A Limit Theorem for Financial Markets with Inert Investors

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Abstract

We study the effect of investor inertia on stock price fluctuations with a market microstructure model comprising many small investors who are inactive most of the time. It turns out that semi-Markov processes are tailor made for modelling inert investors. With a suitable scaling, we show that when the price is driven by the market imbalance, the log price process is approximated by a process with long range dependence and non-Gaussian returns distributions, driven by a fractional Brownian motion. Consequently, investor inertia may lead to arbitrage opportunities for sophisticated market participants. The mathematical contributions are a functional central limit theorem for stationary semi-Markov processes, and approximation results for stochastic integrals of continuous semimartingales with respect to fractional Brownian motion.

Key Words: Semi-Markov processes, fractional Brownian motion, functional central limit theorem, market microstructure, investor inertia.

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1 Introduction

We prove a functional central limit theorem for stationary semi-Markov processes in which the limit process is a stochastic integral with respect to fractional Brownian motion. Our motivation is to develop a probabilistic framework within which to analyze the aggregate effect of investor inertia on asset price dynamics. We show that, in isolation, such infrequent trading patterns can lead to long-range dependence in stock prices and arbitrage opportunities for other more "sophisticated" traders.

In mathematical finance, the dynamics of asset prices are usually modelled by trajectories of a stochastic process defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Geometric Brownian motion has long become the canonical reference model of financial price fluctuations. Since prices are generated by the demand of market participants, it is of interest to support such an approach by a microeconomic model of interacting agents. One possibility is to justify geometric Brownian motion as a rational expectations equilibrium in a market with highly sophisticated and completely rational agents who instantaneously incorporate all available information into the present price; see, for instance, Bick (1987). Föllmer and Schweizer (1993) modelled asset prices as a sequence of temporary price equilibria and in some exogenous random environment. Combining their microeconomic point of view with an invariance principle, they derived a diffusion approximation for the logarithmic price process. Horst (2002) extended this approach by considering a financial market model with many interacting agents that allows for herding and contagion effects.

In recent years, agent-based models of financial markets have attracted much attention. These models are capable of explaining, often through simulations, many facts like the emergence of herding behavior (Lux (1998)), volatility clustering (Lux and Marchesi (2000)) or fat-tailed distributions of stock returns (Cont and Bouchaud (2000)) that are observed in financial data. Brock and Hommes (1997, 1998) proposed to model financial markets as *Adaptive Belief Systems*. In a model with many traders the dynamics of the asset price process can be described by a fairly simple random dynamical system; see Brock *et al.* (2002).

All these discrete-time models assume that the agents trade the asset in each period. At the end of each trading interval, the agents update their expectations for the future evolution of the stock price and formulate their excess demand for the following period. Agents are also supposed to act as price takers. This assumption suggests that investors are small; individual stock holdings do not have an impact on the formation of asset prices. Small investors, however, are typically inert: they are usually inactive, and actually trade only occasionally. This may be because they are waiting to accumulate sufficient capital to make

further stock purchases, or because they tend to monitor their portfolios infrequently.

We study a simple microstructure model for the price evolution of a financial asset in continuous time where the price is driven by the demand of many small investors whose trading behavior exhibits "inertia". To each agent a, we associate a stochastic process $x^a = (x^a_t)_{t\geq 0}$ which represents his propensity for trading. The processes x^a are stationary, non-centered semi-Markov processes on a finite state space with heavy-tailed sojourn times in an inactivity (zero) state, and thin-tailed sojourn times in various active states, which may be positive (buying) or negative (selling). Semi-Markov processes are tailor made to model individual traders' inertia as they generalize Markov processes by removing the requirement of exponentially distributed, and therefore thin-tailed, holding times. In addition, we allow for a market-wide amplitude process Ψ, that describes the evolution of typical trading *size* in the market. It is large on heavy-trading days and small on light trading days. We suggest a non-Walrasian approach to asset pricing and, and assume that prices move in the direction of market imbalance.

We show that in a model with many inert investors long range dependence in the price process emerges. The observation of this phenomenon (sometimes called the Joseph effect) in financial time series motivated the use of fractional Brownian motion as a basis for asset pricing models; see, for instance, Mandelbrot (1997) or Cutland *et al.* (1995). In fact, by our invariance principle, for Ψ a continuous semimartingale, the drift-adjusted logarithmic price process converges weakly to a stochastic integral with respect to a fractional Brownian motion with Hurst coefficient $H > \frac{1}{2}$. Our approach may thus be viewed as a microeconomic foundation for these models. A recent paper that proposes entirely different economic foundations for models based on fractional Brownian motion is Klüppelberg and Kühn (2002).

Evidence of long-range dependence in financial data is discussed in Cutland, Kopp, and Willinger (1995), for example. Bayraktar *et al.* (2002) studied an asymptotically efficient wavelet-based estimator for the Hurst parameter, and applied these techniques on high frequency data set of S&P 500 index over the span of 11.5 years (1989-2000). It was observed that, although the Hurst parameter of this data set is significantly above the efficient markets value of $H = \frac{1}{2}$, it started to approach that level over the period 1997-2000. They suggested that this behaviour of the market might be related to the increase in Internet trading, which had the two-fold effect of increasing the number of small traders and the frequency of trading activity. The model presented in the present work may serve as an explanation of this observation.

As is well-known, see, e.g., Rogers (1997) or Cheridito (2003), fractional Brownian motion processes are not semimartingales, and so these models may allow arbitrage opportunities.

As a result, in our microstructure model, arbitrage opportunities may arise for other, sufficiently sophisticated, market participants. In that sense, inertia leads to arbitrage.

Our mathematical contribution is twofold. First, we establish a functional central limit theorem for semi-Markov processes which extends the results of Taqqu *et al.* (1997) and Taqqu and Levy (1986) who proved a result similar to ours in a situation where the semi-Markov processes take values in a binary state space. Their arguments do not carry over to models with more general state spaces. Our approach builds on Markov renewal theory as well as results reported in Heath *et al.* (1996) and Jelenkovic and Lazar (1998). Taqqu and Levy (1986) considered renewal reward processes with heavy tailed renewal periods and independent and identically distributed rewards. They assume a general state space, but give up the heterogeneity of the distributions of the length of renewal periods. A recent paper by Mikosch *et al.* (2002) studies the binary case under a different limit taking mechanism. Our limit theorem, which is proved in Section 3, also has application to queuing networks and teletraffic, for which the original binary *on/off* theory was developed. We demonstrate (see Example 3.3) that there may be a different limit behaviour when the semi-Markov processes are centered, a situation which does not arise in the binary case.

We also allow for limits which are integrals with respect to fractional Brownian motion, extending the results of Taqqu *et al.* (1997) whose limit is pure fractional Brownian motion. Specifically, in Section 4, we prove an approximation result for stochastic integrals of continuous semimartingales with respect to fractional Brownian motion. We consider $\{\Psi^n\}$ a sequence of good semimartingales and $\{X^n\}$ a sequence of stochastic processes having zero quadratic variation and give sufficient conditions which guarantee that joint convergence of (X^n, Ψ^n) to (B^H, Ψ), where B^H is a fractional Brownian motion process with Hurst parameter $H > \frac{1}{2}$, and Ψ is a continuous semimartingale, implies the convergence of the stochastic integrals $\int \Psi^n dX^n$ to $\int \Psi dB^H$. In addition, we obtain a stability result for the integral of a fractional Brownian motion with respect to itself. These results may be viewed as an extension of Theorem 2.2 in Kurtz and Protter (1991) beyond the semimartingale setting.

2 The microeconomic setup and the main results

We consider a financial market with a set $\mathbb{A} := \{a_1, a_2, \ldots, a_N\}$ of *agents* trading a single risky asset. Our aim is to analyze the effects investor inertia has on the dynamics of stock price processes. For this we choose the simplest possible setup. In particular, as in, e.g., Garman (1976), Föllmer and Schweizer (1993) and Cont and Bouchaud (2000), we do not characterize agents' investment decisions as solutions to individual utility maximization problems.

Instead, we model directly the behavior of individual traders. Specifically, we associate to each agent $a \in \mathbb{A}$ a stochastic process $x^a = (x_t^a)_{t\geq 0}$ on a finite state space E, containing zero. This process describes the agent's *trading mood*. He accumulates the asset at a rate $\Psi_t x_t^a$ at time $t \geq 0$. The random quantity $\Psi_t > 0$ describes the size of a typical trade at time t, and x_t^a may be negative, indicating the agent is selling. Agents do not trade at times when $x_t^a = 0$. We therefore call the state 0 the agents' *inactive state*.

Remark 2.1 *In the simplest setting, $x^a \in \{-1, 0, 1\}$, so that each investor is either buying, selling or inactive, and $\Psi \equiv 1$: there is no external amplification. Even here, the existing results of Taqqu* et al. *(1997) do not apply because the state space is not binary.*

The holdings of the agent $a \in \mathbb{A}$ and the "market imbalance" at time $t \geq 0$ are given by

$$\int_0^t \Psi_s x_s^a ds \qquad \text{and} \qquad I_t^N := \sum_{a\in\mathbb{A}} \int_0^t \Psi_s x_s^a ds, \tag{1}$$

respectively. Hence the process $(I_t^N)_{t\geq 0}$ describes the stochastic evolution of the *market imbalance*. For the quantities I_t^N to be well defined, we need to assume that the asset is in infinite supply.

In our continuous time model buyers and sellers arrive at different points in time. Hence the economic paradigm that a Walrasian auctioneer can set prices such that the markets clear at the end of each trading period does not apply. Rather, temporary imbalances between demand and supply will occur, and prices are assumed to reflect the extent of the current market imbalance. Specifically, we consider the pricing rule

$$dS_t^N = \sum_{a\in\mathbb{A}} \Psi_t x_t^a dt \quad \text{and so} \quad S_t^N = S_0 + I_t^N, \tag{2}$$

for the evolution of the logarithmic stock price process $S^N = (S_t^N)_{t\geq 0}$. In particular, market imbalance is the only component driving the dynamics of stock prices.

