)|\leq C^{H_i}_z(1+|z|^2). \tag{2.11}$$

(A3) The *elastic tensor* $\mathbb{E}:\Omega\to\mathcal{L}(\mathbb{R}^{3\times3}_{\text{sym}},\mathbb{R}^{3\times3}_{\text{sym}})$ is a symmetric positive definite operator such that there exists $c^{\mathbb{E}}>0$ such that for all $\mathrm{e}\in\mathrm{L}^2(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})$ and for all $i,j,k=1,2,3$, we have

$$c^{\mathbb{E}}\|\mathrm{e}\|^2_{\mathrm{L}^2(\Omega)}\leq\int_\Omega\mathbb{E}\mathrm{e}{:}\mathrm{e}\,\mathrm{d}x\quad\text{and}\quad\mathbb{E}(\cdot),\frac{\partial\mathbb{E}_{i,j}(\cdot)}{\partial x_k}\in\mathrm{L}^\infty(\Omega). \tag{2.12}$$

(A4) The tensors $\mathbb{L}$ and $\mathbb{M}$ are symmetric positive definite. This implies that there exist four constants $c^{\mathbb{L}}, C^{\mathbb{L}}, c^{\mathbb{M}}, C^{\mathbb{M}} > 0$ such that for all $\mathrm{e} \in \mathbb{R}^{3\times 3}_{\mathsf{sym}}$ and $z \in \mathbb{R}^{N-1}$:

$$c^{\mathbb{L}}|\mathrm{e}|^2 \leq \mathbb{L}\mathrm{e}{:}\mathrm{e} \leq C^{\mathbb{L}}|\mathrm{e}|^2 \quad \text{and} \quad c^{\mathbb{M}}|z|^2 \leq \mathbb{M}z.z \leq C^{\mathbb{M}}|z|^2. \tag{2.13}$$

(A5) The heat capacity c is continuous from $\mathbb{R}^+$ to $\mathbb{R}^+$, the conductivity κ^c is continuous from $\mathbb{R}^{3\times 3}_{\mathsf{sym}} \times \mathbb{R}^{N-1} \times \mathbb{R}$ to $\mathbb{R}^{3\times 3}_{\mathsf{sym}}$ and there exist $\beta_1 \geq 2$ and $c^c, c^{\kappa^c}, C^{\kappa^c} > 0$ such that for all $\theta \geq 0$, $v \in \mathbb{R}^3$, $(e, z, \vartheta) \in \mathbb{R}^{3\times 3}_{\mathsf{sym}} \times \mathbb{R}^{N-1} \times \mathbb{R}$, we have

$$0 < c^c \leq c^c(1{+}\theta)^{\beta_1 - 1} \leq c(\theta), \tag{2.14a}$$

$$\kappa^c(e, z, \vartheta)v{\cdot}v \geq c^{\kappa^c}|v|^2 \quad \text{and} \quad |\kappa^c(e, z, \vartheta)| \leq C^{\kappa^c}. \tag{2.14b}$$

(A6) The applied loading satisfies

$$\ell \in \mathrm{H}^1(0, T; \mathrm{L}^2(\Omega)). \tag{2.15}$$

Our existence result for problem (2.5)-(2.6) is based on a fixed point argument. More precisely, for any given $\widetilde{\vartheta}$, we define $\theta \stackrel{\text{def}}{=} \zeta(\widetilde{\vartheta})$ and we solve first the system composed by (2.5a)–(2.5b), then we solve (2.7c) with $\kappa^c \stackrel{\text{def}}{=} \kappa^c(\mathrm{e}(u), z, \zeta(\widetilde{\vartheta}))$. This allows us to define a mapping $\phi : \widetilde{\vartheta} \mapsto \vartheta$, and we will prove that this mapping satisfies the assumptions of Schauder's fixed point theorem. Let us observe that, since ζ is a Lipschitz continuous mapping from $\mathbb{R}$ to $\mathbb{R}$, the mapping $\phi_1 : \widetilde{\vartheta} \mapsto \theta$ is also Lipschitz continuous from $\mathrm{L}^{\bar{q}}(0, T; \mathrm{L}^{\bar{p}}(\Omega))$ to $\mathrm{L}^{\bar{q}}(0, T; \mathrm{L}^{\bar{p}}(\Omega))$ for any $\bar{p} \geq 1$ and $\bar{q} \geq 1$. Furthermore (2.14a) implies that for all $\beta \in [1, \beta_1]$ and $\vartheta \in \mathbb{R}$, we have

$$|\zeta(\vartheta)| \leq \big(\tfrac{\beta_1}{c^c}\max(0, \vartheta){+}1\big)^{\frac{1}{\beta}} - 1 \leq \big(\tfrac{\beta_1}{c^c}\max(0, \vartheta)\big)^{\frac{1}{\beta}}. \tag{2.16}$$

Hence, for all $\beta \in [1, \beta_1]$ and for all $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0, T; \mathrm{L}^{\bar{p}}(\Omega))$, we have $\theta \in \mathrm{L}^{\beta\bar{q}}(0, T; \mathrm{L}^{\beta\bar{p}}(\Omega))$ with $\|\theta\|_{\mathrm{L}^{\beta\bar{q}}(0,T;\mathrm{L}^{\beta\bar{p}}(\Omega))} \leq \big(\frac{\beta_1}{c^c}\big)^{\frac{1}{\beta}}\|\widetilde{\vartheta}\|^{\frac{1}{\beta}}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))}$. In the rest of the paper, we will assume that $\bar{q} > 4$ and $\bar{p} = 2$. When there is not any confusion, we will use simply the notation $\mathrm{X}(\Omega)$ instead of $\mathrm{X}(\Omega; \mathrm{Y})$ where X is a functional space and Y is a vector space.

3 Existence and regularity results for the system composed by the momentum equilibrium equation and the flow rule

We focus in this section on existence, uniqueness and regularity results for the system (2.5a)–(2.5b) when $\theta = \zeta(\widetilde{\vartheta})$ is given in a bounded subset of $\mathrm{L}^q(0, T; \mathrm{L}^p(\Omega))$ with $q = \beta_1\bar{q}$ and $p \in \big[4, \min(\beta_1\bar{p}, 6)\big]$. More precisely we look for a solution of the problem (P_{uz}):

$$-\operatorname{div}(\mathbb{E}(\mathrm{e}(u){-}E(z)){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot{u})) = \ell, \tag{3.1a}$$

$$\partial\Psi(\dot{z}) + \mathbb{M}\dot{z} - E_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u){-}E(z)) + \mathrm{D}_z H_1(z) + \theta\mathrm{D}_z H_2(z) - \nu\Delta z \ni 0, \tag{3.1b}$$

with initial and boundary conditions

$$u(\cdot, 0) = u^0, \quad z(\cdot, 0) = z^0, \quad u_{|\partial\Omega} = 0, \quad \nabla z{\cdot}\eta_{|\partial\Omega} = 0. \tag{3.2}$$

As a first step, we use classical results for Partial Differential Equations (PDE) and Ordinary Differential Equations (ODE) to obtain an existence result. In the sequel, the notations for the constants introduced in the proofs are valid only in the proof and we also use the set $\mathcal{Q}_\tau \stackrel{\text{def}}{=} \Omega \times (0, \tau)$ with $\tau \in [0, T]$.

