

A DUALITY THEORY FOR CONTINUOUS-TIME DYNAMIC SHORTEST PATHS WITH NEGATIVE TRANSIT TIMES*

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Abstract. This paper concerns the shortest path problem for a network in which arc costs can vary with time, each arc has a transit time, parking with a corresponding time-varying cost is allowed at the nodes, and time is modeled as a continuum. The resulting problem is called the *continuous-time dynamic shortest path problem*, which is well studied in the literature. However, the problem appears as a subproblem when one wishes to test, via an algorithm for dynamic shortest paths, the presence of negative cycles in the residual network in order to develop continuous-time analogues of several well-known optimality conditions for continuous-time dynamic network flow problems. But, in general, the residual network contains arcs with negative transit times and hence the results in the literature are useless for these purposes since all results are based on the assumption of positive transit times. In this paper, we relax this condition to allow negative transit times. We study a corresponding linear program in space of measures and prove the existence of an optimal extreme point solution. Moreover, we define a dual problem and establish a strong duality result that shows under certain assumptions the value of the linear program equals the value of the dual problem and both values are attained. We also present counterexamples to show that strong duality only holds under these assumptions.

Key words. Shortest Path Problem, Linear Programming in Measure Spaces, Duality Theory, Measure Theory

AMS subject classifications. 05C21, 49J27, 49K27, 90C35, 90C46

1. Introduction. The shortest path problem is among the best studied network optimization problems. A natural extension of this problem is the *dynamic shortest path problem*, whose aim is to find a path with minimum cost through a network, in which each arc has a transit time in addition to arc cost, parking is allowed at the nodes and network characteristics (e.g., arc and parking costs) can change over time. This problem was initially introduced by Cooke and Halsey [5], who presented an algorithm based on the Bellman's principle of optimality. In the model proposed by Cooke and Halsey [5], time is measured in discrete time steps. Research on dynamic shortest path problems has also taken another approach where time is modeled as a continuum.

Work on continuous-time dynamic shortest path problems is conducted by, e.g., Orda and Rom [9, 10], Philpott [11], and Philpott and Mees [13, 14]. In particular, Philpott [11] formulates the problem as a linear program (LP for short) in a space of measures and investigates the relationship between the problem and its LP formulation. Especially, he introduces a dual problem and proves the absence of a duality gap¹. He demonstrates the existence of an optimal extreme point for the LP formula-

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¹There is *no duality gap* between a linear program and its dual if they have the same (finite) value. If this finite value is achieved by feasible solutions of the primal and of the dual program, then *strong duality* holds. In finite-dimensional linear programming, strong duality holds, whenever no duality gap exists and vice versa. In general, this is not the case in infinite-dimensional linear programming (see, e.g., [3]).

tion and derives a correspondence between extreme points and dynamic paths. Moreover, he establishes a strong duality result in case where cost functions are Lipschitz-continuous.

In all of the papers mentioned above, the transit times are supposed to be strictly positive. In particular, Philpott [11] make this condition to abstract certain results from infinite-dimensional linear programming as adapted by e.g., Anderson and Nash [3]). This approach cannot be used if transit times are zero or negative as Philpott [11] writes in the conclusion of his paper, “the assumption that all transit times are strictly positive is central to the arguments presented”.

In this paper, we study the continuous-time dynamic shortest path in a general framework where transit times can be negative. Our main motivation to relax transit times to take negative values is to be able to develop a duality theory and network-related optimality conditions for a general class of dynamic network flows, the so-called *Continuous-time Dynamic Network Flow Problems (CDNFP)*. These problems model the temporal evolution of flows over time and also consider changes of network parameters such as capacities, costs, supplies, and demands over time. CDNFP was first introduced by Anderson [2] and since then has been studied by a number of authors (e.g., Anderson and Philpott [4], Philpott and Craddock [12], Fleischer and Sethuraman [6]).

In the absence of transit times, CDNFP can be treated as special case of *separated continuous linear programs*, for which extensive duality results are developed by Pullan [15] for the problem given piecewise analytic problem data. Pullan [16] also considers a general class of separated continuous linear programs to include time-delays and develops a strong duality for the problem with rational transit times and piecewise constant/linear input functions. However, despite many attempts on dynamic network flows, CDNFP with piecewise analytic input functions still lacks of a duality theory. Therefore, it would be worthwhile to develop a duality theory and network-related optimality conditions in a constructive way as in the static network flows.

We recall that a key step of establishing strong duality for the static minimum cost flow problem is the fact that starting from some feasible flow we can construct an optimal dual solution if the network contains no augmenting cycles with negative cost. More precisely, the shortest distance labels from one specified node to the other nodes in the residual network define a dual feasible solution which is complementary slackness with the given feasible flow. The residual network has a backward arc for each original arc. One possible approach is to derive strong duality for CDNFP is to go along the same lines as in the static network flows. So we require to compute the dynamic shortest distance labels from one node to the other nodes in the residual network. In the setting of dynamic network flows, backward arcs have negative transit times if all original arcs have positive transit times, and hence we cannot use the results in the literature and particularly not those derived by Philpott [11].

Our Contribution. In this paper, we study a general class of dynamic shortest path problems in the more challenging continuous-time model and give a theoretical analysis for such problems. We allow transit times take negative values, not just positive ones. We examine a corresponding linear program in the space of measures and define a dual problem. We show under certain assumptions that the value of LP formulation is the same as the value of the dual problem and both values are attained, leading to a strong duality result. Moreover, we show that the linear program has an optimal extreme point solution.

Our results can be used in developing a duality theory and establishing optimality conditions such as negative cycle optimality condition for CDNFP with piecewise continuous input functions, which has been an open question for a long time (see Anderson and Philpott [4]). Generally speaking, given a dynamic flow, one could make use of the ideas presented in this paper to prove the existence of dynamic shortest distance labels in the residual network satisfying complementary slackness conditions if there are no negative augmenting cycles with negative costs. This would lead to a strong duality result as well as several optimality conditions for CDNFP. Further details can be found in [7].

The rest of the paper is organized as follows. In Section 2, we give a detailed definition of the continuous-time dynamic shortest path problem. In Section 3, we formulate this problem as an infinite-dimensional linear program. In Section 4, we consider a dual problem and give some simple examples to show strong duality does not necessarily hold between the LP formulation and its dual. In Section 5, we make certain assumptions and prove a strong duality result between the infinite-dimensional linear program and its dual.

2. Problem description. Here, we present a precise description of the continuous-time dynamic shortest path problem. To motivate our treatment, we first describe the static shortest path problem.

