Robust families of transmission conditions of high order for thin conducting sheets

Kersten Schmidt[a,b,c], Alexey Chernov[a]

[a] *Hausdorff Center for Mathematics, University of Bonn, 53115 Bonn, Germany*
[b] *Project POEMS, INRIA Paris-Rocqencourt, 78150 Rocquencourt, France*
[c] *currently at TU Berlin and DFG Research center* Matheon, *10623 Berlin, Germany*

Abstract

Three families of transmission conditions of different order are proposed for thin conducting sheets in the eddy current model. Resolving the thin sheet by a finite element mesh is often not possible. With these transmission conditions only the middle curve, but not the thin sheet itself, has not to be resolved by a finite element mesh. The families of transmission conditions are derived by an asymptotic expansion for small sheet thicknesses ε, where each family results from a different asymptotic framework. In the first asymptotic framework the conductivity remains constant, scales with $1/\varepsilon$ in the second and with $1/\varepsilon^2$ in the third. The different asymptotics lead to different limit conditions, namely the vanishing sheet, a non-trivial borderline case, and the impermeable sheet, as well as different transmission conditions of higher orders. We investigated the stability, the convergence of the transmission conditions as well as their robustness. We call transmission conditions robust, if they provide accurate approximation for a wide range of sheet thicknesses and conductivities. We introduce an ordering of transmission conditions for the same sheet with respect to the robustness, and observe that the condition derived for the $1/\varepsilon$ asymptotics is the most robust limit condition, contrary to order 1 and higher, where the transmission conditions derived for the $1/\varepsilon^2$ asymptotics turn out to be most robust.

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2010 MSC: 65N30, 35C20, 35J25,, 41A60, 35B40, 78M30, 78M35

Introduction

The investigated time-harmonic (like $\mathrm{e}^{-i\omega t}$) eddy current problem for the electric field $e(x)$ in the computational domain $\Omega \subset \mathbb{R}^2$ reads

$$-\Delta e(x) + i\omega\mu_0\sigma e(x) = 0, \qquad \text{in } \Omega_{\mathrm{int}}, \tag{0.1a}$$
$$-\Delta e(x) = -i\omega\mu_0 j_0(x), \qquad \text{in } \Omega_{\mathrm{ext}}, \tag{0.1b}$$

where the sheet of thickness d is denoted by Ω_{int} and the remaining exterior domain is Ω_{ext} (see Fig. 1(a)). The angular frequency is ω and positive, $\mu_0 > 0$ is the permeability of air, $\sigma > 0$ the conductivity of sheet and j_0 alternating currents which flow outside the sheet and which are extended by 0 inside the sheet. In Fig. 1 (b) one solution is examplarily shown.

The eddy current problem (0.1) is completed by suitable boundary conditions providing a unique solution in $H^1(\Omega)$. For simplicity we restrict us in this work to "electric" boundary conditions (Dirichlet) on $\Gamma_e \subset \partial\Omega$ of the Lipschitz boundary $\partial\Omega$ and generalised "impedance" boundary conditions (including Neumann boundary conditions) on $\Gamma_i = \partial\Omega\backslash\Gamma_e$,

$$\begin{aligned} e &= e_{\mathrm{imp}} \quad \text{on } \Gamma_e, \\ \nabla e \cdot \underline{n} - \boldsymbol{\beta} e &= \iota_{\mathrm{imp}} \quad \text{on } \Gamma_i. \end{aligned} \tag{0.2}$$

The effective conductivity of the sheet

$$c := i\omega\mu_0\sigma \tag{0.3a}$$

Email addresses: kersten.schmidt@math.tu-berlin.de (Kersten Schmidt), chernov@hcm.uni-bonn.de (Alexey Chernov)

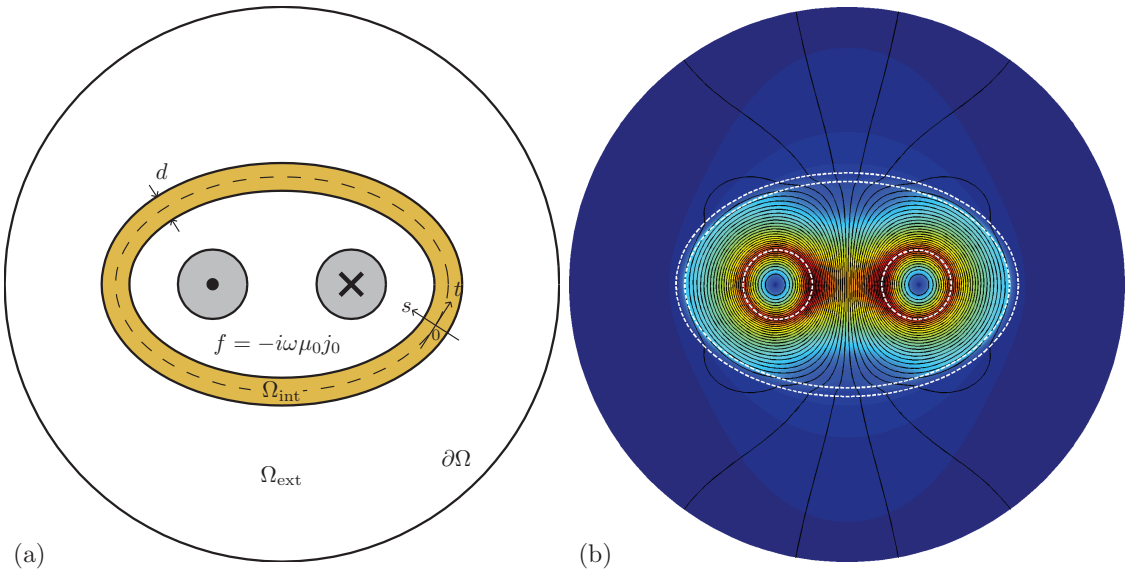


Figure 1: (a) Geometrical setting for the eddy current problem, with two live wires supporting the flux j_0 of angular frequency ω. (b) The magnitude and the flux lines of the in-plane magnetic field for an elliptic shin sheet with semi-major axis $a = 1.2$, semi-minor axis $b = \sqrt{0.6}$, thickness $d = 1/16$, effective conductivity $c = 64i$ and $f = 1$ in the left wire and $f = -1$ in the right one – corresponding to an alternating currents j_0 with opposite direction. The flux lines of the magnetic field compass the wires and are almost trapped in the interior area enclosed by the thin sheet.

leads to a *shielding effect* due to induced currents in the thin sheet, which can be measured by the power loss [10]

$$P := |c| \, \|e\|^2_{L^2(\Omega_{\text{int}})} \sim |c| d, \tag{0.3b}$$

which is for small enough thicknesses approximately proportional to $|c|$ and d (see Lemma 1.1). These induced currents flow mainly in a boundary layer of the sheet, *i. e.*, on its skin. This effect is called *skin effect* where for thin sheets the relevant quantity is

$$\delta := \frac{d_{\text{skin}}}{d} = \frac{1}{d}\sqrt{\frac{2}{\omega\mu_0\sigma}} = \sqrt{\frac{2}{|c|d^2}}, \tag{0.3c}$$

the ratio of the skin depth [10] of solid conductors $d_{\text{skin}} := \sqrt{2/\omega\mu\sigma} = \sqrt{\frac{2}{|c|}}$ and the sheet thickness.

For simulations of the eddy current problem with thin sheets by standard methods, like the finite element method, the smallness of the thin sheet is first a challenge for automatic mesh generators, which nowadays hardly support anisotropic cells. If anyhow anisotropic simplicies (triangles or tetrahedra in 3D) in the sheet are used, the condition number of the (whole) system matrix [31] increase significantly. The thin sheet might be meshed by tensor-product cells (quadrilaterals or hexahedrons in 3D) with almost right angles, and we will use those meshes with high-order polynomial basis functions to obtain reference solutions. For this we create the mesh by hand rather by a mesh generator. Additional refinement inside the sheet, mesh refinement or polynomial degree enlargement, is necessary due to the boundary layer behaviour (or skin effect), especially pronounced for not that small frequencies and sheet thicknesses.

Replacing the thin sheet by an interface and its behaviour by transmission conditions is attractive, as the interface can easily (and usally automatic) be resolved by edges in the finite element mesh, and anisotropic cells with their drawbacks are omitted. These transmission conditions are also known as impedance boundary conditions [15] (the interface is considered as two-sided boundary), as shielding elements [22], as shell models [18], or as shell elements [13].

However, impedance boundary conditions (IBCs) are traditionally called approximative conditions on an artifical boundary which replaces a subdomain of a solid conductor. They have been first proposed by Shchukin [30] and Leontovich [19], and improved by several authors [29, 1, 17, 11]. More similar to transmission conditions for thin sheets are generalised impedance boundary conditions for conducting bodies with dielectric coatings which are derived for high orders [12, 3, 2, 16, 4] and for coatings with a kink [8].

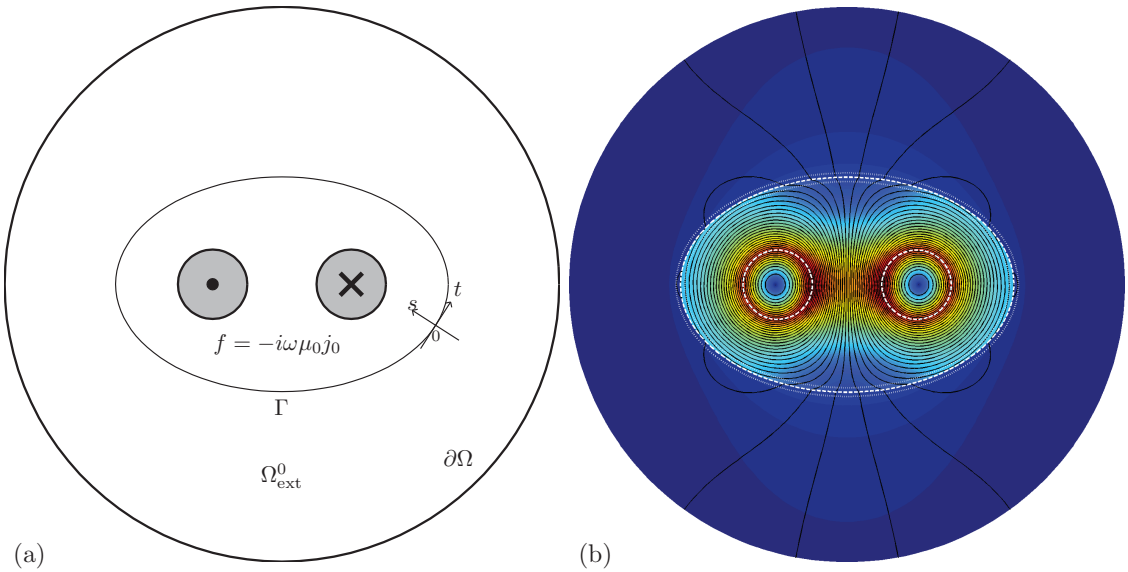


Figure 2: (a) Limit geometry for $\Omega^\varepsilon_{\text{ext}}$ for varying thickness ε is Ω^0_{ext} and $\Omega^\varepsilon_{\text{int}}$ gets in the limit $\varepsilon \to 0$ the midline Γ. The transmission conditions are set on Γ. The local coordinate system determines the direction of the jumps, here the difference of left and right trace when viewing in direction t. (b) The magnitude and the flux lines of the (exterior) in-plane magnetic field for the transmissions conditions of $\alpha = 2$ and order 1 (same parameter as in Fig. 1. The field is defined up to the midline Γ, and its restriction to Ω_{ext} approximates there the exact solution for $\varepsilon = d$. The field in the two stripes $\Omega^0_{\text{ext}} \backslash \Omega_{\text{ext}} = \Omega_{\text{int}} \backslash \Gamma$ is auxilliary and can be used to compute an approximation of the fields in the sheet.

Transmission conditions for thin conducting sheets have mainly be derived by using special functions in thickness direction which are multiplied by (macroscropic) functions along the sheet. Assuming no variation in thickness direction results in a surface quantity [22, 5] and integrals over the interface. Taking two hyberbolic functions in thickness direction which respects the boundary layer behavior an improved transmission condition was derived by Krähenbühl and Muller [18], by Mayergoyz and Bedrosian [20] and other authors [13, 15]. In time domain conditions which extends the Krähenbühl-Mayergoyz conditions have been derived in [21, 7].

The problems for thin conducting sheets differ by the fact that they are in the interior of the domain of interest and can not be approximated together with boundary conditions for solid bodies. As a consequence there are two interfaces of the thin sheet with the surrounding domain and setting a transmission derived for the two distinct interfaces on a single one, *e. g.*, the midline, a modelling error proportional to the thickness [26] is achieved. Thus, those transmission conditions with even more than the two functions underlying the Krähenbühl-Mayergoyz conditions, *e. g.*, in [14], are of order 0.

An asymptotic expansion of the solution for thin (and smooth) conducting sheets in an asymptotic framework in which the conductivity is scaled reciprocically with the thickness is shown for any order by Schmidt and Tordeux [27]. This particular asymptotics was motivated by their non-trivial limit solution. Asymptotic expansion for thin isolating sheets with slowly or fastly varying thickness of order 1 have been derived in [24, 9]. In these asymptotic expansions the solution is the sheet is expanded in scaled coordinates, and the continuity conditions at the two interfaces of the sheet are expanded around its midline to obtain problems for the limit solution and corrector functions on the domain where the sheet is replaced by an interface. For the thin conducting sheets the asymptotic expansions of [27] have been used to derive and verify transmission conditions up to order 3 in this framework [28].

For the thin isolating sheet different asymptotically varying thicknesses are investigated [24, 9]. A similar example is that of two close inclusions for the Laplace equation [6] where different asymptotics of the distance of the inclusions is studied.

Analog to these different asymptotics, we will investigate in this article the complete (*i. e.*, to any order $N \in \mathbb{N}_0$) asymptotic expansion for three different asymptotics of the (effective) conductivity $c(\varepsilon)$ $1/\varepsilon^\alpha$ for $\alpha \in \{0, 1, 2\}$ of the thin conducting sheet of varying thickness ε. In the framework of [28] we derive three different families of transmission conditions, to any order. Transmission conditions, even so derived by asymptotic expansion in a certain asymptotic framework, are used to approximate sheets of a certain thickness d and a certain (effective) conductivity c. We compare the derived transmission conditions related

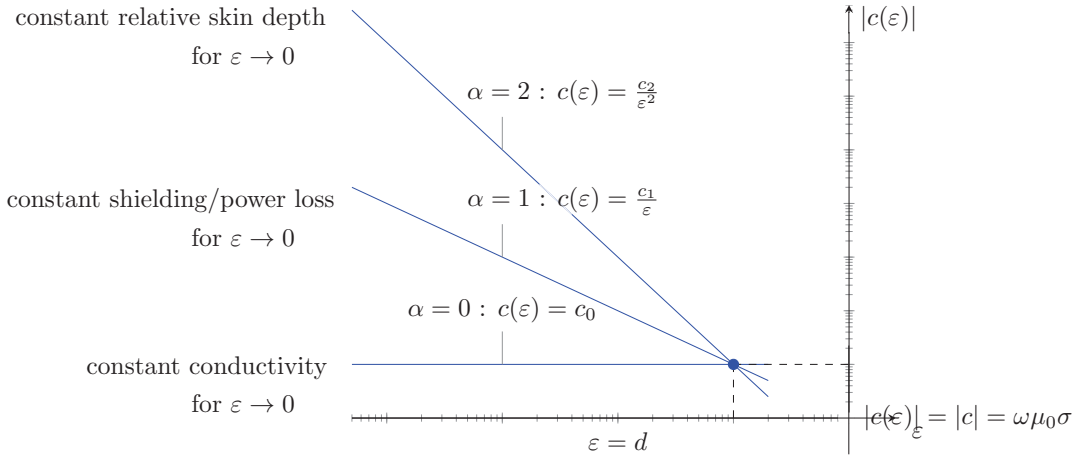


Figure 3: Illustration of the considered asymptotics for the eddy current model. The thickness and the effective conductivity are both in *logarithmic* scales. A sheet of a particular thickness d and effective conductivity c is highlighted with a circle. The three lines corresponds to $c(\varepsilon) \sim 1$, $c(\varepsilon) \sim 1/\varepsilon$ and $c(\varepsilon) \sim 1/\varepsilon^2$, and where $c(d) = i\omega\mu_0\sigma$. The asymptotic expansion is taken around the point when the lines are taken in the limit $\varepsilon \to 0$.

to their robustness, where we call a method robuster than another if it provides accurate approximations for a wide range of thicknesses and conductivities.

In the remainder of the Introduction we will introduce the three asymptotic frameworks more in detail and give the transmission conditions in an abstract form with operators $\mathbf{G}_j^\alpha$, $\mathbf{H}_j^\alpha$, and $\mathbf{J}_j^\alpha$. In Section 1 we derive the asymptotic expansion of the solution up to any order, where the solution in the sheet and outside are coupled. The respective limit problems, which lead to transmission conditions of order 0, are discussed in Section 3. The decoupled asymptotic expansion for any order is given with the operators $\mathbf{G}_j^\alpha$, $\mathbf{H}_j^\alpha$, and $\mathbf{J}_j^\alpha$ in Section 4. Then, in Section 5 we derive the respective transmission conditions including their variational formulations, which are then compared with respect to their robustness.

The asymptotic frameworks for the transmission conditions

In this article we will propose three families of transmission conditions indicated by a parameter $\alpha \in \{0, 1, 2\}$ and their order $N \in \mathbb{N}_0$. To obtain the approximations to the electric field, first, an exterior field $e_{\text{ext}}^{\alpha,N}$ is defined in the domain Ω_{ext}^0, which is the exterior domain extended up to the midline Γ of the sheet (see Fig. 2(a)). This exterior field satisfies the original equation (0.1b), *i. e.*,

$$-\Delta e_{\text{ext}}^{\alpha,N}(x) = -i\omega\mu_0 j_0(x), \qquad \text{in } \Omega_{\text{ext}}^0, \tag{0.4a}$$

together with the transmission conditions on the midline

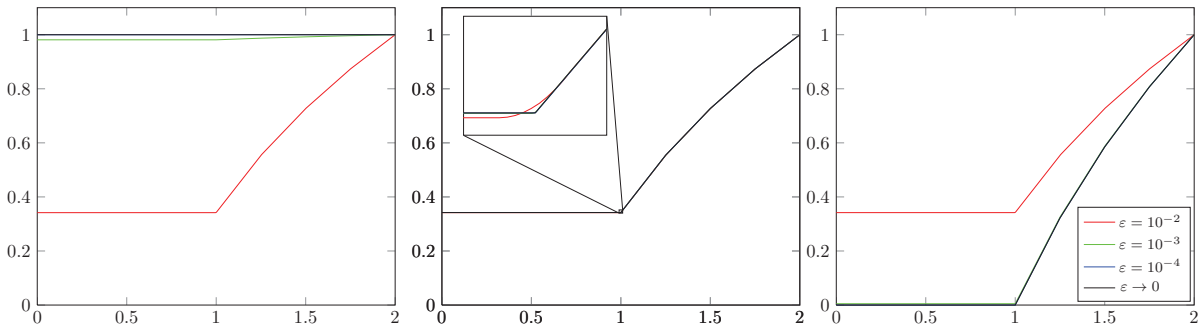
$$\big(\mathbf{G}^{\alpha,N}(d, c = i\omega\mu_0\sigma)\, e_{\text{ext}}^{\alpha,N}\big)(t) = 0, \qquad \text{on } \Gamma, \tag{0.4b}$$

$$\big(\mathbf{H}^{\alpha,N}(d, c = i\omega\mu_0\sigma)\, e_{\text{ext}}^{\alpha,N}\big)(t) = 0, \qquad \text{on } \Gamma, \tag{0.4c}$$

which are completed by the boundary conditions (0.2). The function $e_{\text{ext}}^{\alpha,N}$ restricted to Ω_{ext} serves here as an approximation to the exact solution of (0.1). Often, the field inside the sheet is of no practical interest. If anyhow needed, it can be approximated by

$$e_{\text{int}}^{\alpha,N}(x) = \big(\mathbf{J}^{\alpha,N}(d, c = i\omega\mu_0\sigma) e_{\text{ext}}^{\alpha,N}\big)(x). \tag{0.4d}$$

The transmission conditions are hierarchic meaning that the operators $\mathbf{G}^{\alpha,N}$, $\mathbf{H}^{\alpha,N}$, and $\mathbf{J}^{\alpha,N}$ by which they are represented incorporate only new terms when the order is increased, but all previous terms remain. The conditions and the respective operators will be derived by asymptotic expansions, in which the original problem (0.1) and the original geometry setting (*i. e.*, Ω_{int} and Ω_{ext}) are replaced by families of problems and geometries. These families of problems and geometries are characterised by a parameter ε which stands for the thickness of the sheet. So, Ω_{int} is replaced by a similar sheet $\Omega_{\text{int}}^\varepsilon$ of thickness ε and Ω_{ext} is replaced by $\Omega_{\text{ext}}^\varepsilon = \Omega \backslash \overline{\Omega}_{\text{ext}}^\varepsilon$. The original problem and geometry is member of each of the families, for $\alpha = 0, 1, 2$, with



(a) $\alpha = 0$ (Conductivity is constant). (b) $\alpha = 1$ (Conductivity like $1/\varepsilon$). (c) $\alpha = 2$ (Conductivity like $1/\varepsilon^2$).