Theorem 3.1 (Existence for $(\mathbf{P}_{uz})$) *Assume that* (2.9), (2.10), (2.12), (2.13) *and* (2.15), $u^0 \in \mathrm{H}^1_0(\Omega)$ *and* $z^0 \in \mathrm{H}^1(\Omega)$ *hold. Then for a given* $\theta \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$*, the problem* (3.1)–(3.2) *admits a solution* $(u,z) \in \mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{H}^1(\Omega))$.

Proof. We recall that for all $f \in \mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega))')$ and for all $u^* \in \mathrm{H}^1_0(\Omega)$ the problem

$$-\operatorname{div}(\mathbb{E}\mathrm{e}(u)+\mathbb{L}\mathrm{e}(\dot u)) = f, \quad u(\cdot,0)=u^*\in\mathrm{H}^1_0(\Omega), \quad u_{|\partial\Omega}=0$$

admits a unique solution $u \stackrel{\text{def}}{=} \mathcal{L}(u^*,f) \in \mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)) \cap \mathrm{C}^0([0,T];\mathrm{H}^1_0(\Omega))$. Moreover for all $(f_1,f_2,u^*) \in (\mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega))'))^2\times\mathrm{H}^1_0(\Omega)$, we have $\mathcal{L}(u^*,f_1{+}f_2) = \mathcal{L}(u^*,f_1)+\mathcal{L}(0,f_2)$ and $\mathcal{L}(0,\cdot): f\mapsto u$ is a linear and continuous mapping from $\mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega)'))$ into $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega))\cap\mathrm{C}^0([0,T];\mathrm{H}^1_0(\Omega))$. Then (3.1) can be rewritten as follows

$$\partial\Psi(\dot z)+\mathbb{M}\dot z + E_0^{\mathsf{T}}\mathbb{E}E(z)+\mathrm{D}_zH_1(z)+\theta\mathrm{D}_zH_2(z)-\nu\Delta z+g_1(\theta)+g_2(z)\ni 0, \tag{3.3}$$

with initial and boundary conditions

$$z(\cdot,0)=z^0\in\mathrm{H}^1(\Omega), \quad \nabla z{\cdot}\eta_{|\partial\Omega}=0. \tag{3.4}$$

Here we denoted $g_1(\theta) \stackrel{\text{def}}{=} -E_0^{\mathsf{T}}\mathbb{E}\mathrm{e}(\mathcal{L}(u^0,\ell{+}\operatorname{div}(\alpha\theta\mathrm{I}{-}\mathbb{E}E^N))){\in}\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))$ and

$$\begin{aligned} g_2:\ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)) &\to \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)),\\ z &\mapsto E_0^{\mathsf{T}}\mathbb{E}\mathrm{e}(\mathcal{L}_0(\operatorname{div}(\mathbb{E}E_0(z)))). \end{aligned}$$

Let $\varphi(z) \stackrel{\text{def}}{=} \frac{\nu}{2}\|\nabla z\|^2_{\mathrm{L}^2(\Omega)}$ if $z\in\mathrm{H}^1(\Omega)$ and $\varphi(z)\stackrel{\text{def}}{=}+\infty$ otherwise. Observe that φ is a proper, convex lower semicontinuous function on $\mathrm{L}^2(\Omega)$, which implies that $\partial\varphi$ is a maximal monotone operator on $\mathrm{L}^2(\Omega)$ (see [Bre73]). The resolvant of the subdifferential $\partial\varphi$ is defined by $\mathcal{J}_\epsilon \stackrel{\text{def}}{=} (\mathrm{I}{+}\epsilon\partial\varphi)^{-1}$ where $\epsilon>0$. We also define $\varphi_\epsilon(z) \stackrel{\text{def}}{=} \min_{\bar z\in\mathrm{L}^2(\Omega)}\big\{\frac{1}{2\varepsilon}\|z{-}\bar z\|^2_{\mathrm{L}^2(\Omega)}{+}\varphi(\bar z)\big\}$ for all $z\in\mathrm{L}^2(\Omega)$. It is a convex and Fréchet differentiable mapping from $\mathrm{L}^2(\Omega)$ to $\mathbb{R}$. Furthermore the Yosida approximation of $\partial\varphi$ coincide with the Fréchet differential of φ_ϵ, i.e. $\partial\varphi_\epsilon \stackrel{\text{def}}{=} \frac{1}{\epsilon}(\mathrm{I}{-}\mathcal{J}_\epsilon)$ and it is $\frac{1}{\epsilon}$-Lipschitz continuous on $\mathrm{L}^2(\Omega)$ (see [Bre73]). We approximate the problem (3.3)–(3.4) by

$$\partial\Psi(\dot z_\epsilon)+\mathbb{M}\dot z_\epsilon+\Upsilon(\theta_\epsilon,z_\epsilon)\ni 0, \quad z_\epsilon(\cdot,0)=z^0. \tag{3.5}$$

Here $\Upsilon(\theta_\epsilon,z_\epsilon) \stackrel{\text{def}}{=} \partial\varphi_\epsilon(z_\epsilon)+E_0^{\mathsf{T}}\mathbb{E}E(z_\epsilon)+\mathrm{D}_zH_1(z_\epsilon)+g_1(\theta_\epsilon)+g_2(\mathcal{J}_\epsilon z_\epsilon)+\theta_\epsilon\mathrm{D}_zH_2(\mathcal{J}_\epsilon z_\epsilon)$, where $\theta_\epsilon\in\mathrm{C}^\infty_0(0,T){\otimes}\mathrm{C}^\infty_0(\Omega)$ and $\partial\Psi(\dot z_\epsilon)$ is taken in the sense of the $\mathrm{L}^2(\Omega)$-extension of the subdifferential of the convex function Ψ. Observing that $\partial\Psi+\mathbb{M}$ is a strongly monotone operator on $\mathrm{L}^2(\Omega)$, we rewrite (3.5) as

$$\dot z_\epsilon = (\partial\Psi{+}\mathbb{M})^{-1}(-\Upsilon(\theta_\epsilon,z_\epsilon)). \tag{3.6}$$

We solve this differential equation in $\mathrm{L}^2(\Omega)$ by using the Picard's iteration technique. More precisely, we prove that the mapping Λ_ϵ defined on $\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))$ by

$$\Lambda_\epsilon(z): t\mapsto z^0+\int_0^t(\partial\Psi{+}\mathbb{M})^{-1}(-\Upsilon(\theta_\epsilon(\cdot,s),z(\cdot,s)))\,\mathrm{d}s$$

admits a unique fixed point z_ϵ, which is the unique solution $z_\epsilon\in\mathrm{C}^1([0,T];\mathrm{L}^2(\Omega))$ of (3.6) satisfying $z_\epsilon(\cdot,0)=z^0$ (the verification is left to the reader).