We consider a directed graph $G := (V, E)$ with finite *node set* V and finite *arc set* E . An arc e from a node v to a node w is denoted by $e := (v, w)$ to emphasize that e leaves v and enters w . In this case, we say that node v is the *tail* of e and w is the *head* of e and write $\text{tail}(e) := v$ and $\text{head}(e) := w$. A *walk* P from node v to node w is an alternating sequence of nodes and arcs of the form $P := (v_1, e_1, v_2, \dots, v_n, e_n, v_{n+1})$ such that $v_1 = v$, $e_i = (v_i, v_{i+1})$ for $i = 1, \dots, n$, and $v_{n+1} = w$. Throughout the paper we denote the walk P by the arc sequence (e_1, \dots, e_n) , assuming that $\text{head}(e_i) = \text{tail}(e_{i+1})$ for $i = 1, \dots, n$. Further, we denote by $E(P) := \{e_1, \dots, e_n\}$ and $V(P) := \{v_1, \dots, v_{n+1}\}$ the set of arcs and nodes, respectively, involved in P . The walk P is said to be a *path* from v to w (or simply v - w -path) if v_1, v_2, \dots, v_{n+1} are pairwise distinct, except v_1 and v_{n+1} . If in addition $v_1 = v_{n+1}$, the path P is called a *cycle*. A node w is said to be *reachable* from node v if there exists a v - w -path.

Each arc $e \in E$ has an associated *cost* c_e . The *cost* of a path $P = (e_1, \dots, e_n)$ is defined as the sum of the costs of all arcs in the path, that is, $c_P := \sum_{i=1}^n c_{e_i}$. The network has two distinguished nodes: a *source* s and a *sink* t . We assume without loss of generality that every node of G is reachable from s and that t is reachable from every node. The (static) *shortest path problem* is to find a path from source s to sink t with minimal cost. This problem can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = \begin{cases} 1 & \text{if } v = s \\ 0 & \text{if } v \neq s, t \\ -1 & \text{if } v = t \end{cases} \quad \forall v \in V, \\ & x_e \geq 0 \quad \forall e \in E. \end{aligned} \quad (\text{LP})$$

Here and subsequently, the set of arcs leaving node v and entering node v are denoted by $\delta^+(v) := \{e \in E \mid \text{tail}(e) = v\}$ and $\delta^-(v) := \{e \in E \mid \text{head}(e) = v\}$, respectively. In (LP), the decision variable x_e gives the amount of flow on arc e . It

is well-known that an optimal extreme point of (LP) yields a shortest s - t -path. The dual problem (LP*) of (LP) can be written as follows:

$$\begin{aligned} \max \quad & \pi_s - \pi_t \\ \text{s.t.} \quad & \pi_v - \pi_w \leq c_e \quad \forall e = (v, w) \in E . \end{aligned} \tag{LP*}$$

If π is an optimal solution for (LP*) then for every node $v \in V$ the cost of a shortest s - v -path is equal to $\pi_v - \pi_s$. Moreover, if the network contains a cycle C with negative cost, then (LP) is unbounded because we can send an infinite amount of flow along C and therefore the objective function value goes to $-\infty$. In this case, the dual problem (LP*) is infeasible. The shortest path problem where cycles with negative cost are allowed is difficult to solve. In fact, it is \mathcal{NP} -complete (see [1, page 95]), i.e., no polynomial-time algorithm for this problem exist unless $\mathcal{P} = \mathcal{NP}$. For the case that the network contains no negative cycle, strong duality holds between (LP) and (LP*) and numerous efficient algorithms for solving the shortest path problem exist. A comprehensive discussion and comparison of these algorithms can be found in the textbook by Ahuja, Magnanti, and Orlin [1].

So far we have considered the setting of the *static* shortest path problem. We now turn to the dynamic case in which each arc $e \in E$ has an associated *transit time* τ_e , specifying the required amount of time to travel from the tail to the head of e . More precisely, if we leave node v at time θ along an arc $e = (v, w)$, we arrive at w at time $\theta + \tau_e$. Further, waiting is allowed at the nodes of the network for later departure. In the following we extend the definition of (static) walk, path, and cycle to the dynamic case.

A *dynamic walk* is a pair of a walk $P = (e_1, \dots, e_n)$ together with a family of *waiting times* $(\lambda_1, \dots, \lambda_{n+1})$. For $i = 1, \dots, n+1$ after arriving at node $v_i \in V(P)$ we wait λ_i time units before we leave v_i . Given a *starting time* θ , let α_i be the time when we arrive at node v_i and β_i be the time when we departure from node v_i . For $i = 1, \dots, n+1$, the *arrival time* α_i and the *departure time* β_i can be computed recursively as follows:

$$\alpha_i := \begin{cases} \theta & \text{for } i = 1 \\ \beta_{i-1} + \tau_{e_{i-1}} & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_i := \alpha_i + \lambda_i .$$

A walk P is called a *dynamic path* if P does not revisit any node (except the endpoints) at the same point in time, i.e., $[\alpha_i, \beta_i] \cap [\alpha_j, \beta_j] = \emptyset$ for each $1 \leq i < j \leq n+1$ with $v_i = v_j$ and $(i, j) \neq (1, n+1)$. Note that the underlying (static) walk of a dynamic path need not to be a (static) path since it is allowed that a node can be revisited at different points in time. Moreover, P is said to be a *dynamic cycle* if P is a dynamic path, and in addition $v_1 = v_{n+1}$ and $\alpha_1 = \beta_{n+1}$. Further, we say that the path P has *time horizon* Θ if $\beta_{n+1} = \Theta$.

Each arc e and each node v has a *cost function* $c_e : \mathbb{R} \rightarrow \mathbb{R}$ and $c_v : \mathbb{R} \rightarrow \mathbb{R}$, respectively. For a certain point in time $\theta \in \mathbb{R}$, the cost for leaving the tail of e at time θ and traveling along e is $c_e(\theta)$ and the cost per time unit for the waiting at v at time θ is $c_v(\theta)$. The *cost* of a dynamic walk $P = (e_1, \dots, e_n)$ with arrival times $\alpha_1, \dots, \alpha_{n+1}$ and departure times $\beta_1, \dots, \beta_{n+1}$ is thus given by

$$\text{cost}(P) := \sum_{i=1}^n c_{e_i}(\beta_i) + \sum_{i=1}^{n+1} \int_{\alpha_i}^{\beta_i} c_{v_i}(\theta) d\theta .$$

Here the first sum gives the cost for traveling along arcs in the path and the second one gives the cost for waiting at nodes of the path. A dynamic path P from v to w is called a *dynamic shortest path* if $\text{cost}(P) \leq \text{cost}(P')$ for all dynamic v - w -paths P' with the same starting time and the same time horizon as P .

