

Composite Finite Elements for Elliptic Interface Problems*

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Abstract

A Composite Finite Element Method approximates linear elliptic boundary value problems of Dirichlet type with discontinuous coefficients at possibly high contrast. The challenge is the discontinuity in the coefficient across some interface which is not necessarily resolved by the underlying finite element mesh. The method is non-conforming in the sense that shape functions preserve continuity across the interface only in an approximative way. However, the method allows to balance the non-conformity and the best approximation error in such a way that the total discretization error is optimal with regard to the mesh size and independent of contrast.

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1 Introduction

This research note concerns the construction of a *Composite Finite Element (CFE)* method for Dirichlet problems with discontinuous coefficients across a (possibly rough) interface. Optimal a priori error bounds are derived even on computational grids that do *not* resolve the interface.

Composite finite elements are known to be an efficient tool for the treatment of essential boundary conditions on domains that are not resolved by the computational grid. The key idea is to compose finite element shape functions

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on the coarse (target) level of finite element shape functions of a finer level. In such a two-scale approach, degrees of freedoms are related to the coarse level, whereas the fine scale is exclusively used to define the shape of the ansatz functions close to the boundary. The adaptation of shape is done in such a way that the prescribed boundary condition is fulfilled in an approximative way. The construction of CFE does not rely on smoothness of the domain boundary. Hence, the classical CFE method is especially attractive for problems on domains with rough boundaries, i.e., boundaries which cannot be resolved by the finite element mesh in a reasonable way. We refer to [13] as well as [11, 12] for details.

In the present context of interface problems, we mimic the classical CFE construction. Shape functions are fixed on one side of the interface and adapted on the other side in such a way that the traces match across the interface in an approximative way. The new CFE approach has three main advantages:

1. The definition of basis functions is fully explicit, i.e., no local problems have to be solved.
2. The coarse mesh does not need to be aligned with the interface whereas it is necessary for classical finite element methods (see [9]).
3. The asymptotic order of convergence of the underlying discretization is preserved on coarse meshes which do not resolve the geometry.

Alternative approaches in the literature can be found, for instance, in [15], where another composite finite element method is introduced, in [7, 1], where the interface condition is imposed weakly via penalization, or in [3], where special basis functions are computed by solving local problems on submeshes.

The present method might be useful especially in the context of problems with evolving interfaces. Due to evolution, the interface cannot be well represented by edges resp. faces of a time-independent mesh. In classical finite element methods, an adaptation of the mesh to the interface at every time step is required. Such a mesh adaptation in time is considered to be too costly, especially in three space dimensions. The new CFE approach allows to compute the evolution in time on a fixed (possibly coarse) mesh. Only the shape of the ansatz functions has to be slightly adapted in time. As we will see later, the cost for this shape adaptation is negligible in comparison to the overall cost of updating the solution on the fixed coarse mesh.

Notice finally that, in this paper, we only address the treatment of the singularity caused by the discontinuous coefficient. Singularities which might be caused by a possibly non-smooth interface are not considered. Since the method does not add any degrees of freedom to the coarse finite element space, it cannot be expected to resolve singular behavior caused, e.g., by a kink in the interface. The treatment of singular solutions has to be organized on top by classical techniques, e.g., by enriching the finite element space by certain singular functions, or by mesh adaptivity. In the latter context composite finite elements offer a coarse grid approximation that might serve as the starting

point for an a posteriori-driven adaptive refinement process. They allow the adaptivity towards singularities to start long before the interface is resolved by the underlying finite element mesh.

Notation. We use standard notation for (fractional) Sobolev spaces $W_p^m(\Omega)$, $m \geq 0, p \in \mathbb{N} \cup \{0\}$, and their corresponding norms $\|\cdot\|_{W_p^m(\Omega)}$ and seminorms $|\cdot|_{W_p^m(\Omega)}$. We write short $H^m(\cdot)$ for $W_2^m(\cdot)$ ($m \in \mathbb{N}$) and $L^p(\cdot)$ for $W_p^0(\cdot)$. The space of \mathbb{R} -valued continuous functions on a set Ω is denoted by $C^0(\Omega)$. By $\text{dist}(\cdot, \cdot)$ we refer to the Euclidean distance in \mathbb{R}^d , as well as the distance between closed subsets of \mathbb{R}^d . The symbol $|\cdot|$ without index refers to the volume of a set relative to its dimension, i.e., $|\cdot|$ denotes the length of a curve, the area of a triangle or a surface, or the volume of a simplex. The right interpretation will always be clear from the context.

2 Composite Finite Element Discretization of a Model Poisson Problem

2.1 Model Problem

Consider Poisson's equation $-\text{div}(a\nabla u) = f$ in an open, bounded, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, with homogeneous Dirichlet boundary conditions on $\partial\Omega$. The scalar coefficient a jumps between two disjoint, open Lipschitz subdomains $\Omega_1, \Omega_2 \subset \Omega$, $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$. The interface $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2$ is not assumed to have additional smoothness. The corresponding variational problem reads:

$$\text{Find } u^* \in H_0^1(\Omega) \text{ so that } \int_{\Omega} a \nabla u^* \nabla v = \int_{\Omega} f v \quad \text{for all } v \in H_0^1(\Omega). \quad (1)$$

For simplicity, the coefficient $a : \Omega \rightarrow \mathbb{R}_{>0}$ is chosen piecewise constant,

$$a(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ a_{\text{cont}} > 1 & \text{if } x \in \Omega_2. \end{cases}$$

Thereby $H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \text{ in the sense of traces}\}$ as usual. The parameter a_{cont} is denoted as the contrast. It is supposed to be large in practical applications such as the modeling of heat transfer in particle composite materials.