We choose now a sequence $(\theta_\epsilon)_{\epsilon>0}$ such that θ_ϵ converges strongly to θ in $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$. Define $w_\epsilon(\cdot,t) \stackrel{\text{def}}{=} g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t))-\partial\varphi_\epsilon(z_\epsilon(\cdot,t))-E_0^{\mathsf{T}}\mathbb{E}E(z_\epsilon(\cdot,t))-\mathrm{D}_zH_1(z_\epsilon(\cdot,t))$ with $g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t)) \stackrel{\text{def}}{=}$

$-(g_1(\theta_\epsilon(\cdot,t))+g_2(\mathcal{J}_\epsilon z_\epsilon(\cdot,t))+\theta_\epsilon(\cdot,t)\mathrm{D}_zH_2(\mathcal{J}_\epsilon z_\epsilon(\cdot,t)))$ for all $t\in[0,T]$. We notice that for all $t\in[0,T]$, we have

$$w_\epsilon(\cdot,t)+\partial\varphi_\epsilon(z_\epsilon(\cdot,t))+E_0^{\mathsf{T}}\mathbb{E}E(z_\epsilon(\cdot,t))+\mathrm{D}_zH_1(z_\epsilon(\cdot,t))=g_\epsilon(t,\mathcal{J}_\epsilon z_\epsilon(\cdot,t)), \tag{3.7a}$$
$$\dot z_\epsilon(\cdot,t)=(\partial\Psi+\mathbb{M})^{-1}((w_\epsilon(\cdot,t))). \tag{3.7b}$$

Our goal is to pass to the limit in (3.7) as ϵ tends to 0. As the first step, we may reproduce the same kind of a priori estimates as in the proof of [PaP11, Thm 4.1], we find that z_ϵ is bounded in $\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))\cap\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$, $\mathcal{J}_\epsilon z_\epsilon$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$, w_ϵ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ and $\partial\varphi_\epsilon(z_\epsilon)$ is bounded in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, independently of $\epsilon>0$. Hence, we may extract subsequences, still denoted z_ϵ, $\mathcal{J}_\epsilon z_\epsilon$, w_ϵ and $\partial\varphi_\epsilon(z_\epsilon)$ such that

$$\begin{aligned} z_\epsilon &\rightharpoonup z \ \text{ in } \ \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak and in } \ \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak } *,\\ \mathcal{J}_\epsilon z_\epsilon &\rightharpoonup \widetilde z \ \text{ in } \ \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \ \text{ weak } *,\\ w_\epsilon &\rightharpoonup w, \quad \partial\varphi_\epsilon(z_\epsilon)\rightharpoonup v \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)) \ \text{ weak.}\end{aligned}$$

Moreover, reminding that $\mathcal{J}_\epsilon$ is a contraction on $\mathrm{L}^2(\Omega)$, it is possible to extract another subsequence, still denoted by z_ϵ, such that

$$\mathcal{J}_\epsilon z_\epsilon\to z \ \text{ in } \ \mathrm{C}^0([0,T];\mathrm{L}^4(\Omega)) \quad \text{and} \quad z_\epsilon\to z \ \text{ in } \ \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)).$$

Since the mapping $\mathcal{L}(0,\cdot)$ is linear and continuous, the mappings g_1 and g_2 are also continuous from $\mathrm{L}^2(0,T;L^2(\Omega))$ into $\mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))$, D_zH_i, $i=1,2$, are Lipschitz continuous, it follows that

$$\begin{aligned} g_1(\theta_\epsilon)+g_2(\mathcal{J}_\epsilon z_\epsilon)+\theta_\epsilon\mathrm{D}_zH_2(\mathcal{J}_\epsilon z_\epsilon)&\to g_1(\theta)+g_2(z)+\theta\mathrm{D}_zH_2(z) \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)),\\ \mathrm{D}_zH_1(z_\epsilon)+E_0^{\mathsf{T}}\mathbb{E}E(z_\epsilon)&\to\mathrm{D}_zH_1(z)+E_0^{\mathsf{T}}\mathbb{E}E(z) \ \text{ in } \ \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)),\end{aligned}$$

which allows us to pass to the limit in all the terms of (3.7a), we get

$$w+v+E_0^{\mathsf{T}}\mathbb{E}E(z)+\mathrm{D}_zH_1(z)=-(g_1(\theta)+g_2(z)+\theta\mathrm{D}_zH_2(z)). \tag{3.8}$$

The second step consists in proving that $v(\cdot,t)\in\partial\varphi(z(\cdot,t))$ and $w(\cdot,t)-\mathbb{M}\dot z(\cdot,t)\in\partial\Psi(\dot z(\cdot,t))$ for almost every $t\in[0,T]$ which is obtained by using the lower semicontinuity of φ and [Bre73, Prop. 2.5]. This allows us to deduce that z is a solution of (3.3)–(3.4). ☐

In order to obtain more regularity properties for u and z, we will use maximal regularity results for parabolic systems. Let $\mathcal{A}:\mathrm{H}^1(\Omega)\to(\mathrm{H}^1(\Omega))'$ be the linear continuous mapping defined as follows $\langle\mathcal{A}u,v\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)}\overset{\text{def}}{=}\int_\Omega\nu\mathbb{M}^{-1}\nabla u{:}\nabla v\,\mathrm{d}x$ for all $(u,v)\in(\mathrm{H}^1(\Omega))^2$. Classical results about elliptic operators implies that $\mathcal{A}$ generates an analytic semigroup on $\mathrm{L}^2(\Omega)$, which extends to a C^0-semigroup of contractions on $\mathrm{L}^r(\Omega)$. We denote by $\mathcal{A}_r$ the realization of its generator in $\mathrm{L}^r(\Omega)$ and by $\mathrm{X}_{q,p}(\Omega)\overset{\text{def}}{=}(\mathrm{L}^p(\Omega),\mathcal{D}(\mathcal{A}_p))_{1-\frac{2}{q},\frac{q}{2}}\cap(\mathrm{L}^{p/2}(\Omega),\mathcal{D}(\mathcal{A}_{\frac{p}{2}}))_{1-\frac{1}{q},q}$ where $\mathcal{D}(\mathcal{A}_r)$ is the domain of $\mathcal{A}_r$ with $r=\frac{p}{2},p$ (see [HiR08, PrS01]). Then we observe that (3.1b) can be rewritten as follows

$$\dot z-\nu\mathbb{M}^{-1}\Delta z=\mathbb{M}^{-1}((E_0^{\mathsf{T}}\mathbb{E}(\mathrm{e}(u)-E(z)))-\mathrm{D}_zH_1(z)-\theta\mathrm{D}_zH_2(z)-\psi) \tag{3.9}$$

with $\psi=w-\mathbb{M}\dot z$ and (2.9) implies that $\|\psi(\cdot,t)\|_{\mathrm{L}^\infty(\Omega)}\le C^\Psi$ for almost every $t\in[0,T]$. Since $z\in\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$, we may infer with (2.11) that $\mathrm{D}_zH_i(z)\in\mathrm{L}^\infty(0,T;\mathrm{L}^p(\Omega))$ for $i=1,2$. It follows that the right hand side in (3.9) belongs to $\mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$. We conclude from the maximal regularity result for parabolic systems that $z\in\mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))\cap\mathrm{C}^0([0,T];\mathrm{H}^1(\Omega))$ and $\dot z\in\mathrm{L}^q(0,T;\mathrm{L}^2(\Omega))$ since $z^0\in\mathrm{X}_{q,p}(\Omega)$ (see [Dor93, HiR08, PrS01]).

Next we can prove that (P_{uz}) admits a unique solution.