Given a *source* $s \in V$, a *sink* $t \in V$, and a *time horizon* Θ , the *continuous-time dynamic shortest path problem* is to determine a dynamic shortest path from s to t with starting time 0 and time horizon Θ :

CONTINUOUS-TIME DYNAMIC SHORTEST PATH PROBLEM (CDSP)	
Input:	A network consisting of a directed graph $G := (V, E)$, cost functions $(c_e)_{e \in E}$ and $(c_v)_{v \in V}$, a source $s \in V$, a sink $t \in V$, and a time horizon Θ ,
Task:	Find a dynamic shortest s - t -path with starting time 0 and time horizon Θ .

In the rest of the paper, if not mentioned otherwise, the starting time and the time horizon of a dynamic path from source s to sink t are assumed to be 0 and Θ , respectively.

3. LP formulation in Measure Spaces. As in the static case, we can view CDSP as the problem of sending one unit of flow from source s at time 0 to sink t at time Θ at minimal cost. This problem can be formulated as the following linear program in the space of Borel measures (see Koch and Nasrabadi [8] for a detailed treatment):

$$\begin{aligned}
 \min \quad & \int_{-\infty}^{\infty} \sum_{e \in E} c_e(\theta) dx_e + \int_{-\infty}^{\infty} \sum_{v \in V} c_v(\theta) Y_v(\theta) d\theta \\
 \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} (x_e - \tau_e) + y_v = b_v \quad \forall v \in V, \\
 & x_e \geq 0 \quad \forall e \in E, \\
 & Y_v \geq 0 \quad \forall v \in V.
 \end{aligned} \tag{LPM}$$

Here, the variable x_e is a Borel measure on the real line \mathbb{R} and for each Borel set B the value $x_e(B)$ gives the amount of flow entering arc e within the times in B . Note that $x_e - \tau_e$ is understood to be a Borel measure defined by $(x_e - \tau_e)(B) := x_e(B - \tau_e)$ where $B - \tau_e := \{\theta - \tau_e \mid \theta \in B\}$. The variable y_v is a signed Borel measure with distribution function Y_v . The value $y_v(B)$ gives the overall change in storage at v over the Borel set B and the value $Y_v(\theta)$ the amount of flow stored at node v at time θ . For each $v \in V$, b_v is a signed Borel measure defined as

$$b_v(B) := \begin{cases} 1 & v = s, 0 \in B \\ -1 & v = t, \Theta \in B \\ 0 & \text{otherwise} \end{cases}$$

for every Borel set B . The value $|b_v(B)|$ gives as the amount of supply or demand at node v over the Borel set B depending on whether $b_v(B) > 0$ or $b_v(B) < 0$, respectively.

We require x_e to be a *finite* Borel measure on \mathbb{R} , i.e., $|x_e| := x_e(\mathbb{R}) < \infty$ for all $e \in E$. We refer to $x = (x_e)_{e \in E}$ as *flow* and to $y = (y_v)_{v \in V}$ (or equivalently

the corresponding distribution functions $Y = (Y_v)_{v \in V}$ as *storage*. A flow x with corresponding storage y satisfying the constraints of (LPM) is called a feasible solution or feasible flow. In a similar way as in the finite-dimensional linear programming, a feasible solution of (LPM) is called an extreme point if it cannot be derived from a convex combination of any two other feasible solutions. It is known that every signed Borel measure can be uniquely decomposed into a sum of a discrete and a continuous measure (see [8, Appendix] for more details). This implies that for each arc e the flow x_e is the sum of a continuous flow x_e^c and a discrete flow x_e^d . A flow x is called *discrete* (continuous) if $x_e^c = 0$ ($x_e^d = 0$) for all arcs $e \in E$. Koch and Nasrabadi [8] show that the continuous part of an extreme point is 0 and establish a one-to-one correspondence between extreme points of (LPM) and dynamic paths.

THEOREM 3.1 (Koch and Nasrabadi [8]). *Any extreme point of (LPM) corresponds one-to-one to a dynamic s-t-path. If the cost functions are given, this one-to-one correspondence preserves also costs.*

Our aim is to prove the existence of an optimal extreme point for (LPM). To this end, we require to encode a dynamic path with measures, whereas in the static case a path is identified with its incidence vector whose elements are 0 or 1. Let $P := (e_1, \dots, e_n)$ be a dynamic path with arrival times α_i and departure times β_i for $i = 1, \dots, n+1$. The *incidence vector* χ^P of P is a family $(\chi_e^P)_{e \in E}$ of discrete measures defined by

$$\chi_e^P := \begin{cases} \sum_{i|e_i=e} \chi_i^P & \text{if } e \in E(P) \\ 0 & \text{otherwise} \end{cases} \quad \forall e \in E, \quad (3.1)$$

where

$$\chi_i^P(B) := \begin{cases} 1 & \text{if } \alpha_i \in B \\ 0 & \text{otherwise} \end{cases} \quad \forall i = 1, \dots, n,$$

for each Borel set B . The corresponding storage $\psi^P := (\psi_v^P)_{v \in V}$ is defined by:

$$\psi_v^P := b_v - \sum_{e \in \delta^+(v)} \chi_e^P + \sum_{e \in \delta^-(v)} (\chi_e^P - \tau_e) \quad \forall v \in V. \quad (3.2)$$

For each $v \in V$ let Ψ_v^P denote the distribution function of the measure ψ_v^P , i.e., $\Psi_v^P(\theta) := \psi_v^P((-\infty, \theta])$ for all $\theta \in \mathbb{R}$. It is not hard to observe that

$$\Psi_v^P(\theta) = \begin{cases} \sum_{i|v_i=v} \Psi_i^P(\theta) & \text{if } v \in V(P) \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V,$$

where

$$\Psi_i^P(\theta) := \begin{cases} 1 & \text{for } \theta \in [\alpha_i, \beta_i) \\ 0 & \text{otherwise} \end{cases} \quad \forall i = 1, \dots, n+1.$$

Therefore, $\Psi_v^P \geq 0$ and the incidence vector χ^P is a feasible solution of (LPM). Koch and Nasrabadi [8] show that χ^P is not only a feasible solution, but also an extreme point of (LPM).