Figure 4: Solutions for different asymptotics for $\varepsilon \to 0$ for a sheet of ring shape (inner radius $1 - \frac{\varepsilon}{2}$, outer radius $1 + \frac{\varepsilon}{2}$) inside a circular domain of radius 2. The Dirichlet data at the boundary is 1. Thus, the solutions are radial-symmetric, computed analytically and drawn along the radial coordinate. For this illustration we have chosen an (effective) conductivity $c = 200i$ for $d = 10^{-2}$, where the skin depth $d_{\text{skin}} = 0.1$ is ten times larger than d. For different thicknesses ε we scale $c(\varepsilon)$ like $1/\varepsilon^\alpha$. For $\alpha = 1$ the solution does not change much when ε is decreased, a clipping illustrate this small changes in the vicinity of the sheet.

$\varepsilon = d$. The transmission conditions are called of order N if their error is $O(\varepsilon^{N+1})$ (in the L^2- or H^1-norm in the exterior of the sheet), where the error is meant to be the difference to the exact solution in the particular asymptotics.

We will consider three families of asymptotics (see Fig. 3):

- $\alpha = 0$, the case of constant conductivity c (see (0.3a)),
- $\alpha = 1$, the case of asymptotically constant shielding (and so power loss P, see (0.3b)), and *i. e.*, $c\,\varepsilon$ is constant,
- $\alpha = 2$, the case of asymptotically constant relative skin depth δ (see (0.3c)), *i. e.*, $c\,\varepsilon^2$ is constant.

The case $\alpha = 0$, where the conductivity (and so the material) is not touched when performing the asymptotic expansion, is the choice one would probably think first on. One is familar with statements about the error in this asymptotics, like "the error decreases by a factor of 2 or 4 when the thickness is halved". Deriving an accurate transmission condition this choice might not be the best, and changing (increasing) the conductivity (and so virtually the material) when the thickness ε is decreased, may pay off. Motivated by the fact, that $\alpha = 1$ represents a borderline case where for the limit problem for $\varepsilon \to 0$ shielding is neither vanishing ($\alpha < 1$) nor the sheet gets impermeable ($\alpha > 1$), in [27, 28] the case $\alpha = 1$, *i. e.*, the conductivity is inverse proportional to ε, have been considered. A numerical comparison of the limit problems of the three asymptotics are shown in Fig. 4. We will discuss these limit problems for $\varepsilon \to 0$ more in detail in Section 3. In this article, we will introduce transmission conditions for all the above asymptotics $\alpha = 0, 1, 2$ together, whose operators have the following representations as a power series in the thickness

$$\mathbf{G}^{\alpha,N}(d,c) = \sum_{j=0}^{N} d^j \mathbf{G}_j^\alpha(c\,d^\alpha), \qquad \mathbf{H}^{\alpha,N}(d,c) = \sum_{j=0}^{N} d^j \mathbf{H}_j^\alpha(c\,d^\alpha), \qquad \mathbf{J}^{\alpha,N}(d,c) = \sum_{j=0}^{N} d^j \mathbf{J}_j^\alpha(c\,d^\alpha),$$

with some operators $\mathbf{G}_j^\alpha$, $\mathbf{H}_j^\alpha$, and $\mathbf{J}_j^\alpha$ of order j.

To do so, the asymptotic problems for $\alpha = 0, 1, 2$ will be discussed in Sect. 1. The respective terms of the asymptotic expansions will be derived in Sect. 2, leading to transmission conditions in Sect. 5. The relation of these transmission conditions derived for three different asymptotics will be discussed as well in Sect. 5, on one side by the terms of their expression, and on the other side by numerical experiments. These will eventually enable us to highlight differences in their usability and their areas of validity and optimality.

In this article we deal with three cases $\alpha = 0$ (asymptotically vanishing shielding), $\alpha = 1$ (borderline case) and $\alpha = 2$ (asymptotically impermeable sheet), for all which we will derive the asymptotic expansion and corresponding transmission conditions. This direct comparison enables us to find optimal transmission conditions ($\alpha = 0, 1$ or 2 and order N) for different ranges/magnitudes of conductivity or frequency, and

to compare the models also concerning their usability. In this article we will deal with three cases $\alpha = 0$ (asymptotically vanishing shielding), $\alpha = 1$ (borderline case) and $\alpha = 2$ (asymptotically impermeable sheet), for all which we will derive the asymptotic expansion and corresponding transmission conditions. This direct comparison enables us to find optimal transmission conditions ($\alpha = 0, 1$ or 2 and order N) for different ranges/magnitudes of conductivity or frequency, and to compare the models also concerning their usability. The collective derivation of the asymptotic expansions is in Section 2, where also the well-posedness of the problems for the terms of the expansion and estimates for the modelling error of the truncated series are shown. In Section 5 we define corresponding transmission conditions for up to order 3 for $\alpha = 0$ and $\alpha = 1$ and for up to order 1 for $\alpha = 2$, propose stable variational formulations, and compare in Section 5.6 the transmission conditions by their region of stability, their usability and their optimality regions, for which we use numerical experiments.

1. Asymptotic models

Asymptotic model by introducing the parameter ε for the thickness and scaling the product of conductivity and frequency with $\varepsilon^{-\alpha}$:

$$-\Delta e^\varepsilon(x) + \frac{c_\alpha}{\varepsilon^\alpha} e^\varepsilon(x) = 0, \qquad \text{in } \Omega^\varepsilon_{\text{int}}, \tag{1.1a}$$

$$-\Delta e^\varepsilon(x) = f(x), \qquad \text{in } \Omega^\varepsilon_{\text{ext}}, \tag{1.1b}$$

$$e^\varepsilon = e_{\text{imp}} \qquad \text{on } \Gamma_e, \tag{1.1c}$$

$$\nabla e^\varepsilon \cdot \underline{n} - \boldsymbol{\beta} e^\varepsilon = \iota_{\text{imp}} \qquad \text{on } \Gamma_i. \tag{1.1d}$$

where the real parameter $\alpha \in [0, 2]$, $c_\alpha := id^\alpha \omega \mu_0 \sigma$ for a particular thickness d and so independent of the varying thickness ε, and $f = -i\omega\mu_0 j_0$. Here, ω is the angular frequency, σ the conductivity, μ_0 the permeability and j_0 the injected current whose support is bounded in $\Omega^\varepsilon_{\text{ext}}$ for all $\varepsilon < \varepsilon_{\max}$ for some $\varepsilon_{\max} > d$. We assume the sheet $\Omega^\varepsilon_{\text{ext}}$ of constant thickness ε, which can be described by the mapping $x_\Gamma(t) + sn$, with the local coordinates $(s, t) \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \times \hat{\Gamma}$, $x_\Gamma(t)$ the function describing the mean line $\Gamma \in C^\infty$ with $|x'_\Gamma(t)| = 1$, $\hat{\Gamma} \subset \mathbb{R}$ the ring domain of length $|\Gamma|$, and n the left normalised normal vector to Γ. The curvature $\kappa(t)$ is bounded as $\varepsilon|\kappa(t)| < 2$ for all $\varepsilon < \varepsilon_{\max}$ such that $\Omega^\varepsilon_{\text{int}}$ is C^∞ regular as well. Furthermore, we assume the $\boldsymbol{\beta}$ of the generalised impedance boundary condition to fulfill

$$\operatorname{Re} \langle \boldsymbol{\beta} e, e \rangle_{\Gamma_i} \geq 0, \qquad \operatorname{Im} \langle \boldsymbol{\beta} e, e \rangle_{\Gamma_i} \geq 0$$

for all e in appropriate function spaces $V(\Gamma_i)$ (*e. g.*, $L^2(\Gamma_i)$ if $\boldsymbol{\beta}$ is a multiplicative factor).

Different values of α lead to different asymptotics for $\varepsilon \to 0$, and also statements of the modelling error of approximations to $e^\varepsilon(x)$ in terms of ε are only meaningful when fixing a particular α-asymptotics. The different asymptotic behaviour can be seen exemplarily for the power loss in the sheet [25], which represents the strength of the shielding.

Lemma 1.1. *Let $\alpha \in [0, 2]$, $c_\alpha \in \mathbb{C}$, $c(\varepsilon) = \frac{c_\alpha}{\varepsilon^\alpha}$, Γ to be $C^{1,1}$ continuous and $e^\varepsilon \in H^1(\Omega)$ the unique solution of (1.1). Then, there exists a constant $C > 0$ independent of ε such that*

$$|(c(\varepsilon) e^\varepsilon, e^\varepsilon)_{L^2(\Omega^\varepsilon_{\text{int}})}| \leq C\, \varepsilon^{|1-\alpha|}.$$

For $\alpha < 1$ the sheet has asymptotically no shielding for $\varepsilon \to 0$. For $\alpha > 1$ the sheets gets completely impermeable for $\varepsilon \to 0$ which corresponds to the perfect electric conductor (PEC) boundary condition on both sides of the limit interface Γ. The case $\alpha = 1$ is a borderline case for which a non-trivial limit exists, where the power loss tends to a positive value. As so the limit solution is expected to be already close to the solution for a particular value ε an asymptotic expansion with few corrector terms should describe the solution to a very high precision. The asymptotic expansion for this borderline case has been derived in [27], and was then used in [28] to define transmission conditions up to order 3. We observed well-posedness of the appropiate variational formulations including stability for thicknesses ε which are at the order of the skin depth $d_{\text{skin}} := \sqrt{2/\omega\mu\sigma}$ or smaller.

2. Formal definition of the asymptotic expansions

Analogously to the derivation in [27] we expand the field in the exterior of the sheet in a power series in ε

$$e^{\varepsilon}_{\text{ext}}(x) := e^{\varepsilon}\big|_{\Omega^{\varepsilon}_{\text{ext}}}(x) = \sum_{j=0}^{N} \varepsilon^j u^j_{\text{ext}}(x) + r^{\varepsilon,N}_{\text{ext}}(x), \quad \forall N \in \mathbb{N}, \tag{2.1}$$

where the functions $u^j_{\text{ext}}(x)$ are defined on the limit domain $\Omega^0_{\text{ext}} = \Omega \setminus \Gamma$ and should not depend on ε. The asymptotic expansion will be constructed such that the external resiual $r^{\varepsilon,N}_{\text{ext}}(x)$ is of order $O(\varepsilon^{N+1})$ in the $H^1(\Omega^0_{\text{ext}})$-norm. The field in the interior of the sheet is expanded in a power series in ε with ε-independent functions $U^j_{\text{int}}(S,t)$ defined on a normalised domain, which is the product of $[-\frac{1}{2},\frac{1}{2}]$ and the parameter domain $\hat{\Gamma}$ of the midline,

$$e^{\varepsilon}_{\text{int}}(x) := e^{\varepsilon}\big|_{\Omega^{\varepsilon}_{\text{int}}}(x) = \sum_{j=0}^{N} \varepsilon^j U^j_{\text{int}}\left(t, \frac{s}{\varepsilon}\right), \quad \forall N \in \mathbb{N}. \tag{2.2}$$

Inserting the exterior and interior expansion into the conditions for continuity of $e^{\varepsilon}(x)$ on $\partial\Omega^{\varepsilon}$ and its normal derivative would lead to the equality of $u^j_{\text{ext}}(\pm\frac{\varepsilon}{2},t)$ and $U^j_{\text{int}}(\pm\frac{1}{2},t)$, and of $\partial_s u^j_{\text{ext}}(\pm\frac{\varepsilon}{2},t)$ and $\varepsilon^{-1}\partial_S U^j_{\text{int}}(\pm\frac{1}{2},t)$. These conditions are set on an ε-dependent locus. To obtain conditions independently of ε, we first expand $u^j_{\text{ext}}(\pm\frac{\varepsilon}{2},t)$ with Taylor around $s = \pm 0$ and insert these expansions then into the continuity conditions. Collecting terms of the same order in ε we get

$$U^j_{\text{int}}(t,\pm\tfrac{1}{2}) - u^j_{\text{ext}}(t,\pm 0) = \sum_{\ell=1}^{j} \left(\pm\frac{1}{2}\right)^{\ell} \frac{1}{\ell!} \partial_s^{\ell} u^{j-\ell}_{\text{ext}}(t,\pm 0) \qquad \text{on } \Gamma, \tag{2.3a}$$

$$\partial_S U^j_{\text{int}}(t,\pm\tfrac{1}{2}) = \sum_{\ell=1}^{j} \left(\pm\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!} \partial_s^{\ell} u^{j-\ell}_{\text{ext}}(t,\pm 0), \qquad \text{on } \Gamma. \tag{2.3b}$$

Inserting the far field expansion (2.1) into the equations (1.1b)–(1.1d) in the exterior of the sheet

$$-\Delta u^j_{\text{ext}}(x) = \delta_{j=0} f, \qquad \text{in } \Omega^{\varepsilon}_{\text{int}}, \tag{2.3c}$$

$$u^j_{\text{ext}} = \delta_{j=0} e_{\text{imp}} \qquad \text{on } \Gamma_e, \tag{2.3d}$$

$$\nabla u^j_{\text{ext}} \cdot \underline{n} - \boldsymbol{\beta} u^j_{\text{ext}} = \delta_{j=0}\iota_{\text{imp}} \qquad \text{on } \Gamma_i. \tag{2.3e}$$

We use the Kronecker delta where $\delta_{t(j)} = 1$ if $t(j)$ is true and $\delta_{t(j)} = 0$ otherwise. Inserting the near field expansion (2.2) into (1.1a) and using the expansion of the Laplace operator in curvilinear coordinates (S,t) in powers of ε we get the recurrence relation

$$-\partial_S^2 U^j_{\text{int}}(t,S) + \delta_{\alpha=2} c_2 U^j_{\text{int}}(t,S) = -\delta_{\alpha<2} c_\alpha U^{j+\alpha-2}_{\text{int}}(t,S) + \sum_{\ell=1}^{j} \Delta_\ell U^{j-\ell}_{\text{int}}(t,S), \tag{2.3f}$$

where we used the operators

$$\Delta_\ell(t,S) = -\widehat{\Delta}^0_\ell(t) S^{\ell-2} - \widehat{\Delta}^1_\ell S^{\ell-1}\partial_S$$

and (for later simplifications with different sign as in [27])

$$\widehat{\Delta}^0_\ell(t) := -(-\kappa(t))^{\ell-2}(\ell-1)\left(\partial_t^2 + \frac{\ell-2}{2}\frac{\kappa'(t)}{\kappa(t)}\partial_t\right), \qquad \widehat{\Delta}^1_\ell(t) := (-\kappa(t))^{\ell}. \tag{2.4}$$

In (2.3f) the term $c_\alpha U^{j+\alpha-2}_{\text{int}}$ is of lower order than j for $\alpha = 0,1$ where it therefore appears on the right hand side. For $\alpha = 2$ it is $c_2 U^j_{\text{int}}$, and so of order j and written on the left hand side. Written like it is, the system (2.3) collects conditions for u^j_{int} and U^j_{int} depending on lower order terms. In the following, we first define the interior field U^j_{int} by lower order terms and $\{u^j_{\text{ext}}\}$ for $\alpha = 0,1$. This will be then used to define

equations for the exterior fields u^j_{ext} only. Before doing so we first introduce notations for the mean and the jump of the near field and the far field terms:

$$[V](t) := V(t,\tfrac{1}{2}) - V(t,-\tfrac{1}{2}), \quad \{V\}(t) := \frac{1}{2}\left(V(t,\tfrac{1}{2}) + V(t,-\tfrac{1}{2})\right),$$
$$[v](t) := v(t,0^+) - v(t,0^-), \quad \{v\}(t) := \frac{1}{2}\left(v(t,0^+) + v(t,0^-)\right), \quad [v]^n(t) := \begin{cases} [v](t), & \text{for } n \text{ even}, \\ 2\,\{v\}(t), & \text{for } n \text{ odd}. \end{cases} \tag{2.5}$$

As one result of the asymptotic expansions just defined above are the limit solution u^0_{ext}, U^0_{int} for $\varepsilon \to 0$ for $\alpha = 0, 1, 2$ and the limits models of which they are solutions. This will be discussed in Sec. 3. Second, the asymptotic expansions will be used in Sec. 4 to define transmission conditions on the midline Γ which together with the original equation in the exterior of the sheet provide approximative solutions of order 1 and higher in the respective α-asymptotics.

Remark 2.1 (Asymptotics with $\alpha > 2$ and $\alpha \in \mathbb{R}$)**.** *For $\alpha = 3$ the equation in the interior gets*

$$c_3 U^{j+1}_{\text{int}}(t,S) = \partial_S^2 U^j_{\text{int}}(t,S) + \sum_{\ell=1}^{j} \Delta_\ell U^{j-\ell}_{\text{int}}(t,S),$$

meaning that the interior field is determined by previous terms of the interior field only, and not through boundary data by the exterior field as well. This leads to $U^0_{\text{int}} \equiv 0$ and by induction $U^j_{\text{int}} \equiv 0$ for all $j \in \mathbb{N}_0$. Evaluating (2.3a) for $j = 0$ and (2.3b) for $j = 1$ we get

$$u^0_{\text{ext}}(t,\pm 0) = \partial_s u^0_{\text{ext}}(t,\pm 0) = 0,$$

which are boundary conditions together impossible to satisfy by the exterior field. Similar holds for $3 < \alpha \in \mathbb{N}$.

For $\alpha \in \mathbb{R}$, $\alpha \geq 0$ the ansatz of the asymptotic expansion would be

$$e^\varepsilon_{\text{ext}}(x) = \sum_{j \in \mathbb{N}_0 \cup \alpha\mathbb{N}_0} \varepsilon^j u^j_{\text{ext}}(x) + r^{\varepsilon,N}_{\text{ext}}(x), \qquad e^\varepsilon_{\text{int}}(x) = \sum_{j \in \mathbb{N}_0 \otimes \{1,\alpha\}, j \leq N} \varepsilon^j U^j_{\text{int}}\left(t, \frac{s}{\varepsilon}\right) + r^{\varepsilon,N}_{\text{int}}(x),$$

for which the factorisation might be hardly feasible for higher orders.

We conject that the limit problems (see Sec. 3) for $\alpha < 1$ coincide with that for $\alpha = 0$ and those for $\alpha > 1$ coincide with that for $\alpha = 2$, meaning that $\alpha = 1$ is a borderline case.

3. The limit problems and the transmission conditions of order 0

3.1. The limit problems

Lemma 3.1. *The limit solutions for $\varepsilon \to 0$ satisfy for $\alpha = 0, 1, 2$*

$\alpha = 0$	$\alpha = 1$	$\alpha = 2$	
$[u^0_{\text{ext}}](t) = 0,$	$[u^0_{\text{ext}}](t) = 0,$	$[u^0_{\text{ext}}](t) = 0,$	(3.1a)
$[\partial_s u^0_{\text{ext}}](t) = 0,$	$[\partial_s u^0_{\text{ext}}](t) - c_1\{u^0_{\text{ext}}\}(t) = 0,$	$\{u^0_{\text{ext}}\}(t) = 0,$	(3.1b)
$U^0_{\text{int}}(t,S) = \{u^0_{\text{ext}}\}(t),$	$U^0_{\text{int}}(t,S) = \{u^0_{\text{ext}}\}(t),$	$U^0_{\text{int}}(t,S) = 0.$	(3.1c)

Proof. For $j = 0$ all the terms on the right hand side of (2.3a), (2.3b) and (2.3f) vanish. For $\alpha = 0, 1$ the internal field U^0_{int} is by (2.3f) polynomial of at most degree 1 in S, and by (2.3b) and (2.3a) it follows (3.1a) and (3.1c). Now, evaluating (2.3f) and (2.3b) for $j = 1$ and using the Neumann compatibility condition $\int_{\left(-\frac{1}{2},\frac{1}{2}\right)} \partial_S^2 U^1_{\text{int}}(t,S)\,\mathrm{d}S = [\partial_S U^1_{\text{int}}](t)$ we conclude (3.1b). For $\alpha = 2$ it holds (3.1c) by (2.3f), and consequently (3.1a) and (3.1b) by (2.3a). □

Corollary 3.2. *The point, around which the asymptotic expansions are performed is, for*

- *$\alpha = 0$, the sheet without shielding (which vanishes into "thin air"),*
- *$\alpha = 1$, the sheet of a non-trivial shielding,*

- $\alpha = 2$, *the impermeable (or perfectly shielding) sheet.*

Lemma 3.3 (Stability of the limit problems)**.** *Let for $\alpha = 0$, $\boldsymbol{\beta} = 0$ and $\Gamma_i = \partial\Omega$ (pure Neumann problem) hold the Neumann compatibility condition*

$$\int_{\partial\Omega} \iota_{\text{imp}} \,\mathrm{d}s = \int_{\Omega} i\omega\mu_0 j_0(x) \,\mathrm{d}\underline{x}.$$

Then, the problems (2.3c), (2.3d), (2.3e) with the transmission conditions defined by Tab. 1 for $\alpha = 0, 1, 2$ provide unique solutions $u^0_{\text{ext}} \in H^1(\Omega)$ or $H^1(\Omega)\backslash\mathbb{C}$ for $\alpha = 0$, $\boldsymbol{\beta} = 0$ and $\Gamma_i = \partial\Omega$.