2.1 The dynamics of individual behavior

Next, we specify the probabilistic structure of the processes x^a. We assume that the agents are homogeneous and that all the processes x^a and Ψ are independent. It is therefore enough to specify the dynamics of some reference process $x = (x_t)_{t\geq 0}$. In order to incorporate the idea of market inertia as defined by Assumption 2.4 below, we assume that x is a *semi-Markov* process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space E. Here E may contain both positive and negative values and we assume $0 \in E$. The process

x is specified in terms of random variables $\xi_n : \Omega \to E$ and $T_n : \Omega \to \mathbb{R}_+$ which satisfy $0 = T_0 \leq T_1 \leq \cdots$ almost surely and

$$\mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \big| \xi_0, ..., \xi_n; T_0, ..., T_n\} = \mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t \big| \xi_n\}$$

for each $n \in \mathbb{N}$, $j \in E$ and all $t \in \mathbb{R}_+$ through the relation

$$x_t = \sum_{n \geq 0} \xi_n \mathbf{1}_{[T_n, T_{n+1})}(t). \tag{3}$$

Remark 2.2 *(i) In economic terms, the representative agent's mood in the random time interval $[T_n, T_{n+1})$ is given by ξ_n. The distribution of the length of the interval $T_{n+1} - T_n$ may depend on the sequence $\{\xi_n\}_{n \in \mathbb{N}}$ through the states ξ_n and ξ_{n+1}. This allows us to assume different distributions for the lengths of the agents' active and inactive periods, and in particular to model inertia as a heavy-tailed sojourn time in the zero state.*

(ii) In this first step towards analyzing investor inertia, we do not allow for feedback effects of prices into agents' investment decisions. While such an assumption might be justified for small, non-professional investors, it is clearly desirable to allow active traders' investment decisions to be influenced by asset prices. Allowing for feedback effects into the stock price dynamics leads to a significant increase in the complexity of the system, and is not our focus here. Another extension would be to allow for strategically interacting institutional investors; see Bayraktar and Poor (2002) for a possible game theoretic framework.

Throughout, we assume that x is temporally homogeneous under the measure $\mathbb{P}$, i.e., that

$$\mathbb{P}\{\xi_{n+1} = j,\ T_{n+1} - T_n \leq t \big| \xi_n = i\} = Q(i, j, t) \tag{4}$$

is independent of $n \in \mathbb{N}$. By Proposition 1.6 in Çinlar (1975), this implies that $\{\xi_n\}_{n \in \mathbb{N}}$ is a homogeneous Markov chain on E whose transition probability matrix $P = (p_{ij})$ is given by

$$p_{ij} = \lim_{t \to \infty} Q(i, j, t).$$

Clearly, x is an ordinary temporally homogeneous Markov process if Q takes the form

$$Q(i, j, t) = p_{ij} \left(1 - e^{-\lambda_i t}\right). \tag{5}$$

We assume that the *embedded Markov chain* $\{\xi_n\}_{n \in \mathbb{N}}$ satisfies the following condition.

Assumption 2.3 *For all $i, j \in E$, $i \neq j$ we have that $p_{ij} > 0$. In particular, there exists a unique probability measure π on E such that $\pi P = \pi$.*

The conditional distribution function of the length of the n-th sojourn time, $T_{n+1} - T_n$, given ξ_{n+1} and ξ_n is specified in terms of the *semi-Markov kernel* $\{Q(i,j,t); i,j \in E,\ t \geq 0\}$ and the transition matrix P by

$$G(i,j,t) := \frac{Q(i,j,t)}{p_{ij}} = \mathbb{P}\{T_{n+1} - T_n \leq t | \xi_n = i,\ \xi_{n+1} = j\}. \tag{6}$$

For later reference we also introduce the distribution of the first occurrence of state j under $\mathbb{P}$, given $x_0 = i$. Specifically, for $i \neq j$, we put

$$F(i,j,t) := \mathbb{P}\{\tau_j \leq t | x_0 = i\}, \tag{7}$$

where $\tau_j := \inf\{t \geq 0 : x_t = j\}$. We denote by $F(j,j,\cdot)$ the distribution of the time until the next entrance into state j and by $\eta_j := \int t F(j,j,dt)$ the expected time between two occurrences of state $j \in E$. Further, we recall that a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ is called *slowly varying at infinity* if

$$\lim_{t\to\infty} \frac{L(xt)}{L(t)} = 1 \qquad \text{for all} \qquad x > 0$$

and that $f(t) \sim g(t)$ for two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ means $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

Assumption 2.4 *(i) The average sojourn time at state $i \in E$ is finite:*

$$m_i := \mathbb{E}[T_{n+1} - T_n | \xi_n = i] < \infty. \tag{8}$$

Here $\mathbb{E}$ denotes the expectation operator with respect to $\mathbb{P}$.

(ii) There exists a constant $1 < \alpha < 2$ and a locally bounded function $L : \mathbb{R}_+ \to \mathbb{R}_+$ which is slowly varying at infinity such that

$$\mathbb{P}\{T_{n+1} - T_n \geq t \big| \xi_n = 0\} \sim t^{-\alpha} L(t). \tag{9}$$

(iii) There exists $\beta > \alpha$ such that the distributions of the sojourn times at state $i \neq 0$ satisfy

$$\lim_{t\to 0} \frac{\mathbb{P}\{T_{n+1} - T_n \geq t \big| \xi_n = i\}}{t^{-\beta} L(t)} = 0.$$

(iv) The distribution of the sojourn times in the various states have bounded densities with respect to Lebesgue measure on $\mathbb{R}_+$.

Our condition (9) is satisfied if, for instance, the length of the sojourn time at state $0 \in E$ is distributed according to a Pareto distribution. Assumption 2.4 (iii) reflects the idea of market inertia: the probability of long uninterrupted trading periods is small compared to the probability of an individual agent being inactive for a long time. Such an assumption is appropriate if we think of the agents as being small investors.

2.2 An invariance principle for semi-Markov processes

In this section, we state our main results. With our choice of scaling, the logarithmic price process can be approximated in law by the stochastic integral of Ψ with respect to fractional Brownian motion B^H where the *Hurst coefficient* H depends on α. The convergence concept we use is weak convergence of the Skorohod space $\mathbb{D}$ of all real-valued right continuous processes with left limits. We write $\mathcal{L}\text{-}\lim_{n\to\infty} Z^n = Z$ if a sequence of $\mathbb{D}$-valued stochastic processes $\{Z^n\}_{n\in\mathbb{N}}$, converges in distribution to Z.

In order to derive our approximation result, we assume that the semi-Markov process x is stationary. For this, we apply the following result. Its proof is follows from, e.g., Theorem 4.2.5 in Brandt *et al.* (1990); see also Section 1.4.4 in Baccelli and Bremaud (1994).

Lemma 2.5 *Under Assumption 2.3, there exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ such that $(x_t)_{t\geq 0}$ is stationary under $\mathbb{P}^*$. Under the law $\mathbb{P}^*$ the following holds:*

(i) The joint distribution of the initial state and the initial sojourn time takes the form

$$\mathbb{P}^*\{\xi_0 = k, T_1 > t\} = \frac{\pi_k}{\sum_{j\in E}\pi_j m_j}\int_t^\infty h(k,s)ds. \tag{10}$$

Here m_i denotes the mean sojourn time in state $i \in E$ as defined by (8), and for $i \in E$,

$$h(i,t) = 1 - \sum_{j\in E} Q(i,j,t) \tag{11}$$

is the probability that the sojourn time at state $i \in E$ is at least t.

(ii) The law $\nu = (\nu_k)_{k\in E}$ of x_t in the stationary regime is given by

$$\nu_k = \frac{\pi_k m_k}{\sum_{j\in E}\pi_j m_j}. \tag{12}$$

(iii) The conditional joint distribution of (ξ_1, T_1), given ξ_0 is

$$\mathbb{P}^*\{\xi_1 = j, T_1 < t \mid \xi_0 = k\} = \frac{p_{kj}}{m_{k,j}}\int_0^t [1 - G(k,j,s)]ds. \tag{13}$$

Here $m_{k,j} := \int_0^\infty [1-G(k,j,s)]ds$ denotes the conditional expected sojourn time at state k, given the next state is j, and the functions $G(k,j,\cdot)$ are defined in (6).

Let us now introduce a dimensionless parameter $\varepsilon > 0$, and consider the rescaled processes $x^a_{t/\varepsilon}$. For ε small, $x^a_{t/\varepsilon}$ is a "speeded-up"semi-Markov process. In other words, the investors' individual trading dispensations are evolving on a faster scale than Ψ. Observe, however, that we are not altering the main qualitative feature of the model. That is, agents still remain in the inactive state for relatively much longer times than in an active state.

Mathematically, there is no reason to restrict ourselves to the case where Ψ is non-negative. Hence we shall from now on only assume that Ψ is a continuous semimartingale. Given the processes Ψ and x^a ($a \in \{a_1, \ldots, a_N\}$), the aggregate order rate at time t is given by

$$Y^{\varepsilon,N}_t = \sum_{a \in \mathbb{A}} \Psi_t x^a_{t/\varepsilon}. \tag{14}$$

Let $\mu := \mathbb{E}^* x_t$. For a finite time interval $[0, T]$, let $X^{\varepsilon,N} = (X^{\varepsilon,N}_t)_{0 \le t \le T}$ be the centered process defined by

$$X^{\varepsilon,N}_t := \int_0^t \sum_{a \in \mathbb{A}} \Psi_s (x^a_{s/\varepsilon} - \mu)\, ds = \int_0^t Y^{\varepsilon,N}_s\, ds - \mu N \int_0^t \Psi_s ds. \tag{15}$$

We are now ready to state our main result. Its proof will be carried out in Sections 3 and 4.