In this paper, we focus on the objective function of (LPM) and finding its value. The *value* of (LPM) is the infimum of its objective function over all feasible solutions

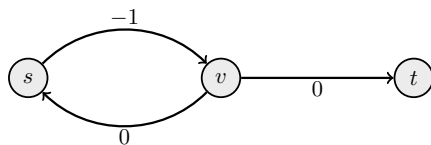


FIG. 4.1. Network for Example 4.1. The transit times are shown on the arcs.

which will be denoted by $V[\text{LPM}]$. Like the static shortest path problem, (LPM) is unbounded (i.e., its value tends to $-\infty$) if the network G contains a negative dynamic cycle (i.e., a dynamic cycle with negative cost). More precisely, let P be a dynamic s - t -path with incidence vector χ^P and C be a negative dynamic cycle with incidence vector χ^C . It is not difficult to see that $\chi^P + \delta \cdot \chi^C$ is a feasible solution of (LPM) for each $\delta \geq 0$ whose objective function value is $\text{cost}(\chi^P) + \delta \text{cost}(\chi^C)$. Therefore if $\text{cost}(\chi^C) < 0$ then $V[\text{LPM}]$ can be made arbitrary negative by making δ sufficiently large. So we give the following assumption.

ASSUMPTION 1. *The network contains no negative dynamic cycle.*

This assumption can be satisfied by making all costs nonnegative or all transit times strictly positive. For the latter case, the number of arcs in any dynamic s - t -path is bounded by a constant independent of the path. Further, the feasible region of (LPM) becomes bounded with respect to a certain norm which makes it possible to apply certain results from the theory of linear programming in infinite-dimensional vector spaces. Philpott [11] assumes all transit times are strictly positive and establishes a duality theory for (LPM) based on the paired-space methodology as adapted by Anderson and Nash [3]. In particular, he develops a dual problem for (LPM) and proves the absence of a duality gap. Further, he shows that the values of (LPM) and its dual are finite and attained in each problem in the case where cost functions satisfy a Lipschitz condition. In what follows, we give some simple examples to show that these results do not necessarily hold for the more general case with arbitrary transit times. Further, we present some necessary and sufficient conditions under which the strong duality result holds between (LPM) and its dual.

4. Dual Formulation. Before formulating a dual problem for (LPM), we consider the following small example.

EXAMPLE 4.1. *Consider the network shown in Figure 4.1 where the transit times are shown on the arcs. The arc cost functions are given by*

$$c_{s,v}(\theta) = c_{v,s}(\theta) = 0, \quad c_{v,t}(\theta) = \theta, \quad \forall \theta \in \mathbb{R}.$$

The node cost functions are supposed to be zero. It is clear that the network contains no negative dynamic cycle. Now let f be a discrete flow concentrated on the singleton $\{0\}$ with $f(\{0\}) = 1$. For each $k \in \mathbb{N}$ we define a feasible solution x^k for (LPM) as follow: The flow f circulates k times around the cycle C induced by s and v and then it is sent along arc (s, v) and (v, t) . This yields the following feasible solution:

$$x_{(s,v)}^k = \sum_{i=0}^k (f + i), \quad x_{(v,s)}^k = (x_{(s,v)}^k + 1) - (f + (k + 1)), \quad x_{(v,t)}^k = f + (k + 1).$$

Obviously, we have $\text{cost}(x^k) = -k - 1$. Hence, (LPM) is unbounded since $\text{cost}(x^k)$ tends to $-\infty$ as k goes to ∞ .

The above example shows that the absence of negative dynamic cycles does not alone guarantee the existence of optimal solutions for (LPM). However, the problem given in Example 4.1 will have an optimal solution if we restrict the feasible region of (LPM) by considering a *time window* for each node. This motivates the following assumption.

ASSUMPTION 2. *For each node $v \in V$ there exists a time window $[a_v, b_v]$ with $a_v > -\infty$ and $b_v < \infty$, within which node v is permitted to be visited.*

Subsequently, the definition of a dynamic path (or cycle) as well as definition of a feasible solution for (LPM) are constrained by time windows at nodes. More precisely, for a dynamic path (or cycle) $P = (e_1, \dots, e_n)$ with arrival time α_i and departure time β_i for node v_i , we assume that $\alpha_i, \beta_i \in [a_{v_i}, b_{v_i}]$ for $i = 1, \dots, n + 1$. Further, for any feasible solution x, y of (LPM) the measures x and y are supposed to be zero at any point out of the time windows. So we let

$$u_e|_{\mathbb{R} \setminus [a_v, b_v]} = 0 \quad \forall e = (v, w) \in E .$$

It is naturally assumed that $0 \in [a_s, b_s]$ and $\Theta \in [a_t, b_t]$. To simplify notation, we suppose that for each node $v \in V$ and each point in time $\theta \in [a_v, b_v]$, the network G contains a dynamic s - v -path with starting time 0 and time horizon θ , and a dynamic v - t -path with starting time θ and time horizon Θ . This assumption imposes no loss of generality because the nodes and times violating this assumption do not appear in any dynamic s - t -path and can therefore be deleted.

Having made Assumption 2, we can formulate a dual problem. For the ease of notation, we assume that the waiting costs are zero, i.e., $c_v(\theta) = 0$ for every $v \in V$ and $\theta \in \mathbb{R}$. We note that this assumption imposes no restriction because the general case can be transformed into this special by integration by parts (see, e.g., [11]). Now by the theory of linear programming in infinite-dimensional spaces (see, e.g., [3]), we can write down a dual problem of (LPM) as follows:

$$\begin{aligned} \min \quad & \rho_s - \rho_t + \int_0^{b_s} \eta_s(\vartheta) d\vartheta - \int_{\Theta}^{b_t} \eta_t(\vartheta) d\vartheta \\ \text{s.t.} \quad & \rho_v - \rho_w + \int_{\theta}^{b_v} \eta_v(\vartheta) d\vartheta \\ & - \int_{\theta + \tau_e}^{b_v} \eta_w(\vartheta) d\vartheta \leq c_e(\theta) \quad \forall e = (v, w) \in E, \theta \in [a_v, b_w - \tau_e] , \\ & \eta_v(\theta) \leq 0 \quad \forall v \in V, \theta \in [a_v, b_v] , \end{aligned} \tag{LPM*}'$$

where $\rho_v \in \mathbb{R}$ and $\eta_v \in L_{\infty}[a_v, b_v]$ for each node $v \in V$. The reader is referred to [11] for a detailed discussion of the above formulation. To derive a similar formulation as (LP*), we consider a more general dual problem. In particular, we shall focus on the following dual problem:

$$\begin{aligned} \max \quad & \pi_s(0) - \pi_t(\Theta) \\ \text{s.t.} \quad & \pi_v(\theta) - \pi_w(\theta + \lambda_e) \leq c_e(\theta) \quad \forall e = (v, w) \in E, \theta \in [a_v, b_w - \tau_e] , \\ & \pi_v \text{ monotonic increasing and} \\ & \text{right continuous on } [a_v, b_v] \quad \forall v \in V . \end{aligned} \tag{LPM*}$$

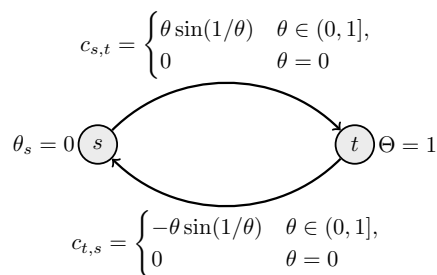


FIG. 4.2. Network for Example 4.3. The transit times are zero.

