

EXTREME POINT CHARACTERIZATION FOR CONTINUOUS-TIME DYNAMIC SHORTEST PATH PROBLEMS WITH ARBITRARY TRANSIT TIMES*

RONALD KOCH[†] AND EBRAHIM NASRABADI[‡]

Abstract. We consider the dynamic shortest path problem in the continuous-time model. This problem has been extensively studied in the literature because of its importance. But so far, all contributions to this problem are based on the assumption that all transit times are strictly positive. However, in order to study dynamic network flow problems it is essential to support negative transit times since they occur quite naturally in residual networks. In this paper we extend the work of Philpott [SIAM J. Control Optim., 1994, pp. 538–552] to the case of arbitrary (also negative and irrational) transit times. We study a corresponding linear program in a space of measures and give a full characterization of its extreme points. In particular, we show a one-to-one correspondence between extreme points and dynamic paths.

Key words. Shortest Path Problem, Linear Programming in Measure Spaces, Extreme Points, Measure Theory

AMS subject classifications. 05C21, 28A25, 49J27, 90C48, 90C49

1. Introduction. The notion of extreme points plays a central role in the theory of linear programming, specifically for the simplex algorithm. Fundamental to this algorithm is that, whenever the problem has an optimum solution, then one can be found among the extreme points of the feasible region. Moreover, extreme points usually have a considerably simpler structure than arbitrary feasible solutions and are more meaningful in practice. This becomes even more important for network flow problems because extreme points of the feasible region correspond to the flows which do not admit cycles, that is, the arcs with flow strictly between their bounds form a forest.

The shortest path problem is one of the most basic and important problems in combinatorial optimization, operations research, and computer sciences. For this problem, the extreme points of the corresponding linear programming problem correspond one-to-one to paths from the source to the sink. An interesting extension of the shortest path problem is the *dynamic shortest path problem*, whose goal is to find a shortest path between two specified nodes in a network where

- each arc has a *transit time* determining the amount of time needed for traversing that arc,
- waiting at the nodes is allowed and causes cost,
- the cost for traversing an arc as well as the cost for waiting vary over time.

In this paper, we concentrate on a continuous-time setting where arcs can be entered and left at arbitrary points in time and assign also negative values to transit times. We study a related linear programming problem formulated in the space of measures, present a full characterization of its extreme points, and conclude a one-to-one correspondence between extreme points and dynamic paths.

*This work is supported by DFG project SK58/7-1. The first author is supported by DFG Research Center MATHEON in Berlin.

[†]Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany, phone: +49 (0)30 314-28641, fax: +49 (0)30 314-25191, ([koch,nasrabadi}@math.tu-berlin.de](mailto:{koch,nasrabadi}@math.tu-berlin.de))

[‡]Present address: Operations Research Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, (nasrabad@mit.edu)

Related Literature. The dynamic shortest path problem was first introduced by Cooke and Halsey [9], who present an algorithm based on Bellman’s principle of optimality. In the model proposed by Cooke and Halsey [9], arcs can be entered and left only at integral points in time, leading to the so-called *discrete-time model*. Since then, a number of authors (see, e.g., [1, 7, 8, 17]) have studied different aspects of the discrete-time dynamic shortest path problem.

Although discretization of time leads to problems that are considerably easier to solve, this approach suffers from a serious drawback: the points in time at which decisions are made are fixed in advance, before the problem is solved. For many applications, this is not a desired feature of the corresponding problem, since we get only approximations of the optimum. In contrast, in the continuous-time model, decisions can be made at arbitrary points in time.

The first work on dynamic network flows in the continuous-time model is due to Philpott [18] and Anderson, Nash and Philpott [5], who study the dynamic maximum flow problem in a network with zero transit times and time-varying transit and storage capacities. Anderson [2] introduces *Continuous-time Dynamic Network Flow Problems (CDNFP)* within the space of bounded measurable functions and characterizes the extreme points for the problem given rational transit times. Anderson, Nash, and Perold [4] characterize extreme points for a general class of CDNFP, but with zero transit time, so-called *Separated Continuous Linear Programs (SCLP)*. Anderson and Pullan [6] develop a purification algorithm for SCLP, that is, the process of turning an arbitrary feasible solution into an extreme point solution whose objective function value is no worse. Hashemi and Nasrabadi [11] present a characterization of extreme points and adapt the purification algorithm developed in [6] for CDNFP with piecewise analytic problem data and rational transit times.