The bounded bilinear form

$$\mathbf{a} := \int_{\Omega} a \nabla \cdot \nabla \cdot = \int_{\Omega_1} \nabla \cdot \nabla \cdot + a_{\text{cont}} \int_{\Omega_2} \nabla \cdot \nabla \cdot$$

induces a norm in $H_0^1(\Omega)$ which is denoted as the energy norm $\|\cdot\| := \|\sqrt{a}\nabla \cdot\|_{L^2(\Omega)}$. Thus, (1) has a unique solution for all $f \in H^{-1}(\Omega) := (H_0^1(\Omega))'$.

Typically, some finite dimensional subspace $V \subset H_0^1(\Omega)$ based on piecewise polynomials replaces $H_0^1(\Omega)$ in a discrete version of (1). However, as long as

the interface is not resolved, such an ansatz suffers from the lack of regularity of the solution at the interface; the solution is continuous across the interface but might have jumps in its gradient.

This issue shall be fixed by considering a discrete space that violates conformity, $V \not\subset H_0^1(\Omega)$. The discrete space is a subset of the larger space

$$H_0^1(\Omega_1 \cup \Omega_2) := \{u \in H^1(\Omega_1 \cup \Omega_2) : u|_{\partial\Omega} = 0\}.$$

In the latter setting, due to the lack of Galerkin orthogonality, the discretization error is not necessarily proportional to the error of the best approximation of u^* . It is bounded by the sum of the best approximation error and the error related to the violation of conformity; this will be made more precise in (3). The aim is to construct V in such a way that the errors due to nonconformity and best approximation are balanced, thereby ensuring an optimal discretization error.

2.2 Construction of the Finite Element Space

The 3-step construction (a)-(c) follows the methodology of composite finite elements [6]:

- (a) **The (piecewise overlapping) 2-scale triangulation.** Let \mathcal{T} be some regular subdivision of $\bar{\Omega}$ into closed non-empty simplices (or triangulation for short) in the sense of Ciarlet [4, 2]. Two non-disjoint distinct simplices in \mathcal{T} share either a common face ($d = 3$), a common edge, or a common vertex. By $V(T)$ we denote the set of vertices (corners) of a simplex $T \in \mathcal{T}$. The union of vertices in a (sub-)triangulation \mathcal{T} is denoted by $V(\mathcal{T}) := \bigcup_{T \in \mathcal{T}} V(T)$. A \mathcal{T} -piecewise mesh width function $h : \bar{\Omega} \rightarrow \mathbb{R}_{>0}$ is given by

$$h(x) := \max_{T \in \mathcal{T} : x \in T} \text{diam}(T).$$

The triangulation \mathcal{T} does not necessarily match the interface Γ , i.e., Γ is not the union of element edges resp. faces. We consider the two sub-triangulations $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}$,

$$\mathcal{T}_k := \{T \in \mathcal{T} : T \cap \Omega_k \neq \emptyset\}, \quad k = 1, 2.$$

\mathcal{T}_1 is further refined (conformingly, e.g., by red-green-refinement or newest vertex bisection) locally near the interface in order to control the non-conformity error. The choice of \mathcal{T}_1 for refinement is due to the smaller coefficient in Ω_1 . The resulting regular triangulation \mathcal{T}_1^Γ is supposed to satisfy

$$\text{dist}(T, \Gamma) \leq C_1 \text{diam}(T)$$

with some universal constant C_1 . The resolution of the interface (see also Lemma 3.2) is reflected by the mesh width function $h^\Gamma : \cup \mathcal{T}_1^\Gamma \rightarrow \mathbb{R}_{>0}$,

$$h^\Gamma(x) := \max_{T \in \mathcal{T}_1^\Gamma : x \in T} \text{diam}(T).$$

The sub-triangulation \mathcal{T}_1^Γ can be computed by successive refinement of those simplices that are intersected by Γ (see [13, Section 2]). In the latter case, the constant $C_1 \approx 2$.

Figure 1 depicts the different grids for some model geometry. We emphasize that the interface is not necessarily resolved by the mesh \mathcal{T}_1^Γ . The analysis of Section 3 will show that h^Γ , locally at the interface, suffices to be of size $h^{3/2}$. This implies that the refinement level in \mathcal{T}_1^Γ is only determined by the mesh size of the coarse mesh \mathcal{T} and not by the roughness of the interface.