It is clear that (LPM^*) is a relaxation of $(LPM^{*'})$ because any feasible solution ρ, η generates one for (LPM^*) of the same objective function value by setting

$$\pi_v(\theta) = \rho_v + \int_{\theta}^{b_v} \eta_v(\vartheta) d\vartheta \quad \forall v \in V, \theta \in [a_v, b_v].$$

Conversely, if π is feasible for (LPM^*) in which π_v is absolutely continuous on $[a_v, b_v]$ for every $v \in V$, then $\rho_v := \pi_v(b_v)$ and $\eta_v(\theta) := \dot{\pi}(\theta)$ for every $\theta \in [a_v, b_v]$, is feasible for $(LPM^{*'})$ and again the two solutions have the same objective function value.

Any π that satisfies the constraints of (LPM^*) is said to be (dual) *feasible*, and the *value* of (LPM^*) , denoted by $V[LPM^*]$, is the supremum over all feasible solutions. The following weak duality result is easily established by integration by parts (see, e.g., [11] for more details).

LEMMA 4.2 (Weak duality). $V[LPM] \leq V[LPM^*]$.

It is of great interest to conjecture whether a strong duality result can be established whereby $V[LPM] = V[LPM^*]$ and these values are attained in each problem. It depends on being able to construct a feasible solution x for (LPM) and a feasible solution π for (LPM^*) for which $V[LPM, x] = V[LPM^*, \pi]$. The following three examples show that in general strong duality may not hold between (LPM) and (LPM^*) if even Assumptions 1 and 2 are fulfilled.

EXAMPLE 4.3. We consider the network shown in Fig. 4.2. The arc cost functions are shown on the arcs and the node cost functions are zero. Moreover, the transit times are zero and a time window $[0, 1]$ is associated to each node. Let $\Theta := 1$ be the time horizon and f be a discrete flow concentrated on the singleton $\{0\}$ with $f(\{0\}) = 1$.

For each $k \in \mathbb{N}$ we define a feasible solution x^k for (LPM) as follow:

$$x_{(s,v)}^k = \sum_{i=1}^k \left(f - \frac{2}{(4i-1)\pi} \right), \quad x_{(v,s)}^k = \sum_{i=1}^{k-1} \left(f - \frac{2}{(4i+1)\pi} \right).$$

Actually the feasible solution x^k is the incidence vector of a dynamic s - t -path P^k derived as follows: We start from node s at time 0 and go from s to t and back from t to s for k times. In addition, we wait for a certain time at nodes s and t whenever we arrive at these nodes. At the end of k th circulation we wait at node t until time 1. More precisely, we have the dynamic s - t -path $P^k = (e_1, \dots, e_{2k-1})$ with

arrival times $\alpha_1, \dots, \alpha_{2k}$ and departure times $\beta_1, \dots, \beta_{2k}$, where

$$\begin{aligned} e_{2i-1} &= (v_{2i-1}, v_{2i}) = (s, t) && \text{for } i = 1, \dots, k, \\ e_{2i} &= (v_{2i}, v_{2i+1}) = (t, s) && \text{for } i = 1, \dots, k-1, \\ \alpha_1 = 0, \alpha_{2i-1} &= \frac{2}{(4k - (4i - 5))\pi} && \text{for } i = 2, \dots, k, \\ \alpha_{2i} &= \frac{2}{(4k - (4i - 3))\pi} && \text{for } i = 1, \dots, k, \\ \beta_{2i-1} &= \frac{2}{(4k - (4i - 3))\pi} && \text{for } i = 1, \dots, k, \\ \beta_{2k} = 1, \beta_{2i} &= \frac{2}{(4k - (4i - 5))\pi} && \text{for } i = 1, \dots, k-1. \end{aligned}$$

We observe that $\text{cost}(x^k) = -\sum_{i=1}^k \frac{2}{(2i+3)\pi}$. So $\text{cost}(x^k)$ tends to $-\infty$ as k goes to ∞ , and hence $V[\text{LPM}] = -\infty$.

The following two examples deal with the situation where the value of (LPM) is finite, but no feasible solution attain this value. Notice that this is not the case for static shortest path problem as it is well known that if the value of (LP) is finite, then this value is attained by some feasible solution.

EXAMPLE 4.4. We consider Example 4.3, but now with the following arc cost functions:

$$c_{t,s}(\theta) = \begin{cases} \theta^2 \sin(1/\theta) & \theta \in (0, 1], \\ 0 & \theta = 0, \end{cases} \quad c_{t,s}(\theta) = \begin{cases} -\theta^2 \sin(1/\theta) & \theta \in (0, 1], \\ 0 & \theta = 0, \end{cases}$$

Then, it holds that

$$V[\text{LPM}] = \lim_{k \rightarrow \infty} \text{cost}(x^k) = -\sum_{i=0}^{\infty} \left(\frac{2}{(2i+3)\pi} \right)^2 < \infty$$

where x^k is a feasible solution of (LPM) as defined in Example 4.3. Here the value of (LPM) is finite, but it is not attained by any feasible solution.

EXAMPLE 4.5. We consider a simple network containing of only one arc $e = (s, t)$ which joins source s to sink t . Let $\Theta := 1$ be the time horizon and c_e be the cost function given by

$$c_e(\theta) = \begin{cases} 1 - \theta & \theta < 1, \\ 1 & \theta \geq 1, \end{cases}$$

There is no waiting costs at the nodes and transit time of e is assumed to be zero. Here we have $V[\text{LPM}] = 1$, but it is not attained by any feasible solution.

The previous two examples show that Assumptions 1 and 2 do not guarantee in general the existence of an optimal solution for (LPM), even for the case that the cost functions satisfy a Lipschitz condition or are piecewise analytic. Actually the problem in Example 4.4 is because of the fact that the cost functions have a infinite number of local optimum and in Example 4.5 due to the fact that the cost functions do not attain its minimum. So it is natural to restrict the cost functions to those that have a finite number of local minima and attain their minima on a closed interval.