Proof. With the conditions $[u^0_{\text{ext}}](t) = 0$ the solution is $C^0(\Omega)$-continuous. For $\alpha = 2$ there holds additionally $\{u^0_{\text{ext}}\}(t) = 0$ and the solution satisfy a homogeneous Dirichlet condition on Γ. Thus, the variational formulations can be stated in V^α_0 for the ansatz and $V^\alpha_{j,0}$ for the test functions, where

$$V^\alpha_{0,0} = H^1_{\Gamma_e}(\Omega), \quad \text{for } \alpha = 0, 1, \qquad \text{and} \qquad V^2_{0,0} = H^1_{\Gamma_e \cup \Gamma}(\Omega_0),$$

and

$$V^\alpha_0 = V^\alpha_{0,0} + E_{\Gamma_e} e_{\text{imp}} \subset H^1(\Omega_0),$$

except for the pure Neumann problem, where it is stated in $V^0_0\backslash\mathbb{C}$ and $V^0_{0,0}\backslash\mathbb{C}$. Here, we used an arbitrary continuous extension operator for the Dirichlet boundary data $E_{\Gamma_e} : H^{1/2}(\Gamma_e) \to H^1(\Omega)$.

For $\alpha = 1$ the variational problem is: Seek $u^0_{\text{ext}} \in V^1_0$ such that for all $v \in V^1_{0,0}$

$$\int_{\Omega^0_{\text{ext}}} \nabla u^0_{\text{ext}} \cdot \nabla v \,\mathrm{d}\underline{x} + c_1 \int_{\Gamma} u^0_{\text{ext}}\, v \,\mathrm{d}t + \int_{\Gamma_i} \boldsymbol{\beta} u^0_{\text{ext}}\, v \,\mathrm{d}\sigma(\underline{x}) = \int_{\Omega^0_{\text{ext}}} f\, v + \int_{\Gamma_i} \iota_{\text{imp}} e_n \,\mathrm{d}\sigma(\underline{x}) =: l(v). \tag{3.2}$$

For $\alpha = 0$ and $\alpha = 2$ a similar formulation as (3.2) holds without the second term in the left-hand side and for the respective spaces V^α_0 and $V^\alpha_{0,0}$. By Poincaré-Friedrichs inequalities the problems provide unique solutions under assumptions of the lemma. □

Lemma 3.4 (Modelling error of the limit problems)**.** *There exists a constant $C > 0$ such that for $\alpha = 0, 1, 2$ it holds*

$$\|e^\varepsilon - u^0_{\text{ext}}\|_{H^1(\Omega^\varepsilon_{\text{ext}})} + \sqrt{\varepsilon}\|e^\varepsilon - \{u^0_{\text{ext}}\}\|_{H^1(\Omega^\varepsilon_{\text{int}})} \le C\,\varepsilon.$$

The lemma justifies that the limit solutions is an approximation of order 0 in the respective asymptotics. It is proved for $\alpha = 1$ in [27]. The proof can be easily adapted for $\alpha = 0$ and $\alpha = 2$.

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
$(\mathbf{G}^\alpha_0(cd^\alpha)\,u)(t)$	$[u](t)$	$[u](t)$	$[u](t)$
$(\mathbf{H}^\alpha_0(cd^\alpha)\,u)(t)$	$[\partial_s u](t)$	$[\partial_s u](t) - cd\{u\}(t)$	$-\{u\}(t)\dfrac{2\sqrt{c}d\sinh(\sqrt{c}\frac{d}{2})}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}$
$(\mathbf{J}^\alpha_0(cd^\alpha)\,u)(t,S)$	$\{u\}(t)$	$\{u\}(t)$	$\{u\}(t)\dfrac{\cosh(\sqrt{c}s)}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}$

Table 1: Operators representing the limit transmission conditions.

3.2. Transmission conditions of order 0

The conditions defined in Lemma 3.1 can be used as the simplest approximative transmission conditions, where for $\alpha = 1$ we have to replace c_1 by cd with d the concrete sheet thickness. These transmission conditions can be written in the form (0.4) with the operators in Tab. 1. The condition (3.1b) ($\{u^0\}(t) = 0$) for $\alpha = 2$ could be written with a simpler operator than $\mathbf{H}^2_0$ introduced in Tab. 1. We have chosen this

operator $\mathbf{H}_0^2$ for the transmission condition as it will be needed to write the transmission conditions for order 1 and higher in Sec. 4 (cf. (4.5c)). Furthermore, this choice these higher order transmission conditions for $\alpha = 2$ will be visibly related to those of the asymptotics $\alpha = 0$ and $\alpha = 1$. With the same reason the operator $\mathbf{J}_0^2$ is chosen in Tab. 1.

As the functions in $\mathbf{H}_0^2$ and $\mathbf{J}_0^2$ are bounded away from zero and with its real and imaginary part from infinity for any fixed $c_2 \in i\mathbb{R}^+$ the solution u_{ext}^0 fulfills (3.1b) and U_{int}^0 (3.1c), and the conditions are equivalent.

As consequence of the uniqueness and existence of u_{ext}^0 by Lemma 3.3 the following corollary holds.

Corollary 3.5 (Stability of the transmission conditions of order 0)**.** *Let the assumption of Lemma 3.3 fulfilled. Then, the problems (0.4) with the transmission conditions defined by Tab. 1 for $\alpha = 0, 1, 2$ provide unique solutions $e_{\text{ext}}^{\alpha,0} \in H^1(\Omega)$ and $e_{\text{ext}}^{0,0} \in H^1(\Omega)\backslash\mathbb{C}$ for $\alpha = 0$, $\boldsymbol{\beta} = 0$ and $\Gamma_i = \partial\Omega$.*

With Lemma 3.4 and Corollary 3.5 the following corollary about the modelling error follows.

Corollary 3.6 (Modelling error of the transmission conditions of order 0)**.** *For the solution of a sequence of problems (1.1) with thickness ε and conductivity $c \sim \varepsilon^{-\alpha}$ it holds for $\alpha = 0, 1, 2$*

$$\|e^\varepsilon - e_{\text{ext}}^{\alpha,0}\|_{H^1(\Omega_{\text{ext}}^\varepsilon)} + \sqrt{\varepsilon}\|e^\varepsilon - \{e_{\text{ext}}^{\alpha,0}\}\|_{H^1(\Omega_{\text{int}}^\varepsilon)} = O(\varepsilon).$$

3.3. The robustness of the transmission conditions of order 0

As the transmission conditions have been derived by asymptotic expansion, they provide the more accurate results the closer the sheet is at the asymptotic limit, this means for sheets with almost no shielding for $\alpha = 0$ (c very small) and almost permeable shielding for $\alpha = 2$ (c very large). But the transmission condition for $\alpha = 0$ fail for almost permeable sheets, and conversely those for $\alpha = 2$ fail for almost vanishing sheets (see first and third row of first column in Fig. 5). As a positive surprise, using the transmission conditions for $\alpha = 1$ we get accurate results for both regions, where the sheet is almost vanishing or where it is almost permeable (see second row of first column in Fig. 5). Note, that the asymptotic limit solution for $\alpha = 1$ is as a border line case non-trivial.

This property of a transmission condition, that can be employed in a wide range of problem parameters, which are here the thickness d and the effective conductivity c, will be called *robustness*.

To compare the robustness of the three families of transmission condition we apply them for a series of sheets with thickness ε and an asymptotics of the conductivity $c(\varepsilon) = c_{\alpha^\star}/\varepsilon^{\alpha^\star}$, where $\alpha^\star \in \{0, 1, 2\}$, *i. e.*, we analyse the transmission conditions derived for asymptotics $\alpha \in \{0, 1, 2\}$ for any (other) asymptotics $\alpha^\star \in \{0, 1, 2\}$. By derivation, the transmission condition derived for $\alpha \in \{0, 1, 2\}$ are of order 0 (Corollary 3.6) in the asymptotics $\alpha^\star = \alpha$.

If we use the transmission condition for $\alpha = 1$ for a series of sheets with $\alpha^\star = 0$, *i. e.*, it is used for sheets of vanishing shielding and we have to replace $c_1 \to c_0\varepsilon$, and for $\varepsilon \to 0$ the term responsible for the shielding vanishes and we obtain in the limit the transmission conditions for $\alpha = 0$. When using the same transmission condition for $\alpha^\star = 2$, *i. e.*, for the asymptotics versus perfectly shielding sheets, we have to replace $c_1 \to c_2/\varepsilon$. For $\varepsilon \to 0$ the term $\int_\Gamma u_{\text{ext}}^0 v \, \mathrm{d}t$ has to vanish such that it stays bounded when weighed with $1/\varepsilon$. This implies that $u^0 \to 0$ and we obtain in the limit for $\varepsilon \to 0$ the transmission condition of $\alpha = 2$. Consequently, this transmission condition of $\alpha = 1$ is robust. It will obtain for sheets with very small and very large shielding the same asymptotic accuracy as the for this constructed transmission conditions.

Contrary, the transmission conditions for $\alpha = 0$ and $\alpha = 2$ do not depend on the effective conductivity and consequently do not tend to one of the other limit models. These conditions are less robust, and obtain therefore a reduced accuracy in the respective other asymptotic framework, which we observed in numerical experiments.

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
$\alpha^\star = 0$	robust	robust	
$\alpha^\star = 1$		robust	
$\alpha^\star = 2$		robust	robust

Table 2: Robustness of the transmission conditions of order 0.

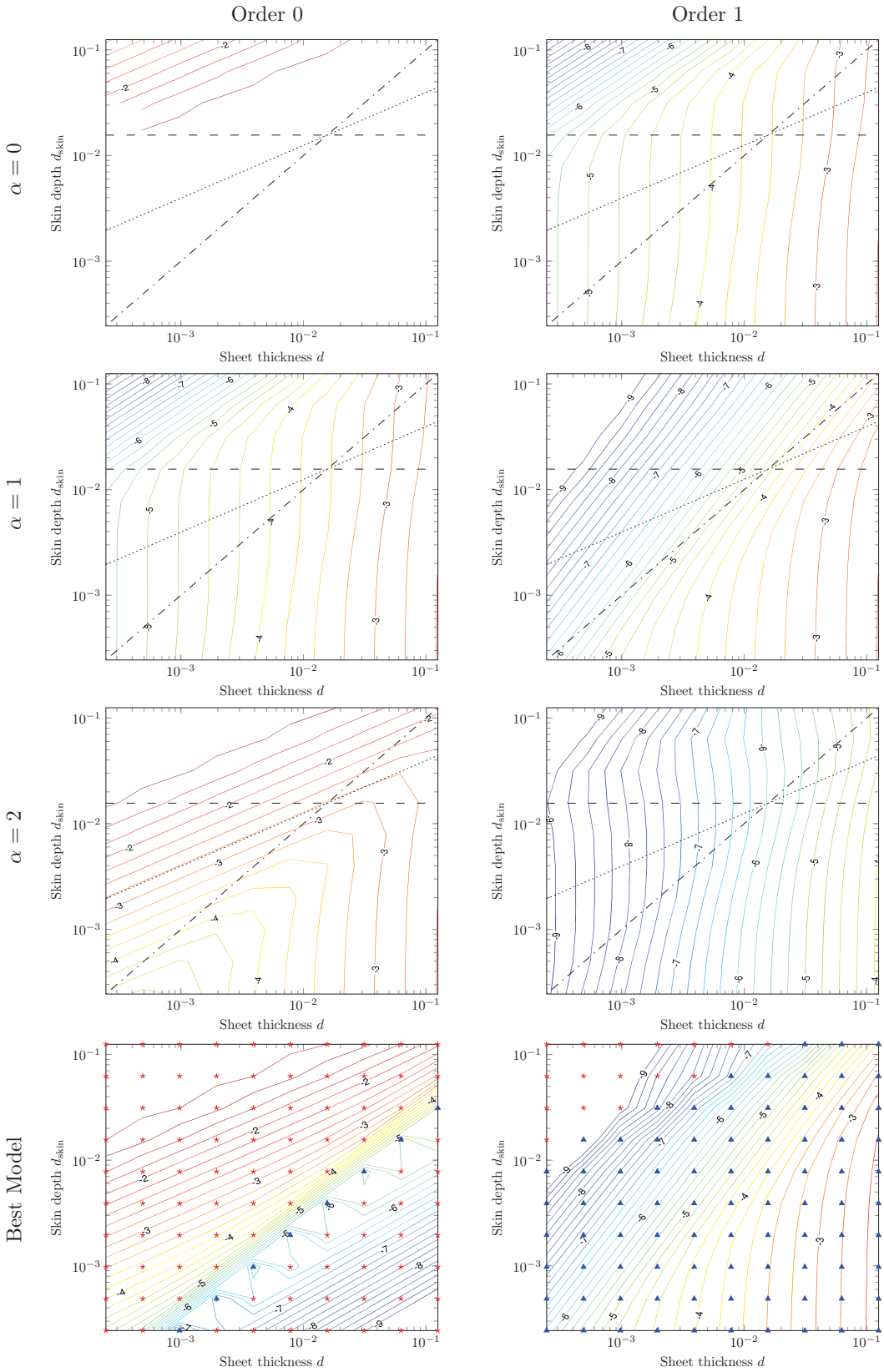


Figure 5: The (decadic logarithmic of the) L^2-error outside the sheet $\|e - e_{\text{ext}}^{\alpha,N}\|_{L^2(\Omega_{\text{ext}})}$ for the limit model ($N = 0$) and the model of order 1 ($N = 1$) for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$. At the abscissa is the thickness d and at the ordinate the skin depth $d_{\text{skin}} = \sqrt{2/|c|}$. The (diagonal) dash-dotted line is where the skin depth coincides with the thickness ($\alpha^\star = 2$). The horizontal dashed line represents constant conductivity ($\alpha^\star = 0$), whereas along the dotted line where the conductivity times the thickness is a constant ($\alpha^\star = 1$). The best model for each point is highlighted with a coloured mark, this is $\star$ for $\alpha = 1$ and $\star$ for $\alpha = 2$, and the difference between the limit models of $\alpha = 1$ and $\alpha = 2$ is shown with contour lines.

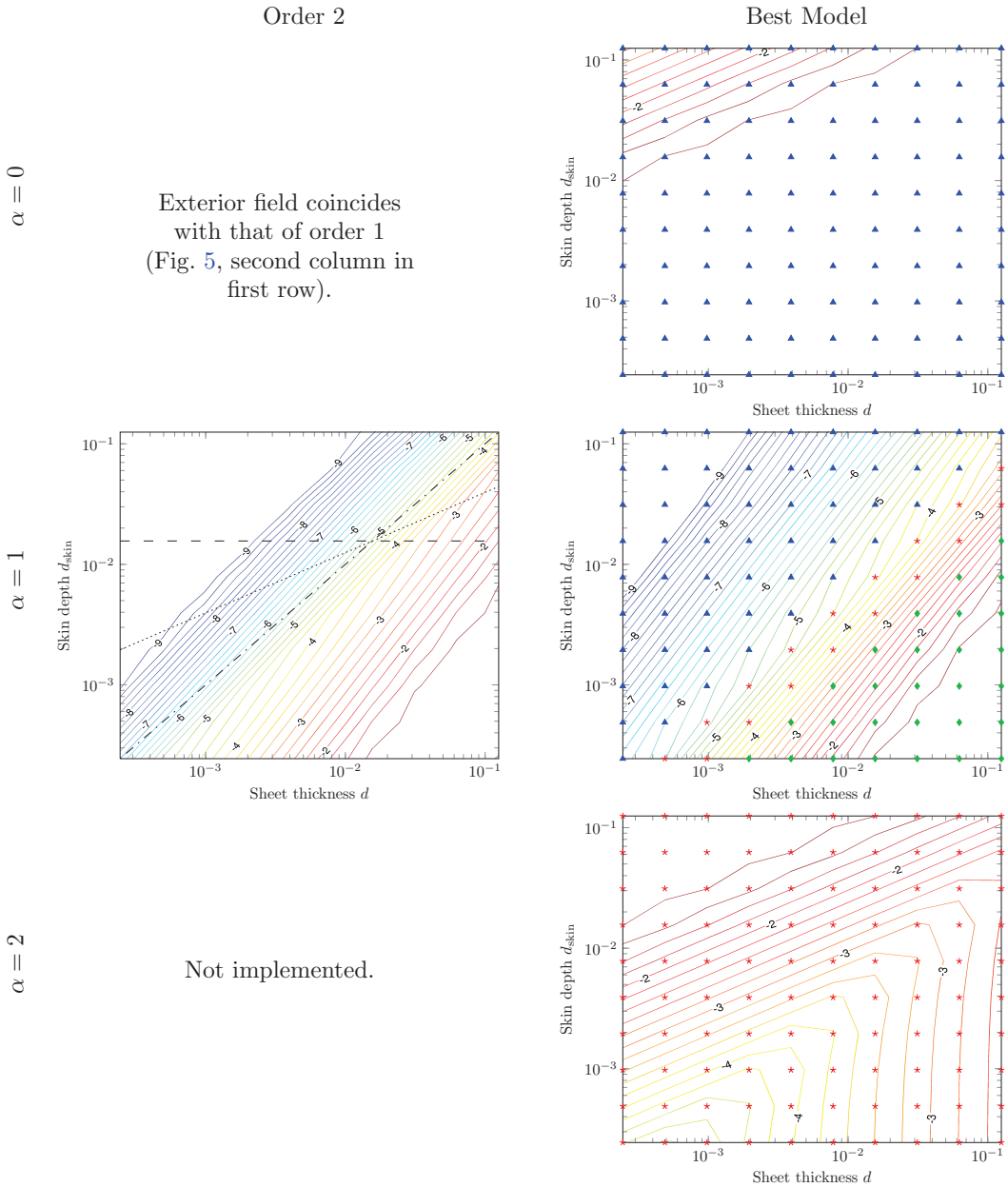


Figure 6: The (decadic logarithmic of the) L^2-error outside the sheet for the model of order 2 for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$ (first column), similarly presented as in Fig. 5. The best model related to each α-framework for each point is highlighted with a coloured mark, this is $\star$ for order 0, $\star$ for order 1, and $\star$ for order 2, and the difference between the model of order 2 and the model of order 1 is shown with contour lines (except for $\alpha = 2$ where the difference is between order 1 and order 0).

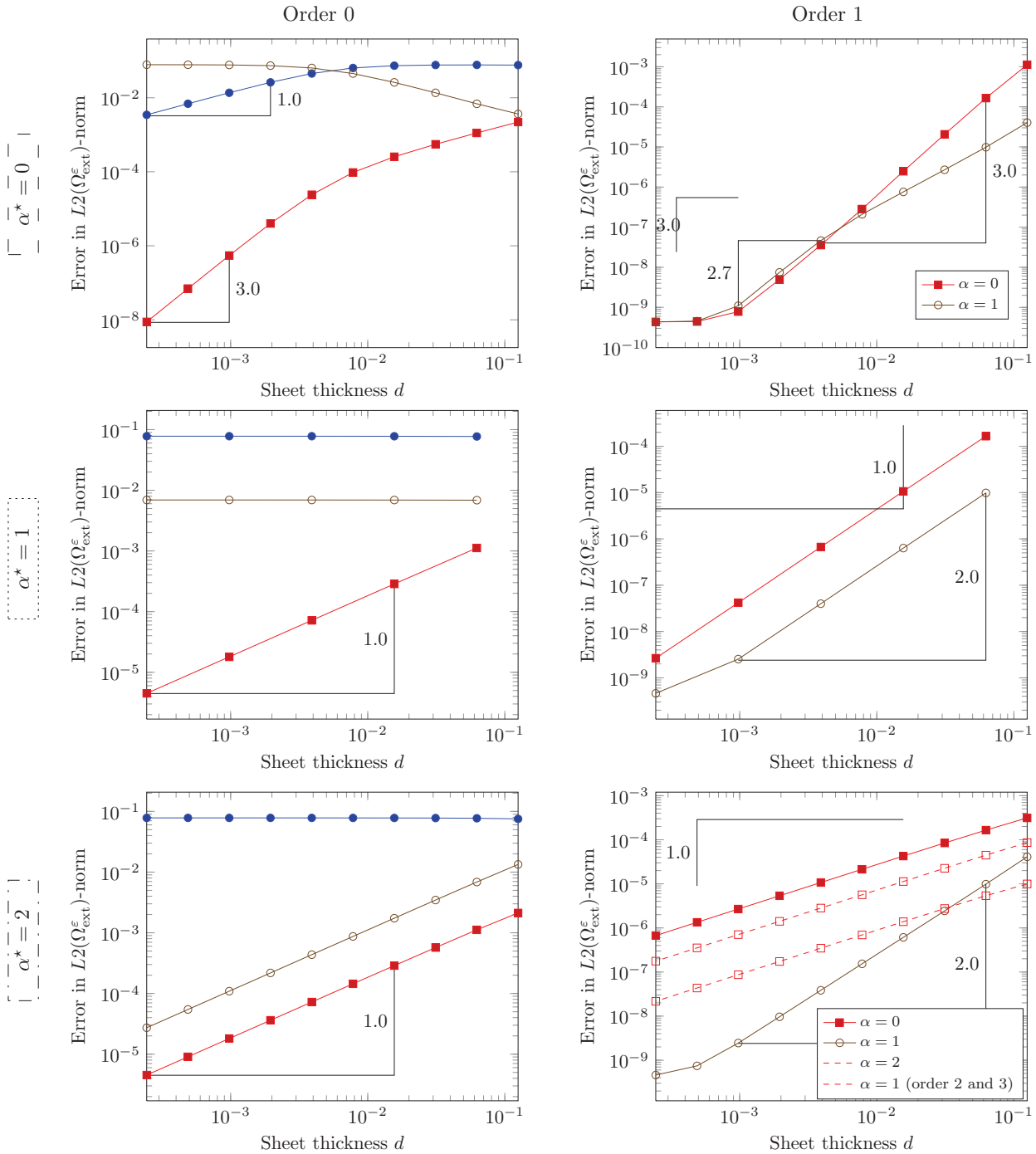


Figure 7: The L^2-error outside the sheet for the limit model and the model of order 1 for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, evaluated along the dashed, dotted and dash-dotted lines in Fig. 5, *i. e.*, in the asymptotics $c(\varepsilon) = c_{\alpha^\star}/\varepsilon^{\alpha^\star}$ for $\alpha^\star = 0$ (constant conductivity), $\alpha^\star = 1$ (asymptotically constant shielding) and $\alpha^\star = 2$ (asymptotically constant skin depth). The dashed, dotted and dash-dotted on the left side indicate the relation to the corresponding lines in Fig. 5. The convergence in the sheet thickness depends notably if one compares sheets of constant conductivity, or conductivities scaled with the inverse or squared inverse of the thickness. The two red dashed lines in second column of the last row are for the transmission conditions of order 2 and order 3 for $\alpha = 1$. Increasing the order of these transmission conditions leads for sheets of asymptotically constant skin depth only lowers the convergence curves, but does not increase their rates.