Theorem 2.6 *Let $\Psi = (\Psi_t)_{t \ge 0}$ be a continuous semimartingale on $(\Omega, \mathcal{F}, \mathbb{P}^*)$. If assumptions 2.3 and 2.4 are satisfied, and if $\mu \sum_{k \in E} k \frac{m_k}{\eta_k^2} > 0$, then there exists $c > 0$ such that the process $X^{\varepsilon,N}$ satisfies*

$$\mathcal{L}\text{-}\lim_{\varepsilon \downarrow 0} \mathcal{L}\text{-}\lim_{N \to \infty} \left(\frac{1}{\varepsilon^{1-H} \sqrt{NL(\varepsilon^{-1})}} X^{\varepsilon,N}_t \right)_{0 \le t \le T} = \left(c \int_0^t \Psi_s dB^H_s \right)_{0 \le t \le T}. \tag{16}$$

Here the Hurst coefficient of the fractional Brownian motion process B^H is $H = \frac{3-\alpha}{2} > \frac{1}{2}$.

The definition of the stochastic integral with respect to fractional Brownian motion is given in Section 4.

Observe that Theorem 2.6 does not apply to the case $\mu = 0$. For centered semi-Markov processes x^a, Example 3.3 below illustrates that the limiting process depends on the tail structure of the waiting time distribution in the various active states. This phenomenon does not appear in the case of binary state spaces.

Remark 2.7 *(i) Theorem 2.6 says the drift-adjusted logarithmic price process in our model of inert investors can be approximated in law by the stochastic integral of Ψ with respect to a fractional Brownian motion process with Hurst coefficient $H > \frac{1}{2}$.*

(ii) In a situation where the processes x^a are independent, stationary and ergodic Markov processes on E, i.e., in cases where the semi-Markov kernel takes the form (5), it is easy to show that

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0}\mathcal{L}\text{-}\lim_{N\to\infty}\left(\frac{1}{\sqrt{\varepsilon N}}X_t^{\varepsilon,N}\right)_{0\le t\le T}=\left(c\int_0^t\Psi_s dW_s\right)_{0\le t\le T}$$

where $(W_t)_{t\ge 0}$ is a standard Wiener process. Thus, if the market participants are not inert, i.e., if the distribution of the lengths of the agents' inactivity periods is thin-tailed, no arbitrage opportunities emerge because the limit process is a semimartingale.

The proof of Theorem 2.6 will be carried out in two steps. In Section 3 we prove a functional central limit theorem for stationary semi-Markov processes on finite state spaces. In Section 4 we combine our central limit theorem for semi-Markov processes with extensions of arguments given in Kurtz and Protter (1991) to obtain (16).

2.3 Markets with both Active and Inert Investors

It is simple to extend the previous analysis to incorporate both active and inert investors. Let ρ be the ratio of active to inert investors. We associate to each *active* trader $b\in\{1,2,\dots,\rho N\}$ a stationary Markov chain $y^b=(y_t^b)_{t\ge 0}$ on the state space E. The processes y^b are independent and identically distributed and independent of the processes x^a. The thin-tailed sojourn time in the zero state of y^b reflects the idea that, as opposed to inert investors, these agents frequently trade the stock. We assume for simplicity that $\Psi\equiv 1$. With $\hat{Y}_t^{\varepsilon,N}=\sum_{b=1}^{\rho N}\left(y_{t/\varepsilon}^b-\mathbb{E}^* y_0\right)$ and $\hat{X}_t^{\varepsilon,N}:=\int_0^t\hat{Y}_s^{\varepsilon,N}ds$, it is straightforward to prove the following modification of Theorem 2.6.

Proposition 2.8 *Let x^a $(a=1,2,\dots,N)$ be semi-Markov processes that satisfy the assumption of Theorem 2.6. If y^b $(b=1,2,\dots,\rho N)$ are independent stationary Markov processes on E, then there exist constants $c_1,c_2>0$ such that*

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0}\mathcal{L}\text{-}\lim_{N\to\infty}\left(\frac{1}{\varepsilon^{1-H}\sqrt{NL(\varepsilon^{-1})}}X_t^{\varepsilon,N}+\frac{1}{\sqrt{N\varepsilon}}\hat{X}_t^{\varepsilon,N}\right)_{0\le t\le T}=\left(c_1B_t^H+c_2\sqrt{\rho}\,W_t\right)_{0\le t\le T}.$$

Here, $W=(W_t)_{t\ge 0}$ is a standard Wiener process.

Thus, in a financial market with both active and inert investors, the dynamics of the asset price process can be approximated in law by a stochastic integral with respect to a superposition, $B^H+\delta W$, of a fractional and a regular Brownian motion. It is known

(Cheridito (2001)) that $B^H + \delta W$ is a semimartingale for any $\delta \neq 0$, if $H > \frac{3}{4}$, i.e., if $\alpha < \frac{3}{2}$, but not if $H \in (\frac{1}{2}, \frac{3}{4}]$. Thus, no arbitrage opportunities arise if the small investors are "sufficiently inert." The parameter α can also be viewed as a measure for the fraction of small investors that are active at any point in time. Hence, independent of the actual trading volume, the market is arbitrage free in periods where the fraction of inert investors who are active on the financial market is small enough.

3 A limit theorem for stationary semi-Markov processes

This section establishes Theorem 2.6 for the special case $\Psi \equiv 1$. We approach the general case where Ψ is a continuous semimartingale in Section 4. Here we consider the situation where

$$Y_t^{\varepsilon,N} = \sum_{a \in \mathbb{A}} x_{t/\varepsilon}^a \quad \text{and where} \quad X_t^{\varepsilon,N} = \int_0^t Y_s^{\varepsilon,N}\, ds - N\mu t,$$

and prove a functional central limit theorem for stationary semi-Markov processes. Our Theorem 3.1 below extends the results of Taqqu *et al.* (1997) to situations where the semi-Markov process takes values in an arbitrary finite state space. The arguments given there are based on results from ordinary renewal theory, and do not carry over to models with more general state spaces. The proof of the following theorem will be carried out through a series of lemmas.

Theorem 3.1 *Let $H = \frac{3-\alpha}{2}$. Under the assumptions of Theorem 2.6,*

$$\mathcal{L}\text{-}\lim_{\varepsilon \downarrow 0} \mathcal{L}\text{-}\lim_{N \to \infty} \left(\frac{1}{\varepsilon^{1-H} \sqrt{NL(\varepsilon^{-1})}} X_t^{\varepsilon,N} \right)_{0 \leq t \leq T} = \left(cB_t^H \right)_{0 \leq t \leq T}. \tag{17}$$

Let γ be the covariance function of the semi-Markov process $(x_t)_{t \geq 0}$ under $\mathbb{P}^*$, and consider the case $\varepsilon = 1$. By the Central Limit Theorem, and because x is stationary, the process $Y = (Y_t)_{t \geq 0}$ defined by

$$Y_t = \mathcal{L}\text{-}\lim_{N \to \infty} \frac{1}{\sqrt{N}} (Y_t^{1,N} - N\mu)$$

is a stationary zero-mean Gaussian process. It is easily checked that the covariance function of the process $(\frac{1}{\sqrt{N}} Y_t^{1,N})$ is also γ for any N, and hence for Y_t. By standard calculations, the variance $\mathrm{Var}(t)$ of the aggregate process $(\int_0^t Y_s\, ds)$ at time $t \geq 0$ is given by

$$\mathrm{Var}(t) := \mathrm{Var}\left(\int_0^t Y_s\, ds \right) = 2 \int_0^t \left(\int_0^v \gamma(u) du \right) dv. \tag{18}$$

In the first step towards the proof of Theorem 3.1, we can proceed by analogy to Taqqu *et al.* (1997). We are interested in the asymptotics as $\varepsilon \downarrow 0$ of the process

$$X_t^\varepsilon := \int_0^t Y_{s/\varepsilon}\, ds, \tag{19}$$

which can be written $X_t^\varepsilon = \varepsilon \int_0^{t/\varepsilon} Y_s\, ds$. Therefore the object of interest is the large t behavior of $\mathrm{Var}(t)$. Suppose that we can show

$$\mathrm{Var}(t) \sim c^2 t^{2H} L(t) \quad \text{as} \quad t \to \infty. \tag{20}$$

Then the mean-zero Gaussian processes $X^\varepsilon = (X_t^\varepsilon)_{t\geq 0}$ have stationary increments and satisfy

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^* \left(\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}} X_t^\varepsilon \right)^2 = c^2 t^{2H}. \tag{21}$$

Since the variance characterizes the finite dimensional distributions of a mean-zero Gaussian process with stationary increments, we see that the finite dimensional distributions of the process $\left(\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}} X_t^\varepsilon \right)_{t\geq 0}$ converge to $\left(cB_t^H\right)_{t\geq 0}$ whenever (20) holds. The following lemma gives a sufficient condition for (20) in terms of the covariance function γ.

Lemma 3.2 *For (20) to hold, it suffices that*

$$\gamma(t) \sim c^2 H(2H-1) t^{2H-2} L(t) \quad \textit{as} \quad t \to \infty. \tag{22}$$

Proof: By Proposition 1.5.8 in Bingham *et al.* (1987), every slowly varying function L which is locally bounded on $\mathbb{R}_+$ satisfies

$$\int_0^t \tau^\beta L(\tau) d\tau \sim \frac{t^{\beta+1} L(t)}{\beta+1}$$

if $\beta > -1$. Applying this proposition to the slowly varying function

$$\tilde{L}(t) := \frac{\gamma(t)}{c^2 H(2H-1) t^{2H-2}},$$

we conclude

$$\int_0^t \int_0^v \gamma(u) du\, dv \sim \frac{c^2}{2} t^{2H} L(t),$$

and so our assertion follows from (18). □

Before we proceed with the proof of our main result, let us briefly consider the case $\mu = 0$. If $\mathbb{E}^* x_0 = 0$, then the structure of the limit process depends on the distribution of the sojourn times in the various active states.[1]

[1] We thank Chris Rogers for Example 3.3 (i).