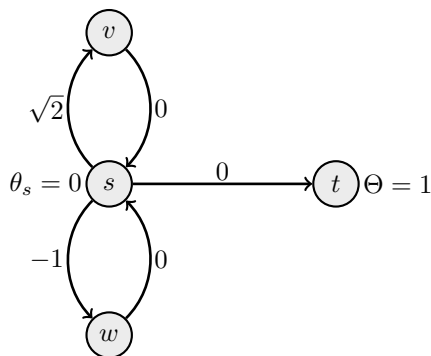


FIG. 4.3. Network for Example 4.6. The transit times are shown on the arcs.

ASSUMPTION 3. For each arc $e = (v, w) \in E$, the cost function c_e is both piecewise analytic and lower semi-continuous on $[a_v, b_v - \tau_e]$.

Notice that a function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *piecewise analytic* if there exists a partition $\{\theta_0, \theta_1, \dots, \theta_m\}$ of $[a, b]$ (i.e., $a = \theta_0 < \theta_1 < \dots < \theta_m = b$), $\epsilon > 0$, and g_k analytic on $(\theta_{k_1} - \epsilon, \theta_k + \epsilon)$ with $g_k(t) = f(t)$ for $\theta \in [\theta_{k_1}, \theta_k]$, $k = 1, \dots, m$. Hence, a piecewise analytic function can be discontinuous at a finite number of points and such a function may not attain its minimum over a closed interval. That is why we require that the cost functions are both piecewise analytic and lower semi-continuous. It is well known that a lower semi continuous function attains its minimum on a compact set. We shall use this fact later on to prove the existence of dynamic shortest paths.

The following example shows that not only the structure of cost functions, but also of transit times are important.

EXAMPLE 4.6. Consider the network shown in 4.3 with cost functions as given below:

$$\begin{aligned} c_{s,t}(\theta) = 1 - \theta & & c_{s,v}(\theta) = c_{s,w}(\theta) = c_{v,s}(\theta) = c_{w,s}(\theta) = 0 & & \forall \theta \in \mathbb{R} . \\ & & c_s(\theta) = c_v(\theta) = c_w(\theta) = c_t(\theta) = 1 & & \forall \theta \in \mathbb{R} . \end{aligned}$$

We associate a time window $[-1, 1]$ with each node and let $\Theta := 1$ be the time horizon. We observe that $V[LPM] = 0$, but no feasible solution attains this value. This is because of the fact that

$$\sup_{-1 < S < 1} \{S = m\sqrt{2} - n \mid m, n \in \mathbb{N} \cup \{0\}\} = 0 ,$$

but this value is not reached by any finite n and m .

In Example 4.6 the value of (LPM) is finite, but the problem has no optimal solution. The reason here is that greatest common factor of a set of numbers including irrational numbers does not exist. This is not the case for rational numbers. Hence, we give the following assumption.

ASSUMPTION 4. The transit times $(\tau_e)_{e \in E}$ as well as the time horizon Θ are all rational.

So far, we have observed that strong duality does not necessarily hold if at least one of the Assumptions 1, 2, 3, and 4 is not fulfilled. Throughout the rest of the paper we suppose that these assumptions hold and prove a strong duality result.

5. Strong Duality. The basic idea for establishing a strong duality result between (LPM) and (LPM*) goes along the same lines as in the static case. Therefore, we first show that the network G contains a dynamic shortest s - v -path with starting time 0 for each node $v \in V$ and for each time horizon $\theta \in [a_v, b_v]$.

Let P be a dynamic s - t -path. By Theorem 3.1 the incidence vector χ^P of P is an extreme point of (LPM). Note that, for each $e \in E$, the *support*² $\text{supp}(\chi_e^P)$ of χ_e^P is finite and, for each $\theta \in \text{supp}(\chi_e^P)$, it holds that $\chi_e^P(\theta) = 1$. Note that in a dynamic path no node is revisited at the same point in time. Hence, χ_e^P is uniquely defined by $\text{supp}(\chi_e^P)$ and is therefore interpretable as a (finite) vector $\chi_e^P \in \mathbb{R}^{|\text{supp}(\chi_e^P)|}$. The entries of the vector χ_e^P are exactly the times when we leave the tail of e along the path P . In the following χ_e^P denotes also this vector and we assume that entries are ordered monotonically increasing.

In order to define locally shortest paths we define, for all $\epsilon > 0$, the ϵ -neighborhood of a dynamic s - t -path P as the set of all dynamic s - t -paths P' satisfying:

$$|\text{supp}(\chi_e^P)| = |\text{supp}(\chi_e^{P'})| \quad \text{and} \quad \|\chi_e^P - \chi_e^{P'}\|_\infty < \epsilon \quad \forall e \in E .$$

Then, P is a *locally shortest path* if there exists an $\epsilon > 0$ such that $\text{cost}(P) \leq \text{cost}(P')$ for all paths P' in the ϵ -neighborhood of P . In the following, we show that the set of locally shortest paths is finite and hence a dynamic shortest always exists under the assumptions 1–4. For this, we give an alternative characterization of locally shortest paths and start with the definition of nonstop paths.

Let $P = (e_1, \dots, e_n)$ be a dynamic s - t -path with waiting times $(\lambda_1, \dots, \lambda_{n+1})$. A subsequence $P' = (e_k, \dots, e_\ell)$ of consecutive arcs in P is called a *nonstop subpath* of P if $\lambda_i = 0$ for $i = k+1, \dots, \ell$. If, in addition, $\lambda_k > 0$ and $\lambda_{\ell+1} > 0$ holds then the nonstop subpath P' is called *maximal*. In particular, P' is not maximal if P' starts at s at time 0 or ends at t at time Θ .

For any $\epsilon \in [-\lambda_k, \lambda_{\ell+1}]$ the arc sequence (e_1, \dots, e_n) with starting time 0 and waiting times

$$(\lambda_1, \dots, \lambda_{k-1}, \lambda_k + \epsilon, \underbrace{0, \dots, 0}_{\ell - k \text{ times}}, \lambda_{\ell+1} - \epsilon, \lambda_{\ell+2}, \dots, \lambda_{n+1})$$

is a dynamic s - t -path denoted by $P|_{P'}(\epsilon)$. Let β_k and $\alpha_{\ell+1}$ be the departure time at v_k and the arrival time at $v_{\ell+1}$ in P , respectively, and $\tau_{P'} := \sum_{i=k}^{\ell} \tau_{e_i}$ be the transit time of P' . Then, $P|_{P'}(\epsilon)$ is obtained by leaving node v_k at time $\beta_k + \epsilon$ instead of time β_k and arriving at node $v_{\ell+1}$ at time $\beta_k + \epsilon + \tau_{P'} = \alpha_{\ell+1} + \epsilon$ instead of time $\alpha_{\ell+1}$. Roughly speaking, $P|_{P'}(\epsilon)$ is obtained by shifting P' within path P by ϵ time units.