Proposition 3.2 (Uniqueness for $(\mathbf{P}_{uz})$) *Assume that* (2.9), (2.10), (2.12), (2.13) *and* (2.15), $u^0 \in \mathrm{H}^1_0(\Omega)$ *and* $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *hold. Then for any given* $\theta \in \mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$*, the problem* (3.1)–(3.2) *admits a unique solution.*

Proof. Let θ be given in $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ and denote by (u_i,z_i), $i=1,2$, two solutions of (P_{uz}). Let $C^{H_1}>0$ and define $h_1(z) \overset{\text{def}}{=} H_1(z) - C^{H_1}|z|^2$ for all $z \in \mathbb{R}^{N-1}$ and

$$\begin{aligned}\gamma(t) \overset{\text{def}}{=} \tfrac{1}{2}\int_\Omega \mathbb{E}((\mathrm{e}(u_1){-}E(z_1)) - (\mathrm{e}(u_2){-}E(z_2))){:}((\mathrm{e}(u_1){-}E(z_1)){-}(\mathrm{e}(u_2){-}E(z_2)))\,\mathrm{d}x \\ -\tfrac{\nu}{2}\int_\Omega \Delta(z_1{-}z_2).(z_1{-}z_2)\,\mathrm{d}x + C^{H_1}\int_\Omega |z_1{-}z_2|^2\,\mathrm{d}x \quad \forall t\in[0,T].\end{aligned}$$

By using (2.12) and Korn's inequality, we infer that there exists $C_\gamma>0$ such that

$$\gamma(t) \geq C_\gamma\big(\|u_1(\cdot,t){-}u_2(\cdot,t)\|^2_{\mathrm{H}^1(\Omega)}{+}\|z_1(\cdot,t){-}z_2(\cdot,t)\|^2_{\mathrm{H}^1(\Omega)}\big) \text{ for all } t\in[0,T]. \tag{3.10}$$

We can obtain an estimate of $\dot\gamma(t)$ by using $\dot u_i - \dot u_{3-i}$ as a test-function in (3.1a) and the definition of the subdifferential $\partial\Psi(\cdot)$ to rewrite(3.1b) as a variational inequality associated with the test-function $\dot z_{3-i}$. Then adding these expressions, we find

$$\begin{aligned}&\int_\Omega (\mathbb{E}(\mathrm{e}(u_i){-}E(z_i)){+}\alpha\theta\mathrm{I}{+}\mathbb{L}\mathrm{e}(\dot u_i)){:}(\mathrm{e}(\dot u_i){-}\mathrm{e}(\dot u_{3-i}))\,\mathrm{d}x \\ &+\int_\Omega (\mathbb{M}\dot z_i{-}E_0^\top\mathbb{E}(\mathrm{e}(u_i){-}E(z_i)){+}\mathrm{D}_zH_1(z_i){+}\theta\mathrm{D}_zH_2(z_i)).(\dot z_i{-}\dot z_{3-i})\,\mathrm{d}x \\ &+\nu\int_\Omega \Delta z_i{\cdot}(\dot z_i{-}\dot z_{3-i})\,\mathrm{d}x \leq \int_\Omega (\ell{\cdot}(\dot u_i{-}\dot u_{3-i}){+}\Psi(\dot z_{3-i}){-}\Psi(\dot z_i))\,\mathrm{d}x\end{aligned} \tag{3.11}$$

for $i=1,2$. We add these two inequalities and we get

$$\begin{aligned}&\dot\gamma(t) + c^{\mathbb{M}}\int_\Omega |\dot z_1{-}\dot z_2|^2\,\mathrm{d}x + c^{\mathbb{L}}\int_\Omega |\mathrm{e}(\dot u_1){-}\mathrm{e}(\dot u_2)|^2\,\mathrm{d}x \\ &\leq -\int_\Omega (\mathrm{D}_zh_1(z_1){-}\mathrm{D}_zh_1(z_2)).(\dot z_1{-}\dot z_2)\,\mathrm{d}x - \int_\Omega \theta(\mathrm{D}_zH_2(z_1){-}\mathrm{D}_zH_2(z_2)).(\dot z_1{-}\dot z_2)\,\mathrm{d}x\end{aligned}$$

for almost every $t\in[0,T]$. Then (2.10), (2.13) and the continuous embedding $\mathrm{H}^1(\Omega)\hookrightarrow\mathrm{L}^4(\Omega)$ imply that there exists $C>0$ depending on $c^{\mathbb{M}}$, C^{H_1},$C^{H_1}_{zz}$ and $C^{H_2}_{zz}$ such that

$$\dot\gamma(t) \leq C\big(1{+}\|\theta(\cdot,t)\|^2_{\mathrm{L}^4(\Omega)}\big)\|z_1(\cdot,t){-}z_2(\cdot,t)\|^2_{\mathrm{H}^1(\Omega)} \text{ for almost every } t\in[0,T]. \tag{3.12}$$

We insert (3.10) into (3.12) and we conclude with Grönwall's lemma. $\square$

Lemma 3.3 *Assume that* (2.9), (2.10), (2.12), (2.13), (2.15), $u^0\in\mathrm{H}^1_0(\Omega)$, $z^0\in\mathrm{X}_{q,p}(\Omega)$ *hold. Then the mapping* $\vartheta \mapsto (u,z)$ *is continuous from* $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ *into* $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega))\cap\mathrm{L}^\infty(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{H}^1(\Omega))$ *and maps any bounded subset of* $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ *into a bounded subset of* $\mathrm{H}^1(0,T;\mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega))$.

Proof. Reminding that $\phi_1 : \widetilde\vartheta \mapsto \theta$ is continuous from $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$, we need to prove that the mapping $\theta\mapsto(u,z)$ is continuous from $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$. We consider $\vartheta_i\in\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))$ and for $i=1,2$, we define $\theta_i \overset{\text{def}}{=} \zeta(\vartheta_i)\in\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ and

(u_i, z_i) the solution of the problem (3.1)–(3.2) with $\theta = \theta_i$. Therefore we reproduce the same kind of computations as in the proof of Proposition 3.2, we find by using (2.11) and (2.13) that

$$\dot{\gamma}(t) + c^{\mathbb{M}}\|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)} + c^{\mathbb{L}}\|\mathrm{e}(\dot{\bar{u}})\|^2_{\mathrm{L}^2(\Omega)} \leq -\int_\Omega \alpha\bar{\theta}\mathrm{tr}(\mathrm{e}(\dot{\bar{u}}))\,\mathrm{d}x$$
$$-\int_\Omega (\theta_1 \mathrm{D}_z H_2(z_1) - \theta_2 \mathrm{D}_z H_2(z_2)).\dot{\bar{z}}\,\mathrm{d}x - \int_\Omega (\mathrm{D}_z h_1(z_1) - \mathrm{D}_z h_1(z_2)).\dot{\bar{z}}\,\mathrm{d}x$$

where $\bar{u} \stackrel{\text{def}}{=} u_1 - u_2$, $\bar{z} \stackrel{\text{def}}{=} z_1 - z_2$ and $\bar{\theta} \stackrel{\text{def}}{=} \theta_1 - \theta_2$. The first and the third terms on the right hand side are estimated by using Cauchy-Schwarz's inequality, while the second term is estimated by employing the decomposition $\theta_1 \mathrm{D}_z H_2(z_1) - \theta_2 \mathrm{D}_z H_2(z_2)).\dot{\bar{z}} = (\bar{\theta}\mathrm{D}_z H_2(z_1) + \theta_2(\mathrm{D}_z H_2(z_1) - \mathrm{D}_z H_2(z_2))).\dot{\bar{z}}$ combined with Young's inequality. Then using (3.10) and the continuous embeddings $\mathrm{H}^2(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$ and $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$, we deduce that there exists a generic constant $C > 0$ depending only on the data such that

$$\dot{\gamma}(t) + \|\dot{\bar{z}}\|^2_{\mathrm{L}^2(\Omega)} + \|\mathrm{e}(\dot{\bar{u}})\|^2_{\mathrm{L}^2(\Omega)} \leq C\big(1 + \|z_1\|^2_{\mathrm{L}^\infty(\Omega)}\big)\|\bar{\theta}\|^2_{\mathrm{L}^2(\Omega)} + C\big(1 + \|\theta_2\|^2_{\mathrm{L}^4(\Omega)}\big)\gamma(t)$$

for almost every $t \in [0, T]$, which allows us to conclude by using once again Grönwall's lemma. $\square$