- (b) **The (discontinuous) finite element spaces.** By S we denote the \mathcal{T} -piecewise affine finite element space which is continuous with respect to each subdomain

$$S := \{u : \bar{\Omega} \rightarrow \mathbb{R} : u|_{\Omega_k} \in C^0(\Omega_k), \\ u|_{T \cap \Omega_k} \in \mathbb{P}^1 \text{ for all } T \in \mathcal{T}_k, k = 1, 2\}. \quad (2)$$

By S^Γ we denote the $\mathcal{T}_1^\Gamma \cup \mathcal{T}_2$ -piecewise affine finite element space which is continuous with respect to each subdomain

$$S^\Gamma := \{u : \bar{\Omega} \rightarrow \mathbb{R} : u|_{\Omega_k} \in C^0(\Omega_k), \\ u|_{T \cap \Omega_k} \in \mathbb{P}^1 \text{ for all } T \in \mathcal{T}_1^\Gamma \cup \mathcal{T}_2, k = 1, 2\}. \quad (3)$$

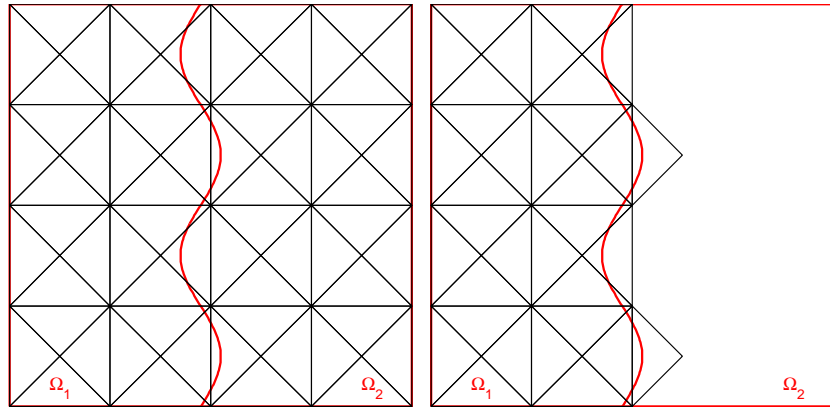
Note that the space S^Γ will be used exclusively to define shape functions while the degrees of freedom will be related to the (coarse) space S .

- (c) **The composite finite element space.** The composite finite element space S^{cfe} is given as the image of S under a projection operator $\mathcal{P}^{\text{cfe}} : S \rightarrow S^\Gamma$, i.e., $S^{\text{cfe}} := \mathcal{P}^{\text{cfe}}(S)$. The definition of \mathcal{P}^{cfe} is based on two mappings that relate the different meshes and the interface:

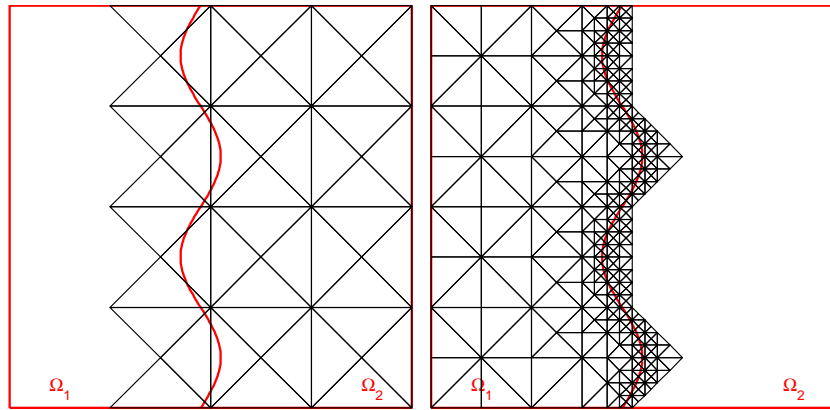
Closest inner simplex. The mapping $T_{(\cdot)}^1 : V(\mathcal{T}_1^\Gamma) \rightarrow \mathcal{T}_1 \setminus \mathcal{T}_2$ is chosen such that $T_x^1 \in \operatorname{argmin}_{T \in \mathcal{T}_1 \setminus \mathcal{T}_2} \operatorname{dist}(x, T)$, i.e., $T_{(\cdot)}^1$ assigns a closest inner simplex (fully contained in Ω_1) to every vertex $x \in V(\mathcal{T}_1^\Gamma)$. $\mathcal{I}_{T_x^1} u \in \mathbb{P}_1(\mathbb{R}^2)$ denotes the globally affine function which interpolates u in the vertices of T_x^1 . Accordingly, $T_{(\cdot)}^2 : V(\mathcal{T}_2) \rightarrow \mathcal{T}_2 \setminus \mathcal{T}_1$ and $\mathcal{I}_{T_x^2} u$ are defined.

Interface projection. The projection operator $(\cdot)^\Gamma : \mathbb{R}^d \rightarrow \Gamma$ is chosen so that $x^\Gamma \in \operatorname{argmin}_{y \in \Gamma} \operatorname{dist}(x, y)$. This projection encodes the geometrical information about the interface that is required by our method.

Based on the above mappings, \mathcal{P}^{cfe} is defined in the two subdomains as follows:



(a) The coarse triangulation \mathcal{T} . The interface is not well represented. (b) The overlapping triangulation \mathcal{T}_1 related to Ω_1 .



(c) The overlapping triangulation \mathcal{T}_2 related to Ω_2 . (d) The refined triangulation $\mathcal{T}_1^{\text{ref}}$ related to Ω_1 . The interface is better represented by this mesh but still not resolved.

Figure 1: The (piecewise overlapping) 2-scale triangulation as it is introduced in Section 2.2(a).