We observe that for a given $\epsilon \in [-\lambda_k, \lambda_{\ell+1}]$, the dynamic s - t -path $P|_{P'}(\epsilon)$ is contained in the $|\epsilon|$ -neighborhood of P . We are now interested to compute the difference in costs between P and $P|_{P'}(\epsilon)$. For this, we define a cost function $c_{P'} : [a_{v_k}, b_{v_k}] \rightarrow \mathbb{R}$ where $[a_{v_k}, b_{v_k}]$ is the time window of v_k by

$$c_{P'}(\theta) := \sum_{i=k}^{\ell} c_{e_i} \left(\theta + \sum_{j=k}^{i-1} \tau_{e_j} \right) . \quad (5.1)$$

²The *support* of a measure μ is defined to be the set of all points in \mathbb{R} with a neighborhood of positive measure, that is, $\text{supp}(\mu) := \{\theta \in \mathbb{R} : \mu(U) > 0 \text{ for every open neighborhood } U \text{ of } \theta\}$ (see [8, Appendix] for more details).

Thus, for a point in time θ the value $c_{P'}(\theta)$ determines the cost for traveling along P' without waiting and with starting time θ . Hence, the cost for moving from P to $P|_{P'}(\theta)$ is given by $c_{P'}(\beta_k + \epsilon) - c_{P'}(\beta_k)$. The following lemma gives a necessary condition for locally shortest paths.

LEMMA 5.1. *Let $P = (e_1, \dots, e_n)$ be a locally dynamic shortest s - t -path with departure times $\beta_1, \dots, \beta_{n+1}$. Then for each maximal nonstop subpath $P' = (e_k, \dots, e_\ell)$ of P the cost function $c_{P'}$ is locally minimized at the point β_k .*

Proof. Follows from the above discussion. \square

Let $\bar{\mathcal{P}}_{\text{loc}}$ be the set of dynamic s - t -paths P where each maximal nonstop subpath P' starting at θ locally minimizes $c_{P'}$, i.e., $c_{P'}$ has a local minimum at θ . In addition, we assume that $c_{P'}$ is not constant on any open neighborhood containing θ . Further, we say that two s - t -paths P_1 and P_2 are equivalent if they differ only in the starting time θ_1 and θ_2 , respectively, of one common nonstop subpath P' and $c_{P'}$ is constant over $[\theta_1, \theta_2]$. Note that in this case P_1 and P_2 have cost. Then, for all locally shortest paths, an equivalent path is contained in $\bar{\mathcal{P}}_{\text{loc}}$. Hence, the following lemma shows that the set of locally shortest s - t -paths is in essence finite.

LEMMA 5.2. *The set $\bar{\mathcal{P}}_{\text{loc}}$ is finite.*

Proof. Let $P \in \bar{\mathcal{P}}_{\text{loc}}$ be a dynamic s - t -path and P' be a nonstop subpath of P . Note that P' contains no dynamic cycles. First we show that there are only finitely many possible arc sequences for the nonstop subpath P' . Let $I_{\max} := \max_{v \in V} \{b_v - a_v\}$ be the maximum length of time windows $[a_v, b_v]$ over all nodes $v \in V$ (see Assumption 2) and let $\hat{\tau}$ be the greatest common factor of transit times, i.e.,

$$\hat{\tau} := \min\{S > 0 \mid S \text{ is a finite sum of elements of the form } -\tau_e \text{ or } \tau_e\}.$$

Note that $\hat{\tau}$ exists and is greater than 0 because of Assumption 4. Since P' contains no dynamic cycle it visits any node at most $\lceil \frac{I}{\hat{\tau}} \rceil$ times. Consequently, the number of possible arc sequences for P' is bounded by a constant. This implies that $c_{P'}$ is the finite sum of piecewise analytic functions. Hence, $c_{P'}$ is also piecewise analytic.

In the the following let P' be a maximal nonstop subpath of P . Because of Assumption 3 and equation (5.1), $c_{P'}$ is piecewise analytic and lower semi-continuous. Hence, $c_{P'}$ has only a finite number of local minima (at points where c'_P is not constant) and attains all of them by some real value. Therefore there are only finite number of possible start times for P' . This implies that the number of maximal nonstop subpath with the same arc sequence as P' is bounded by a constant. Otherwise P contains a dynamic cycle. Therefore the length of the arc sequence of P is bounded by a constant.

Since P is chosen arbitrary at the beginning of this proof, any arc sequence of a path in $\bar{\mathcal{P}}_{\text{loc}}$ is bounded by the same constant. Hence, the cardinality of $\bar{\mathcal{P}}_{\text{loc}}$ is finite. \square

The next lemma shows that $\bar{\mathcal{P}}_{\text{loc}}$ contains the dynamic shortest s - t -path.

LEMMA 5.3. *Let P be a dynamic s - t -path. Then there exists an s - t -path $\bar{P} \in \bar{\mathcal{P}}_{\text{loc}}$ with $\text{cost}(\bar{P}) \leq \text{cost}(P)$.*

Proof. If $P \in \bar{\mathcal{P}}_{\text{loc}}$, then we are done. So we consider the case where P is not in $\bar{\mathcal{P}}_{\text{loc}}$. In this case we iteratively apply the following procedure to construct a dynamic s - t -path $\bar{P} \in \bar{\mathcal{P}}_{\text{loc}}$:

- (i) Let $P' = (v_k, \dots, v_\ell)$ be a maximal nonstop subpath of P such that the cost function $c_{P'}$ does not have a local minimum (or is constant) at β_k . Notice that such a path path exists because of the definition of $\bar{\mathcal{P}}_{\text{loc}}$ and the fact that P is not in $\bar{\mathcal{P}}_{\text{loc}}$. Further, choose P' such that it contains a minimal number of arcs.

- (ii) Since the functions c_e , $e \in E$, are lower semi-continuous by Assumption 3, $c_{P'}$ is also lower semi-continuous. Thus, $c_{P'}$ takes its minimum over $[\beta_k - \lambda_k, \beta_k + \lambda_k]$ at some point θ . If there are several local minima choose θ maximal.
- (iii) Let $P|_{P'}(\beta_k - \theta)$ be the dynamic s - t -path obtained from P by shifting the nonstop subpath P' by $\beta_k - \theta$ time units. Since $P|_{P'}(\theta)$ may contain dynamic cycles, we delete all dynamic cycles in $P|_{P'}(\theta)$.
- (iv) Set $P := P|_{P'}(\theta)$. If P is not in $\bar{\mathcal{P}}_{\text{loc}}$, then go to (i).