For the geometrical setting and source terms of Fig. 1 with Neumann boundary conditions on $\partial\Omega$ we performed numerical experiments to compare the exact solution, computed with high-order FEM on a curved mesh, which resolves the geometry exactly, with the solutions of the three transmission conditions of order 0. For the transmission conditions of $\alpha = 0$, where the sheet vanishes, a vanishing mean condition is added. The experiments have been realised for $d = 2^{-12}, 2^{-11}, \dots, 2^{-3}$ and $c = 2^6 i, 2^7 i, \dots, 2^{24} i$. The results for the L^2-error in the exterior Ω_{ext} of the sheet are shown in Fig. 5 (left column), where on the ordinate the effective conductivity is represented by the skin depth. At the diagonal line the skin depth and the thickness coincide. The accuracy of the transmission condition of $\alpha = 1$ outperforms the very poor results of those of $\alpha = 0$ in the whole parameter domain. It also outperforms the results of the transmission conditions of $\alpha = 2$ when the skin depth is not exceeding the sheet thickness. Anyhow, if the skin depth exceeds the sheet thickness the transmission conditions for $\alpha = 1$ are only slightly inferior, as shown in the direct comparison in the last row of Fig. 5, and justifying the robustness of this model.

It is visible when following the dashed lines in the Fig. 5 (first column in first row) that for $\alpha^\star = 1$ no convergence is obtained, and for $\alpha^\star = 2$ the error for decreasing d even increases. For the transmission condition of $\alpha = 2$ (see Fig. 5) convergence for $\alpha^\star = 0, 1$ is obtained only for larger d, asymptotically there is no convergence for $\alpha^\star = 1$ and the error even increases for $\alpha^\star = 0$. For the transmission condition of $\alpha = 1$ the error decreases for $\alpha^\star = 0$ and 2, also asymptotically.

Fig. 7 shows the error along the dashed lines Fig. 5, the convergence rate of 1 of each of the transmission condition in their own asymptotics. The transmission condition for $\alpha = 1$ converge in the asymptotics $\alpha^\star = 0$ with a rate of 3, and for $\alpha^\star = 2$ at least with a rate 1.

4. Deriving the asymptotic expansion of higher orders

The transmission conditions we are going to derive are based on the asymptotic expansion of Sect. 2. To do so, we will solve for the terms $(u^j_{\text{ext}}, U^j_{\text{int}})$ of the asymptotic expansion where the internal solution and the conditions on Γ are written in operator form

$$\sum_{\ell=0}^{j} (\mathbf{G}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) = 0 \quad \text{on } \Gamma, \tag{4.1a}$$

$$\sum_{\ell=0}^{j} (\mathbf{H}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) = 0 \quad \text{on } \Gamma, \tag{4.1b}$$

$$U^j_{\text{int}}(t, S) = \sum_{\ell=0}^{j} (\mathbf{J}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) \qquad \text{in } \Gamma \times (-\tfrac{1}{2}, \tfrac{1}{2}). \tag{4.1c}$$

The families of operators $\{\mathbf{G}^\alpha_\ell\}_{\ell=0}^\infty$, $\{\mathbf{H}^\alpha_\ell\}_{\ell=0}^\infty$, and $\{\mathbf{J}^\alpha_\ell\}_{\ell=0}^\infty$ to be consistent with the asymptotic expansion of Sect. 2 are *not* unique and, consequently not the transmission conditions. The reason for this ambiguity is, that the same condition for the asymptotic expansion terms can be written in several forms, *e. g.*, one may add always a function of $[u^0_{\text{ext}}]$ which equals zero by (3.1a). Transmission conditions for those two choices of operators would differ by $O(\varepsilon^M)$ where $M > N$ with N is the order of the condition. Another example is related to the second normal derivative $\partial_s^2 u^j_{\text{ext}}$ to Γ which can be written in terms of tangential derivatives if u^j_{ext} is harmonic in the neighbourhood of Γ. This change of the operators and so of the transmission condition is practically useful when dealing with variational methods (*e. g.*, discretisation by the finite element method) or boundary integral formulations. When using numerical methods based on the strong form of the PDE, like the finite difference method, the change of (higher) normal to tangential derivatives might be not urgent/stringent.

We will define a particular choice of the operators $\mathbf{G}^\alpha_\ell$, $\mathbf{H}^\alpha_\ell$, and $\mathbf{J}^\alpha_\ell$. Order by order we will

(A) define the operator $\mathbf{J}^\alpha_\ell$ of the internal field, and then
(B) the operators $\mathbf{G}^\alpha_\ell$ and $\mathbf{H}^\alpha_\ell$ using $\mathbf{J}^\alpha_\ell$,

where in each of the steps (A) and (B)

(C) the jumps $[u^m_{\text{ext}}]$, $m = 0, \dots, \ell - 1$ in the conditions are replaced using (4.1a) and Tab. 1, and
(D) the jumps $[\partial_s u^m_{\text{ext}}]$, $m = 0, \dots, \ell - 1$ in the conditions are replaced using (4.1b) and Tab. 1.
(E) the mean or the jump of normal derivatives $\partial_s^k u^m_{\text{ext}}$, with $k \geq 2$ are replaced by means or jumps of tangential derivatives where the expressions derived in steps (C) and (D) are used.

The step (E) is essential for variational formulations, and the steps (C) and (D) will potentially simplify the concrete expressions of the operators (less terms to implement) while the complexity of their abstract definitions increase. The possibility of the replacement of $[u^m_{\text{ext}}]$ and of $[\partial_s u^m_{\text{ext}}]$ for $\alpha = 0, 1$ is obvious by Tab. 1, while that of $[\partial_s u^m_{\text{ext}}]$ for $\alpha = 2$ will use counterintuitively terms of higher order. This procedure corresponds to the derivations for $\alpha = 1$ in [27] and [28].

A major result of the article is the transformation of the system (2.3) for the terms of the asymptotic expansions u^j_{ext}, U^j_{int} in the form (4.1) of a discrete convolution, where we introduce in the following of this section a procedure to obtain the involved operators $\mathbf{G}^\alpha_\ell$, $\mathbf{H}^\alpha_\ell$, and $\mathbf{J}^\alpha_\ell$ to any order. The equivalence of the transformed system is stated in the following theorem.

Theorem 4.1. *The system (4.1), (2.3c), (2.3d), (2.3e), where the operators $\mathbf{G}^\alpha_\ell$, $\mathbf{H}^\alpha_\ell$, and $\mathbf{J}^\alpha_\ell$ are defined in Tab. 1, Def. 4.9, Def. 4.10, and Def. 4.11 is equivalent to the problem (2.3).*

The proof uses the definition of the operators and will be given in Sec. 4.7.

Lemma 4.2 (Convolution of a convolution)**.** *It holds for two sequences of operators $\mathbf{A}_\ell$, $\mathbf{B}_\ell$ and the sequence of functions v^m that*

$$\sum_{\ell=\ell_0}^{j} \mathbf{A}_\ell \sum_{m=m_0}^{j-\ell} \mathbf{B}_m v^{j-\ell-m} = \sum_{\ell=\ell_0+m_0}^{j} \Big(\sum_{m=\ell_0}^{\ell-m_0} \mathbf{A}_m \mathbf{B}_{\ell-m}\Big) v^{j-\ell} = \sum_{\ell=\ell_0+m_0}^{j} \Big(\sum_{m=m_0}^{\ell-\ell_0} \mathbf{A}_{\ell-m} \mathbf{B}_m\Big) v^{j-\ell}.$$

4.1. Variational problem for the exterior expansion functions

With the conditions (4.1b) and (4.1c) we are in the position to collect the problems for the exterior expansion functions only. Taking (2.3c), (2.3d), (2.3e), and the defintions for $\mathbf{G}^\alpha_0$ and $\mathbf{H}^\alpha_0$ (see Tab. 1) we obtain the following system, written in general from for all three asymptotic frameworks.

$$\begin{cases} -\Delta u^j_{\text{ext}}(x) = \delta_{j=0} f, & \text{in } \Omega^0_{\text{ext}}, \\ u^j_{\text{ext}} = \delta_{j=0} e_{\text{imp}} & \text{on } \Gamma_e, \\ \nabla u^j_{\text{ext}} \cdot \underline{n} - \boldsymbol{\beta} u^j_{\text{ext}} = \delta_{j=0} \iota_{\text{imp}} & \text{on } \Gamma_i, \\ [u^j_{\text{ext}}](t) = -\sum_{\ell=1}^{j} (\mathbf{G}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) & \text{on } \Gamma, \\ \left.\begin{array}{ll} \text{for } \alpha = 0, & [\partial_s u^j_{\text{ext}}](t) \\ \text{for } \alpha = 1, & [\partial_s u^j_{\text{ext}}](t) - c_1 \{u^j_{\text{ext}}\}(t) \\ \text{for } \alpha = 2, & \{u^j_{\text{ext}}\}(t) \end{array}\right\} = -\sum_{\ell=1}^{j} (\mathbf{H}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) & \text{on } \Gamma. \end{cases} \tag{4.2}$$

For the variational formulation we incorporate the essential condition for the jumps $[u^j_{\text{ext}}]$ and the mean value $\{u^j_{\text{ext}}\}$ (for $\alpha = 2$) in the solution space, similarily to the Dirichlet boundary condition at Γ_e. Let the vector spaces $V^\alpha_{j,0}$ for the test functions are

$$V^\alpha_{j,0} = H^1_{\Gamma_e}(\Omega), \quad \text{for } \alpha = 0, 1, \qquad \text{and} \qquad V^2_{j,0} = H^1_{\Gamma_e \cup \Gamma}(\Omega_0),$$

where the subscript denotes the part of the boundary with homogeneous Dirichlet data. The affine solution spaces V^α_j for the ansatz functions are

$$V^\alpha_j = V^\alpha_{j,0} - E_{\Gamma_e} \delta_{j=0} e_{\text{imp}} - E_{[\Gamma]} \sum_{\ell=1}^{j} (\mathbf{G}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) + \delta_{\alpha=2} E_{\{\Gamma\}} \sum_{\ell=1}^{j} (\mathbf{H}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t) \subset H^1(\Omega_0).$$

Here, we used besides the extension operator for the Dirichlet boundary data E_{Γ_e} arbitrary continuous jump and mean extension operators $E_{[\Gamma]}, E_{\{\Gamma\}} : H^{k-1/2}(\Gamma) \to H^k(\Omega_0)$, for any $k \in \mathbb{N}$. As the framework for $\alpha = 2$ has two essential conditions (Dirichlet condition from both sides of Γ), the solution in the sub-domains separated by Γ decouples.

In contrast, the conditions, in which the jump of the normal derivative $[\partial_s u^j_{\text{ext}}]$ is involved ($\alpha = 0, 1$), are natural, *i. e.*, they are taken in weak form, so by inserting in the formulation. The natural condition for $\alpha = 0, 1$ leads to a coupling between both sides of Γ.

The variational formulation for (4.2) reads as follows: Seek $u^j_{\rm ext} \in V^\alpha_j$ such that for all $v \in V^\alpha_{j,0}$ it holds

$$\begin{aligned}\mathsf{a}^\alpha_{\rm ext}(u^j_{\rm ext}, v) &= \int_\Omega \nabla u^j_{\rm ext} \cdot \nabla v \mathrm{d}x + \int_{\Gamma_i} \boldsymbol{\beta} u^j_{\rm ext} v \,\mathrm{d}\sigma(\underline{x}) + c_1 \delta_{\alpha=1} \int_\Gamma \{u^j_{\rm ext}\} v \mathrm{d}t \\ &= \delta_{j=0} \left(\int_{\Gamma_i} \iota_{\rm imp} v \,\mathrm{d}\sigma(\underline{x}) + \int_\Omega f v \mathrm{d}x \right) - \delta_{\alpha<2} \int_\Gamma \sum_{\ell=1}^{j} (\mathbf{H}^\alpha_\ell u^{j-\ell}_{\rm ext})(t) v \mathrm{d}t = \ell^\alpha_j(v). \quad (4.3)\end{aligned}$$

It is not difficult to prove that under proper assumptions on $\boldsymbol{\beta}$ and c_1 the bilinear form (see *e. g.* [28]) is $\mathsf{a}^\alpha_{\rm ext}$ is $V^\alpha_{j,0}$-elliptic and continuous in $V^\alpha_{j,0}$. The continuity of ℓ^α_j in $V^\alpha_{j,0}$ requires the assumption of bounded $\iota_{\rm imp}$ and f as well as sufficient regularity of $u^0_{\rm ext}, \dots, u^{j-1}_{\rm ext}$. The latter can be shown for $\alpha = 0, 1, 2$ under the assumption of a smooth sheet Γ as it has been done in [27] for $\alpha = 1$.

Lemma 4.3 (Well-posedness of the exterior problem)**.** *Let Γ be C^∞, $|(\boldsymbol{\beta} v, v)_{L^2(\Gamma_i)}| \geq 0$ for all $v \in C^\infty(\Gamma_i)$, and for $\alpha = 0$ no pure Neumann boundary conditions, i. e., $|\Gamma_e| > 0$ or $|(\boldsymbol{\beta} v, v)_{L^2(\Gamma_i)}| > 0$. Then, the variational formulation (4.3) of the systems (4.2) provide unique solutions $u^j_{\rm ext} \in V^\alpha_j(\Omega)$.*

Lemma 4.4 (Modelling error of the asymptotic expansion of higher orders)**.** *Let the assumption of Lemma 4.3 hold. Then, there exists a constant $C > 0$ such that for $\alpha = 0, 1, 2$ it holds*

$$\|e^\varepsilon - \sum_{j=0}^{N} \varepsilon^j u^j_{\rm ext}\|_{H^1(\Omega^\varepsilon_{\rm ext})} + \sqrt{\varepsilon} \|e^\varepsilon(t,s) - \sum_{j=0}^{N} \varepsilon^j \sum_{\ell=0}^{j} (\mathbf{J}^\alpha_\ell u^{j-\ell}_{\rm ext})(t, \frac{s}{\varepsilon})\|_{H^1(\Omega^\varepsilon_{\rm int})} \leq C\, \varepsilon^{N+1}.$$

The proof is given for $\alpha = 1$ in [27], where the proof for $\alpha = 0, 2$ differs only in the position of the relative conductivity in the interior equations.

4.2. Replacing the jumps $[u^j_{\rm ext}]$ and $[\partial_s u^j_{\rm ext}]$

Using (4.1a) and $(\mathbf{G}^\alpha_0 u)(t) = [u](t)$ (see Tab. 1) we can write

$$[u^j_{\rm ext}](t) = -\sum_{\ell=1}^{j} (\mathbf{G}^\alpha_\ell u^{j-\ell}_{\rm ext})(t) \quad (4.4)$$

We can replace $[\partial_s u^j_{\rm ext}]$ for $\alpha = 0$ with $(\mathbf{H}^0_0 u)(t) = [\partial_s u](t)$ (see Tab. 1) by

$$[\partial_s u^j_{\rm ext}](t) = -\sum_{\ell=1}^{j} (\mathbf{H}^0_\ell u^{j-\ell}_{\rm ext})(t), \quad (4.5a)$$

which are all terms of lower order. For $\alpha = 1$ it is $(\mathbf{H}^1_0 u)(t) = [\partial_s u](t) - c_1\{u\}(t)$ and so

$$[\partial_s u^j_{\rm ext}](t) = c_1\{u^j_{\rm ext}\}(t) - \sum_{\ell=1}^{j} (\mathbf{H}^1_\ell u^{j-\ell}_{\rm ext})(t). \quad (4.5b)$$

The replacement (4.5b) includes besides terms of lower order with a single term of the same order.

For $\alpha = 2$ it seems to be natural to replace $\{u^j_{\rm ext}\}$ as this correspond to $\mathbf{H}^2_0$. We prefer to apply replace as for $\alpha = 0$ and $\alpha = 1$ the term $[\partial_s u^j_{\rm ext}]$, such that the derived transmission conditions take similar forms. To do so we have to anticipate that the $\mathbf{H}$-operator of order 1 is $(\mathbf{H}^2_1 u)(t) = [\partial_s u](t)$. With $(\mathbf{H}^2_0 u)(t) = -h(c_2)\{u\}(t)$ (see Tab. 1) we can therefore write

$$[\partial_s u^j_{\rm ext}](t) = h(c_2)\{u^{j+1}_{\rm ext}\}(t) - \sum_{\ell=1}^{j} (\mathbf{H}^2_{\ell+1} u^{j-\ell}_{\rm ext})(t). \quad (4.5c)$$

Here, we use the function (cf. Fig. 8).

$$h(c_2) = \frac{2\sqrt{c_2}\sinh(\sqrt{c_2}\frac{1}{2})}{\cosh(\sqrt{c_2}\frac{1}{2}) - \frac{1}{2}\sqrt{c_2}\sinh(\sqrt{c_2}\frac{1}{2})}. \quad (4.5d)$$

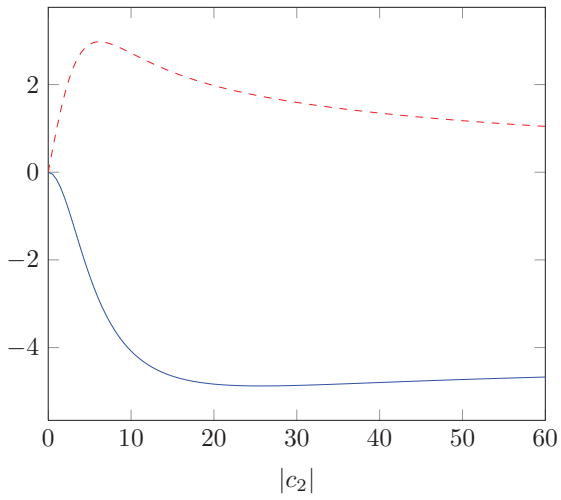


Figure 8: The function $h(c_2)$ defined in (4.5d) with its real (——) and imaginary (- - -) part for $c_2 \in i\mathbb{R}^+$.

The replacement (4.5c) includes besides terms of lower order also a term of an order higher. As we will use this replacement only for the source terms on the right hand sides of at least two orders lower, the asymptotic expansion is still iteratively (cf. Lemma 4.12).

The reader may note, that

$$h(c_2) = c_2 + \frac{1}{6}c_2^2 + O(c_2^3).$$

4.3. Writing higher normal traces in terms of Dirichlet and Neumann traces

The higher normal traces $\partial_s^j u$, which appear in the approximate Dirichlet (2.3a) and Neumann transmission condition (2.3b) are unfavourable in variational formulations where we would like to deal with Sobolev spaces with bounded first weak derivatives where only Dirichlet and Neumann traces can be incorporated. These Dirichlet or Neumann traces may even be stated with higher tangential derivatives where the Sobolev space may be enriched with Dirichlet traces of higher regularity. We therefore will rewrite the higher normal traces in terms of Dirichlet and Neumann traces using the fact that the far field terms are harmonic (cf. (2.3c)). This will be then used in Section 5 to derive the variational formulation of the transmission conditions.

Lemma 4.5 (Higher normal traces of harmonic functions)**.** *Let $\Delta u = 0$ in a neighbourhood of Γ. Then, we can write for $\ell \geq 2$ the recurrence relation*

$$\partial_s^\ell u(t, \pm 0) = (\ell - 2)! \sum_{m=0}^{\ell-1} \left(\frac{\widehat{\Delta}_{\ell-m}^0}{m!} \delta_{m<\ell-1} + \frac{\widehat{\Delta}_{\ell-m}^1}{(m-1)!} \delta_{m>0} \right) \partial_s^m u(t, \pm 0), \tag{4.6}$$

with the operators $\widehat{\Delta}_\ell^0$ and $\widehat{\Delta}_\ell^1$ defined in (2.4).