Example 3.3 *We consider the case $E = \{-1, 0, 1\}$, and assume that $p_{-1,0} = p_{1,0} = 1$ and that $p_{0,-1} = p_{0,1} = \frac{1}{2}$. With $\nu_1 = \mathbb{P}^*\{x_t = 1\} > 0$, we obtain*

$$\gamma(t) = \nu_1 \left(\mathbb{E}^*[x_t x_0 | x_0 = 1] + \mathbb{E}^*[x_t x_0 | x_0 = -1] \right).$$

(i) Suppose that the sojourns in the inactive state be heavy tailed, and that the waiting times in the active states are exponentially distributed with parameter 1. In such a symmetric situation

$$\mathbb{E}^*[x_t x_0 | x_0 = \pm 1] = \mathbb{P}^*\{T_1 \geq t | x_0 = \pm 1\} = e^{-t}.$$

Therefore, $\gamma(t) = 2\nu_1 e^{-t}$. In view of (21), this yields $c > 0$ such that

$$\mathcal{L}\text{-}\lim_{\varepsilon \downarrow 0} \mathcal{L}\text{-}\lim_{N \to \infty} \left(\frac{1}{\sqrt{\varepsilon N}} X_t^{\varepsilon, N} \right)_{0 \leq t \leq T} = (cW_t)_{0 \leq t \leq T}$$

for some standard Wiener process W.

(ii) Assume now that the sojourn times in the active states have Pareto distributions:

$$\mathbb{P}\{T_{n+1} - T_n \geq t | \xi_n = \pm 1\} \sim L(t) t^{-\beta} \quad \text{for some } 1 < \alpha < \beta < 2.$$

In this case $\gamma(t) \sim 2\nu_1 L(t) t^{-\beta}$. Hence (21) yields $c > 0$ such that

$$\mathcal{L}\text{-}\lim_{\varepsilon \downarrow 0} \mathcal{L}\text{-}\lim_{N \to \infty} \left(\frac{1}{\varepsilon^{1-H} \sqrt{N L(\varepsilon^{-1})}} X_t^{\varepsilon, N} \right)_{0 \leq t \leq T} = \left(c B_t^H \right)_{0 \leq t \leq T}.$$

Here, B^H is a fractional Brownian motion process with $H = \frac{3-\beta}{2}$.

Our Theorem 3.1 is proven as soon as we can show that (22) holds. For this, the following representation of the covariance function turns out to be useful: in terms of the marginal distribution $\nu_i = \mathbb{P}^*\{x_t = i\}$ $(i \in E)$ of the stationary semi-Markov process given in Lemma 2.5 (i), and in terms of the conditional probabilities

$$P_t^*(i, j) := \mathbb{P}^*\{x_t = j | x_0 = i\},$$

we have

$$\gamma(t) = \sum_{i,j \in E} ij\nu_i \left(P_t^*(i, j) - \nu_j \right). \tag{23}$$

It follows from, e.g., Proposition 6.12 in Çinlar (1975) that $P_t^*(i, j) \to \nu_j$ as $t \to \infty$. Hence $\lim_{t \to \infty} \gamma(t) = 0$. In order to prove Theorem 3.1, however, we also need to show that this convergence is sufficiently slow. We shall see that the agents' inertia accounts for the slow

decay of correlations. It is thus the agents' inactivity that is responsible for that fact that the logarithmic price process is not approximated by a stochastic integral with respect to a Wiener process, but by an integral with respect to fractional Brownian motion.

We are now going to determine the rate of convergence of the covariance function to 0. To this end, we show that $P_t^*(i,j)$ can be written as a convolution of a renewal function with a slowly decaying function plus a term which has asymptotically, i.e., for $t \to \infty$, a vanishing effect compared to the first term; see Lemma 3.5 below. We will then apply results by Heath *et al.* (1996) and Jelenkovic and Lazar (1998) to analyze the tail structure of the convolution term.

Let

$$R(i,j,t) := \mathbb{E}\left\{ \sum_{n=0}^{\infty} \mathbf{1}_{\{\xi_n = j, T_n \le t\}} \mid |x_0 = \xi_0 = i \right\},$$

the expected number of visits of the process $(x_t)_{t\ge 0}$ to state j up to time t in the *non-stationary* situation, i.e., under the measure $\mathbb{P}$, given $x_0 = i$. For fixed $i, j \in E$, the function $t \mapsto R(i,j,t)$ is a renewal function. If, under $\mathbb{P}$, the initial state is j, then the entrances to j form an ordinary renewal process and

$$R(j,j,t) = \sum_{n=0}^{\infty} F^n(j,j,t). \tag{24}$$

Here $F(j,j,\cdot)$ denotes the distribution of the travel time between to occurrences of state $j \in E$ as defined in (7), and $F^n(j,j,\cdot)$ is the n-fold convolution of $F(j,j,\cdot)$. On the other hand, if $i \neq j$, the time until the first visit to j has distribution $F(i,j,\cdot)$ under $\mathbb{P}$ which might be different from $F(j,j,\cdot)$. In this case $R(i,j,\cdot)$ satisfies a delayed renewal equation, and we have

$$R(i,j,t) = \int_0^t R(j,j,t-u) F(i,j,du). \tag{25}$$

We refer the interested reader to Çinlar (1975) for a survey on Markov renewal theory.

Let us now return to the stationary setting and derive a representation for the expected number $R^*(i,j,t)$ of visits of the process $(x_t)_{t\ge 0}$ to state j up to time t under $\mathbb{P}^*$, given $x_0 = i$. To this end, we denote by $F^*(i,j,\cdot)$ the distribution function in the stationary setting of the first occurrence of j, given $x_0 = i$ and put

$$P_t(i,j) := \mathbb{P}\{x_t = j | x_0 = i\}.$$

Given the first jump time T_1 and given that $x_{T_1} = i$ we have that

$$\mathbb{P}^*\{x_t = j | x_{T_1} = i\} = P_{t-T_1}(i,j) \quad \text{on} \quad \{t \ge T_1\}. \tag{26}$$

Thus,

$$R^*(i,j,t) = \int_0^t R(j,j,t-u)F^*(i,j,du). \tag{27}$$

3.1 A representation for the conditional transition probabilities

In this section we derive a representation for $P_t^*(i,j)$ which will allow us to analyze the asymptotic behavior of $P_t^*(i,j) - \nu_j$. To this end, we recall the definition of the joint distribution of the initial state and the initial sojourn time and the definition of the conditional joint distribution of (ξ_1, T_1), given ξ_0 from (10) and (13) respectively. We define

$$s(i,t) := \mathbb{P}^*\{\xi_0 = i, T_1 > t\} \quad \text{and} \quad \hat{s}(i,j,t) = \mathbb{P}^*\{\xi_1 = j, T_1 \leq t | \xi_0 = i\}.$$

In terms of these quantities, the transition probability $P_t^*(i,j)$ can be written as

$$P_t^*(i,j) = \frac{s(i,t)}{\nu_i}\delta_{ij} + \sum_{k\in E}\int_0^t P_{t-u}(k,j)\hat{s}(i,k,du). \tag{28}$$

Here the first term on the right-hand-side of (28) accounts for the $\mathbb{P}^*$-probability that $x_0 = i$ and that the state i survives until time t. The quantity $\int_0^t P_{t-u}(k,j)\hat{s}(i,k,du)$ captures the conditional probability that the first transition happens to be to state k before time t, given $\xi_0 = i$. Observe that we integrate the conditional probability $P_{t-u}(i,j)$ and not $P^*_{t-u}(i,j)$: conditioned on the value of semi-Markov process at the first renewal instance the distributions of $(x_t)_{t\geq 0}$ under $\mathbb{P}$ and $\mathbb{P}^*$ are the same; see (26).

In the sequel it will be convenient to have the following convolution operation: let $\tilde{h}$ be a locally bounded function, and $\tilde{F}$ be a distribution function both of which are defined on $\mathbb{R}_+$. The convolution $\tilde{F} * \tilde{h}$ of $\tilde{F}$ and $\tilde{h}$ is given by

$$\tilde{F} * \tilde{h}(t) := \int_0^t \tilde{h}(t-x)\tilde{F}(dx) \quad \text{for } t \geq 0. \tag{29}$$

Remark 3.4 *Since $\tilde{F} * \tilde{h}$ is locally bounded, the map $t \mapsto G * (\tilde{F} * \tilde{h})(t)$ is well defined for any distribution G on $\mathbb{R}_+$. Moreover, $G * (\tilde{F} * \tilde{h})(t) = \tilde{F} * (G * \tilde{h})(t) = (G * \tilde{F}) * \tilde{h}(t)$. In this sense distributions acting on the locally bounded function can commute. Thus, for the renewal function $R = \sum_{n=0}^{\infty} \tilde{F}^n$ associated to $\tilde{F}$, as defined in (24), the integral $R * \tilde{h}(t)$ is well defined and $R * (G * \tilde{h})(t) = G * (R * \tilde{h})(t) = (R * G) * \tilde{h}(t)$.*

We are now going to establish an alternative representation for the conditional probability $P_t^*(i,j)$ that turns out to be more appropriate for our subsequent analysis.