We establish now some further regularity properties for the solutions of the system (P_{uz}). Let us define $\mathrm{V}^p(\Omega) \stackrel{\text{def}}{=} \big\{u \in \mathrm{L}^2(\Omega;\mathbb{R}^3) : \ \nabla u \in \mathrm{L}^p(\Omega;\mathbb{R}^{3\times 3})\big\}$ endowed with the norm $\|u\|_{\mathrm{V}^p(\Omega)} \stackrel{\text{def}}{=} \|u\|_{\mathrm{L}^2(\Omega)} + \|\nabla u\|_{\mathrm{L}^p(\Omega)}$ and $\mathrm{V}^p_0(\Omega) \stackrel{\text{def}}{=} \big\{u \in \mathrm{V}^p(\Omega;\mathbb{R}^3) : \ u_{|\partial\Omega} = 0\big\}$.

Lemma 3.4 *Assume that* (2.9)*,* (2.10)*,* (2.12)*,* (2.13)*,* (2.15)*,* $u^0 \in \mathrm{V}^p_0(\Omega)$*,* $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *hold. Then* $\mathrm{e}(u)$ *belongs to* $\mathrm{W}^{1,q}(0,T;\mathrm{L}^p(\Omega))$ *and* $\theta \mapsto \mathrm{e}(u)$ *maps any bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ *into a bounded subset of* $\mathrm{W}^{1,q}(0,T;\mathrm{L}^p(\Omega))$*.*

Proof. The key-point consists in interpreting (3.1a) as an ODE for u in an appropriate Banach space. More precisely, let $\mathcal{F}^p(\Omega) \stackrel{\text{def}}{=} \mathrm{L}^2(\Omega;\mathbb{R}^3) \times \mathrm{L}^p(\Omega;\mathbb{R}^{3\times 3}_{\mathsf{sym}})$ be endowed with the norm $\|\varphi\|_{\mathcal{F}^p(\Omega)} \stackrel{\text{def}}{=} \|\varphi_1\|_{\mathrm{L}^2(\Omega)} + \|\varphi_2\|_{\mathrm{L}^p(\Omega)}$ with $\varphi \stackrel{\text{def}}{=} (\varphi_1, \varphi_2)$. Since $\mathbb{L}$ is a symmetric, positive definite tensor then classical results about PDE in Banach spaces give that for all $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}^p(\Omega)$, there exists a unique $u \in \mathrm{V}^p_0(\Omega)$, denoted by $u \stackrel{\text{def}}{=} \Lambda_p(\varphi)$, satisfying

$$\int_\Omega \mathbb{L}\mathrm{e}(u){:}\mathrm{e}(v)\,\mathrm{d}x = \int_\Omega \varphi_1 {\cdot} v\,\mathrm{d}x + \int_\Omega \varphi_2{:}\mathrm{e}(v)\,\mathrm{d}x$$

for all $v \in \mathcal{D}(\Omega)$. Furthermore Λ_p is linear continuous from $\mathcal{F}^p(\Omega)$ to $\mathrm{V}^p_0(\Omega)$ (see [Val88]). It comes that (3.1a) can be rewritten as

$$\dot{u} = \Lambda_p(\ell, \mathbb{E}E(z) - \alpha\theta\mathrm{I}) - \Lambda_p(0, \mathbb{E}\mathrm{e}(u)). \tag{3.13}$$

Then using (2.12), (2.15) and the continuous embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$, we infer that $(\ell, \mathbb{E}E(z) - \alpha\theta\mathrm{I})$ belongs to $\mathrm{L}^q(0,T;\mathcal{F}^p(\Omega))$ and (3.13) is an ODE for u in $\mathrm{V}^p_0(\Omega)$. Hence classical results about ODE in Banach spaces and Lemma 3.3 allow us to conclude. $\square$

Finally, by using (3.9), the regularity results for (u, z) previously obtained and the maximal regularity results for parabolic systems (see [Dor93, HiR08, PrS01]), we can easily deduce the Lemma 3.5. The reader is referred to [PaP11] for technical details.

Lemma 3.5 *Assume that* (2.9), (2.10), (2.12), (2.13), (2.15), $u^0 \in \mathrm{V}_0^p(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *hold. Then* $(\dot{z}, \Delta z) \in (\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega)))^2$ *and* $z \in \mathrm{C}^0([0,T], \mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega))$. *Moreover* $\theta \mapsto (\dot{z}, \Delta z, z)$ *maps any bounded subset of* $\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))$ *into a bounded subset of* $(\mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega)))^2 \times (\mathrm{C}^0([0,T];\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega)))$.

4 Local existence result

We establish here a local existence result for (2.7)–(2.8) by using a fixed-point argument. To this aim, for any given $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$, we consider the solutions of (P_{uz}) with $\theta = \zeta(\widetilde{\vartheta})$ and we define $\widetilde{\kappa}^c \overset{\text{def}}{=} \kappa^c(\mathrm{e}(u), z, \theta)$ and $f^{\widetilde{\vartheta}} \overset{\text{def}}{=} \mathbb{L}\mathrm{e}(\dot{u}){:}\mathrm{e}(\dot{u}) + \theta(\alpha \mathrm{tr}(\mathrm{e}(\dot{u})) + \mathrm{D}_z H_2(z).\dot{z}) + \Psi(\dot{z}) + \mathbb{M}\dot{z}.\dot{z}$. We already know from Section 3 that $f^{\widetilde{\vartheta}} \in \mathrm{L}^{q/4}(0,T;\mathrm{L}^{p/2}(\Omega))$. Since $p \geq 4$ and $q > 8$, we infer that $f^{\widetilde{\vartheta}} \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. We assume that $\vartheta^0 \in \mathrm{L}^2(\Omega)$ and (2.14) hold. By using [Lio68] we infer that there exists a unique $\vartheta \in \mathrm{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ with $\dot{\vartheta} \in \mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega)'))$ such that

$$\dot{\vartheta} - \mathrm{div}(\widetilde{\kappa}^c \nabla \vartheta) = f^{\widetilde{\vartheta}}, \quad \vartheta(\cdot,0) = \vartheta^0, \quad \widetilde{\kappa}^c \nabla \vartheta \cdot \eta_{|\partial\Omega} = 0. \tag{4.1}$$

Moreover, for all $\tau \in [0,T]$, we have

$$\|\vartheta(\tau)\|^2_{\mathrm{L}^2(\Omega)} + 2c^{\kappa^c} \int_0^\tau \|\nabla \vartheta(t)\|^2_{\mathrm{L}^2(\Omega)} \,\mathrm{d}t \leq \mathrm{e}^\tau \big(\|\vartheta^0\|^2_{\mathrm{L}^2(\Omega)} + \|f\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \big). \tag{4.2}$$

Proposition 4.1 *The mapping* $\phi : \widetilde{\vartheta} \mapsto \vartheta$ *is continuous from* $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ *to* $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$.