Proof. First, we observe that the Laplace operator can be written in a neighbourhood of Γ in curvilinear coordinates as [27]

$$\Delta = \partial_s^2 - \sum_{k=1}^{\infty} s^{k-1} \widehat{\Delta}_k^1(t)\, \partial_s - \sum_{k=2}^{\infty} s^{k-2} \widehat{\Delta}_k^0(t).$$

Applying this expression to the harmonic function u and taking the $(\ell-2)$-th derivative in s we get

$$\partial_s^\ell u = \partial_s^{\ell-2} \partial_s^2 u = \partial_s^{\ell-2} \sum_{\sigma=0,1} \sum_{k=2-\sigma}^{\infty} s^{k-2+\sigma} \widehat{\Delta}_k^\sigma(t)\, \partial_s^\sigma u$$

and with the Leibniz rule of differentiation we obtain

$$\partial_s^\ell u = \sum_{\sigma=0,1}\sum_{k=2-\sigma}^{\infty} \widehat{\Delta}_k^\sigma(t) \sum_{m=0}^{\ell-2} \binom{\ell-2}{m} \partial_s^{\ell-2-m}(s^{k-2+\sigma})\partial_s^{m+\sigma} u.$$

Taking the limit $s \to \pm 0$ we have $\partial_s^{\ell-2-m}(s^{k-2+\sigma}) \to \delta_{k=\ell-m-\sigma}(\ell-2-m)!$, *i. e.*, only terms for which $k = \ell - m - \sigma$ are non-zero. Thus, we get

$$\partial_s^\ell u(t,\pm 0) = \sum_{\sigma=0,1}\sum_{m=0}^{\ell-2} \binom{\ell-2}{m} (\ell-2-m)!\, \widehat{\Delta}_{\ell-m-\sigma}^\sigma(t) \partial_s^{m+\sigma} u(t,\pm 0)$$

which easily simplifies to the expression in the statement of the lemma. □

Resolving the recurrence relation given by Lemma 4.5 we get the following corollary (cf. [23]).

Corollary 4.6 (Higher normal traces of harmonic functions)**.** *Let u like in Lemma 4.5. Then, we can write*

$$\partial_s^\ell u(t,\pm 0) = \ell! \left(\mathbf{R}_N^\ell(\partial_s u(t,\pm 0)) + \mathbf{R}_D^\ell(u(t,\pm 0))\right) \tag{4.7}$$

with $\mathbf{R}_D^0 = \mathbf{R}_N^1 = Id$, $\mathbf{R}_N^0 = \mathbf{R}_D^1 = 0$ and for all $\ell \geq 2$

$$\mathbf{R}_{D/N}^\ell = \frac{1}{\ell(\ell-1)} \sum_{m=0}^{\ell-1} \left(\widehat{\Delta}_{\ell-m}^0 \delta_{m<\ell-1} + m\widehat{\Delta}_{\ell-m}^1 \delta_{m>0}\right) \mathbf{R}_{D/N}^m. \tag{4.8}$$

The same holds by linearity for the mean and jumps of higher derivatives. Hence, we can write

$$\frac{1}{\ell!}[\partial_s^\ell u]^m(t) = \mathbf{R}_N^\ell([\partial_s u]^m(t)) + \mathbf{R}_D^\ell([u]^m(t)). \tag{4.9}$$

4.4. Replacing sums of alternating means and jumps of the increasing normal derivatives

In the Dirichlet continuity (2.3a) between external and internal and the Neumann continuity (2.3b) appear terms of the Taylor expansion of the external fields around the midline of the sheet Γ. Thus, there are sums of alternating means and jumps of the increasing normal derivatives of terms of the external expansion of decreasing order. Using the above derived $\mathbf{R}_D$- and $\mathbf{R}_N$-operators for replacing the higher normal derivatives and using the conditions for the jumps of the Dirichlet and Neumann trace we can simplify these sums, or more precisely their mean and jumps. This simplification will be expressed $\mathbf{T}$-operators which we will define in the following.

Definition 4.7. *Let the operators $\mathbf{T}_{\ell,\ell_0}^{\alpha,a,b,c}$ with $\ell_0 \in \{1,2\}$, $a+1 \in \mathbb{N}_0$, $b \in \mathbb{N}_0$, $c \in \mathbb{Z}$, $b = 0$ for c odd be defined for $\ell = 0$ by*

$$(\mathbf{T}_{0,\ell_0}^{\alpha,a,b,c} u)(t) = \delta_{a=2}\delta_{1+c \text{ is even}}\delta_{\ell_0=1} h(c_2) \frac{1}{2(1+a)!}\{u\}(t), \tag{4.10}$$

and for $\ell > 0$ by

$$\begin{aligned}
(\mathbf{T}_{\ell,\ell_0}^{\alpha,a,b,c} u)(t) = {} & \delta_{\ell+c \text{ is odd}} \left(\frac{1}{2}\right)^{\ell-1} \frac{(\ell+b)!}{(\ell+a)!} \left(\mathbf{R}_N^{\ell+b}\{\partial_s u\}(t) + \mathbf{R}_D^{\ell+b}\{u\}(t)\right) \\
& - \sum_{m=\ell_0}^{\ell-1} \delta_{m+c \text{ is even}} \left(\frac{1}{2}\right)^m \frac{(m+b)!}{(m+a)!} (\mathbf{R}_D^{m+b} \mathbf{G}_{\ell-m}^\alpha u)(t) \\
& + \delta_{\alpha=1}\delta_{\ell+c \text{ is even}} \left(\frac{1}{2}\right)^\ell \frac{(\ell+b)!}{(\ell+a)!} c_1 \mathbf{R}_N^{\ell+b}\{u\}(t) \\
& + \delta_{\alpha=2}\delta_{\ell+1+c \text{ is even}} h(c_2) \left(\frac{1}{2}\right)^{\ell+1} \frac{(\ell+1+b)!}{(\ell+1+a)!} \mathbf{R}_N^{\ell+1+b}\{u\}(t) \\
& - \delta_{\alpha<2} \sum_{m=\ell_0}^{\ell-1} \delta_{m+c \text{ is even}} \left(\frac{1}{2}\right)^m \frac{(m+b)!}{(m+a)!} (\mathbf{R}_N^{m+b} \mathbf{H}_{\ell-m}^\alpha u)(t) \\
& - \delta_{\alpha=2} \sum_{m=\ell_0-1}^{\ell-2} \delta_{m+1+c \text{ is even}} \left(\frac{1}{2}\right)^{m+1} \frac{(m+1+b)!}{(m+1+a)!} (\mathbf{R}_N^{m+1+b} \mathbf{H}_{\ell-m}^2 u)(t).
\end{aligned} \tag{4.11}$$

Note, that all operators $\mathbf{T}^{\alpha,a,b,c}_{\ell,\ell_0}$ contain no term $\mathbf{H}^\alpha_0 u$, no term $\mathbf{G}^\alpha_0 u$ and for $\alpha = 2$ even no term $\mathbf{H}^2_1 u$. This is important as they are or will be defined explicitely and not in this iterative derivation.
For $\ell = 0$ we have the following operators

$$(\mathbf{T}^{2,-1,0,-1}_{0,1}u)(t) = (\mathbf{T}^{2,0,0,1}_{0,1}u)(t) = \tfrac{1}{2}h(c_2)\{u\}(t).$$

For $\ell = 1$ we have the operators

$$(\mathbf{T}^{0,0,0,1}_{1,1}u)(t) = 0, \qquad (\mathbf{T}^{1,0,0,1}_{1,1}u)(t) = \tfrac{c_1}{2}\{u\}(t), \qquad (\mathbf{T}^{2,-1,0,\pm1}_{1,\ell_0}u)(t) = (\mathbf{T}^{2,0,0,1}_{1,\ell_0}u)(t) = 0, \text{ for } \ell_0 = 1, 2.$$

Furthermore, for $\alpha = 0, 1$ we have

$$(\mathbf{T}^{\alpha,-1,0,0}_{1,1}u)(t) = (\mathbf{T}^{\alpha,0,0,0}_{1,1}u)(t) = \{\partial_s u\}(t), \qquad (\mathbf{T}^{\alpha,0,1,0}_{1,1}u)(t) = -\kappa(t)\{\partial_s u\}(t) - \partial_t^2\{u\}.$$

and for $\alpha = 2$

$$(\mathbf{T}^{2,-1,0,0}_{1,1}u)(t) = \{\partial_s u\}(t) - \tfrac{1}{4}h(c_2)\kappa(t)\{u\}(t), \qquad (\mathbf{T}^{2,0,0,0}_{1,1}u)(t) = \{\partial_s u\}(t) - \tfrac{1}{8}h(c_2)\kappa(t)\{u\}(t).$$

The operators $\mathbf{T}^{\alpha,a,b,c}_{\ell,\ell_0}$, $\ell_0 = 1, 2$ are constructed such that they fulfill the following lemma.

Lemma 4.8. *Let $\alpha = 0, 1, 2$, $j \in \mathbb{N}$. Then, it holds for a sequence of functions $(v^\ell_{\text{ext}})_{\ell=1}^j$ being harmonic in a neighbourhood of Γ, fulfilling (4.4) and (4.5) for some operators $\mathbf{G}^\alpha_\ell$ and $\mathbf{H}^\alpha_\ell$, $\ell = 1, \dots, j-1$ and $\{v^0_{\text{ext}}\}(t) = 0$ for $\alpha = 2$, that*

$$\sum_{\ell=1}^{j}\left(\frac{1}{2}\right)^\ell \frac{1}{(\ell+a)!}[\partial_s^{\ell+b} v^{j-\ell}_{\text{ext}}]^{\ell+c}(t) = \sum_{\ell=0}^{j}(\mathbf{T}^{\alpha,a,b,c}_{\ell,1} v^{j-\ell}_{\text{ext}})(t), \tag{4.12}$$

and for c odd

$$\sum_{\ell=2}^{j}\left(\frac{1}{2}\right)^\ell \frac{1}{(\ell+a)!}[\partial_s^{\ell+b} v^{j-\ell}_{\text{ext}}]^{\ell+c}(t) = \sum_{\ell=2}^{j}(\mathbf{T}^{\alpha,a,b,c}_{\ell,2} v^{j-\ell}_{\text{ext}})(t). \tag{4.13}$$

Proof. With the notation (2.5) and as v^ℓ_{ext} are harmonic we can write with (4.9)

$$\begin{aligned}\sum_{\ell=\ell_0}^{j}\left(\frac{1}{2}\right)^\ell \frac{1}{(\ell+a)!}[\partial_s^{\ell+b} v^{j-\ell}_{\text{ext}}]^{\ell+c}(t) = \sum_{\ell=\ell_0}^{j}\left(\frac{1}{2}\right)^\ell \frac{(\ell+b)!}{(\ell+a)!}\Big(&2\delta_{\ell+c \text{ is odd}}(\mathbf{R}^{\ell+b}_N\{\partial_s v^{j-\ell}_{\text{ext}}\}(t) + \mathbf{R}^{\ell+b}_D\{v^{j-\ell}_{\text{ext}}\}(t)) + \\ &\delta_{\ell+c \text{ is even}}(\mathbf{R}^{\ell+b}_N[\partial_s v^{j-\ell}_{\text{ext}}](t) + \mathbf{R}^{\ell+b}_D[v^{j-\ell}_{\text{ext}}](t))\Big).\end{aligned} \tag{4.14}$$

Using (4.4) and Lemma 4.2 we can replace the last term in (4.14)

$$\sum_{\ell=\ell_0}^{j}\delta_{\ell+c \text{ is even}}\left(\frac{1}{2}\right)^\ell \frac{(\ell+b)!}{(\ell+a)!}\mathbf{R}^{\ell+b}_D[v^{j-\ell}_{\text{ext}}](t)) = -\sum_{\ell=\ell_0+1}^{j}\sum_{m=\ell_0}^{\ell-1}\delta_{m+c \text{ is even}}\left(\frac{1}{2}\right)^m \frac{(m+b)!}{(m+a)!}(\mathbf{R}^{m+b}_D\mathbf{G}^\alpha_{\ell-m} v^{j-\ell}_{\text{ext}})(t).$$

The latter expression does not change when the first index of the sum over ℓ is replaced by 0 for $\ell_0 = 1$ and 2 for $\ell_0 = 2$ (instead $\ell_0 + 1$).

To replace the second last term in (4.14) we consider the following cases:

(i) Let $\alpha = 0, 1$. Then, with (4.5a), (4.5b), and Lemma 4.2 we get

$$\begin{aligned}\sum_{\ell=\ell_0}^{j}\delta_{\ell+c \text{ is even}}\left(\frac{1}{2}\right)^\ell \frac{(\ell+b)!}{(\ell+a)!}\mathbf{R}^{\ell+b}_N[\partial_s v^{j-\ell}_{\text{ext}}](t) = &\sum_{\ell=\ell_0}^{j}\delta_{\ell+c \text{ is even}}\delta_{\alpha=1}\left(\frac{1}{2}\right)^\ell \frac{(\ell+b)!}{(\ell+a)!}c_1\mathbf{R}^{\ell+b}_N\{\partial_s v^{j-\ell}_{\text{ext}}\}(t) \\ &- \sum_{\ell=\ell_0+1}^{j}\sum_{m=\ell_0}^{\ell-1}\delta_{m+c \text{ is even}}\left(\frac{1}{2}\right)^m \frac{(m+b)!}{(m+a)!}(\mathbf{R}^{m+b}_N\mathbf{H}^\alpha_{\ell-m} v^{j-\ell}_{\text{ext}})(t).\end{aligned}$$

Once again, the first index of the sum over ℓ can be replaced by 0 for $\ell_0 = 1$ and 2 for $\ell_0 = 2$.

(ii) Let $\alpha = 2$. Then, with (4.5c), Lemma 4.2 and $\{v_{\text{ext}}^0\}(t) = 0$ by assumption

$$\begin{aligned}
&\sum_{\ell=\ell_0}^{j} \delta_{\ell+c \text{ is even}} \left(\frac{1}{2}\right)^{\ell} \frac{(\ell+b)!}{(\ell+a)!} \mathbf{R}_N^{\ell+b} [\partial_s v_{\text{ext}}^{j-\ell}](t) \\
&\qquad = \sum_{\ell=\ell_0}^{j} \delta_{\ell+c \text{ is even}} h(c_2) \left(\frac{1}{2}\right)^{\ell} \frac{(\ell+b)!}{(\ell+a)!} \mathbf{R}_N^{\ell+b} \{v_{\text{ext}}^{j+1-\ell}\}(t) \\
&\qquad\quad - \sum_{\ell=\ell_0+1}^{j} \sum_{m=\ell_0}^{\ell-1} \delta_{m+c \text{ is even}} \left(\frac{1}{2}\right)^{m} \frac{(m+b)!}{(m+a)!} (\mathbf{R}_N^{m+b} \mathbf{H}_{\ell+1-m}^2 v_{\text{ext}}^{j-\ell})(t). \\
&\qquad = \sum_{\ell=\ell_0-1}^{j} \delta_{\ell+1+c \text{ is even}} h(c_2) \left(\frac{1}{2}\right)^{\ell} \frac{(\ell+1+b)!}{(\ell+1+a)!} \mathbf{R}_N^{\ell+1+b} \{v_{\text{ext}}^{j-\ell}\}(t) \\
&\qquad\quad - \sum_{\ell=\ell_0+1}^{j} \sum_{m=\ell_0-1}^{\ell-2} \delta_{m+1+c \text{ is even}} \left(\frac{1}{2}\right)^{m+1} \frac{(m+1+b)!}{(m+1+a)!} (\mathbf{R}_N^{m+1+b} \mathbf{H}_{\ell-m}^2 v_{\text{ext}}^{j-\ell})(t).
\end{aligned}$$

The sum over ℓ in the second line can be started at 0 for $\ell_0 = 1$ and 2 for $\ell_0 = 2$ without changing anything. For $\ell_0 = 2$ and c odd the first term in the first sum vanishes and the sum can be started at $\ell = 2$ as well.

With the definition of the $\mathbf{T}_{\ell,\ell_0}^{\alpha,a,b,c}$ the lemma holds for the two cases $\ell_0 = 1$ and $\ell_0 = 2$. □

4.5. Definition of the interior field operators of the asymptotic expansion

Definition 4.9. *Let for $\ell \in \mathbb{N}_0$ the operators $\mathbf{J}_\ell^\alpha$ with $(\mathbf{J}_\ell^\alpha u)(t,\cdot) \in \mathbb{P}_{(\alpha+1)\ell}$ for $\alpha = 0, 1$ and $(\mathbf{J}_\ell^2 u)(t,\cdot) \in \mathbb{V}_{\ell-1}$ for any u smooth enough in a neighbourhood of Γ be defined as the solution of the following differential equations in S, for $\alpha = 0, 1$*

$$\begin{cases}
\partial_S^2 (\mathbf{J}_\ell^\alpha u)(t,S) = \displaystyle\sum_{m=1}^{\ell} (\delta_{m=2-\alpha} c_\alpha - \Delta_m)(\mathbf{J}_{\ell-m}^\alpha u)(t,S), \\
\{\partial_S (\mathbf{J}_\ell^\alpha u)\}(t) = (\mathbf{T}_{\ell,1}^{\alpha,-1,0,0} u)(t), \\
\{\mathbf{J}_\ell^\alpha u\}(t) = \frac{1}{2} (\mathbf{T}_{\ell,1}^{\alpha,0,0,1} u)(t),
\end{cases} \tag{4.15a}$$

and for $\alpha = 2$

$$\begin{cases}
(-\partial_S^2 + c_2)(\mathbf{J}_\ell^2 u)(t,S) = \displaystyle\sum_{m=1}^{\ell} \Delta_m (\mathbf{J}_{\ell-m}^2 u)(t,S), \\
\{\partial_S (\mathbf{J}_\ell^2 u)\}(t) = (\mathbf{T}_{\ell,1}^{2,-1,0,0} u)(t), \\
\{\mathbf{J}_\ell^2 u\}(t) - \frac{1}{4} [\partial_S (\mathbf{J}_\ell^2 u)](t) = \delta_{\ell=0} \{u\}(t) + \dfrac{1}{2} ((\mathbf{T}_{\ell,2}^{2,0,0,-1} - \mathbf{T}_{\ell,2}^{2,-1,0,-1}) u)(t).
\end{cases} \tag{4.15b}$$

For $\alpha = 0, 1, 2$ the systems (4.15) are recurrence relations in ℓ, beginning with $\ell = 0$. The existence and uniqueness of the operators due to ellipticity for $\alpha = 0, 1$ (with assumed $c_2 \in i\mathbb{R}$) and due to Fredholm's alternative for $\alpha = 2$, for which the bilinear form of the respective weak formulation is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_S u \partial_S v + c_2 uv \, \mathrm{d}S - 4\{u\}\{v\},$$

follows by induction in ℓ. Concurrently, by induction in ℓ it results the mapping of the operators to $\mathbb{P}_{(\alpha+1)\ell}$ for $\alpha = 0, 1$ and to $\mathbb{V}_{\ell-1}$ for $\alpha = 2$. When applying $\mathbf{J}_\ell^\alpha$ regularity in t is lost due to the higher normal derivatives and the operators Δ_m. The solutions are even so defined as we assumed enough regularity of the sheet which results in higher regularity of the right hand sides. When resolving these recurrences we can define $(\mathbf{J}_\ell^\alpha u)(t,S)$ in terms of c_α, $\widehat{\Delta}_m^0$ and $\widehat{\Delta}_m^1$ (within Δ_m) applied to $\{u\}(t)$ and $\{\partial_s u\}(t)$.

It is important to note, that the operators $\mathbf{J}_\ell^2$ depend through $\mathbf{T}_{\ell,2}^{2,-1/0,0,-1}$ on $\mathbf{H}_2^2, \ldots, \mathbf{H}_{\ell-2}^2$ only.

The operators for $\ell = 0$ introduced in Tab. 1 fulfill Def. 4.9 and those for $\ell = 1$ are given by

$$(\mathbf{J}_1^\alpha u)(t,S) = \begin{cases} \{\partial_s u\}(t) S, & \text{for } \alpha = 0, \\ \{\partial_s u\}(t) S + \frac{c_1}{2}\{u\}(t)(S^2 + \frac{1}{4}) & \text{for } \alpha = 1, \\ \left(\{\partial_s u\}(t) + \frac{\kappa(t)}{2}\{u\}(t)\right) \frac{\sinh(\sqrt{c_2}S)}{\sqrt{c_2}\cosh(\sqrt{c_2}\frac{1}{2})} - \frac{\kappa(t)}{2}\{u\}(t) \frac{\cosh(\sqrt{c_2}S)S}{\cosh(\sqrt{c_2}\frac{1}{2}) - \frac{1}{2}\sqrt{c_2}\sinh(\sqrt{c_2}\frac{1}{2})} & \text{for } \alpha = 2. \end{cases}$$

4.6. Definition of the exterior fields

Definition 4.10. *Let for $\ell \in \mathbb{N}$ the operators $\mathbf{G}_\ell^\alpha$ for any u smooth enough in a neighbourhood of Γ be defined for $\alpha = 0, 1, 2$ as*

$$(\mathbf{G}_\ell^\alpha u)(t) = -[\mathbf{J}_\ell^\alpha u](t) + (\mathbf{T}_{\ell,1}^{\alpha,0,0,0} u)(t). \tag{4.16}$$

The operators for $\ell = 1$ are

$$(\mathbf{G}_1^\alpha u)(t) = \begin{cases} 0, & \text{for } \alpha = 0, 1, \\ \left(\{\partial_s u\}(t) + \frac{\kappa(t)}{2}\{u\}(t)\right)\left(1 - \frac{2\sinh(\sqrt{c_2}\frac{1}{2})}{\sqrt{c_2}\cosh(\sqrt{c_2}\frac{1}{2})}\right) & \text{for } \alpha = 2. \end{cases}$$

Definition 4.11. *Let for $\ell \in \mathbb{N}$ the operators $\mathbf{H}_\ell^\alpha$ for any u smooth enough in a neighbourhood of Γ be defined for $\alpha = 0, 1$ as*

$$(\mathbf{H}_\ell^\alpha u)(t) = (\mathbf{T}_{\ell,1}^{\alpha,0,1,0} u)(t) - \sum_{m=0}^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta_{m=1-\alpha} c_\alpha - \Delta_{m+1})(\mathbf{J}_{\ell-m}^\alpha u)(t,S)\,\mathrm{d}S, \tag{4.17a}$$

for $\alpha = 2$ and for $\ell = 1$

$$(\mathbf{H}_1^2 u)(t) = [\partial_s u](t). \tag{4.17b}$$

$\ell > 1$ as

$$(\mathbf{H}_\ell^2 u)(t) = -4\{\mathbf{J}_\ell^2 u\}(t) + 2(\mathbf{T}_{\ell,2}^{2,0,0,1} u)(t). \tag{4.17c}$$

It is important to note, that the definition for $\alpha = 2$ and $\ell = 1$ is that was we have used in Sec. 4.2. No other of the defined operators $\mathbf{T}_{\ell,\ell_0}^{2,a,b,c}$, $\mathbf{G}_\ell^2$, $\mathbf{H}_\ell^2$, or $\mathbf{J}_\ell^2$ depend on $\mathbf{H}_1^2$ such that none of them possess a term $[\partial_s u]$.