In Ω_2 , \mathcal{P}^{cfe} determines the degrees of freedom which are related to the vertices of $\mathcal{T}_2 \setminus \mathcal{T}_1$ (the vertices that belong to elements fully contained in Ω_2). The operator is uniquely defined through the values of its images at vertices $x \in V(\mathcal{T}_2)$:

$$\mathcal{P}^{\text{cfe}}u(x) = \begin{cases} u(x) & \text{if } x \in V(\mathcal{T}_2 \setminus \mathcal{T}_1), \\ \mathcal{I}_{T_x^2}u(x) & \text{else.} \end{cases} \quad (4)$$

In Ω_1 , \mathcal{P}^{cfe} fulfills two tasks, it determines the degrees of freedom as in Ω_2 and, supported by the submesh \mathcal{T}_1^Γ , it approximately matches the traces of $\mathcal{P}^{\text{cfe}}u|_{\Omega_1}$ and $\mathcal{P}^{\text{cfe}}u|_{\Omega_2}$. The function $\mathcal{P}^{\text{cfe}}u|_{\Omega_1}$ is the restriction (to Ω_1) of the unique continuous \mathcal{T}_1^Γ -piecewise affine function taking the following values at vertices $x \in V(\mathcal{T}_1^\Gamma)$:

$$\mathcal{P}^{\text{cfe}}u(x) = \begin{cases} (\mathcal{P}^{\text{cfe}}u|_{\Omega_2})(x^\Gamma) + \langle \nabla \mathcal{I}_{T_x^1}u, x - x^\Gamma \rangle & \text{if } x \in V(\mathcal{T}_1^\Gamma \setminus \mathcal{T}_1), \\ u(x) & \text{else.} \end{cases} \quad (5)$$

In short, \mathcal{P}^{cfe} picks up a piecewise affine function which is discontinuous across the interface, fixes it on Ω_2 , and ‘‘smears’’ it in Ω_1 towards the interface in such a way that it becomes almost continuous. Theoretically, in the limit $h^\Gamma \rightarrow 0$, S^{cfe} is conforming.

Remark 2.1 *There is some algorithmic freedom in the above construction:*

1. *The subtriangulations $\mathcal{T}_1, \mathcal{T}_2$ could have been chosen to be independent overlapping triangulations of their corresponding subdomains.*
2. *The choice of the closest inner simplex in part (c) of the construction does not need to be restricted to elements $T \in \mathcal{T}_1 \setminus \mathcal{T}_2$ fully contained in Ω_1 . All elements that have sufficiently large intersection with Ω_1 , e.g., all T that fulfill $\frac{|T \cap \Omega_1|}{|T|} > c$ with some positive parameter c , could be considered. This strategy might be especially useful in the case where Ω_2 is multiply connected.*
3. *It is not essential, that the definition of the mappings in part (c) of the construction is based on the minimality of certain distances. Points resp. simplices close by would do the job as well.*

Remark 2.2 *Notice that, by the use of extensions in the above construction, the degrees of freedom are either related to the vertices of $\mathcal{T}_1 \setminus \mathcal{T}_2$ or to the vertices of $\mathcal{T}_2 \setminus \mathcal{T}_1$. Thus every vertex in \mathcal{T} represents at most one basis function of S^{cfe} and all basis functions are well supported. The condition number of the method differs only slightly from the one related to a conforming first order finite element method based on the coarse mesh \mathcal{T} .*

2.3 Discrete Problem

The discrete variational formulation of (1) reads:

$$\text{Find } u^{\text{cfe}} \in S^{\text{cfe}} \text{ so that } \int_{\Omega} a \nabla u^{\text{cfe}} \nabla v = \int_{\Omega} f v, \quad \text{for all } v \in S^{\text{cfe}}. \quad (6)$$

Remark 2.3 *The solution of (6) requires the evaluation of integrals over intersections $T \cap \Omega_k$ which is beyond the scope of this note; it is assumed that all integrals are evaluated exactly. We refer to [5, 10, 13] for a practicable resolution of this issue.*

The following theorem addresses the solvability of (6) and gives an optimal a priori error bound in energy norm. Besides parameters already mentioned the constant in the estimate depends, as usual, on ρ_T , $T \in \mathcal{T}$, which is the ratio between the diameter of the largest ball that can be inscribed in T and the diameter of T . The triangulations \mathcal{T} resp. \mathcal{T}_1^Γ are assumed to be non-degenerate, i.e., $\rho_{\mathcal{T}} := \min_{T \in \mathcal{T}} \rho_T > 0$ (resp. $\rho_{\mathcal{T}_1^\Gamma} := \min_{T \in \mathcal{T}_1^\Gamma} \rho_T > 0$).

Theorem 2.4 *The discrete problem (6) has always a unique solution $u^{\text{cfe}} \in S^{\text{cfe}}$. If the solution of (1) $u^* \in H^2(\Omega_1 \cup \Omega_2)$ and if*

$$\|h^\Gamma / h^{3/2}\|_{L^\infty(\cup\{t \in \mathcal{T}_1^\Gamma : t \cap \Gamma \neq \emptyset\})} \leq C_2 \quad (7)$$

with some constant C_2 then the following a priori error estimate holds:

$$\|u^* - u^{\text{cfe}}\| \leq C \|h\|_{L^\infty(\Omega)} \|\sqrt{a} \nabla^2 u\|_{L^2(\Omega_1 \cup \Omega_2)}.$$

The constant $C = C(\rho_{\mathcal{T}}, \rho_{\mathcal{T}_1^\Gamma}, C_1, C_2)$ does neither depend on the mesh width functions h, h^Γ nor the contrast parameter a_{cont} .