Lemma 3.5 *In terms of the quantities $s(i,t)$ and $h(i,t)$ in (11) and $R^*(i,j,t)$, we have*

$$P_t^*(i,j) = \frac{s(i,t)}{\nu_i}\delta_{ij} + \int_0^t h(j,t-s)R^*(i,j,ds). \tag{30}$$

PROOF: In view of (28), it is enough to show

$$R^*(i,j,t) * h(j,t) = \sum_{k\in E}\int_0^t P_{t-u}(k,j)\hat{s}(i,k,du).$$

To this end, observe first that $F^*(i,j,t)$ can be decomposed as

$$F^*(i,j,t) = \hat{s}(i,j,t) + \sum_{k\neq j}\int_0^t F(k,j,t-u)\hat{s}(i,k,du). \tag{31}$$

Indeed, $\hat{s}(i,j,t)$ is the probability that the first transition takes place before time t and happens to be to state $j \in E$, and

$$\int_0^t F(k,j,t-u)\hat{s}(i,k,du) = \mathbb{P}^*\{x_v = j \text{ for some } v \leq t,\ x_{T_1} = k | x_0 = i\}.$$

In view of (27) and (31), Remark 3.4 yields

$$\begin{aligned} R^*(i,j,t) * h(j,t) &= R(j,j,t) * F^*(i,j,t) * h(j,t) \\ &= R(j,j,t) * \hat{s}(i,j,t) * h(j,t) + \sum_{k\neq j} F(k,j,t) * \hat{s}(i,k,t) * R(j,j,t) * h(j,t). \end{aligned}$$

Now recall from, e.g., Proposition 6.3 in Çinlar (1975) that

$$P_t(i,j) = \int_0^t h(j,t-s)R(i,j,ds).$$

Thus, we obtain

$$\begin{aligned} R^*(i,j,t) * h(j,t) &= \hat{s}(i,j,t) * P_t(j,j) + \sum_{k\neq j} R(j,j,t) * F(k,j,t) * \hat{s}(i,k,t) * h(j,t) \\ &= \hat{s}(i,j,t) * P_t(j,j) + \sum_{k\neq j} R(k,j,t) * \hat{s}(i,k,t) * h(j,t) \\ &= \sum_k \hat{s}(i,k,t) * P_t(k,j). \end{aligned}$$

This proves our assertion. □

3.2 The rate of convergence to equilibrium

Now, our goal is to derive the rates of convergence of the mappings $t \mapsto s(i,t)$ to 0 and $t \mapsto R^*(i,j,t) * h(j,t)$ to ν_j, respectively. Due to (23) it is enough to analyze the case $i, j \neq 0$. In a first step we are going to show that the function $s(i,\cdot)$ converges to 0 sufficiently fast.

Lemma 3.6 *For all $i \neq 0$ we have*

$$\lim_{t\to\infty} \frac{s(i,t)}{t^{-\alpha+1}L(t)} = 0.$$

PROOF: By Assumption 2.4 (iii) there exists $1 < \alpha < \beta$ such that

$$\lim_{t\to\infty} \frac{h(i,t)}{t^{-\beta}L(t)} = 0 \quad \text{for all } i \neq 0.$$

In particular,

$$s(i,t) = \int_t^\infty h(i,s)ds \leq \int_t^\infty s^{-\beta}L(s)ds \tag{32}$$

for all sufficiently large $t > 0$. Now, we apply Proposition 1.5.10 in Bingham, Goldie, and Teugels (1987): if g is a function on $\mathbb{R}_+$ that satisfies $g(t) \sim t^{-\beta}L(t)$ for $\beta > 1$, then

$$\int_t^\infty g(s)ds \sim \frac{t^{-\beta+1}}{\beta - 1}L(t).$$

Thus, it follows from (10) and (32) that

$$\limsup_{t\to\infty} \frac{s(i,t)}{t^{-\alpha+1}L(t)} \leq \frac{\pi_i}{\sum_j \pi_j m_j} \limsup_{t\to\infty} \frac{\int_t^\infty s^{-\beta}L(s)ds}{t^{-\alpha+1}L(t)} = \limsup_{t\to\infty} t^{-\beta+\alpha} = 0.$$

This proves our assertion. □

In order to derive the rate of convergence of the function $t \mapsto R^*(i,j,t) * h(i,t)$, we shall first study the asymptotic behavior of the map $t \mapsto R^*(i,j,t)$. Since $R(i,j,\cdot)$ is a renewal function, we see from (27) and (30) that the asymptotics of $R^*(i,j,\cdot)$ can be derived by Lemma A.1 if we can show that

$$F^*(i,j,t) * h(j,t) = o(\bar{F}(i,j,t)) \quad \text{as} \quad t \to \infty. \tag{33}$$

3.2.1 The tail structure of the travel times

Let us first deal with the issue of finding the convergence rate of $\bar{F}(j,j,t) = 1 - F(j,j,t)$ to 0. To this end, we introduce the family of random variables

$$\Theta = \{(\theta_{i,j}^\ell), i, j \in E, \ell = 1, 2, ...\}, \tag{34}$$

such that any two random variable in Θ are independent, and for fixed pair (i,j) the random variables $\theta_{i,j}^k$ have $G(i,j,\cdot)$ as their common distribution function. To ease the notational complexity we assume that the law of $T_{n+1}-T_n$ only depends on ξ_n. We shall therefore drop the second sub-index from the elements of Θ. The random variables $(\theta_i^\ell)_{\ell\in\mathbb{N}}$ are independent copies of the sojourn time in state i. We shall prove Lemma 3.8 below under this additional assumption. The general case where $T_{n+1}-T_n$ depends both on ξ_n and ξ_{n+1} can be analyzed by similar means.

Let $N_k^{i,j}$ denote the number of times the embedded Markov chain $\{\xi_n\}_{n\in\mathbb{N}}$ visits state $k\in E$ before it visits state j, given $\xi_0=i$. By definition, $N_j^{i,j}=0$ with probability one. We denote by $\boldsymbol{N}^{i,j}$ the vector of length $|E|$ with entries $N_k^{i,j}$, and by $\boldsymbol{n}=(n_k)_{k\in E}$ an element of $\mathbb{N}^{|E|}$. Then we have

$$\bar{F}(i,j,t)=\mathbb{P}\left\{\theta_i^1+\sum_{k\neq j}\sum_{\ell=1}^{N_k^{i,j}}\theta_k^\ell\geq t\right\}. \tag{35}$$

With $G(k,t):=\mathbb{P}\{T_{n+1}-T_n\leq t\big|\xi_n=k\}$, we can rewrite (35) as

$$\begin{aligned}\bar{F}(i,j,t) &= \sum_{\boldsymbol{n}}\mathbb{P}\left\{\theta_i^1+\sum_{k\neq j}\sum_{\ell=1}^{n_k}\theta_k^\ell\geq t\Big|\boldsymbol{N}^{i,j}=\boldsymbol{n},\right\}\mathbb{P}\{\boldsymbol{N}^{i,j}=\boldsymbol{n}\}\\ &= 1-G(i,t)*\sum_{\boldsymbol{n}}\underset{k\neq j}{*}G^{n_k}(k,t)\mathbb{P}\{\boldsymbol{N}^{i,j}=\boldsymbol{n}\}.\end{aligned}$$

Now, our goal is to show that

$$\lim_{t\to\infty}\frac{\bar{F}(i,j,t)}{t^{-\alpha}L(t)}=\sum_{n\geq 0}n\mathbb{P}\{N_0^{i,j}=n\}+\delta_{i,0} \tag{36}$$

for $i,j\neq 0$. Here, $\sum_{n\geq 0}n\mathbb{P}\{N_0^{i,j}=n\}$ is the expected number of occurrences of state 0 under $\mathbb{P}$ before the first visit to state j, given $\xi_0=x_0=i$. This quantity is positive, due to Assumption 2.3. In order to prove (36), we need the following results which appear as Lemma 10 in Jelenkovic and Lazar (1998).

Lemma 3.7 *Let $F_1,...,F_m$ be probability distribution functions such that, for all $j\neq i$, we have $\bar{F}_j(t)=o(\bar{F}_i(t))$ as $t\to\infty$. Then for any positive integers $n_1,\ldots,n_m$,*

$$1-F_1^{n_1}*...*F_m^{n_m}(t)\sim n_i\bar{F}_i(t).$$

Moreover, for each $u>0$, there exists some $K_u<\infty$ such that

$$\frac{1-F_1^{n_1}*...*F_m^{n_m}(t)}{1-F_i^{n_i}(t)}\leq K_u(1+u)^{n_i}$$

for all $t\geq 0$.

We are now ready to prove (36).

Lemma 3.8 *Under the assumptions of Theorem 3.1 we have, for $j \neq 0$,*

$$\lim_{t\to\infty} \frac{\bar{F}(i,j,t)}{t^{-\alpha}L(t)} = \sum_{n\geq 0} n\mathbb{P}\{N_0^{i,j} = n\} + \delta_{i,0} > 0.$$

PROOF: Let us first prove that the expected number of occurrences of state 0 before the first return to state j occurs is finite. To this end, we put $p = \min\{p_{ij} : i, j \in E, i \neq j\} > 0$. Since

$$\mathbb{P}\{N_0^{i,j} = n\} \leq \mathbb{P}\{\xi_m \neq j \text{ for all } m \leq n\} \leq (1-p)^n,$$

we obtain

$$\sum_{n\geq 0} n\mathbb{P}\{N_0^{i,j} = n\} \leq \sum_{n\geq 0} n(1-p)^n < \infty. \tag{37}$$

Now, we define a probability measure $\bar{\mu}$ on $\mathbb{N}^{|E|}$ by

$$\bar{\mu}\{\boldsymbol{n}\} = \mathbb{P}\{\boldsymbol{N}^{i,j} = \boldsymbol{n}\}$$

and put

$$A_{\boldsymbol{n}}(t) = G(i,t) \underset{k\neq j}{*} G^{n_k}(k,t).$$

Since $\frac{1-G(0,t)}{t^{-\alpha}L(t)} \to 1$ as $t \to \infty$, the first part of Lemma 3.7 yields

$$\lim_{t\to\infty} \frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} = \lim_{t\to\infty} \left(\frac{1 - G(i,t) \underset{k\neq j}{*} G^{n_k}(k,t)}{1 - G(0,t)} \right) \left(\frac{1-G(0,t)}{t^{-\alpha}L(t)} \right) = n_0 + \delta_{i,0}.$$

From the definition of the measure $\bar{\mu}$, we obtain

$$\frac{1 - \sum_{\boldsymbol{n}} A_{\boldsymbol{n}}(t)\mathbb{P}\{\boldsymbol{N}^{i,j} = \boldsymbol{n}\}}{t^{-\alpha}L(t)} = \mathbb{E}_{\bar{\mu}} \left\{ \frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \right\},$$

and so our assertion follows from the dominated convergence theorem if we can show that

$$\sup_t \frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \in L^1(\bar{\mu}). \tag{38}$$