Proof. The proof is obtained by the same techniques detailed in [PaP11, Prop. 6.1]. Since it is quite a routine to adapt this proof to our case, the verification is left to the reader. ☐

It remains to prove that the mapping ϕ fulfills the other assumptions of the Schauder's fixed point theorem. To do so, we define the following functional space

$$\mathcal{W}_\tau \overset{\text{def}}{=} \{\vartheta \in \mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega)) : \ \dot{\vartheta} \in \mathrm{L}^2(0,\tau;(\mathrm{H}^1(\Omega)'))\}$$

endowed with the norm $\|\vartheta\|_{\mathcal{W}_\tau} \overset{\text{def}}{=} \|\vartheta\|_{\mathrm{L}^2(0,\tau;\mathrm{H}^1(\Omega))} + \|\vartheta\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} + \|\dot{\vartheta}\|_{\mathrm{L}^2(0,\tau;(\mathrm{H}^1(\Omega)')}$ for all $\vartheta \in \mathcal{W}_\tau$ with $\tau \in (0,T]$. We know that $\mathcal{W}_\tau$ is compactly embedded in $\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$ (see [Sim87]). From the previous results, we may infer that ϕ maps any bounded subset of $\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ into a bounded subset of $\mathcal{W}_T$. More precisely, for any $R^\vartheta > 0$ and for any $\widetilde{\vartheta}$ such that $\|\widetilde{\vartheta}\|_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))} \leq R^\vartheta$, we have

$$\|\zeta(\widetilde{\vartheta})\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))} = \|\theta\|_{\mathrm{L}^q(0,T;\mathrm{L}^p(\Omega))} \leq R^\theta \overset{\text{def}}{=} (\tfrac{\beta_1}{c^c} R^\vartheta)^{\frac{1}{\beta_1}} |\Omega|^{\frac{\beta_1 \bar{p} - p}{\beta_1 \bar{p} p}}$$

and there exists a constant $C = C(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{X}_{q,p}(\Omega)}, \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}, R^\theta)$, depending only on $\|u^0\|_{\mathrm{V}^p(\Omega)}$, $\|z^0\|_{\mathrm{X}_{q,p}(\Omega)}$, $\|\vartheta^0\|_{\mathrm{L}^2(\Omega)}$, $\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}$ and R^θ such that

$$\|\phi(\widetilde{\vartheta})\|_{\mathcal{W}_T} = \|\vartheta\|_{\mathcal{W}_T} \leq C(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{X}_{q,p}(\Omega)}, \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}, R^\theta).$$

Now let $0 < \tau \leq T$. For any $\widetilde{\vartheta} \in \mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^{\bar{p}}(\Omega))$, we define $\widetilde{\vartheta}_{\text{ext}} \in \mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^{\bar{p}}(\Omega))$ by $\widetilde{\vartheta}_{\text{ext}} = \widetilde{\vartheta}$ on $[0,\tau]$ and $\widetilde{\vartheta}_{\text{ext}} = 0$ on $(\tau,T]$ and $\phi_\tau(\widetilde{\vartheta})$ as the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ to $[0,\tau]$. From Proposition 4.1, it is clear that ϕ_τ is continuous from $\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$ to $\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$. Furthermore, reminding that $\bar p = 2$ and observing that $\|\widetilde{\vartheta}_{\text{ext}}\|_{\mathrm{L}^{\bar q}(0,T;\mathrm{L}^{\bar p}(\Omega))} = \|\widetilde{\vartheta}\|_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))}$, we get

$$\begin{aligned}\|\phi_\tau(\widetilde{\vartheta})\|_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))} &\le \tau^{\frac{1}{\bar q}}\|\phi(\widetilde{\vartheta}_{\text{ext}})\|_{\mathrm{L}^{\infty}(0,T;\mathrm{L}^{\bar p}(\Omega))} \le \tau^{\frac{1}{\bar q}}\|\phi(\widetilde{\vartheta}_{\text{ext}})\|_{\mathcal{W}_T}\\ &\le \tau^{\frac{1}{\bar q}} C(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{X}_{q,p}(\Omega)}, \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}, R^\theta).\end{aligned}$$

It follows that, for any $R^\vartheta > 0$, there exists $\tau \in (0,T]$ such that ϕ_τ maps $\bar B_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))}(0,R^\vartheta)$ into itself. Moreover, for any $\widetilde{\vartheta} \in \bar B_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))}(0,R^\vartheta)$, $\phi(\widetilde{\vartheta}_{\text{ext}})$ belongs to a bounded subset of $\mathcal{W}_T$, thus the image of $\bar B_{\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))}(0,R^\vartheta)$ by ϕ_τ is included into a bounded subset of $\mathcal{W}_\tau$ and is relatively compact in $\mathrm{L}^{\bar q}(0,\tau;\mathrm{L}^{\bar p}(\Omega))$ ([Sim87]). Consequently, we may apply Schauder's fixed point theorem to ϕ_τ and we conclude that the problem (2.7)–(2.8) possesses a local solution (u,z,ϑ) defined on $[0,\tau]$ such that $u \in \mathrm{W}^{1,q}(0,\tau;\mathrm{V}^p_0(\Omega))$, $z \in \mathrm{L}^\infty(0,\tau;\mathrm{H}^1(\Omega)) \cap \mathrm{H}^1(0,\tau;\mathrm{L}^2(\Omega)) \cap \mathrm{C}^0([0,\tau];\mathrm{X}_{q,p}(\Omega)) \cap \mathrm{L}^q(0,\tau;\mathrm{H}^2(\Omega))$, $\dot z, \Delta z \in \mathrm{L}^{q/2}(0,\tau;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,\tau;\mathrm{L}^{p/2}(\Omega))$ and $\vartheta \in \mathcal{W}_\tau$.

In order to go back to problem (2.5)–(2.6), we observe that the mappings g and ζ are two C^1-diffeomorphism from $(0,+\infty)$ into $(0,+\infty)$ and any solution of (2.7)–(2.8) gives a solution of (2.5)–(2.6) as soon as the enthalpy remains positive. So we assume in the sequel that there exists $\bar\vartheta > 0$ such that

$$\vartheta^0(x) \ge \bar\vartheta > 0 \tag{4.3}$$

for almost every $x \in \Omega$. The local solution for the problem (2.5)–(2.6) is obtained by using the Stampacchia's truncation method.

Theorem 4.2 (Local existence result) *Assume that* (2.9), (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Then, for any initial data* $u^0 \in \mathrm{V}^p_0(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *and* $\vartheta^0 \in \mathrm{L}^2(\Omega)$ *satisfying* (4.3), *there exists* $\tau \in (0,T]$ *such that the problem* (2.5)–(2.6) *admits a solution on* $[0,\tau]$.