The proof of Theorem 2.4 will be given in Section 3. Considering a uniform coarse-scale grid of width h , shape functions are adapted on a mesh of size $h^\Gamma \approx h^{3/2}$ close to the interface to preserve an optimal estimate. Thus, the additional cost caused by the submesh is at most proportional to a pure coarse scale solve [11, 12].

Even though the constant in the above theorem does not depend on the contrast parameter, the norm on the right hand side does. However, the regularity results of [8, Theorem 2.1, Corollary 2.1] (see also [3]),

$$\|\nabla^2 u^*\|_{L^2(\Omega_1)} \leq C_{\text{reg}} \|f\|_{L^2(\Omega)}, \quad \|\nabla^2 u^*\|_{L^2(\Omega_2)} \leq \frac{C_{\text{reg}}}{a_{\text{cont}}} \|f\|_{L^2(\Omega)}, \quad (8)$$

show that this dependence is not critical and allow to give an error bound that does not depend on the contrast parameter.

Theorem 2.5 *Under the assumptions of Theorem 2.4 there holds*

$$\|u^* - u^{\text{cfe}}\| \leq C \|h\|_{L^\infty(\Omega)} \|f\|_{L^2(\Omega)}.$$

The constant $C = C(\rho_{\mathcal{T}}, \rho_{\mathcal{T}_1^\Gamma}, C_1, C_2, C_{\text{reg}})$ does neither depend on the mesh width functions h, h^Γ nor the contrast a_{cont} .

Remark 2.6 *Even though it is not explicitly assumed that the interface has to be smooth, such a condition is implicitly hidden in the assumption on the solution to be in $H^2(\Omega_1 \cup \Omega_2)$. If the interface has a corner than the solution can only be expected to be in $H^{3/2}$, at least on the side of the interface where the corner is re-entrant. Such a lack of regularity is not addressed by the proposed method. Singularities need to be treated separately by standard techniques, e.g., by adding the singular function to the approximation space, or by adaptive refinement of the coarse mesh \mathcal{T} towards the singularity.*

3 Error Analysis

This section addresses the proof of Theorem 2.4. Unique solvability of (6) follows from the fact that $\|\cdot\|$ is a norm in S^{cfe} . Since, in the limit $h_1^\Gamma \rightarrow 0$, S^{cfe} is conforming, the latter is quite obvious if $h_1^\Gamma|_{(\cup\{t \in \mathcal{T}_1^\Gamma : t \cap \Gamma \neq \emptyset\})}$ is sufficiently small. Otherwise, this property can be proved along the lines of [11, Lemma 4.10].

The error of the composite finite element approximation can be estimated as in [2, Lemma 10.1.7] by

$$\|u^* - u^{\text{cfe}}\| \leq \inf_{v \in S^{\text{cfe}}} \|u^* - v\| + \sup_{0 \neq v \in S^{\text{cfe}}} \frac{|\mathbf{a}(u^* - u^{\text{cfe}}, v)|}{\|v\|}.$$

The first term in the above estimate reflects the best approximation error which is further addressed in Section 3.1. The additional second term is due to non-conformity (see Section 3.2).

3.1 Approximability

Let $\mathcal{I}_{\mathcal{T}_k} u$ denote the unique \mathcal{T}_k -piecewise affine functions that interpolates a sufficiently smooth function u at the vertices of \mathcal{T}_k . A solution $u \in H^2(\Omega_1 \cup \Omega_2)$ of (1) is well approximated by $u_S \in S$, where $u_S|_{\Omega_k} = (\mathcal{I}_{\mathcal{T}_k} u)|_{\Omega_k}$, $k = 1, 2$. The error in energy norm is proportional to h . This property is preserved when u_S is suitably mapped onto the finite element space S^{cfe} as the following lemma states.

Lemma 3.1 (Approximation property of S^{cfe}) *There is a constant $C = C(\rho_{\mathcal{T}}, \rho_{\mathcal{T}_1^\Gamma}, C_1, C_2) > 0$ which does not depend on h or h^Γ such that*

$$\sup_{u \in H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)} \|u - \mathcal{P}^{\text{cfe}} u\| \leq C \|ah \nabla^2 u\|_{L^2(\Omega_1 \cup \Omega_2)}.$$