To verify (38), we will use the second part of Lemma 3.7. For each $u > 0$ there exists a constant K_u such that

$$\frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} = \left(\frac{1 - A_{\boldsymbol{n}}(t)}{1 - G(0,t)} \right) \left(\frac{1-G(0,t)}{t^{-\alpha}L(t)} \right) \leq K_u(1+u)^{n_0+\delta_{i,0}} \sup_t \frac{1-G(0,t)}{t^{-\alpha}L(t)}.$$

Since

$$\lim_{t\to\infty} \frac{1-G(0,t)}{t^{-\alpha}L(t)} = 1,$$

and because we are only interested in the asymptotic behavior of the function $t \mapsto \frac{\bar{F}(i,j,t)}{t^{-\alpha}L(t)}$, we may with no loss of generality assume that

$$\sup_t \frac{1 - G(0,t)}{t^{-\alpha}L(t)} = 1.$$

This yields

$$\sup_t \frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \leq K_u(1+u)^{n_0+\delta_{i,0}}. \tag{39}$$

From (37) and (39) we get

$$\mathbb{E}_{\bar{\mu}} \left\{ \sup_t \frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \right\} \leq K_u(1+u)^{\delta_{i,0}} \sum_{k=0}^{\infty} (1-p)^k (1+u)^k.$$

Choosing $u < \frac{p}{1-p}$ we obtain $\beta := (1-p)(1+u) < 1$ and so the assertion follows from

$$\mathbb{E}_{\bar{\mu}} \left\{ \sup_t \frac{1 - A_{\boldsymbol{n}}(t)}{t^{-\alpha}L(t)} \right\} \leq K_u(1+u)^{\delta_{i,0}} \sum_{n \geq 0} \beta^n < \infty.$$

□

So far, we have shown that $\bar{F}(i,j,t) \sim t^{-\alpha}L(t) \left(\sum_{n \geq 0} n\mathbb{P}\{N_0^{i,j} = n\} + \delta_{i,0} \right)$ for $j \neq 0$. In view of Lemmas 3.7 and 3.8, the representation (31) of $F^*(i,j,t)$ yields a similar result for the stationary setting.

Corollary 3.9 *For all $i, j \neq 0$ we have*

$$\lim_{t \to \infty} \frac{\bar{F}^*(i,j,t)}{t^{-\alpha}L(t)} = \sum_{n \geq 0} n\mathbb{P}^*\{N_0^{i,j} = n\} < \infty.$$

3.2.2 The tail structure of $R^* * h$

So far, we have analyzed the tail structure of the distribution of the travel time between states i and j $(i, j \neq 0)$ in the stationary regime. In this section we are now going to study the tail structure of $R^*(i,j,t) * h(j,t)$. Having recalled to notion of virtually non-increasing functions from the appendix, let us now prove the following result which turns out to be central to the proof of our main theorem.

Lemma 3.10 *Let A, B be distribution functions on $\mathbb{R}_+$. Assume that A and B have bounded densities a and b, respectively, and $\bar{B}(t) = o(\bar{A}(t))$ as $t \to \infty$.*

(i) We have $\lim_{t\to\infty} \frac{A*\bar{B}(t)}{1-A(t)} = 0$. In particular, $\lim_{t\to\infty} A * \bar{B}(t) = 0$.

(ii) The map $z(t) := A * \bar{B}(t)$ allows the representation

$$A * \bar{B}(t) = -\bar{A}(t) + \int_t^\infty \int_0^s b(s-u)a(u)\,du\,ds, \tag{40}$$

and

$$\lim_{t\to\infty} \frac{\int_t^\infty \int_0^s b(s-u)a(u)\,du\,ds}{1-A(t)} = 1. \tag{41}$$

(iii) The map $z(t)$ is of bounded variation and virtually non-increasing in the sense of (51).

PROOF: Since $\bar{B}(t) = o(\bar{A}(t))$ as $t \to \infty$, the first assertion follows from Lemma 3.7 because

$$\bar{A}(t) + A * \bar{B}(t) = 1 - A * (1 - \bar{B})(t) = 1 - A * B(t) \sim \bar{A}(t).$$

The second assertion follows from (i) along with an application of Leibniz' rule:

$$\frac{\partial}{\partial t} A * \bar{B}(t) = a(t) - \int_0^t b(t-s)a(s)ds.$$

By (41), there exists T such that

$$\int_t^\infty \int_0^s b(s-u)a(u)\,du\,ds \le 2\bar{A}(t) \quad \text{for all } t \ge T.$$

Thus, z is of bounded variation. In order to prove that z is also virtually non-increasing, we introduce a continuous and *decreasing* probability density function $\tilde{a} : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies

$$\tilde{a}(0) > 0, \quad \text{and} \quad a(t) \sim \tilde{a}(t).$$

We put $\tilde{A}(t) := \int_0^t \tilde{a}(s)ds$, and introduce the differentiable function $\tilde{z}(t) := \tilde{A} * \bar{B}(t)$. Since $\tilde{a}$ is decreasing, Lemma 4.1 in Heath *et al.* (1996) yields

$$\tilde{z}(t) \le \tilde{z}^*(t) \le \tilde{z}(t) + \delta(t) \quad \text{where} \quad \delta(t) = 2\int_t^\infty \bar{B}(s)\tilde{a}(s)ds$$

and $\tilde{z}^*$ is defined as in (50). In a first step we are now going to show that $\tilde{z}$ is virtually non-increasing. Then we apply a comparison argument in order to prove that not only $\tilde{z}$, but also z is virtually non-increasing.

a) Let us rewrite $\tilde{z}$ as $\tilde{z}(t) = \int_0^t \bar{B}(s)\tilde{a}(t-s)\,ds$. Hence, for every $\epsilon > 0$, we obtain

$$\tilde{z}(t) \;=\; \int_0^{\epsilon t} \bar{B}(s)\tilde{a}(t-s)ds + \int_{\epsilon t}^t \bar{B}(s)\tilde{a}(t-s)ds := \tilde{z}_1(t) + \tilde{z}_2(t).$$

As shown by Heath *et al.* (1996), there exist a constant $c_1 < \infty$ such that

$$\tilde{z}_1(t) \sim \tilde{a}(t) \quad \text{and such that} \quad \tilde{z}_2(t) \le c_1\bar{B}(t) \quad \text{as } t \to \infty. \tag{42}$$

On the other hand, since the map $t \mapsto \bar{B}(t)$ is decreasing, since $\tilde{a}(0) > 0$ and because $\tilde{a}$ is continuous, there is $c_2 > 0$ such that

$$\tilde{z}_2(t) = \int_{\epsilon t}^{t} \bar{B}(s)\tilde{a}(t-s)ds \geq \bar{B}(t) \int_{\epsilon t}^{t} \tilde{a}(t-s)ds \geq c_2 \bar{B}(t). \tag{43}$$

¿From (42) along with (43), it follows that the function $\tilde{z}_2$ converges to 0 at the same rate as the map $\bar{B}$. Hence,

$$\lim_{t\to\infty} \frac{\delta(t)}{\tilde{z}(t)} = \lim_{t\to\infty} \frac{\delta(t)}{\tilde{z}_1(t) + \tilde{z}_2(t)} \leq \lim_{t\to\infty} \frac{2\int_t^\infty \bar{B}(s)\tilde{a}(s)ds}{\tilde{a}(t) + c_2\bar{B}(t)} = 0.$$

Therefore, the map $\tilde{z}$ is virtually non-increasing.

b) We are now ready to show that the map z itself is virtually non-increasing. To this end, it is enough to show that

$$\lim_{t\to\infty} \left| \int_t^\infty (\tilde{z}'(s) - z'(s))\, ds \right| = 0 \quad \text{and} \quad \lim_{t\to\infty} \left| \int_t^\infty (|\tilde{z}'(s)| - |z'(s)|)\, ds \right| = 0. \tag{44}$$

Since

$$\int_t^\infty \tilde{z}'(s)ds = -1 + \tilde{A}(t) + \int_t^\infty \int_0^s b(s-u)\tilde{a}(u)\, du\, ds$$

and because

$$\int_t^\infty z'(s)ds = -1 + A(t) + \int_t^\infty \int_0^s b(s-u)a(u)\, du\, ds,$$

part (ii) yields the first equation in (44). In order to prove second relation, it suffices to show that

$$\lim_{t\to\infty} \int_t^\infty \int_0^s b(s-u)|a(u) - \tilde{a}(u)|\, du\, ds = 0.$$

Since $a(t) \sim \tilde{a}(t)$, this can be achieved by defining a density function $g \geq |a - \tilde{a}|$ whose tail structure takes the form $g(t) \sim 2a(t)$. Then (ii) allows to conclude.

□

Let us now fix $i, j \in E$ with $j \neq 0$. In order to prove the main result of this section, we need to apply the key renewal theorem to the function

$$z(t) = F^*(i,j,t) * h(j,t). \tag{45}$$

Corollary 3.11 *Under the assumptions of Theorem 3.1 the function z defined by (45) is of bounded variation, $z(t) = o(\bar{F}(i,j,t))$ as $t \to \infty$, and z is virtually non-increasing.*

PROOF: Since $i, j \neq 0$ and because $p_{i,0} > 0$ the probability that the semi-Markov process x visits the state 0 before it reaches the state j is positive. Thus, it follows from Assumption 2.4 (iii) and from Corollary 3.9 that

$$h(j,t) = o(\bar{F}^*(i,j,t)) \quad \text{as} \quad t \to \infty.$$

Hence Lemma 3.10 (i) and (iii) shows that $F^*(i,j,t) * h(j,t) = o(\bar{F}(i,j,t))$ as $t \to \infty$ and that z is of bounded variation, respectively. It follows from Assumption 2.4 (iv) and the representation (35) that the distribution function $F^*(i,j,\cdot)$ has a density with respect to Lebesgue measure. Therefore, z is virtually non-increasing by Lemma 3.10 (iv). □

We are now going to specify the asymptotic behavior of the function $R^*(i,j,t) * h(j,t)$.

Lemma 3.12 *There exists $C_j > 0$ such that, for $j \neq 0$,*

$$\lim_{t\to\infty} \frac{R^*(i,j,t) * h(j,t) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{C_j}{\alpha - 1}$$

for all $i \in E$, $i \neq 0$, where the map $h(j,\cdot)$ has been defined in (11).