The operators $\mathbf{H}_\ell^2$ for $\ell > 1$ are well defined by (4.17c) as $\mathbf{J}_\ell^2$ and $\mathbf{T}_{\ell,2}^{2,0,0,1}$ depend only on $\mathbf{H}_m^2$, $m < \ell - 1$. The remaining operators for $\ell = 1$ are

$$(\mathbf{H}_1^\alpha u)(t) = \begin{cases} -c_0\{u\}(t), & \text{for } \alpha = 0, \\ -\dfrac{c_1^2}{6}\{u\}(t), & \text{for } \alpha = 1. \end{cases} \tag{4.18}$$

Lemma 4.12. *Let $\alpha = 0, 1, 2$, $\ell \in \mathbb{N}_0$ and u any smooth enough function in Ω_{ext}^0. The following functions in u are well-defined linear functions in $\{u\}$ and $\{\partial_s u\}$ only:*

1. *$\mathbf{J}_\ell^\alpha u$ as defined in Def. 4.9 for all $\ell \geq 0$,*
2. *$\mathbf{G}_\ell^\alpha u$ as defined in Def. 4.10 for all $\ell \geq 1$,*
3. *$\mathbf{H}_\ell^\alpha u$ as defined in Def. 4.11 for all $\ell \geq 1$ in case of $\alpha < 2$ and for all $\ell \geq 2$ in case of $\alpha = 2$.*

Proof. We first note by their respective definitions, that

- $\mathbf{T}_{\ell,\ell_0}^{\alpha,a,b,c} u$ is for $\alpha < 2$ a linear function in $\{u\}$, $\{\partial_s u\}$, $\mathbf{G}_1^\alpha u, \ldots, \mathbf{G}_{\ell-\ell_0}^\alpha u$, and $\mathbf{H}_1^\alpha u, \ldots, \mathbf{H}_{\ell-\ell_0}^\alpha u$.
- $\mathbf{T}_{\ell,1}^{2,a,b,0} u$ and $\mathbf{T}_{\ell,2}^{2,a,b,c} u$ are linear functions in $\{u\}$, $\{\partial_s u\}$, $\mathbf{G}_1^2 u, \ldots, \mathbf{G}_{\ell-1}^2 u$, and $\mathbf{H}_2^2 u, \ldots, \mathbf{H}_{\ell-1}^2 u$,

and consequently

- $\mathbf{J}^\alpha_\ell u$, $\mathbf{G}^\alpha_\ell u$, and $\mathbf{H}^\alpha_\ell u$ are linear functions in $\{u\}$, $\{\partial_s u\}$, $\mathbf{J}^\alpha_0 u, \dots, \mathbf{J}^\alpha_{\ell-1} u$, $\mathbf{G}^\alpha_1 u, \dots, \mathbf{G}^\alpha_{\ell-1} u$, and $\mathbf{H}^\alpha_{1+\delta_{\alpha=2}} u, \dots, \mathbf{H}^\alpha_{\ell-1} u$.

Now, it is easy to verify that under the assumption that the statement of the Lemma is true for all $\ell' < \ell$ that it is true for ℓ as well, and so by complete induction in ℓ the lemma is proved. □

As a consequence of Lemma 4.12 the function $\mathbf{T}^{\alpha,a,b,c}_{\ell,\ell_0} u$ as defined in Def. 4.7 is also a linear function in $\{u\}$ and $\{\partial_s u\}$ only.

4.7. Proof of Theorem 4.1

Proof of Theorem 4.1. The conditions (2.3c), (2.3d), (2.3e) are included in both problems, and we will use out of these conditions only that all the function u^j_{ext} are harmonic in a neighbourhood of Γ.

The proof is by complete induction in j. For some order $j > 0$, we assume the equivalence for all orders till $j-1$ including for $\alpha = 0, 1$ the Neumann compability condition for U^j_{int}. We show that the same holds order j.

Since (4.1c) and the respective first equation of (4.15a) and (4.15b) hold up to order $j-1$, using Lemma 4.2 and $(-\partial_S^2 + \delta_{\alpha=2} c_2)(\mathbf{J}^\alpha_0 u^j_{\text{ext}})(t) = 0$ (see Tab. 1) it follows

$$\begin{aligned}(\partial_S^2 - \delta_{\alpha=2} c_2) U^j_{\text{int}}(t,S) &\overset{(2.3f)}{=} \sum_{\ell=1}^{j} (\delta_{\ell=2-\alpha} c_\alpha - \Delta_\ell) U^{j-\ell}_{\text{int}})(t,S) \overset{(4.1c)}{=} \sum_{\ell=1}^{j} (\delta_{\ell=2-\alpha} c_\alpha - \Delta_\ell) \sum_{m=0}^{j-\ell} (\mathbf{J}^\alpha_m u^{j-\ell-m}_{\text{ext}})(t,S) \\ &= \sum_{\ell=1}^{j} \sum_{m=1}^{\ell} (\delta_{m=2-\alpha} c_\alpha - \Delta_m)(\mathbf{J}^\alpha_{\ell-m} u^{j-\ell}_{\text{ext}})(t,S) = (\partial_S^2 - \delta_{\alpha=2} c_2) \sum_{\ell=0}^{j} (\mathbf{J}^\alpha_\ell u^{j-\ell}_{\text{ext}})(t,S).\end{aligned}$$

Applying the mean operator to (2.3b), using subsequently Lemma 4.8 due to the harmonicity of $u^{j-\ell}_{\text{ext}}$, $\ell = 1, \dots, j$ in a neighbourhood of Γ, and using the respective second equations in (4.15a), (4.15b) together with $\{\partial_S \mathbf{J}^\alpha_0 u^j_{\text{ext}}\} = 0$ by definition (see Tab. 1) we get

$$\{\partial_S U^j_{\text{int}}\}(t) \overset{(2.3b)}{=} \sum_{\ell=1}^{j} \left(\frac{1}{2}\right)^\ell \frac{1}{(\ell-1)!} [\partial_s^\ell u^{j-\ell}_{\text{ext}}]^\ell(t) = \sum_{\ell=1}^{j} (\mathbf{T}^{\alpha,-1,0,0}_\ell u^{j-\ell}_{\text{ext}})(t) = \{\partial_S \sum_{\ell=0}^{j} (\mathbf{J}^\alpha_\ell u^{j-\ell}_{\text{ext}})\}(t).$$

For $\alpha = 0, 1$, applying the mean operator to (2.3a), using $\{\mathbf{J}^\alpha_0 u^j_{\text{ext}}\}(t) = \mathbf{J}^\alpha_0 u^j_{\text{ext}}(t) = \{u^j_{\text{ext}}\}(t)$ by definition (see Tab. 1), Lemma 4.8 and the first equation in (4.15a) we get

$$\begin{aligned}\{U^j_{\text{int}}\}(t) &\overset{(2.3a)}{=} \{u^j_{\text{ext}}\}(t) + \sum_{\ell=1}^{j} \left(\frac{1}{2}\right)^{\ell+1} \frac{1}{\ell!} [\partial_s^\ell u^{j-\ell}_{\text{ext}}]^{\ell+1}(t) \\ &= \{\mathbf{J}^\alpha_0 u^j_{\text{ext}}\}(t) + \frac{1}{2} \sum_{\ell=1}^{j} (\mathbf{T}^{\alpha,0,0,1}_\ell u^{j-\ell}_{\text{ext}})(t) = \{\sum_{\ell=0}^{j} (\mathbf{J}^\alpha_\ell u^{j-\ell}_{\text{ext}})\}(t).\end{aligned}$$

Applying the jump operator to (2.3b) for $\alpha = 2$ we get

$$\begin{aligned}[\partial_S U^j_{\text{int}}](t) &\overset{(2.3b)}{=} \sum_{\ell=1}^{j} \left(\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!} [\partial_s^\ell u^{j-\ell}_{\text{ext}}]^{\ell-1}(t) = 2 \sum_{\ell=0}^{j} (\mathbf{T}^{2,-1,0,-1}_\ell u^{j-\ell}_{\text{ext}})(t) \\ &= h(c_2)\{u^j_{\text{ext}}\}(t) + [\partial_S \sum_{\ell=1}^{j} (\mathbf{J}^2_\ell u^{j-\ell}_{\text{ext}})](t) = [\partial_S \sum_{\ell=1}^{j} (\mathbf{J}^2_\ell u^{j-\ell}_{\text{ext}})](t),\end{aligned}$$

where we used Lemma 4.8, the last equation in (4.15b) and $[\partial_S \mathbf{J}^2_0 u^j_{\text{ext}}](t) = h(c_2)\{u^j_{\text{ext}}\}(t)$ by definition (see Tab. 1).

Due to the unique definition of $U^j_{\text{int}}(t,S)$ by the above four equations for all cases $\alpha = 0, 1, 2$ it follows (4.1c) for order j from (2.3a), (2.3b) and (2.3f). We have also shown that given (4.1c) it follows (2.3f), the mean operator applied to (2.3b), the mean operator applied to (2.3a) for $\alpha = 0, 1$ and the jump operator applied to (2.3b) for $\alpha = 2$.

Applying the jump operator to (2.3a), using Lemma 4.8, and the definition of $\mathbf{G}_\ell^\alpha$ by (4.16) and Tab. 1 we get

$$[U_{\text{int}}^j](t) \overset{(2.3a)}{=} [u_{\text{ext}}^j](t) + \sum_{\ell=1}^{j} \left(\frac{1}{2}\right)^\ell \frac{1}{\ell!} [\partial_s^\ell u_{\text{ext}}^{j-\ell}]^\ell(t) = (\mathbf{G}_0^\alpha u_{\text{ext}}^j)(t) + \sum_{\ell=1}^{j} (\mathbf{T}_\ell^{\alpha,0,0,0} u_{\text{ext}}^{j-\ell})(t)$$

$$= (\mathbf{G}_0^\alpha u_{\text{ext}}^j)(t) + \sum_{\ell=1}^{j} (\mathbf{G}_\ell^\alpha u_{\text{ext}}^{j-\ell})(t) + [\mathbf{J}_\ell^\alpha u_{\text{ext}}^{j-\ell}](t) = \sum_{\ell=0}^{j} (\mathbf{G}_\ell^\alpha u_{\text{ext}}^{j-\ell})(t) + [\mathbf{J}_\ell^\alpha u_{\text{ext}}^{j-\ell}](t),$$

where we used $[\mathbf{J}_0^\alpha u_{\text{ext}}^j] = 0$ by definition. So, (2.3) entails with (4.1c) and so $[U_{\text{int}}^j](t) = \sum_{\ell=0}^{j} [\mathbf{J}_\ell^\alpha u_{\text{ext}}^{j-\ell}](t)$ in (4.1a) for order j. Contrary, if (4.1) is fulfilled up to order j then it follow that (2.3a) for order j holds when the jump operator is applied.

For $\alpha = 2$, applying the mean operator to (2.3a), using Lemma 4.8, and the definition of $\mathbf{H}_\ell^\alpha$ by (4.16) and Tab. 1 and $\{\mathbf{J}_0^2 u_{\text{ext}}^j\} = (1 + \frac{1}{4}h(c_2))\{u_{\text{ext}}^j\}$, $\{\mathbf{J}_1^2 u_{\text{ext}}^{j-1}\} = 0$ by definition we get

$$\{U_{\text{int}}^j\}(t) \overset{(2.3a)}{=} \{u_{\text{ext}}^j\}(t) + \sum_{\ell=1}^{j} \left(\frac{1}{2}\right)^{\ell+1} \frac{1}{\ell!} [\partial_s^\ell u_{\text{ext}}^{j-\ell}]^{\ell+1}(t)$$

$$= \{\mathbf{J}_0^2 u_{\text{ext}}^j\}(t) + \{\mathbf{J}_1^2 u_{\text{ext}}^{j-1}\} - \tfrac{1}{4}h(c_2)\{u_{\text{ext}}^j\}(t) + \tfrac{1}{4}[\partial_s u_{\text{ext}}^{j-1}](t) + \sum_{\ell=2}^{j} \left(\frac{1}{2}\right)^{\ell+1} \frac{1}{\ell!} [\partial_s^\ell u_{\text{ext}}^{j-\ell}]^{\ell+1}(t)$$

$$\overset{(4.5c)}{=} \{\mathbf{J}_0^2 u_{\text{ext}}^j\}(t) + \{\mathbf{J}_1^2 u_{\text{ext}}^{j-1}\} - \tfrac{1}{4}\sum_{\ell=2}^{j} (\mathbf{H}_\ell^2 u_{\text{ext}}^j)(t) + \frac{1}{2}\sum_{\ell=2}^{j} (\mathbf{T}_{\ell,2}^{2,0,0,1} u_{\text{ext}}^{j-\ell})(t) = \sum_{\ell=0}^{j} \{\mathbf{J}_\ell^2 u_{\text{ext}}^{j-\ell}\}(t).$$

Thus, (2.3) for $\alpha = 2$ entails with (4.1c) and so $\{U_{\text{int}}^j\}(t) = \sum_{\ell=0}^{j} \{\mathbf{J}_\ell^2 u_{\text{ext}}^{j-\ell}\}(t)$ in (4.1b) for order j. Contrary, if (4.1) is fulfilled up to order j then it follow that (2.3a) for order j holds for $\alpha = 2$ when the mean operator is applied.

For $\alpha = 0, 1$, using (2.3f), Lemma 4.8, $(\mathbf{J}_0^\alpha u_{\text{ext}}^j)(t, S) = \{u_{\text{ext}}^j\}(t)$ by definition, and $\Delta_1\{u_{\text{ext}}^j\}(t) = 0$ we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_S^2 U_{\text{int}}^{j+1}(t, S),\, \mathrm{d}S \overset{(2.3f)}{=} \sum_{\ell=1}^{j+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta_{\ell=2-\alpha} c_\alpha - \Delta_\ell) U_{\text{int}}^{j+1-\ell}(t, S)\, \mathrm{d}S$$

$$\overset{(4.1c)}{=} \sum_{\ell=1}^{j+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta_{\ell=2-\alpha} c_\alpha - \Delta_\ell) \sum_{m=0}^{j+1} (\mathbf{J}_\ell^\alpha u_{\text{ext}}^{j+1-\ell-m})(t, S)\, \mathrm{d}S$$

$$= \sum_{\ell=1}^{j+1} \sum_{m=1}^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta_{m=2-\alpha} c_\alpha - \Delta_m)(\mathbf{J}_{\ell-m}^\alpha u_{\text{ext}}^{j+1-\ell})(t, S)\, \mathrm{d}S$$

$$= \delta_{\alpha=1} c_1 \{u_{\text{ext}}^j\}(t) + \sum_{\ell=1}^{j} \sum_{m=0}^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta_{m=1-\alpha} c_\alpha - \Delta_{m+1})(\mathbf{J}_{\ell-m}^\alpha u_{\text{ext}}^{j-\ell})(t, S)\, \mathrm{d}S.$$

With (2.3b) we have

$$[\partial_S U_{\text{int}}^{j+1}](t) = \sum_{\ell=1}^{j+1} \left(\frac{1}{2}\right)^{\ell-1} \frac{1}{(\ell-1)!} [\partial_s^\ell u_{\text{ext}}^{j+1-\ell}]^{\ell-1}(t) = [\partial_s u_{\text{ext}}^j](t) + \sum_{\ell=1}^{j} \left(\frac{1}{2}\right)^\ell \frac{1}{\ell!} [\partial_s^{\ell+1} u_{\text{ext}}^{j-\ell}]^\ell(t)$$

and so with Lemma 4.8 and the definition of $\mathbf{H}_\ell^\alpha$ in Tab. 1 and in (4.17a)

$$0 = [\partial_S U_{\text{int}}^{j+1}](t) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial_S^2 U_{\text{int}}^{j+1}(t, S)\, \mathrm{d}S$$

$$= [\partial_s u_{\text{ext}}^j](t) - \delta_{\alpha=1} c_1 \{u_{\text{ext}}^j\}(t) + \sum_{\ell=1}^{j} (\mathbf{T}_\ell^{\alpha,0,1,0} u_{\text{ext}}^{j-\ell})(t) - \sum_{m=0}^{\ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} (\delta_{m=1-\alpha} c_\alpha - \Delta_{m+1})(\mathbf{J}_{\ell-m}^\alpha u_{\text{ext}}^{j-\ell})(t, S)\, \mathrm{d}S$$

$$= (\mathbf{H}_0^\alpha u_{\text{ext}}^j)(t) + \sum_{\ell=1}^{j} (\mathbf{H}_\ell^\alpha u_{\text{ext}}^{j-\ell})(t).$$

Thus, the Neumann compatibility condition of (2.3) for U_{int}^{j+1} in case of $\alpha = 0, 1$ and (2.3b) entails with (4.1c) in (4.1b) for order j. Contrary, if (4.1) is fulfilled up to order j then it follow that (2.3b) holds for order j if the jump operator is applied. Additionally, the Neumann compatibility condition of (2.3) holds for U_{int}^{j+1}.

We have shown that the remaining equations are equivalent for order j and the Neumann compatibility condition for order $j+1$. This completes the induction step, and the statement of the lemma holds. □

5. The transmission conditions of higher orders

5.1. Derivation of the transmission conditions

To derive the transmission conditions of higher orders we write the formal Taylor series of $e_{\text{ext}}^{\varepsilon}$ by (2.1) and $e_{\text{int}}^{\varepsilon}$ by (2.2),

$$e_{\text{ext}}^{\varepsilon}(x) \sim \sum_{j=0}^{+\infty} \varepsilon^j u_{\text{ext}}^j(x) \qquad\qquad e_{\text{int}}^{\varepsilon}(t,s) \sim \sum_{j=0}^{+\infty} \varepsilon^j U_{\text{int}}^j(t, \frac{s}{\varepsilon}).$$

We use the symbol "$\sim$" to highlight that these series' converge for $\varepsilon \to 0$, but they may diverge or converge not to $e_{\text{ext}}^{\varepsilon}$ and $e_{\text{int}}^{\varepsilon}$, when ε is fixed and the order N tends to infinity.

We multiply for all $j \in \mathbb{N}$ the conditions (4.1) by ε^j and summing we get

$$\sum_{j=0}^{+\infty} \varepsilon^j \sum_{\ell=0}^{j} (\mathbf{G}_\ell^\alpha u_{\text{ext}})(t) = 0 \quad \text{on } \Gamma, \tag{5.1a}$$

$$\sum_{j=0}^{+\infty} \varepsilon^j \sum_{\ell=0}^{j} (\mathbf{H}_\ell^\alpha u_{\text{ext}})(t) = 0 \quad \text{on } \Gamma, \tag{5.1b}$$

$$\sum_{j=0}^{+\infty} \varepsilon^j U_{\text{int}}^j(t,S) = \sum_{j=0}^{+\infty} \varepsilon^j \sum_{\ell=0}^{j} (\mathbf{J}_\ell^\alpha u_{\text{ext}})(t,S) \qquad \text{in } \Gamma \times (-\tfrac{1}{2}, \tfrac{1}{2}). \tag{5.1c}$$

Interchanging the sums in j and ℓ and identifying $e_{\text{ext}}^{\varepsilon}$ and $e_{\text{int}}^{\varepsilon}$, we get

$$\mathbf{G}^{\alpha,\varepsilon} e_{\text{ext}}^{\varepsilon}(t) \sim 0 \quad \text{on } \Gamma, \tag{5.2a}$$

$$\mathbf{H}^{\alpha,\varepsilon} e_{\text{ext}}^{\varepsilon}(t) \sim 0 \quad \text{on } \Gamma, \tag{5.2b}$$

$$e_{\text{int}}^{\varepsilon}(t) \sim \mathbf{J}^{\alpha,\varepsilon} e_{\text{ext}}^{\varepsilon}(t) \qquad \text{in } \Gamma \times (-\tfrac{1}{2}, \tfrac{1}{2}), \tag{5.2c}$$

with the formal operator series' $\mathbf{G}^{\alpha,\varepsilon} := \lim_{N\to\infty} \mathbf{G}^{\alpha,\varepsilon,N}$, $\mathbf{H}^{\alpha,\varepsilon} := \lim_{N\to\infty} \mathbf{H}^{\alpha,\varepsilon,N}$ and $\mathbf{H}^{\alpha,\varepsilon} := \lim_{N\to\infty} \mathbf{J}^{\alpha,\varepsilon,N}$ given by

$$\mathbf{G}^{\alpha,\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j \mathbf{G}_j^\alpha, \qquad \mathbf{H}^{\alpha,\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j \mathbf{H}_j^\alpha, \qquad \mathbf{J}^{\alpha,\varepsilon,N} = \sum_{j=0}^{N} \varepsilon^j \mathbf{J}_j^\alpha. \tag{5.3}$$

Although, the conditions (5.1) seem to be perfect, the question of convergence of the operator series' (5.3) for $N \to \infty$ remains. We are in doubt that they converge for all combinations of α, c_α, and ε, as some some combination the modelling error in numerical experiments increase when increasing the order N of the transmission condition.