Proof. The proof picks up some standard techniques for composite finite elements as they are used, e.g., in the proof of Theorem 4.4 in [11]. In addition, we will frequently make use of classical estimates of the error of nodal interpolation with respect to simplices. Following [4, Theorem 16.1], there exists a universal constant C_{ip} such that

$$|u - \mathcal{I}_t u|_{W_p^m(t)} \leq \frac{C_{\text{ip}}}{\rho_t} \text{diam}(t)^{2 - \frac{d}{2} - \frac{d}{p} - m} |u|_{H^2(t)} \quad (9)$$

for all $u \in H^2(t)$, $m \in \{0, 1\}$, provided $W_p^m(t) \subset H^2(t)$; $\mathcal{I}_t u$ denotes the affine interpolant of u at the vertices of a triangle t . The main tool for exploiting the piecewise regularity is a suitable extension operator. It is known that, if Ω_k is bounded and Lipschitz, there exists a continuous, linear extension operator $\mathfrak{E}_k : H^2(\Omega_k) \cap H_0^1(\Omega) \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$, $k \in \{1, 2\}$, such that

$$\text{for all } u \in H^2(\Omega) : \quad \mathfrak{E}_k u|_{\Omega_k} = u \quad \text{and} \quad \|\nabla^2 \mathfrak{E} u\|_{L^2(\Omega)} \leq C_{\text{ext}} \|\nabla^2 u\|_{L^2(\Omega_k)} \quad (10)$$

with a constant C_{ext} depending only on Ω_k and Ω [16]. Moreover, C_{ext} is moderately small under mild assumptions on the geometry [14]. We write short $u_k := \mathfrak{E}_k u$, $k = 1, 2$.

Our proof rests upon the splitting

$$u = u_2 + (u - u_2)$$

and the observation that $(u - u_2)|_{\Omega_1} \in H_0^1(\Omega_1) \cap H^2(\Omega_1)$ and $(u - u_2)|_{\Omega_2} = 0$. The splitting and the linearity of \mathcal{P}^{cfe} lead to the following upper bound

$$\|u - \mathcal{P}^{\text{cfe}} u\|^2 = \|u_2 - \mathcal{P}^{\text{cfe}} u_2\|^2 + \|\nabla(u - u_2 - \mathcal{P}^{\text{cfe}}(u - u_2))\|_{L^2(\Omega_1)}^2. \quad (11)$$

The second term on the right hand side of (11) can be bounded by classical techniques for the analysis of composite finite elements in [11]. More precisely, Theorem 4.4 in [11], together with (10), shows that

$$\|\nabla((u - u_2) - \mathcal{P}^{\text{cfe}}(u - u_2))\|_{L^2(\Omega_1)}^2 \leq C \|h \nabla^2 u\|_{H^2(\Omega_1)} \quad (12)$$

with some constant C that depends only on $\rho_{\mathcal{T}}$, $\rho_{\mathcal{T}_1^\Gamma}$, C_1 , C_2 , and C_{ext} .

Thus, we are left to bound the first term on the right hand side of (11). The advantage of the splitting is that, compared to the initial assertion, we can now make use of the fact that u_2 is smooth regardless of the interface, that is $u_2 \in H^2(\Omega)$. By repeated use of the triangle inequality we separate the elements where standard estimates apply from those where more involved techniques are required:

$$\begin{aligned} \|u_2 - \mathcal{P}^{\text{cfe}} u_2\|^2 &= \|\nabla(u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\Omega_1)}^2 + a_{\text{cont}} \|\nabla(u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\Omega_2)}^2 \\ &\leq \|\nabla(u_1 - \mathcal{P}^{\text{cfe}} u_1)\|_{L^2(\cup \mathcal{T}_1^\Gamma)}^2 + a_{\text{cont}} \|\nabla(u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup \mathcal{T}_2)}^2 \\ &\stackrel{(4),(5)}{\leq} \|\nabla(u_2 - \mathcal{I}_{\mathcal{T}_1^\Gamma} u_2)\|_{L^2(\cup \mathcal{T}_1^\Gamma)}^2 + a_{\text{cont}} \|\nabla(u_2 - \mathcal{I}_{\mathcal{T}_2} u_2)\|_{L^2(\cup \mathcal{T}_2)}^2 \\ &\quad + \|\nabla(\mathcal{I}_{\mathcal{T}_1^\Gamma} u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup(\mathcal{T}_1^\Gamma \setminus \mathcal{T}_1))}^2 + a_{\text{cont}} \|\nabla(\mathcal{I}_{\mathcal{T}_2} u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup(\mathcal{T}_2 \cap \mathcal{T}_1))}^2 \\ &\stackrel{(9),(10)}{\leq} C_{\text{ext}} C_{\text{ip}} \left(\|h^\Gamma \nabla^2 u_2\|_{L^2(\Omega_1)}^2 + a_{\text{cont}} \|h \nabla^2 u_2\|_{L^2(\Omega_2)}^2 \right) \\ &\quad + \|\nabla(\mathcal{I}_{\mathcal{T}_1^\Gamma} u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup(\mathcal{T}_1^\Gamma \setminus \mathcal{T}_1))}^2 + a_{\text{cont}} \|\nabla(\mathcal{I}_{\mathcal{T}_2} u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup(\mathcal{T}_2 \cap \mathcal{T}_1))}^2. \end{aligned} \quad (13)$$

During the rest of the proof, we will write $a \lesssim b$ short for $a \leq Cb$ with some constant C which depends only on the constants $C_1, C_2, C_{\text{ip}}, C_{\text{ext}}, \rho_{\mathcal{T}}$, and $\rho_{\mathcal{T}_1^\Gamma}$.