Proof. Let $\varphi(t) \overset{\text{def}}{=} \bar\vartheta \mathrm{e}^{-\frac{\beta_1}{c^c}\int_0^t (C+\frac{(C_z^{H_2})^2}{c^{\mathbb{M}}}\|z(\cdot,s)\|^2_{\mathrm{L}^\infty(\Omega)})\,\mathrm{d}s}$ for all $t \in [0,\tau]$ where $C \overset{\text{def}}{=} \frac{(3\alpha)^2}{2c^{\mathbb{L}}} + \frac{(C_z^{H_2})^2}{c^{\mathbb{M}}}$. Let $G \in \mathrm{C}^1(\mathbb{R})$ be is strictly increasing on $(0,\infty)$ such that there exists $C^G > 0$ such that $|G'(\sigma)| \le C^G$ for all $\sigma \in \mathbb{R}$ and $G(\sigma) = 0$ for all $\sigma \le 0$. Then we define $H(\sigma) \overset{\text{def}}{=} \int_0^\sigma G(s)\,\mathrm{d}s$ for all $\sigma \in \mathbb{R}$ and $h(t) \overset{\text{def}}{=} \int_\Omega H(-\vartheta+\varphi)\,\mathrm{d}x$. Clearly $h(0) = 0$. Since $\vartheta \in \mathcal{W}_\tau$ and $\varphi \in \mathrm{H}^1(0,\tau;\mathbb{R})$, we infer that h is absolutely continuous and it follows from (2.11), (2.13), (2.14) and Cauchy-Schwarz's inequality that

$$\dot h(t) \le \int_\Omega G(-\vartheta+\varphi)\big(|\theta|^2\big(C+\frac{(C_z^{H_2})^2}{c^{\mathbb{M}}}|z|^2\big)+\dot\varphi\big)\,\mathrm{d}x$$

for almost every $t \in [0,\tau]$. But $\theta = \zeta(\vartheta)$ and, since $\beta_1 \ge 2$, we infer from (2.16) that $|\theta|^2 \le \frac{\beta_1}{c^c}\max(0,\vartheta)$. Observing that $G(-\vartheta+\varphi) = 0$ whenever $-\vartheta+\varphi \le 0$, we get

$$\dot h(t) \le \int_\Omega G(-\vartheta+\varphi)\big(\varphi\big(C+\frac{(C_z^{H_2})^2}{c^{\mathbb{M}}}|z|^2\big)+\dot\varphi\big)\,\mathrm{d}x \le 0$$

for almost every $t \in [0,\tau]$. We deduce that $h(t) \le h(0) = 0$ for all $t \in [0,\tau]$ and it follows that $-\vartheta+\varphi \le 0$ for almost every $(x,t) \in \Omega \times (0,\tau)$ which proves the theorem. □

5 Global existence result

We establish some a priori estimates for the solutions of the problem (2.7)–(2.8) which are relied on an energy balance combined with Grönwall's lemma. Then by using a contradiction argument together with the results obtained in the previous sections, the global existence result is obtained.

Proposition 5.1 *Assume that* (2.9), (2.10), (2.12), (2.13), (2.14) *and* (2.15) *hold. Assume moreover that* $u^0 \in \mathrm{V}_0^p(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$ *and* $\vartheta^0 \in \mathrm{L}^2(\Omega)$ *such that* (4.3) *is satisfied. Then, there exists a constant* $\widetilde{C} > 0$, *depending only on* $\|u^0\|_{\mathrm{H}^1(\Omega)}$, $\|z^0\|_{\mathrm{H}^1(\Omega)}$, $\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}$ *and the data such that for any solution* (u, z, ϑ) *of problem* (2.7)–(2.8) *defined on* $[0, \tau]$, $\tau \in (0, T]$, *we have* $\|u(\cdot, \widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|z(\cdot, \widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \|\vartheta(\cdot, \widetilde{\tau})\|_{\mathrm{L}^1(\Omega)} \le \widetilde{C}$ *for all* $\widetilde{\tau} \in [0, \tau]$.

Proof. We test (2.5a) with $\dot{u}$, (2.5b) with $\dot{z}$ and (2.5c) with the test-function equal to 1. Then we add these equalities and we integrate over $[0, \widetilde{\tau}]$, with $\widetilde{\tau} \in (0, \tau]$. We get

$$\frac{1}{2}\int_\Omega \mathbb{E}(\mathrm{e}(u(\cdot,\widetilde{\tau}))-E(z(\cdot,\widetilde{\tau}))):(\mathrm{e}(u(\cdot,\widetilde{\tau}))-E(z(\cdot,\widetilde{\tau})))\,\mathrm{d}x + \frac{\nu}{2}\|\nabla z(\cdot,\widetilde{\tau})\|^2_{\mathrm{L}^2(\Omega)}$$
$$+\int_\Omega H_1(z(\cdot,\widetilde{\tau}))\,\mathrm{d}x + \int_\Omega \vartheta(\cdot,\widetilde{\tau})\,\mathrm{d}x = C_0 + \int_{\mathcal{Q}_{\widetilde{\tau}}} \ell\cdot\dot{u}\,\mathrm{d}x\,\mathrm{d}t$$

with $C_0 \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega \mathbb{E}(\mathrm{e}(u^0)-E(z^0)):(\mathrm{e}(u^0)-E(z^0))\,\mathrm{d}x+\frac{\nu}{2}\|\nabla z^0\|^2_{\mathrm{L}^2(\Omega)}+\int_\Omega H_1(z^0)\,\mathrm{d}x+\|\vartheta^0\|_{\mathrm{L}^1(\Omega)}$. We estimate from below the first term of the left hand side and we integrate by parts the last term of the right hand side: there exist two generic constants $C_1, C_2 > 0$ depending only on $c^{\mathbb{E}}$, $\|\mathbb{E}\|_{\mathrm{L}^\infty(\Omega)}$, ν, C^{Korn}, c^{H_1}, $\widetilde{c}^{H_1}$, $\|E_0\|$ and $\|E^N\|$ such that

$$\frac{C_1}{2}\|u(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + C_1\|z(\cdot,\widetilde{\tau})\|^2_{\mathrm{H}^1(\Omega)} + \int_\Omega \vartheta(\cdot,\widetilde{\tau})\,\mathrm{d}x \le C_0 + \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}\|u^0\|_{\mathrm{L}^2(\Omega)}$$
$$+ C_2 + \frac{1}{2C_1}\|\ell\|^2_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))} + \frac{1}{2}\|\dot{\ell}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \frac{1}{2}\int_0^{\widetilde{\tau}} \|u\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}t.$$

This allows us to conclude by using Grönwall's lemma since $\vartheta \ge 0$ almost everywhere on $\mathcal{Q}_{\widetilde{\tau}}$. ☐