PROOF: Let us fix $i, j \in E$, $i, j \neq 0$. Using the representation (27) for the function $R^*(i,j,\cdot)$ it is enough to show that

$$\lim_{t\to\infty} \frac{R(j,j,t) * F^*(i,j,t) * h(j,t) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{C_j}{\alpha - 1}.$$

By Lemma 3.11, $F^*(i,j,t) * h(j,t) = o(\bar{F}(i,j,t))$ as $t \to \infty$. Observe that $t \mapsto F^*(i,j,t) * h(j,t)$ is a bounded continuous non-negative function of bounded variation. It follows from Fubini's theorem that

$$\begin{aligned}
\int_0^\infty F^*(i,j,t) * h(j,t)dt &= \int_0^\infty \int_0^\infty h(j,t-s)\mathbf{1}_{\{s\leq t\}}F^*(i,j,ds)dt \\
&= \int_0^\infty \int_s^\infty h(j,t-s)dtF^*(i,j,ds) \\
&= \int_0^\infty h(j,t)dt \\
&= m_j,
\end{aligned}$$

where m_j is the mean sojourn time at state $j \in E$ as defined in (8).

Since $R(j,j,t) = \sum_{n\geq 0} F^n(j,j,t)$ is a renewal function, because $F(j,j,\cdot)$ is nonsingular and because $\bar{F}(j,j,t) \sim t^{-\alpha}L(t)$ for $j \neq 0$ it follows from Corollary 3.11 and theorem A.1 that

$$\frac{m_j}{\eta_j} - \int_0^t F^*(i,j,t-s) * h(j,t-s)R(j,j,ds) \sim -\frac{m_j}{(\alpha-1)\eta_j^2}t^{-\alpha+1}L(t).$$

By Propositions 5.5 and 6.12 in Çinlar (1975), $\nu_j = \frac{m_j}{\eta_j}$. Therefore,

$$\lim_{t\to\infty} \frac{R^*(i,j,t) * h(j,t) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{m_j}{(\alpha-1)\eta_j^2}.$$

This proves our assertion with $C_j := \frac{m_j}{\eta_j^2}$. □

3.2.3 Proof of the central limit theorem for semi-Markov processes

We are now ready to prove the main result of this section.

Proof of Theorem 3.1: By (30) we have the representation

$$P_t^*(i,j) = \frac{s(i,t)}{\nu_i}\delta_{ij} + \int_0^t h(j,t-s)R^*(i,j,ds).$$

for the conditional probability that $x_t = j$, given $x_0 = i$. Due to Lemmas 3.6 and 3.12,

$$\lim_{t\to\infty} \frac{P_t^*(i,j) - \nu_j}{t^{-\alpha+1}L(t)} = \frac{C_j}{\alpha-1} \qquad (i,j \neq 0).$$

With $H = \frac{3-\alpha}{2}$ it follows from (23) that

$$\lim_{t\to\infty} \frac{\gamma(t)}{t^{2H-2}L(t)} = \frac{1}{(2-2H)} \sum_{i,j\in E} ij\nu_i C_j.$$

By Lemma 3.2 this proves the existence of a constant c such that the finite dimensional distributions of the processes $\left(\frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}} X_t^\varepsilon\right)_{0\leq t<\infty}$ converge weakly to the finite dimensional distributions of the fractional Brownian motion process cB^H as $\varepsilon \downarrow 0$, and c is given by

$$c^2 = \frac{1}{2H(1-H)(2H-1)} \sum_{i,j\in E} ij\nu_i \frac{m_j}{\eta_j^2} = \frac{1}{2H(1-H)(2H-1)} \mu \sum_{j\in E} j\frac{m_j}{\eta_j^2}.$$

It remains to prove tightness. The arguments given in Taqqu *et al.* (1997) for the case $|E| = 2$ carry over to our case and we repeat them here for the sake of completeness. Let us denote by $\{X^{\varepsilon_n}\}_{n\in\mathbb{N}}$, a subsequence of continuous stochastic processes. Then by Theorem 12.3 in Billingsley (1968) it is sufficient to show the following moment condition in order to conclude tightness of the sequence $\{X^{\varepsilon_n}\}_{n\in\mathbb{N}}$:

$$\mathbb{E}[X_{t_1}^\varepsilon - X_{t_2}^\varepsilon]^\gamma \leq C|t_2 - t_1|^\delta \tag{46}$$

for some $\gamma \geq 0$ and $\delta > 1$. Note that X^ε is a process with stationary increments whose variance is given by $\frac{\mathrm{Var}(t/\varepsilon)}{\varepsilon^{-2H}L(\varepsilon^{-1})}$ where $\mathrm{Var}(t)$ is defined by (18). Thus, (19) yields

$$\mathbb{E}[X_{t_2}^\varepsilon - X_{t_1}^\varepsilon]^2 = L(\varepsilon^{-1})^{-1}\varepsilon^{2H}\mathrm{Var}\left(\frac{t_2-t_1}{\varepsilon}\right).$$

Since $\mathrm{Var}(t)$ satisfies (20), we see that for any $u > 0$, there exists T_u such that for all $T > T_u$

$$\mathbb{E}[X^\varepsilon_{t_2} - X^\varepsilon_{t_1}]^2 \leq (1+u)c^2(t_2 - t_1)^{2H}\frac{L(\varepsilon^{-1}(t_2-t_1))}{L(\varepsilon^{-1})}.$$

Since $\frac{L(\varepsilon^{-1}(t_2-t_1))}{L(\varepsilon^{-1})}$ tends to 1 and is bounded by kn^{-u_0} for any $u_0 > 0$ and some $k > 0$, choosing u_0 such that $\delta = 2H - u_0 > 1$ we get (46) with $C = (1+u)kc^2$. □.

4 A limit theorem for stationary semi-Markov processes in a random environment

In this section we prove an approximation result for stochastic integrals which contains Theorem 2.6 as a special case. More precisely we give conditions which guarantee that for a sequence of processes $\{(\Psi^n, Z^n)\}_{n\in\mathbb{N}}$ the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}(\Psi^n, Z^n) = (\Psi, B^H)$ implies the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi, B^H, \int \Psi dB^H\right)$.

All stochastic integrals in this section are understood as probabilistic limits of Stieltjes-type sums. That is, given stochastic processes ϕ and Z, such that ϕ is adapted to the filtration generated by Z, we say that the integral $\int \phi dZ$ exists if for any $T < \infty$ and for each sequence of partitions $\{\tau^l\}_{l\in\mathbb{N}}$, $\tau^l = (\tau^l_1, \tau^l_2, \ldots, \tau^l_{N_l})$, of the interval $[0,T]$ that satisfies $\lim_{l\to\infty}\max_i|\tau^l_{i+1} - \tau^l_i| = 0$,

$$\int_0^T \phi_s dZ_s = \mathbb{P}\text{-}\lim_{l\to\infty}\sum_i \phi_{\tau^l_i}(Z_{\tau^l_{i+1}} - Z_{\tau^l_i}). \tag{47}$$

This definition of stochastic integrals applies to the usual semimartingale setting where ϕ is a process in $\mathbb{D}$ and where Z is a semimartingale. If $Z = B^H$ is a fractional Brownian motion process with Hurst coefficient $H > \frac{1}{2}$ the limit in (47) exists for a large classes of integrands, including continuous semimartingales and C^1-functions of fractional Brownian motion. In particular, the stochastic integral $\int B^H dB^H$ exists in the sense of (47), and for the continuous semimartingale Ψ we have the following integration by parts formula, due to Lin (1995):

$$-\int B^H d\Psi + \Psi B^H = \int \Psi dB^H. \tag{48}$$

Before we state the main result of this section, we recall that a sequence $\{\Psi^n\}_{n\in\mathbb{N}}$ of semimartingales defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ is called *"good"* in the sense of Duffie and Protter (1992) if, for any sequence $\{Z^n\}_{n\in\mathbb{N}}$ of càdlàg adapted processes, the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}(\Psi^n, Z^n) = (\Psi, Z)$ implies the convergence

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int Z^n_- d\Psi^n\right) = \left(\Psi, Z, \int Z_- d\Psi\right).$$

In the following we will denote by $\{\Psi^n\}_{n\in\mathbb{N}}$ a sequence of "good" semimartingales and by $\{Z^n\}_{n\in\mathbb{N}}$ a sequence of $\mathbb{D}$-valued stochastic processes defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that the following conditions are satisfied.

Assumption 4.1 *(i) The sample paths of the processes Z^n are almost surely of zero quadratic variation on compact sets, and $\mathbb{P}\{Z_0^n = 0\} = 1$.*

(ii) The stochastic integrals $\int \Psi^n dZ^n$ and $\int Z^n dZ^n$ exist in the sense of (47), and the sample paths $t \mapsto \int_0^t Z_s^n dZ_s^n$ and $t \mapsto \int_0^t \Psi_s^n dZ_s^n$ are càdlàg.

We are now ready to state the main theorem of this section. Its proof requires some preparation which will be carried out below.

Theorem 4.2 *Let $\{\Psi^n\}_{n\in\mathbb{N}}$ be a sequence of good semimartingales and let $\{Z^n\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{D}$-valued stochastic processes that satisfy Assumption 4.1. If Ψ is a continuous semimartingale and if B^H is a fractional Brownian motion process with Hurst parameter $H > \frac{1}{2}$, then the convergence $\mathcal{L}\text{-}\lim_{n\to\infty}(\Psi^n, Z^n) = (\Psi, B^H)$ implies the convergence*

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi, B^H, \int \Psi dB^H\right).$$

Before we turn to the proof of Theorem 4.2, we consider an example where Assumption 4.1 can indeed be verified.