Let $t \in \mathcal{T}_1^\Gamma \setminus \mathcal{T}_1$. Then, by an inverse estimate,

$$\|\nabla(\mathcal{I}_t u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(t)} \leq 2 \text{diam}(t)^{d/2-1} \|\mathcal{I}_t u_2 - \mathcal{P}^{\text{cfe}} u_2\|_{L^\infty(t)}. \quad (14)$$

We fix $x \in V(t)$ so that $\|\mathcal{I}_t u_2 - \mathcal{P}^{\text{cfe}} u_2\|_{L^\infty(t)} = |\mathcal{I}_t u_1(x) - \mathcal{P}^{\text{cfe}} u_1(x)|$ and define $T_t^1 := T_x^1, T_t^2 := T_x^2$. In addition we introduce environments

$$\omega_{T_t^k} := \cup\{T \in \mathcal{T}_k : T_t^k \cap T \neq \emptyset\}.$$

The definition of \mathcal{P}^{cfe} and the application of Lemma 1 from [11] lead to

$$\begin{aligned} & \|\mathcal{I}_t u_2 - \mathcal{P}^{\text{cfe}} u_2\|_{L^\infty(t)} \stackrel{(5)}{=} \left| \mathcal{I}_t u_2(x) - \mathcal{I}_{T_t^2} u_2(x^\Gamma) - \langle \nabla \mathcal{I}_{T_t^1} u_2, x - x^\Gamma \rangle \right| \\ & \lesssim \left| \mathcal{I}_t u_2(x) - \mathcal{I}_{T_t^2} u_2(x) \right| + \left| \langle \nabla(\mathcal{I}_{T_t^2} u_2 - \mathcal{I}_{T_t^1} u_2), x - x^\Gamma \rangle \right| \\ & \lesssim \text{diam}(t)^{-d/2} \left(\|\mathcal{I}_t u_2 - \mathcal{I}_{T_t^2} u_2\|_{L^2(t)} + \text{diam}(t) \|\nabla(\mathcal{I}_t u_2 - \mathcal{I}_{T_t^2} u_2)\|_{L^2(t)} \right. \\ & \quad \left. + \text{diam}(t) \|\nabla(\mathcal{I}_{T_t^2} u_2 - \mathcal{I}_{T_t^1} u_2)\|_{L^2(t)} \right) \\ & \stackrel{[11, L1]}{\lesssim} \text{diam}(t)^{1-d/2} \max\{\text{diam}(T_t^1), \text{diam}(T_t^2)\} \|\nabla^2 u\|_{L^2((\omega_{T_t^1} \cup \omega_{T_t^2}) \cap \Omega_2)}. \quad (15) \end{aligned}$$

Summation over all $t \in \mathcal{T}_1^\Gamma \setminus \mathcal{T}_1$ yields

$$\begin{aligned} & \left\| \nabla(\mathcal{I}_{\mathcal{T}_1^\Gamma} u_2 - \mathcal{P}^{\text{cfe}} u_2) \right\|_{L^2(\cup(\mathcal{T}_1^\Gamma \setminus \mathcal{T}_1))}^2 \leq \sum_{t \in \mathcal{T}_1^\Gamma \setminus \mathcal{T}_1} \|\nabla(\mathcal{I}_t u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(t)}^2 \\ & \leq \sum_{T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2} \sum_{t \in \mathcal{T}_1^\Gamma \setminus \mathcal{T}_1 : T_t^k = T_k} \|\nabla(\mathcal{I}_t u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(t)}^2 \\ & \stackrel{(14), (15)}{\lesssim} \sum_{T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2} \left(\sum_{t \in \mathcal{T}_1^\Gamma \setminus \mathcal{T}_1 : T_t^k = T_k} |t| \text{diam}(T_t^1)^{d-2} \right) \|\nabla^2 u\|_{L^2((\omega_{T_1} \cup \omega_{T_2}) \cap \Omega_2)}^2 \\ & \lesssim \sum_{T_1 \in \mathcal{T}_2} h^2 |T_2| \|\nabla^2 u\|_{L^2(\omega_{T_2} \cap \Omega_2)}^2 \lesssim \|h \nabla^2 u\|_{L^2(\cup(\mathcal{T}_1 \setminus \mathcal{T}_2) \cap \Omega_2)}^2. \quad (16) \end{aligned}$$

Applying the arguments of (14), (15), and (16) to the remaining term on the right hand side of (13), $\|\nabla(\mathcal{I}_{\mathcal{T}_2} u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup(\mathcal{T}_2 \cap \mathcal{T}_1))}$, yields

$$\|\nabla(\mathcal{I}_{\mathcal{T}_2} u_2 - \mathcal{P}^{\text{cfe}} u_2)\|_{L^2(\cup(\mathcal{T}_1 \cap \mathcal{T}_2))}^2 \lesssim \|h \nabla^2 u\|_{L^2(\cup(\mathcal{T}_1 \cap \mathcal{T}_2) \cap \Omega_2)}^2. \quad (17)$$

The combination of (11), (12), (13), (16), and (17) finally proves the assertion. \blacksquare

3.2 Nonconformity

If the solution is sufficiently smooth, i.e., $u^* \in H^{3/2}(\Omega_1 \cup \Omega_2)$, the second term in (3) can be estimated using Greens's identity, (1), (6), the classical jump relation, and Cauchy-Schwarz inequality as follows:

$$\sup_{0 \neq v \in S^{\text{cfe}}} \frac{|\mathbf{a}(u^* - u^{\text{cfe}}, v)|}{\|v\|} \leq C \left\| \frac{\partial u^*}{\partial \nu_{\Omega_1}} \right\|_{L^2(\Gamma)} \sup_{0 \neq v \in S^{\text{cfe}}} \frac{\|[[v]]_{\Gamma}\|_{L^2(\Gamma)}}{\|v\|}. \quad (18)$$

Here, ν_{Ω_1} denotes the outer normal of Ω_1 and $[[v]]_{\Gamma}$ denotes the jump of v across Γ . By picking up ideas from [11, Lemma 4.9] the discontinuity $[[v]]_{\Gamma}$ can be proved to be small.