The limit operators $\mathbf{G}^{\alpha,\varepsilon}$, $\mathbf{H}^{\alpha,\varepsilon}$, and $\mathbf{J}^{\alpha,\varepsilon}$ for $N \to \infty$ seem not to be possible to write in a simple form, by what we are using truncated series' at a certain order N.

We use the transmission conditions derived for varying and small sheet thicknesses ε for sheets of a particular of thickness d and conductivity c. This means first that we replace c_α by cd^α and S by s/d. Then, we omit the superscript ε in the operators (5.3) which even so depend on c and d. We also omit ε as superscript in the approximative solutions, while keeping the superscripts α for the asymptotics in which the transmission conditions have been derived and the approximation order N.

Thus, the problems for the approximative solutions $e_{\text{ext}}^{\alpha,N}$ and $e_{\text{int}}^{\alpha,N}$ reads

$$
\begin{aligned}
-\Delta e_{\text{ext}}^{\alpha,N}(x) &= f(x), && \text{in } \Omega_{\text{ext}}^0, && (5.4a)\\
(\mathbf{G}^{\alpha,N} e_{\text{ext}}^{\alpha,N})(t) &= 0, && \text{on } \Gamma, && (5.4b)\\
(\mathbf{H}^{\alpha,N} e_{\text{ext}}^{\alpha,N})(t) &= 0, && \text{on } \Gamma, && (5.4c)\\
e_{\text{ext}}^{\alpha,N}(x) &= e_{\text{imp}}(x) && \text{on } \Gamma_e, && (5.4d)\\
\nabla e_{\text{ext}}^{\alpha,N}(x)\cdot \underline{n} - \boldsymbol{\beta} e_{\text{ext}}^{\alpha,N}(x) &= \iota_{\text{imp}}(x) && \text{on } \Gamma_i, && (5.4e)\\
e_{\text{int}}^{\alpha,N}(t,s) &= (\mathbf{J}^{\alpha,N} e_{\text{ext}}^{\alpha,N})(t,\tfrac{s}{d}). && && (5.4f)
\end{aligned}
$$

5.2. Variational formulation

The derived transmission conditions (5.4) for the eddy current model can be used in different representations. We will discuss variational formulations which can discretised by finite elements. Alternatively, the transmission conditions can be represented by boundary integral formulations or discretised by finite difference methods.

The variational formulations in Sec. 4.1, where for each of exterior expansion function is solved, have all the same structure and prescribed jumps $[u_{\text{ext}}^j](t)$ and mean values $\{u_{\text{ext}}^j\}(t)$ on the interface Γ are incorporated in the solution space. The transmission conditions (5.4) have no source terms on Γ and are not the same for all orders. These conditions are natural, *i. e.*, they are incorporated in weak form by inserting the conditions into the variational problems. After multiplying (5.4a) by a test function e', integrating over Ω_{ext}^0 and applying Green's theorem we get

$$
\int_{\Omega_{\text{ext}}^0} \nabla e_{\text{ext}}^{\alpha,N}\cdot\nabla e'\,\mathrm{d}\underline{x} + \int_{\Gamma_i} \boldsymbol{\beta} e_{\text{ext}}^{\alpha,N}\, e'\,\mathrm{d}\sigma(\underline{x}) + \int_{\Gamma} [\partial_s e_{\text{ext}}^{\alpha,N}]\{e'\} + \{\partial_s e_{\text{ext}}^{\alpha,N}\}[e']\,\mathrm{d}t = \int_{\Omega_{\text{ext}}^0} f e'\,\mathrm{d}\underline{x} + \int_{\Gamma_i} \iota_{\text{imp}} e'\,\mathrm{d}\sigma(\underline{x})
$$

We may obtain variational formulations of the same form for all orders and α-asymptotics when $[\partial_s e_{\text{ext}}^{\alpha,N}]$ and $\{\partial_s e_{\text{ext}}^{\alpha,N}\}$ are replaced by Lagrange multiplyers and both conditions (5.4b) and (5.4c) are taken as additional equations in their respective weak forms. In contrast, we will prefer to introduce simpler variational formulations where $[\partial_s e_{\text{ext}}^{\alpha,N}]$ is replaced using (5.4c). This is possible for all transmission conditions except for that of order 0 for $\alpha = 2$. An varational formulation for this transmission condition of $\alpha = 2$ coincides with (4.3) for the limit solution u_{ext}^0.

The mean functions $\{\partial_s e_{\text{ext}}^{\alpha,N}\}$ in the variational formulation as well as in the operators $\mathbf{G}^{\alpha,\varepsilon,N}$ and $\mathbf{H}^{\alpha,\varepsilon,N}$ will be replaced by Lagrange multiplyers $\lambda^{\varepsilon,N}$.

By Lemma 4.12 and with $\mathbf{G}_0^\alpha u = [u]$, $\mathbf{H}_{\delta_{\alpha=2}}^\alpha u = [\partial_s u] - \delta_{\alpha=1} c_1 \{u\}$ we write similarly to [28, Sec. 4.3]

$$
\begin{aligned}
0 = (\mathbf{G}^{\alpha,N} e_{\text{ext}}^{\alpha,N})(t) &:= (\mathbf{G}_0^{\alpha,N}\{e_{\text{ext}}^{\alpha,N}\})(t) + (\mathbf{G}_1^{\alpha,N}\{\partial_s e_{\text{ext}}^{\alpha,N}\})(t) + [e_{\text{ext}}^{\alpha,N}](t),\\
0 = (\mathbf{H}^{\alpha,N} e_{\text{ext}}^{\alpha,N})(t) &:= (\mathbf{H}_0^{\alpha,N}\{e_{\text{ext}}^{\alpha,N}\})(t) + (\mathbf{H}_1^{\alpha,N}\{\partial_s e_{\text{ext}}^{\alpha,N}\})(t) + [\partial_s e_{\text{ext}}^{\alpha,N}](t),
\end{aligned}
\qquad (5.5)
$$

In $\mathbf{G}_0^{\alpha,N}$ and $\mathbf{H}_0^{\alpha,N}$ the part of the respective operators acting on $\{e_{\text{ext}}^{\alpha,N}\}$ is collected, and $\mathbf{G}_1^{\alpha,N}$ and $\mathbf{H}_1^{\alpha,N}$ are the parts acting on $\{\partial_s e_{\text{ext}}^{\alpha,N}\}$.

The solution space V^N for $e_{\text{ext}}^{\alpha,N}$ is a subspace of $H^1(\Omega_{\text{ext}}^0)$ with Dirichlet data $e_{\text{imp}}(\mathrm{x})$ on Γ_e, whose trace to Γ_i is in the domain $V(\Gamma_i)$ of $\boldsymbol{\beta}$ and exhibits bounded k-th tangential derivatives when $\mathbf{G}_0^{\alpha,N}$ and $\mathbf{H}_0^{\alpha,N}$ involves at most $2k$-th tangential derivatives. The respective space for test functions V_0^N have a homogeneous Dirichlet condition on Γ_e. The solution space W^N for the Lagrange multiplyer $\lambda^{\varepsilon,N}$ is $H^k(\Gamma)$ when $\mathbf{G}_1^{\alpha,N}$ and $\mathbf{H}_1^{\alpha,N}$ are differential operators of degree $2k-1$ or $2k$ with $k \in \mathbb{N}_0$.

Then, the variational formulations for (5.4) reads: Seek $(e^{\varepsilon,N}, \lambda^{\varepsilon,N}) \in V^N \times W^N$ such that

$$
\begin{cases}
\displaystyle\int_{\Omega_{\text{ext}}^0} \nabla e_{\text{ext}}^{\alpha,N}\cdot\nabla e'\,\mathrm{d}\underline{x} + \int_{\Gamma_i} \boldsymbol{\beta} e_{\text{ext}}^{\alpha,N}\, e'\,\mathrm{d}\sigma(\underline{x}) + \int_{\Gamma} (\mathbf{H}_0^{\alpha,N}\{e_{\text{ext}}^{\alpha,N}\} + \mathbf{H}_1^{\alpha,N}\lambda_{\text{ext}}^{\varepsilon,N})\{e'\} + \lambda_{\text{ext}}^{\varepsilon,N}[e']\,\mathrm{d}t \\
\qquad\qquad = \displaystyle\int_{\Omega_{\text{ext}}^0} f e'\,\mathrm{d}\underline{x} + \int_{\Gamma_i} \iota_{\text{imp}} e'\,\mathrm{d}\sigma(\underline{x}) \quad \forall\, e' \in V_0^N,\\
\displaystyle\int_{\Gamma} ([e_{\text{ext}}^{\alpha,N}] + \mathbf{G}_0^{\alpha,N}\{e_{\text{ext}}^{\alpha,N}\} + \mathbf{G}_1^{\alpha,N}\lambda_{\text{ext}}^{\varepsilon,N})\lambda'\,\mathrm{d}t = 0. \qquad \forall\, \lambda' \in W^N,
\end{cases}
\qquad (5.6)
$$

in which all $2k$- and $(2k+1)$-tangential derivatives are meant to be replaced by k-times integration by parts along Γ. Then, by definition the bilinear forms are continuous in the respective spaces, and the linear form on the right hand side as well.

5.3. Explicit determination of the first transmission conditions

With the operators $\mathbf{G}_\ell^\alpha$ and $\mathbf{H}_\ell^\alpha$, which are specified in Appendix A.3 we can now give the transmission conditions of order 1 and higher, and with $\mathbf{J}_\ell^\alpha$ the respective interior solution. They are extend the limit conditions (3.1) to higher orders.

For a comparison let us write the conditions for the particular sheet of thickness d and conductivity c, *i. e.*, we replace c_α by cd^α and S by s/d.

The transmission conditions for $\alpha = 0$ up to order 2 and the respective interior field can be written by

$$\begin{aligned}
0 &= [e_{\text{ext}}^{\alpha,N}](t) \\
0 &= [\partial_s e_{\text{ext}}^{\alpha,N}](t) \;\; - cd\{e_{\text{ext}}^{\alpha,N}\}(t) \\
e_{\text{int}}^{\alpha,N}(t,s) &= \underbrace{\{e_{\text{ext}}^{\alpha,N}\}(t)}_{\text{order 0}} \;\; \underbrace{+\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t)\, s}_{\text{order 1}} \;\; \underbrace{+\tfrac{1}{2}\{e_{\text{ext}}^{\alpha,N}\}(t)(s^2+d^2/4) - \tfrac{1}{2}(\kappa(t)\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t) + \partial_t^2\{e_{\text{ext}}^{\alpha,N}\}(t))s^2}_{\text{order 2}}
\end{aligned}$$

where for order N the respective terms up to this order have to be considered. For $\alpha = 1$ the respective equations are

$$\begin{aligned}
0 &= [e_{\text{ext}}^{\alpha,N}](t) \\
0 &= [\partial_s e_{\text{ext}}^{\alpha,N}](t) - cd\{e_{\text{ext}}^{\alpha,N}\}(t) \;\; - \tfrac{1}{6}c^2d^3\{e_{\text{ext}}^{\alpha,N}\}(t) \\
e_{\text{int}}^{\alpha,N}(t,s) &= \underbrace{\{e_{\text{ext}}^{\alpha,N}\}(t)}_{\text{order 0}} \;\; \underbrace{+\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t)\, s + \tfrac{1}{2}\{e_{\text{ext}}^{\alpha,N}\}(t)(s^2+d^2/4)}_{\text{order 1}},
\end{aligned}$$

and for $\alpha = 2$

$$\begin{aligned}
0 &= [e_{\text{ext}}^{\alpha,N}](t) \qquad\qquad + d\left(\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t) + \tfrac{\kappa(t)}{2}\{e_{\text{ext}}^{\alpha,N}\}(t)\right)\left(1 - \tfrac{\sinh(\sqrt{c}\frac{d}{2})}{\sqrt{c}\frac{d}{2}\cosh(\sqrt{c}\frac{d}{2})}\right) \\
0 &= -\frac{2\sqrt{c}\sinh(\sqrt{c}\frac{d}{2})}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}\{e_{\text{ext}}^{\alpha,N}\}(t) \;\; + \;[\partial_s e_{\text{ext}}^{\alpha,N}](t) \\
e_{\text{int}}^{\alpha,N}(t,s) &= \underbrace{\frac{\cosh(\sqrt{c}s)}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}\{e_{\text{ext}}^{\alpha,N}\}(t)}_{\text{order 0}} \\
&\qquad \underbrace{+\Big(\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t) + \tfrac{\kappa(t)}{2}\{e_{\text{ext}}^{\alpha,N}\}(t)\Big)\frac{\sinh(\sqrt{c}s)}{\sqrt{c}\cosh(\sqrt{c}\frac{d}{2})} - \tfrac{\kappa(t)}{2}\{e_{\text{ext}}^{\alpha,N}\}(t)\frac{\cosh(\sqrt{c}s)s}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}}_{\text{order 1}},
\end{aligned}$$

where we divided the second equation by d. Applying the Taylor expansion in ε before replacing ε by d, we get

$$\begin{aligned}
0 &= [e_{\text{ext}}^{\alpha,N}](t) \qquad\qquad + (\tfrac{1}{24}cd^3(\kappa(t)\{e_{\text{ext}}^{\alpha,N}\}(t) + 2\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t)) + \ldots) \\
0 &= (-cd - \tfrac{1}{6}c^2d^3 + ...)\{e_{\text{ext}}^{\alpha,N}\}(t) \qquad + [\partial_s e_{\text{ext}}^{\alpha,N}](t) \\
e_{\text{int}}^{\alpha,N}(t,s) &= \underbrace{(1 + \tfrac{c}{2}(s^2+d^2/4) + \ldots)\{e_{\text{ext}}^{\alpha,N}\}(t)}_{\text{order 0}} \;\; \underbrace{+(\{\partial_s e_{\text{ext}}^{\alpha,N}\}(t)\, s + \ldots)}_{\text{order 1}},
\end{aligned}$$

where higher order terms in $\sqrt{c}d$ (or $\sqrt{c}s$) are indicated by dots. Nevertheless, we will apply the transmission conditions for $\alpha = 2$ always directly, their presentation after application of the Taylor expansion is here convenient for the comparison of the models.

Comparing the terms of the models, we observe that the same terms appear for all asymptotics, but mostly at different orders. In general the same terms appear for $\alpha = 1$ at smaller or equal order than for $\alpha = 0$. From order 1 on also the same terms appear for $\alpha = 2$ at smaller or equal order than for $\alpha = 1$ (and so for $\alpha = 0$). An exception is the term $[\partial_s e_{\text{ext}}^{\alpha,N}]$ which appear for $\alpha = 2$ not before order 1 where already at order 0 for $\alpha = 1$ (and $\alpha = 0$). This signifies that (from order 1 on) transmission conditions with higher α (and also the respective approximation of the interior field) are in many situations of higher accuracy and more robust.

5.4. Stability of the transmission conditions

We will analyse the stability inside the own asymptotics in the sheet thickness ε for each transmission condition, and will indicate this by using the varying thickness ε instead the particular thickness d.

Lemma 5.1 (Stability of the problems of higher orders)**.** *The problems* (5.6) *provide for* $\alpha = 0$ *and* $0 < N \leq 3$*,* $\alpha = 1$ *and* $0 < N \leq 3$ *unique solutions* $(e_{\text{ext}}^{\varepsilon,N}, \lambda_{\text{ext}}^{\varepsilon,N}) \in V^N \times W^N$ *for all* $\varepsilon < \varepsilon_0$ *for some* $\varepsilon_0 > 0$*, and for* $|c_2| < 63.74$ *the problem for* $\alpha = 2$ *and* $N = 1$ *has unique solutions* $(e_{\text{ext}}^{\varepsilon,N}, \lambda_{\text{ext}}^{\varepsilon,N}) \in V^{\varepsilon.N} \times W^N$*, where* $\|e_{\text{ext}}^{\varepsilon,N}\|_{H^1(\Omega_{\text{ext}}^0)}$ *is bounded by the right hand side with a constant independent of* ε*.*

Proof. For $\alpha = 1$ the lemma has been proven in [28, Lemma 4.10, Corollary 5.2]. The proof for $\alpha = 0$ is unessentially different, and will be let to the reader. Although the proofs for $\alpha = 2$ have similarities too, we detail them in the following for the sake of completeness and for later comparison. For simplicity we use for the rest of the proof e and λ instead of $e_{\text{ext}}^{\varepsilon,N}$ and $\lambda_{\text{ext}}^{\varepsilon,N}$.

In all cases the linear form is

$$\ell(e') = \int_{\Omega_{\text{ext}}^0} f e' \,\mathrm{d}\underline{x} + \int_{\Gamma_i} \iota_{\text{imp}} e' \,\mathrm{d}\sigma(\underline{x})$$

which is bounded in (all subspaces of) $H^1(\Omega_{\text{ext}}^0)$.

For simplicity we assume $\boldsymbol{\beta}$ to be a multiplicative factor, and so for $\alpha = 2$ and $N = 0, 1$ we have $V_0^N = H^1_{\Gamma_e}(\Omega_{\text{ext}}^0)$ and $V^N = H^1_{\Gamma_e}(\Omega_{\text{ext}}^0) + E_{\Gamma_e} e_{\text{imp}}$, both with the $H^1(\Omega_{\text{ext}}^0)$-norm. The space for the Lagrange multiplicator is $W^N = L^2(\Gamma)$.

The model for $\alpha = 2$ of order $N = 0$ is trivially bounded independent of ε and provides a unique solution, as both Dirichlet traces are zero on Γ.

For $\alpha = 2$ of order $N = 1$ the problem (5.6) reads: Seek $(e, \lambda) \in V^1 \times W^1$ such that

$$\begin{cases} \displaystyle\int_{\Omega_{\text{ext}}^0} \nabla e \cdot \nabla e' \,\mathrm{d}\underline{x} + \int_{\Gamma_i} \boldsymbol{\beta} e\, e' \,\mathrm{d}\sigma(\underline{x}) + \int_\Gamma \frac{1}{\varepsilon} h(c_2)\{e\}\{e'\} + \lambda[e'] \,\mathrm{d}t = \ell(e'), & \forall\, e' \in V_0^1, \\ \displaystyle\int_\Gamma ([e] - \tfrac{1}{2}\varepsilon\kappa(t) j(c_2)\{e\} - \varepsilon j(c_2)\lambda)\lambda' \,\mathrm{d}t = 0, & \forall\, \lambda' \in W^1, \end{cases} \tag{5.7}$$

in which we used the function $j(c_2) := \frac{\sinh(\sqrt{c_2}\frac{1}{2})}{\sqrt{c_2}\frac{1}{2}\cosh(\sqrt{c_2}\frac{1}{2})} - 1$. Summing the two equations for $e' = \overline{e}$ and $\lambda' = \varepsilon^{-1} j^{-1}(c_2)[\overline{e}]$ we get

$$|e|^2_{H^1(\Omega_{\text{ext}}^0)} + \big\langle \boldsymbol{\beta} e, e\big\rangle_{\Gamma_i} + \frac{1}{\varepsilon} h(c_2)\|\{e\}\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon} j^{-1}(c_2)\|[e]\|^2_{L^2(\Gamma)} - \big\langle \tfrac{\kappa(t)}{2}\{e\}, [e]\big\rangle_\Gamma = \ell(\overline{e}). \tag{5.8}$$

Using $\varepsilon|\kappa(t)| < 2$, the Cauchy-Schwarz and Youngs inequality we can bound the last term of the left hand side as

$$|\big\langle \tfrac{\kappa(t)}{2}\{e\}, [e]\big\rangle_\Gamma| \leq \frac{1}{\varepsilon}\|\{e\}\|_{L^2(\Gamma)}\|[e]\|_{L^2(\Gamma)} \leq \frac{1}{\varepsilon}\big((1-a)\mathrm{Im}\, h(c_2)\|\{e\}\|^2_{L^2(\Gamma)} + \frac{1}{4(1-a)\mathrm{Im}\, h(c_2)}\|[e]\|^2_{L^2(\Gamma)}\big),$$

where $a < 1$ is some real constant we will fix later.