Example 4.3 *Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers with $H_n > \frac{1}{2}$, and assume that $\lim_{n\to\infty} H_n = H > \frac{1}{2}$. Let Z^n be a fractional Brownian motion process with Hurst parameter H_n and let Ψ be a continuous semimartingale. Since $H_n > \frac{1}{2}$, the processes Z^n have zero quadratic variation. Moreover, $\mathcal{L}\text{-}\lim_{n\to\infty} Z^n = B^H$ because the centered Gaussian processes Z^n and B^H are uniquely determined by their covariation functions and all stochastic integrals exists in the sense of (47). Thus, Theorem 4.2 yields*

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(Z^n, \int \Psi dZ^n\right) = \left(B^H, \int \Psi dB^H\right).$$

We prepare the proof of Theorem 4.2 with the following simple lemma.

Lemma 4.4 *Under the assumptions of Theorem 4.2 the processes $[Z^n]$ and $[Z^n, \Psi^n]$ defined by*

$$[Z^n]_t := (Z_t^n)^2 - 2\int_0^t Z_{s-}^n dZ_s^n \quad \text{and} \quad [Z^n, \Psi^n]_t := Z_t^n \Psi_t^n - \int_0^t Z_{s-}^n d\Psi_s^n - \int_0^t \Psi_{s-}^n dZ_s^n,$$

have $\mathbb{P}$-a.s. sample paths which are equal to zero.

PROOF: It follow from the representation of the stochastic integrals $\int Z^n dZ^n$, $\int \Psi^n dZ^n$ and $\int Z^n d\Psi^n$ as probabilistic limits of Stieltjes-type sums that, for any t and each sequence of partitions $\{\tau^l\}_{l\in\mathbb{N}}$ of $[0,t]$ with $\lim_{l\to\infty} \max_i |\tau^l_{i+1} - \tau^l_i|$,

$$[Z^n] = \mathbb{P}\text{-}\lim_{l\to\infty} \sum_i (Z^n_{\tau^l_{i+1}} - Z^n_{\tau^l_i})^2 \quad \text{and} \quad [Z^n, \Psi^n] = \mathbb{P}\text{-}\lim_{l\to\infty} \sum_i (Z^n_{\tau^l_{i+1}} - Z^n_{\tau^l_i})(\Psi^n_{\tau^l_{i+1}} - \Psi^n_{\tau^l_i}).$$

Since a typical sample path of the stochastic integrals $\int \Psi^n dZ^n$ and $\int Z^n d\Psi^n$ is in $\mathbb{D}$, we can apply the same arguments as in the proof of Theorem II.6.25 in Protter (1990) in order to obtain the Cauchy-Schwartz-type inequality $\mathbb{P}\{[Z^n, \Psi^n]^2_t \leq [Z^n]_t [\Psi^n]_t\} = 1$. Thus, our assertion follows from $\mathbb{P}\{[Z^n]_t = 0 \text{ for all } t \geq 0\} = 1$. □

For the proof of Theorem 4.2 we will also need the following simple lemma.

Lemma 4.5 *(i) Let $\mathbb{C}$ be the space of all real valued continuous functions. For $n \in \mathbb{N}$, let $\alpha_n, \beta_n \in \mathbb{D}$ and assume that the sequence $\{(\alpha_n, \beta_n)\}_{n\in\mathbb{N}}$ converges in the Skorohod topology to $(\alpha, \beta) \in \mathbb{C}$. Then, on compact intervals, the process*

$$\gamma_n = (\gamma_n(t))_{t\geq 0} \quad \textit{defined by} \quad \gamma_n(t) = \alpha_n(t)\beta_n(t)$$

converges to $\alpha\beta = (\alpha(t)\beta(t))_{t\geq 0}$ in the Skorohod topology on $\mathbb{D}$.

(ii) Let $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$ be a sequence of $\mathbb{D}$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that converges in law to (Y, Z). If $\mathbb{P}\{(Y, Z) \in \mathbb{C} \times \mathbb{C}\} = 1$, then

$$\mathcal{L}\text{-}\lim_{n\to\infty} \{(Y^n_t Z^n_t)_{0\leq t\leq T}\} = (Y_t Z_t)_{0\leq t\leq T}$$

holds for all $T < \infty$.

PROOF: Since α and β are continuous, (i) follows from Lemma 2.1 in Kurtz and Protter (1991). The second assertion follows from (i) and Skorohod's representation theorem. □

We are now ready to finish the proof of Theorem 4.2.

PROOF OF THEOREM 4.2: Since $\{\Psi^n\}_{n\in\mathbb{N}}$ is a sequence of good semimartingales and because a typical sample path of a fractional Brownian motion process is continuous, we deduce from Theorem 2.2 in Kurtz and Protter (1991) and from Lemma 4.5 (ii) that

$$\mathcal{L}\text{-}\lim_{n\to\infty} \left(\Psi^n, Z^n, \int Z^n d\Psi^n\right) = \left(\Psi, B^H, \int B^H d\Psi\right) \text{ and } \mathcal{L}\text{-}\lim_{n\to\infty} (\Psi^n Z^n) = \left(\Psi B^H\right), \quad (49)$$

respectively. By the continuous mapping theorem, it follows from (49), from Lemma 4.4 and from the integration by parts formula for fractional Brownian motion (48) that the finite dimensional distributions of the processes

$$\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi^n, Z^n, -\int Z^n d\Psi^n + \Psi^n Z^n\right)$$

converge weakly to the finite dimensional distributions of the process

$$\left(\Psi, B^H, -\int B^H d\Psi + \Psi B^H\right) = \left(\Psi, B^H, \int \Psi dB^H\right).$$

It also follows from (49) that the sequences $\left\{\int Z^n d\Psi^n\right\}_{n\in\mathbb{N}}$ and $\{\Psi^n Z^n\}_{n\in\mathbb{N}}$ are tight. By Corollary VI.3.33 in Jacod and Shiryaev (1987) the sum of tight sequences of stochastic processes with continuous sample paths is tight. Thus, continuity of the processes ΨB^H and $\int B^H d\Psi$ yields tightness of the sequence $\left\{\int \Psi^n dZ^n\right\}_{n\in\mathbb{N}}$. This shows that

$$\mathcal{L}\text{-}\lim_{n\to\infty}\left(\Psi^n, Z^n, \int \Psi^n dZ^n\right) = \left(\Psi, B^H, \int \Psi dB^H\right).$$

□

In view of Lemma 4.4 and the integration by parts formula for fractional Brownian motion processes we also have the following approximation result for the integral of fractional Brownian motion process with respect to itself.

Proposition 4.6 *Under the assumption of Theorem 4.2 it holds that*

$$\mathcal{L}\text{-}\lim_{n\to\infty}\int Z^n dZ^n = \int B^H dB^H.$$

Proof: By Lemma 4.4

$$(Z_t^n)^2 = 2\int_0^t Z_s^n dZ_s^n \qquad \mathbb{P}\text{-a.s.}$$

Thus, in view of Lemma 4.5 (ii) and the Itô formula for fractional Brownian motion the sequence $\{(Z_t^n)^2\}_{n\in\mathbb{N}}$ converges in distribution to

$$(B_t^H)^2 = 2\int_0^t B_s^H dB_s^H.$$

This yields the assertion. □

We finish this section with the proof of Theorem 2.6.

Proof of Theorem 2.6: In terms of the processes X^ε introduced in (19), we have

$$\mathcal{L}\text{-}\lim_{N\to\infty}\frac{1}{\sqrt{N}}(X_t^{\varepsilon,N})_{0\le t\le T} = \left(\int_0^t \Psi_s dX_s^\varepsilon\right)_{0\le t\le T} \text{ and } \int_0^t X_s^\varepsilon dX_s^\varepsilon = \int_0^t X_s^\varepsilon Y_{s/\varepsilon}\, ds$$

where the stochastic integrals have continuous sample paths. By Theorem 3.1,

$$\mathcal{L}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^{1-H}\sqrt{L(\varepsilon^{-1})}} X^{\varepsilon} = c\, B^{H}$$

with $H > \frac{1}{2}$, and so the assertion follows from Theorem 4.2 if we can show that the processes X^{ε} have zero quadratic variation on compact time intervals and that the stochastic integrals $\int \Psi dX^{\varepsilon}$ and $\int X^{\varepsilon}\, dX^{\varepsilon}$ exist as the probabilistic limits of Stieltjes-type sums. These properties, however, follow from (19) by direct computation. □

A The key renewal theorem in the heavy tailed case

In order to keep the paper as self contained as possible, we recall here a result of Heath *et al.* (1996) on the rate of convergence in the key renewal theorem in the heavy tailed case.

Let z be a continuous, non-negative function of bounded variation on $[0,\infty)$, such that $\lim_{t\to\infty} z(y) = 0$. That is, $z(t) = \int_t^{\infty} \zeta(dy)$ for some finite signed measure ζ on $[0,\infty)$. Let z^* denote the total variation function of ζ. That is,

$$z^*(t) = \int_t^{\infty} |\zeta|(dy). \tag{50}$$

Following Heath *et al.* (1996), we say that z is *virtually non-increasing* if

$$\lim_{t\to\infty} \frac{z(t)}{z^*(t)} = 1. \tag{51}$$

The following extension of the key renewal theorem appears as Theorem 3.10 (iii) in Heath *et al.* (1996).

Theorem A.1 *Let F be a distribution with domain $[0,\infty)$ satisfying*

$$\bar{F}(t) = 1 - F(t) \sim t^{-\alpha} L(t)$$

for some $1 \le \alpha < 2$, and L be a slowly varying function at infinity. Assume that F^n is nonsingular for some $n \ge 1$. Let $\zeta = \int_0^{\infty} \bar{F}(x)dx$ be the expected value and denote by U the renewal function associated with F, i.e.,

$$U = \sum_{n=0}^{\infty} F^n.$$

Let z be a virtually non-increasing function on $[0,\infty)$, such that $z(t) = o(\bar{F}(t))$ as $t \to \infty$. Let $\lambda = \int_0^{\infty} z(t)dt < \infty$. Then the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$h(t) = \frac{\lambda}{\zeta} - \int_0^t z(t-s)U(ds)$$

satisfies

$$h(t) \sim -\frac{\lambda}{(\alpha-1)\zeta^2} t^{-\alpha+1} L(t).$$

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