Lemma 3.2 (Nonconformity) *There is a constant $C = C(C_1, C_3) > 0$, $C_3 := \max_{T \in \mathcal{T}_1^{\Gamma}: T \cap \Gamma \neq \emptyset} |\Gamma \cap T| / \text{diam}(T)^{(d-1)}$, such that*

$$\|[[u]]_{\Gamma}\|_{L^2(\Gamma)} \leq C \|h\|_{L^{\infty}(\Omega)} \|h_1^{\Gamma}/h^{3/2}\|_{L^{\infty}(\cup\{T \in \mathcal{T}_1^{\Gamma}: T \cap \Gamma \neq \emptyset\})} \|u\| \quad \text{for all } u \in S^{\text{cfe}}.$$

Proof. Let $t \in \mathcal{T}_1^{\Gamma}$ such that $t \cap \Gamma \neq \emptyset$. We start with some pointwise estimate of the jump of $\mathcal{P}^{\text{cfe}} u \in S^{\text{cfe}}$ on t :

$$\|[[u]]\|_{L^{\infty}(\Gamma \cap t)} = \|u|_{\Omega_2} - u|_{\Omega_1}\|_{L^{\infty}(\Gamma \cap t)} \leq \|u|_{\Omega_2} - u|_{\Omega_1}\|_{L^{\infty}(t)},$$

where $u|_{\Omega_1}$ (resp. $u|_{\Omega_2}$) is identified with its unique affine extension onto t . By definition (see (5), (4)) we further get

$$\begin{aligned} \|[[u]]\|_{L^{\infty}(\Gamma \cap t)} &= \max_{y \in V(t)} \left| u|_{\Omega_2}(y) - u|_{\Omega_2}(y^{\Gamma}) - \left\langle \nabla \mathcal{I}_{T_y^1}, y - y^{\Gamma} \right\rangle \right| \\ &\lesssim \text{diam}(t) \left\| h^{-d/2} \nabla u \right\|_{L^2(T_y^1 \cup T_y^2)}. \end{aligned} \quad (19)$$

Finally, the L^2 -norm of u on $\partial\Omega$ is estimated as follows:

$$\begin{aligned} \|u\|_{L^2(\Gamma)}^2 &\leq \sum_{T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2} \sum_{t \in \mathcal{T}_1^{\Gamma}: t \cap \Gamma \neq \emptyset, T_t^k = T_k} |\Gamma \cap t| \|[[u]]\|_{L^{\infty}(t)}^2 \\ &\stackrel{(19)}{\lesssim} \sum_{T_1 \in \mathcal{T}_1, T_2 \in \mathcal{T}_2} \sum_{t \in \mathcal{T}_1^{\Gamma}: t \cap \Gamma \neq \emptyset, T_t^k = T_k} \frac{|\Gamma \cap t| \text{diam}(t)^2}{\text{diam}(T)^d} \|\nabla u\|_{L^2(T_1 \cup T_2)}^2 \\ &\lesssim \|h_1^{\Gamma}/\sqrt{h}\|_{L^{\infty}(\cup\{t \in \mathcal{T}_1^{\Gamma}: t \cap \Gamma \neq \emptyset\})}^2 \|\nabla u\|_{L^2(\Omega)}^2. \quad \blacksquare \end{aligned}$$

If h^{Γ} is chosen proportional to $h^{(3/2)}$, as it is assumed in (7), Theorem 2.4 finally follows from (3), Lemma 3.1, Lemma 3.2, and (8).

Remark 3.3 *The constant C_3 introduced in Lemma 3.2 reflects the smoothness of the interface Γ . Notice, that C_3 might be large if Γ is highly oscillating. However, the proof of Lemma 3.2 shows that a possibly large constant can be controlled by simply choosing h^{Γ} small enough. This modification concerns only the submesh \mathcal{T}_1^{Γ} and does not affect the overall number of degrees of freedom.*

4 Conclusion

We have described a finite element method for the Poisson equation with discontinuous diffusion coefficient across some interface. The method does not require the underlying finite element mesh to resolve the physical domain exactly. Overlapping, and possibly structured, simplicial meshes can be used instead. Moreover, the definition of the basis functions is fully explicit, no local problems have to be solved. On a quasi uniform coarse grid of width h , the complexity of our method is proportional to h^{-d} while the error is proportional to h . This is optimal in comparison to the approximation of a Poisson problem with overall constant coefficient on the same mesh.

This paper focused on the difficulty of treating discontinuous coefficients. In order to keep notation and technicalities at minimum the simplest possible setting has been chosen. Let us point out that the method is not based on these simplifications. Generalizations to general linear elliptic problems but also to saddle point problems such as the Stokes problem are straight forward with regard to the previous work done in [11, 12].

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