Since $\mathrm{Im}\big\langle \boldsymbol{\beta} e, e\big\rangle_{\Gamma_i}$ is non-negative by assumption, we get for the imaginary part of (5.8) the bound

$$\frac{a}{\varepsilon}\mathrm{Im}\, h(c_2)\|\{e\}\|^2_{L^2(\Gamma)} + \frac{1}{\varepsilon}\tilde{\jmath}_a(c_2)\|[e]\|^2_{L^2(\Gamma)} \leq \mathrm{Im}\, \ell(\overline{e}),$$

with $\tilde{\jmath}_a(c_2) = \mathrm{Im}(j^{-1}(c_2)) - (4(1-a)\mathrm{Im}\, h(c_2))^{-1}$. The continuous functions $\mathrm{Im}(j^{-1}(c_2))$ and $(\mathrm{Im}\, h(c_2))^{-1}$ are positive and for any $|c_2| \leq 63.74$ it is $\tilde{\jmath}_{a=0}(c_2) > 0$. For each c_2 with $|c_2| < 63.74$ we can fix a constant $1 > a > 0$ such that $\tilde{\jmath}_a(c_2) > 0$, where all terms on the left hand side are positive. Thus, both the mean and the jump are bounded by the right hand side, namely with a constant C independent of ε

$$\|\{e\}\|_{L^2(\Gamma)} + \|[e]\|_{L^2(\Gamma)} \leq C\sqrt{\varepsilon}\|\ell\|_{(V^N)'}. \tag{5.9}$$

Now, taking the real part of (5.8) we find the boundness of $|e|_{H^1(\Omega_{\text{ext}}^0)}$ by the right hand side with a constant independent of ε. As the traces from both sides on Γ are bounded and the $H^1(\Omega_{\text{ext}}^0)$-seminorm as well, applying Poincaré-Friedrichs inequality leads to the boundness of $\|e\|_{H^1(\Omega_{\text{ext}}^0)}$ as well.

Choosing $\lambda' = \overline{\lambda}$ in (5.7), taking the imaginary part, using the Cauchy-Schwartz inequality and (5.9) the Lagrange multiplier is bounded as well.

Hence, it holds uniqueness and stability of the model.

To prove existence, we have to show that $e' = 0$ and $\lambda' = 0$ is the only solution of

$$\mathsf{a}(e, \lambda; e', \lambda') = 0 \quad \forall (e, \lambda) \in W^N \times V^N, \tag{5.10}$$

with the bilinear form on the product space $V^N \times W^N$ associated to the variational formulation (5.7)

$$\begin{aligned}\mathsf{a}(e, \lambda; e', \lambda') := &\int_{\Omega^0_{\text{ext}}} \nabla e \cdot \nabla e' \, \mathrm{d}\underline{x} + \int_{\Gamma_i} \boldsymbol{\beta} e \, e' \, \mathrm{d}\sigma(\underline{x}) \\ &+ \int_\Gamma \frac{1}{\varepsilon} h(c_2)\{e\}\{e'\} + \lambda[e'] + [e]\lambda' + (-\tfrac{1}{2}\varepsilon\kappa(t) j(c_2)\{e\} - \varepsilon j(c_2)\lambda)\lambda' \, \mathrm{d}t,\end{aligned}$$

Choosing $e = \overline{e'}$ and $\lambda = \varepsilon^{-1} j^{-1}(c_2)[\overline{e'}] - \frac{\kappa}{2}\{\overline{e'}\}$ in (5.10) we get

$$|e'|^2_{H^1(\Omega^0_{\text{ext}})} + \left\langle \boldsymbol{\beta} e', e' \right\rangle_{\Gamma_i} + \frac{1}{\varepsilon} h(c_2) \|\{e'\}\|_{L^2(\Gamma)} + \frac{1}{\varepsilon} j^{-1}(c_2) \|[e']\|^2_{L^2(\Gamma)} - \left\langle \tfrac{\kappa(t)}{2}\{e'\}, [e'] \right\rangle_\Gamma = 0,$$

which is (5.8) with vanishing right hand side and where e is replaced by e'. Consequently, $e' = 0$. Now, choosing $e = 0$ and $\lambda = \overline{\lambda'}$ in (5.10) we get $\lambda' = 0$ as well, and a solution of (5.7) exists.

□

5.5. Estimates of the modelling error

As for Corollary 3.6 the modelling error for higher orders follow from the stability result, here by Lemma 5.1, and the construction of the terms of asymptotic expansion u^j_{ext}, U^j_{int}.

Lemma 5.2. *Let the assumptions for stability of Lemma 5.1 be fulfilled for the problems (5.6) for $\alpha = 0$ and $0 < N \leq 3$, $\alpha = 1$ and $0 < N \leq 3$, $\alpha = 2$ and $N = 1$. Then, the modelling error can be bounded as*

$$\|e^\varepsilon_{\text{ext}} - e^{\varepsilon,N}_{\text{ext}}\|_{H^1(\Omega^\varepsilon_{\text{ext}})} \leq C(c_\alpha)\, \varepsilon^{N+1},$$

where the constant $C(c_\alpha)$ does not depend on ε.

For $\alpha = 1$ the lemma has been proven in [28, Lemma 5.1, Theorem 5.3]. The proof for $\alpha = 0$, and the original models for $\alpha = 2$ are unessentially different.

5.6. The robustness of the transmission conditions of higher orders

Similarly to Sec. 3.3 for the transmission conditions of order 0 we investigate here the robustness of the transmission conditions of higher orders, *i. e.*, their accuracy for a range of the problem parameters thickness d and effective conductivity c.

To illustrate the approximation properties and their robustness, we first show the results of numerical experiments similarly to those described in Sec. 3.3, but now also for order 1 and 2. In Fig. 5 on page 11 the L^2-error outside the sheet is shown for the models of order 0 and order 1, and in Fig. 6 on page 12 for order 2.

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
$\alpha^\star = 0$	robust	robust	robust
$\alpha^\star = 1$	robust	robust	robust
$\alpha^\star = 2$	robust	robust	robust

Table 3: Robustness of the transmission conditions of order 1 and higher.

What changes when going to higher orders?. The first observation when going within the transmission conditions for a particular α first order is a remarkable decrease of the error, which is apparent for $\alpha = 0$ and $\alpha = 2$ in the whole investigated range of parameters. We have noticed already in Sec. 3.3 that the limit models for $\alpha = 0$ and $\alpha = 2$ are only robust in their own asymptotics, in contrast to $\alpha = 1$. At order 1 the transmission conditions for $\alpha = 0$ and $\alpha = 2$ lead also to a change to a robustness in all other $\alpha^\star$-asymptotics (Tab. 3). This gets visible when following the dashed lines, for all of them the error decreases for smaller sheet thickness d. The convergence is for all transmission conditions for $\alpha = 0, 1, 2$ fastest in the asymptotics $\alpha^\star = 0$, then within $\alpha^\star = 1$, and slowest convergence is within $\alpha^\star = 2$ (cf. Fig. 7). The transmission condition for $\alpha = 0$ are the same for order 1 and order 2 (as $\mathbf{G}_2^0 = 0$ and $\mathbf{H}_2^0 = 0$, see Appendix A.3), and for $\alpha = 2$ we have implemented up to order 1. For transmission condition for $\alpha = 1$, which is the only non-trivial one for order 0, gets more accurate at order 1 and even more for order 2, but not in the whole investigated range of thicknesses d and skin depths d_{skin}. The model of order 1 gets almost no improvement for very small relative skin depths d_{skin}/d, where for the model of order 2 the error even increases in this region. This is exactly why the expansion is called *asymptotic*, and not convergent. It converges in the small parameter ε, but not necessarily in the order N. We conject that there is a parameter region where the transmission conditions for $\alpha = 0, 1$ are convergent, and that they are convergent for $\alpha = 2$.

In this example the model of order 0 is the best for $d_{\text{skin}} \gg d$, for $d_{\text{skin}} \approx d$ all three orders give about the same results, and for $d_{\text{skin}} \ll d$ it is counterproductive to go to order 1 or 2, here the limit model performs best.

What changes with transmission conditions of different α ?. The model of order 0 and $\alpha = 0$ provides virtually for no set of parameters a decent error. The model of order 1 improves the accuracy clearly, but is equivalent to the model of order 0 for $\alpha = 1$ (at least for the exterior field), as well as to the model of order 2 and $\alpha = 0$ (at least for the exterior field). The transmission condition of order 2 for $\alpha = 0$ stays at the accuracy of that order 0 for $\alpha = 1$, whereas for $\alpha = 1$ the accuracy is improved notably for order 1 and even more for order 2 (for smaller skin depth). This shows that the choice in [28] to use the non-trivial asymptotics $\alpha = 1$ for transmission conditions pays off. As already mentioned, the accuracy is not improved for larger skin depths. Here, only the model for $\alpha = 2$ — it was yet derived for asymptotically constant relative skin depth — obtains an improvement at order 1. In comparison to the model for $\alpha = 1$ the error is virtually identical for very small relative skin depth, and in almost all configurations the model of $\alpha = 2$ is better. The model of order 2 for $\alpha = 1$ has already a substantial number of terms, however, even higher orders for $\alpha = 1$ does not achieve better results in case of small relative skin depths (cf. second row in first column in Fig 6).

Convergence rates for higher orders. In the second column of Fig. 7 the L^2-error is shown along the different contour lines for the transmission conditions of order 1. In the asymptotics of an asymptotically constant skin depth ($\alpha^\star = 2$) with the respective transmission condition a convergence rate of 2 is obtained, for the transmission condition for $\alpha = 0$ and $\alpha = 1$ it is only 1, even for order 2 and 3 for $\alpha = 1$ (dashed lines). In the latter case only the constant is improved, but the optimal convergence rate is not obtained. The picture for $\alpha^\star = 0$ is different. All three transmission conditions lead asymptotically to a convergence rate of (about) 3. This rate is higher as expected for $\alpha = 0$ as the models of order 1 and 2 coincide in the exterior field (see Sec. 5.3). Furthermore, we observe that the transmission conditions for $\alpha = 1$ and $\alpha = 2$ get identically for smaller sheet thicknesses. As in the asymptotics of $\alpha^\star = 1$ the transmission condition for $\alpha = 0$ of order 1 and order 2 (due to coincidence) results as well in a lower convergence rate, namely 1, we can conclude in the following conjecture about ordering the models with respect to their of robustness.

Definition 5.3 (Ordering with respect to the robustness)**.** *We call a transmission condition for $\alpha = \alpha_R$ more robust than another derived for $\alpha = \alpha_L$, if it results asymptotically in all asymptotics with $\alpha^\star \neq \alpha_R$ at least the same convergence rates, and the other has a lower convergence rate for $\alpha^\star = \alpha_R$.*

Conjecture 5.4 (Ordering the derived transmission conditions with respect to their robustness)**.** *The transmission condition for $\alpha = 2$ of order 1 is more robust than transmission conditions of any order for $\alpha = 1$. The transmission condition for $\alpha = 1$ of order 1 is more robust that the transmission condition for $\alpha = 0$ of order 1 and 2.*

Concluding remarks

In the article the asymptotic expansion for thin conducting sheets for varying thickness ε for the time-harmonic eddy current model have been derived to arbitrary order for three different asymptotic frameworks,

the constant conductivity (scaled like 1), the conductivity scaled like $1/\varepsilon$ and like $1/\varepsilon^2$. To these asymptotics a parameter α has been assigned which take the values 0, 1 and 2 for conductivities scaled like $1/\varepsilon^\alpha$. We have seen, that these different asymptotical frameworks possess different limit solutions for $\varepsilon \to 0$, namely the vanishing sheet for $\alpha = 0$, a non-trivial limit for $\alpha = 1$ and the impermeable sheet for $\alpha = 2$. The expansion of the solution inside the sheet is in scaled coordinates, the solution in its exterior in original coordinates, and the continuity conditions between both are expanded around the midline. This leads to iterative definition of the terms of the expansion, where the problems for the exterior terms are stated on the domain without the thin sheet. For a particular sheet of thickness d and effective conductivity c the terms of the expansion multiplied with powers of d are added. A main achievement of the article is the explicit derivation of operators for any order which define as a discrete convolution conditions for the expansion terms of any order of the midline Γ. We have derived for each $\alpha = 0, 1, 2$ the respective first operators.

The asymptotic expansion for the three asymptotic frameworks serve as a starting point for the derivation of three families of transmission conditions, where a solution of an arbitrary order N is computed in one step, contrary to the iterative definition of the $N + 1$ individual terms of the asymptotic expansion. The transmission conditions are hierarchic, meaning that terms are added when increasing the order. The transmission conditions for the three α-asymptotics and for different orders are compared with each other with respect to their robustness. The model is robust if it has a high accuracy for a wide range of sheet thicknesses d and conductivities c. We assess the robustness by inserting a transmission conditions for a certain α into an asymptotic framework related to $\alpha^\star$ and by numerical experiments for a rather general geometrical configuration and where ratios of skin depth and thickness take very small up to very large values. For the limit models (order 0) that for $\alpha = 1$ turns out to be the only robust. For higher orders the transmission conditions for all α-asymptotics are robust, where those of higher α are robuster than those of smaller α. Those higher α-transmission conditions turn out to be generalisations of the lower α ones which coincide for small relative skin depth.

We expect that the results related to the α-asymptotics transfer to the eddy current model for smooth sheets in 3D, where however the derivation will be for vectorial electromagnetic fields and similar transmission conditions will incorporate the curvatures and tangential derivatives of a 2D surface.

We have shown the stability of the transmission conditions for each α up to a certain order, partly under assumption on the relative skin depth. Although transmission conditions of higher orders differ only by small terms (they are weighted by powers of the small thickness) there is no general procedure to proof of stability to any order. Such a general procedure is wanted also as stability is necessary for the estimates of the modeling error.

The transmission conditions of high orders will not provide the same accuracy for thin sheets with kinks or ends. A consideration of these corners in the asymptotic expansion, as derived for example for thin layers on the boundary in [8], is in that case needed.

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A. Appendix

A.1. The operators of the development of the Laplace operator

By (2.4) we can evaluate the first operators

$$\begin{aligned}
\widehat{\Delta}_1^0 &= 0, & \widehat{\Delta}_1^1 &= -\kappa(t),\\
\widehat{\Delta}_2^0 &= -\partial_t^2, & \widehat{\Delta}_2^1 &= \kappa^2(t),\\
\widehat{\Delta}_3^0 &= 2\kappa(t)\partial_t^2 + \kappa'(t)\partial_t, & \widehat{\Delta}_3^1 &= -\kappa^3(t),\\
\widehat{\Delta}_4^0 &= -3\kappa^2(t)\partial_t^2 - 3\kappa(t)\kappa'(t)\partial_t, & \widehat{\Delta}_4^1 &= \kappa^4(t),\\
\widehat{\Delta}_5^0 &= 4\kappa^3(t)\partial_t^2 + 6\kappa^2(t)\kappa'(t)\partial_t, & \widehat{\Delta}_5^1 &= -\kappa^5(t),\\
\widehat{\Delta}_6^0 &= -5\kappa^4(t)\partial_t^2 - 10\kappa^3(t)\kappa'(t)\partial_t, & \widehat{\Delta}_6^1 &= \kappa^6(t).
\end{aligned} \tag{A.1}$$

A.2. The operators for higher normal derivatives

Inserting (A.1) into the recurrence relation (4.8) for $\mathbf{R}^{\ell}_{D/N}$ we get

$$\begin{aligned}
&\mathbf{R}^0_D = Id, && \mathbf{R}^0_N = 0,\\
&\mathbf{R}^1_D = 0, && \mathbf{R}^1_N = Id,\\
&\mathbf{R}^2_D = -\tfrac{1}{2}\partial_t^2, && \mathbf{R}^2_N = -\tfrac{1}{2}\kappa(t)\\
&\mathbf{R}^3_D = \tfrac{1}{2}\kappa(t)\partial_t^2 + \tfrac{1}{6}\kappa'(t)\partial_t, && \mathbf{R}^3_N = \tfrac{1}{6}\left(-\partial_t^2 + 2\kappa^2(t)\right),\\
&\mathbf{R}^4_D = \tfrac{1}{24}\left(\partial_t^4 - 11\kappa^2(t)\partial_t^2 - 7\kappa(t)\kappa'(t)\partial_t\right), && \mathbf{R}^4_N = \tfrac{1}{24}\left(6\kappa(t)\partial_t^2 + 4\kappa'(t)\partial_t - 6\kappa^3(t) + \kappa''(t)\right).
\end{aligned}$$

A.3. The operators of the transmission conditions

For $\alpha = 0$. The non-zero operators up to order 3 are

$$(\mathbf{G}^0_0 u)(t) = [u](t), \qquad (\mathbf{G}^0_3 u)(t) = \frac{c_0}{24}(\kappa(t)u(t) + 2\{\partial_s u\}(t)),$$

$$(\mathbf{H}^0_0 u)(t) = [\partial_s u](t), \qquad (\mathbf{H}^0_1 u)(t) = -c_0\{u\}(t), \qquad (\mathbf{H}^0_3 u)(t) = \frac{c_0}{24}(\partial_t^2\{u\}(t) - \kappa(t)\{\partial_s u\}(t)) - \frac{c_0^2}{6}\{u\}(t),$$

$$(\mathbf{J}^0_0 u)(t,S) = \{u\}(t), \qquad (\mathbf{J}^0_1 u)(t,S) = \{\partial_s u\}(t)S,$$

$$(\mathbf{J}^0_2 u)(t,S) = \tfrac{1}{2}\{u\}(t)(S^2 + \tfrac{1}{4}) - \tfrac{1}{2}(\kappa(t)\{\partial_s u\}(t) + \partial_t^2\{u\}(t))S^2,$$

$$\begin{aligned}
(\mathbf{J}^0_3 u)(t,S) = &-\frac{c_0}{6}(S^2 + \tfrac{4}{3})S\kappa(t)\{u\}(t) + \tfrac{1}{6}S^3(3\kappa(t)\partial_t^2 + \kappa'(t)\partial_t)\{\partial_s u\}(t)\\
&+ \frac{c_0}{6}(S^2 - \tfrac{4}{3})S\{\partial_s u\}(t) \;+ \tfrac{1}{6}S^3(2\kappa^2(t) - \partial_t^2)\{\partial_s u\}(t)
\end{aligned}$$

For $\alpha = 1$. The operators are (up to different definition of the sign) equivalent to those given up to order 3 in [28, Appendix], for completeness we affiliate the non-zero operators up to order 2

$$(\mathbf{G}^1_0 u)(t) = [u](t), \qquad (\mathbf{G}^1_2 u)(t) = \frac{c_1}{24}\big(\kappa(t)\{u\}(t) + 2\{\partial_s u\}(t)\big),$$

$$(\mathbf{H}^1_0 u)(t) = [\partial_s u](t) - c_1\{u\}(t), \qquad (\mathbf{H}^1_1 u)(t) = \frac{c_1^2}{6}\{u\}(t),$$

$$(\mathbf{H}^1_2 u)(t) = \frac{c_1}{12}\left(\frac{7}{20}c_1^2 - \partial_t^2\right)\{u\}(t) + \frac{c_1}{24}\kappa(t)\{\partial_s u\}(t),$$

$$(\mathbf{J}^1_0 u)(t) = \{u\}(t), \qquad (\mathbf{J}^1_1 u)(t) = \frac{c_1}{2}\{u\}(t)\left(S^2 + \tfrac{1}{4}\right) + \{\partial_s u\}(t)\,S,$$

$$\begin{aligned}
(\mathbf{J}^1_2 u)(t) = &\frac{c_1}{2}\{u\}(t)\left(S^2 + \tfrac{1}{4}\right) + \{\partial_s u\}(t)\,S\big(S^2 + \tfrac{3}{4}\big)^2 + \frac{c_0}{6}\{\partial_n u\}(t)\big(S^3 - \tfrac{3}{4}S\big) - \frac{c_0}{6}\kappa(t)\{u\}(t)\big(S^3 + \tfrac{3}{4}S\big)\\
&- \frac{1}{2}\left(\kappa(t)\{\partial_n u\}(t) + \partial_t^2\{u\}(t)\right)S^2.
\end{aligned}$$

For $\alpha = 2$. The non-zero operators up to order 1 are

$$(\mathbf{G}^2_0 u)(t) = [u](t), \qquad (\mathbf{G}^2_1 u)(t) = \left(\{\partial_s u\}(t) + \tfrac{\kappa(t)}{2}\{u\}(t)\right)\left(1 - \tfrac{\sinh(\sqrt{c}\frac{d}{2})}{\sqrt{c}\frac{d}{2}\cosh(\sqrt{c}\frac{d}{2})}\right),$$

$$(\mathbf{H}^2_0 u)(t) = -\frac{2\sqrt{c}\sinh(\sqrt{c}\frac{d}{2})}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}\{u\}(t), \qquad (\mathbf{H}^2_1 u)(t) = [\partial_s u](t),$$

$$(\mathbf{J}^2_0 u)(t,S) = \frac{\cosh(\sqrt{c}s)}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}\{u\}(t)$$

$$(\mathbf{J}^2_1 u)(t,S) = \left(\{\partial_s u\}(t) + \tfrac{\kappa(t)}{2}\{u\}(t)\right)\tfrac{\sinh(\sqrt{c}s)}{\sqrt{c}\cosh(\sqrt{c}\frac{d}{2})} - \tfrac{\kappa(t)}{2}\{u\}(t)\tfrac{\cosh(\sqrt{c}s)s}{\cosh(\sqrt{c}\frac{d}{2}) - \sqrt{c}\frac{d}{2}\sinh(\sqrt{c}\frac{d}{2})}.$$

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