Notice that the number of arcs in P' is bounded by $|E(P)|$ and increases after at most $|E(P)|$ iterations. Hence, the above procedure terminates after a finite number of iterations and the resulting dynamic s - t -path P is contained in $\bar{\mathcal{P}}_{\text{loc}}$. Further, in each iteration the cost of P does not increase which proves this lemma. \square

Lemmas 5.2 and 5.3 show that a dynamic shortest s - t -path exists. More precisely, a dynamic shortest s - t -path is one in $\bar{\mathcal{P}}_{\text{loc}}$ with minimal cost. Further, Lemma 5.2 as well as Lemma 5.3 remain valid if the sink t is replaced by any node $v \in V$ and if the time horizon Θ is replaced by any point in time $\theta \in [a_v, b_v]$. Furthermore, we obtain the following result.

LEMMA 5.4. *For each node $v \in V$ and each point in time $\theta \in [a_v, b_v]$, let $d_v(\theta)$ be the cost of a dynamic shortest s - v -path with starting time 0 and time horizon θ . Then, for each node $v \in V$, the label $d_v(\theta)$ exists for all θ and the function $d_v : [a_v, b_v] \rightarrow \mathbb{R}$ is piecewise analytic and monotonic decreasing.*

Proof. As discussed above the existence of $d_v(\theta)$ follows from 5.2 and 5.3. Furthermore, since there are no waiting costs the function d_v is monotonic decreasing. Hence, it thus remains to show that d_v is piecewise analytic on $[a_v, b_v]$ for each $v \in V$.

In the following we fix a node $v \in V$. Similar to the definition of $\bar{\mathcal{P}}_{\text{loc}}$ before Lemma 5.2 define $\bar{\mathcal{P}}_{\text{loc}}(\theta)$ as the set of dynamic s - v -paths P with starting time 0 and with time horizon θ where each maximal nonstop subpath P' of P starting at ϑ locally minimizes $c_{P'}$. In addition, we assume that $c_{P'}$ is not constant on any open neighborhood containing ϑ . Then $\mathcal{P}_v := \cup_{\theta \in [a_v, b_v]} \bar{\mathcal{P}}_{\text{loc}}(\theta)$ contains (nearly) all dynamic shortest s - v -paths for any feasible time horizon θ .

Next we define an equivalence relation \sim on the set of all dynamic s - v -paths. Let $P = (e_1, \dots, e_{n_P})$ and $\bar{P} = (\bar{e}_1, \dots, \bar{e}_{n_{\bar{P}}})$ be two dynamic s - v -paths with waiting times $(\lambda_1, \dots, \lambda_{n_P})$ and $(\bar{\lambda}_1, \dots, \bar{\lambda}_{n_{\bar{P}}})$. Then \sim is defined by

$$\begin{aligned}
 P \sim \bar{P} \quad & :\iff & (e_1, \dots, e_{n_P}) = (\bar{e}_1, \dots, \bar{e}_{n_{\bar{P}}}) , \\
 & & \exists k \in \{1, \dots, n_P + 1\} : \quad \lambda_i = \bar{\lambda}_i \quad \forall i < k , \\
 & & \lambda_i, \bar{\lambda}_i > 0 \quad i = k , \\
 & & \lambda_i = \bar{\lambda}_i = 0 \quad \forall i > k .
 \end{aligned}$$

Hence, P and \bar{P} are equivalent if they differ only in the last positive waiting time. For an equivalence class $[P]$ we denote by P_1 the path consisting of the first $k - 1$ arcs of P without waiting at the end and by P_2 the path consisting of the last $n_P - k + 1$ arcs of P without waiting at the beginning. Note that P_1 and P_2 can be the empty path. Further, P_1 and P_2 are well-defined in the sense that they coincide for any member of $[P]$. On the other hand, any dynamic path in $[P]$ is obtained by concatenating P_1 and P_2 and introducing some positive waiting between them. (If P is a nonstop path without waiting at all we put it in the equivalence class $P_1 = \emptyset$ and $P_2 = P$.)

Consider the quotient set \mathcal{P}_v / \sim and an equivalence class $[P] \in \mathcal{P}_v / \sim$. Then each maximal nonstop subpath P' and also the last nonstop subpath of P locally minimizes $c_{P'}$. Further, P_2 is a nonstop subpath itself. Hence, along the same lines

as in the proof of Lemma 5.2 we obtain that there exists only a finite number of possibilities for P_1 and P_2 . Hence, \mathcal{P}_v / \sim is a finite set. In order to get an expression for d_v we define a cost function $c_{[P]} : [a_v, b_v] \rightarrow \mathbb{R}$ by

$$c_{[P]}(\theta) := \text{cost}(P_1) + \begin{cases} c_{P_2}(\theta - \tau_{P_2}) & \text{if } \theta > \tau_{P_1} + \tau_{P_2} , \\ \infty & \text{if } \theta \leq \tau_{P_1} + \tau_{P_2} , \end{cases}$$

Then, for every $P \in \mathcal{P}_v$ we have $\text{cost}(P) = c_{[P]}(\theta)$ where θ is the time horizon of P . Thus we obtain $d_v = \min\{c_{[P]}\}$. Therefore d_v is piecewise analytic since it is the minimum of a finite number of piecewise analytic functions. \square

From the above discussion we obtain the main result of this section.

THEOREM 5.5 (Strong duality). *There exist an extreme point x for (LPM) and a piecewise analytic solution π for (LPM*) so that*

$$V[\text{LPM}, x] = V[\text{LPM}^*, \pi].$$

Proof. Following Lemma 5.4, we define for each $v \in V$ and each $\theta \in [a_v, b_v]$ the *shortest label* $d_v(\theta)$ to be the cost of a dynamic shortest s - v -path with starting time 0 and time horizon θ . Obviously, we have $d_s(0) = 0$ since the network contains no negative dynamic cycle due to Assumption 1. In the following we show that the shortest path labels define a dual feasible solution whose value equals to the cost of some feasible solution for (LPM).

Let $\pi_v(\theta) = -d_v(\theta)$ for any $v \in V$ and every $\theta \in [a_v, b_v]$. It follows from Lemma 5.4 that π is a piecewise analytic solution for (LPM*) Now let P be a dynamic shortest s - t -path with starting time 0 and time horizon Θ and χ^P denote its corresponding incidence vector. We know from Theorem 3.1 that χ^P is an extreme point of (LPM) whose value is equal to the cost of P . Summarizing, we can conclude that

$$V[\text{LPM}, \chi^P] = \text{cost}(P) = d_t(\Theta) = -d_s(0) + d_t(\Theta) = \pi_s(0) - \pi_t(\Theta) = V[\text{LPM}^*, \pi] .$$

It now follows from Lemma 4.2 that x is optimal for (LPM) and π is optimal for (LPM*). This yields the desired result. \square

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