Let us assume now that $\beta_1 \ge 4$ and let (u, z, ϑ) be a solution of problem (2.7)–(2.8) defined on $[0, \tau] \subset (0, T]$. We have $\|\theta = \zeta(\vartheta)\|_{\mathrm{L}^q(0,\tau;\mathrm{L}^4(\Omega))} \le \bar{R}^\theta \stackrel{\text{def}}{=} T^{\frac{1}{q}}|\Omega|^{\frac{\beta_1-4}{4\beta_1}}\big(\frac{\beta_1}{c^c}\widetilde{C}\big)^{\frac{1}{\beta_1}}$. By using Lemmas 3.4, 3.5 and estimate (4.2), we infer that there exists a constant $\bar{R}^\vartheta_\infty$, depending only on $\widetilde{C}$, $\|u^0\|_{\mathrm{V}^p(\Omega)}$, $\|z^0\|_{\mathrm{X}_{q,p}(\Omega)}$, $\|\vartheta^0\|_{\mathrm{L}^2(\Omega)}$ and $\|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}$, but independent of τ, such that $\|\vartheta = \phi_\tau(\vartheta)\|_{\mathrm{L}^\infty(0,\tau;\mathrm{L}^2(\Omega))} \le \bar{R}^\vartheta_\infty$. Then we can check that there exists $\widetilde{\tau}_0 > 0$, independent of τ, such that problem (2.7)–(2.8) admits a solution on the extended time-interval $[0, \min(T, \tau+\widetilde{\tau}_0)]$. Indeed, let us assume that $\tau \in (0, T)$ (otherwise there is noting to prove) and define $\bar{R}^\vartheta \stackrel{\text{def}}{=} T^{\frac{1}{\bar{q}}}\bar{R}^\vartheta_\infty + 1$, $\widetilde{R}^\vartheta \stackrel{\text{def}}{=} ((\bar{R}^\vartheta)^{\bar{q}} - T(\bar{R}^\vartheta_\infty)^{\bar{q}})^{\frac{1}{\bar{q}}} > 0$. For any $\widetilde{\vartheta} \in \bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$, we define $\widetilde{\vartheta}_{\text{ext}}$ by $\widetilde{\vartheta}_{\text{ext}} \stackrel{\text{def}}{=} \vartheta$ on $[0, \tau]$, $\widetilde{\vartheta}_{\text{ext}} \stackrel{\text{def}}{=} \widetilde{\vartheta}$ on $(\tau, \tau+\widetilde{\tau}]$ and $\widetilde{\vartheta}_{\text{ext}} \stackrel{\text{def}}{=} 0$ on $(\tau+\widetilde{\tau}, T]$. Clearly, we have

$$\|\widetilde{\vartheta}_{\text{ext}}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(0,T;\mathrm{L}^2(\Omega))} = \|\vartheta\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(0,\tau;\mathrm{L}^2(\Omega))} + \|\widetilde{\vartheta}\|^{\bar{q}}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))} \le \tau(\bar{R}^\vartheta_\infty)^{\bar{q}} + (\widetilde{R}^\vartheta)^{\bar{q}} \le (\bar{R}^\vartheta)^{\bar{q}},$$

and the mapping $\widetilde{\vartheta} \mapsto \widetilde{\vartheta}_{\text{ext}}$ is a contraction on $\mathrm{L}^{\bar{q}}(\tau, \tau+\widetilde{\tau}; \mathrm{L}^2(\Omega))$. Let $\widetilde{\theta} = \zeta(\widetilde{\vartheta}_{\text{ext}})$. By definition of ζ, we have $\widetilde{\theta} = \zeta(\vartheta) = \theta$ on $[0, \tau]$, $\widetilde{\theta} = \zeta(\widetilde{\vartheta})$ on $(\tau, \tau+\widetilde{\tau}]$ and $\widetilde{\theta} = \zeta(0) = 0$ on $(\tau+\widetilde{\tau}, T]$. Hence

$\widetilde{\theta} \in \mathrm{L}^q(0,T;\mathrm{L}^4(\Omega))$ and

$$\|\widetilde{\theta}\|^q_{\mathrm{L}^q(0,T;\mathrm{L}^4(\Omega))} \le (\bar{R}^\theta)^q + \int_\tau^{\tau+\widetilde{\tau}} \|\zeta(\widetilde{\vartheta})\|^q_{\mathrm{L}^4(\Omega)}\,\mathrm{d}t \le (\widetilde{R}^\theta)^q,$$

with $\widetilde{R}^\theta \overset{\text{def}}{=} \big((\bar{R}^\theta)^q + \big(\frac{\beta_1}{c^c}\big)^{\frac{q}{\beta_1}} |\Omega|^{\frac{\beta_1-2}{4\beta_1}q} (\widetilde{R}^\vartheta)^{\bar{q}}\big)^{\frac{1}{q}}$. By definition of ϕ, we get immediately that the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ on $[0,\tau]$ coincide with $\phi_\tau(\vartheta) = \vartheta$ and we define $\widetilde{\phi}_{\widetilde{\tau}}(\widetilde{\vartheta})$ as the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ to $[\tau, \tau+\widetilde{\tau}]$. Furthermore, with the estimates of Section 4, we have $\phi(\widetilde{\vartheta}_{\text{ext}}) \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$ and

$$\|\phi(\widetilde{\vartheta}_{\text{ext}})\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \le C\big(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{X}_{q,p}(\Omega)}, \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}), \widetilde{R}^\theta\big).$$

Let $\widetilde{\tau}_0 > 0$ be such that $\widetilde{\tau}_0^{\frac{1}{q}} C(\|u^0\|_{\mathrm{V}^p(\Omega)}, \|z^0\|_{\mathrm{X}_{q,p}(\Omega)}, \|\vartheta^0\|_{\mathrm{L}^2(\Omega)}, \|\ell\|_{\mathrm{C}^0([0,T];\mathrm{L}^2(\Omega))}), \widetilde{R}^\theta) \le \widetilde{R}^\vartheta$. Then $\widetilde{\phi}_{\widetilde{\tau}}$ maps $\bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$ into itself for all $\widetilde{\tau} \in (0, \min(\widetilde{\tau}_0, T-\tau)]$. By using the same arguments as in section 4, we can check that $\widetilde{\phi}_{\widetilde{\tau}}$ satisfies the other assumptions of Schauder's fixed point theorem. Hence $\widetilde{\phi}_{\widetilde{\tau}}$ admits a fixed point $\widetilde{\vartheta}$ in $\bar{B}_{\mathrm{L}^{\bar{q}}(\tau,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \widetilde{R}^\vartheta)$. But, by construction of $\widetilde{\phi}_{\widetilde{\tau}}$, the restriction of $\phi(\widetilde{\vartheta}_{\text{ext}})$ to $[0,\tau+\widetilde{\tau}]$ is also a fixed point of $\phi_{\tau+\widetilde{\tau}}$ in $\bar{B}_{\mathrm{L}^{\bar{q}}(0,\tau+\widetilde{\tau};\mathrm{L}^2(\Omega))}(0, \bar{R}^\vartheta)$. By choosing $\widetilde{\tau} = \min(\widetilde{\tau}_0, T-\tau)$, we get a solution (2.7)–(2.8) on $[0, \min(\tau+\widetilde{\tau}_0, T)]$. Since $\widetilde{\tau}_0$ does not depend on τ, we may reproduce this argument to obtain finally a global solution of (2.7)–(2.8) on $[0,T]$. Therefore we conclude with the following theorem:

Theorem 5.2 (Global existence result) *Assume that* (2.9)*,* (2.10)*,* (2.12)*,* (2.13)*,* (2.14) *and* (2.15) *hold. Assume moreover that $\beta_1 \ge 4$, $u^0 \in \mathrm{V}^p_0(\Omega)$, $z^0 \in \mathrm{X}_{q,p}(\Omega)$, $\vartheta^0 \in \mathrm{L}^2(\Omega)$ such that* (4.3) *is satisfied. Then the problem* (2.7)–(2.8) *admits a global solution (u,z,ϑ) such that $u \in \mathrm{W}^{1,q}(0,T;\mathrm{V}^p_0(\Omega))$, $z \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega) \cap \mathrm{X}_{q,p}(\Omega)) \cap \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega))$, $\dot{z}, \Delta z \in \mathrm{L}^{q/2}(0,T;\mathrm{L}^p(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{L}^{p/2}(\Omega))$ and $\vartheta \in \mathcal{W}_T$. Moreover ϑ remains strictly positive and $(u, z, \theta = \zeta(\vartheta))$ is a solution of problem* (2.5)–(2.6) *on $[0,T]$.*

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