Pullan [22] examines a larger class of SCLP to include transit-times. For the case that transit times are rational, he transforms the problem into a larger problem which is very close to a special class of SCLP and develops interesting results. In particular, Pullan [22] characterizes the extreme points and proves the existence of piecewise analytic optimum extreme point solutions for the case where input functions are piecewise analytic.

Research on the continuous-time version of the dynamic shortest path problem is conducted by, e.g., Orda and Rom [15, 16], Philpott [19], and Philpott and Mees [20, 21]. In particular, Philpott [19] formulates the problem as a linear program (LP for short) in a space of measures and investigates the relationship between the problem and its LP formulation. Especially, he introduces a dual problem and proves the absence of a duality gap. He also demonstrates the existence of an optimal extreme point for the LP formulation and derives a correspondence between extreme points and dynamic paths. Moreover, he establishes a strong duality result in the case where the cost functions are Lipschitz-continuous.

In all of the work mentioned above, it is assumed that the transit times are strictly positive. In particular, this assumption is critical to the arguments presented by Philpott [19]. In this case, the feasible region of the LP formulation becomes bounded with respect to a certain norm. This makes it possible to apply certain results from the theory of linear programming in infinite-dimensional spaces (see, e.g., Anderson and Nash [3]). This method no longer works for the case of nonpositive transit times since the feasible region of the corresponding LP formulation may be unbounded. Philpott [19] writes in the conclusion of his paper, “the assumption that all transit times are strictly positive is central to the arguments presented”.

In this paper we extend Philpott's work [19] to the case where transit times can be also negative. Notice that the assumption that all transit times are strictly positive is not too restrictive in direct applications of the dynamic shortest path problem. For instance, when some material (e.g., a vehicle or a message) has to be sent between two specified points in a network as quickly or as cheaply as possible. However, like classical shortest path problems, instances with negative transit times arise in solving more complicated problems. For example, verifying whether a given dynamic flow has minimum cost, we have to scan the residual network for negative cycles (see Chapter 6 in [14]). But, in general, the residual network contains arcs with negative transit times. Hence, we cannot use the results in the literature and particularly not those derived by Philpott [19]. This is our main motivation to study the dynamic shortest path problem in the continuous-time setting, but – in contrast to Philpott's work – with possibly negative transit times.

As already mentioned, research on dynamic flows has been pursued in two different and mainly independent directions with respect to time modeling: discrete-time and continuous-time models. However, in many applications a joint combination of both models is desirable. Measure theory provides an appreciate tool to unify discrete-time and continuous-time dynamic network flows into a single model. In this regard, Koch, Nasrabadi, and Skutella [13] study the dynamic maximum flow problem formulated in a network where capacities on arcs are given as Borel measures and storage is allowed at the nodes of the network. They generalize the concept of cuts to the case of these *Borel Flows* and extend the famous MaxFlow-MinCut Theorem. In addition to the paper [13], this paper provides a basic conceptual framework for Borel flows, but concentrated on the dynamic shortest path problem, and lays the ground for further research on dynamic network flows formulated in the space of measures.

Our Contribution. We advance the state of the art for dynamic shortest path problems by bridging the gap between them and linear programming. Our results generalize those achieved by Philpott [19] for the case of positive transit times, but the ideas of the underlying analysis and proofs do not follow those of Philpott. In particular, Philpott [19] appeals to abstract results from infinite-dimensional linear programming, whereas our results are more constructive and can be used to characterize extreme points for dynamic minimum cost flow problems in a space of measures.

In Section 2, a detailed definition of the continuous-time dynamic shortest path problem is given. Like in the static case, the problem is formulated as the minimum cost flow problem, but in contrast, the flow on each arc is modeled as a finite *Borel measure* on the real line (time axis) indicating the amount of flow entering the arc over time. This idea is due to Philpott [19], who gives an LP formulation of the problem in space of measures. In Section 3, we give an alternative LP formulation, which differs slightly from that formulated by Philpott [19].

The main results of this paper appear in Sections 4 and 5 where we show that extreme points of the LP formulation are measures with finite support corresponding to dynamic paths. To this end, in Section 4, we prove that the continuous part of any extreme point solution is zero. The main idea of the proof is to start at a node with some stored flow and then construct a search tree that tracks where this flow goes (while preserving measure). We then use the assumption that the flow on each arc is a finite measure to show that this construction terminates at another node with storage after a finite number of steps. In Section 5, we extend this construction for discrete solutions to show that extreme points correspond to dynamic paths.

Our results are based on novel ideas and new techniques, which, among others,

include a fair amount of advanced measure theory. For the convenience of the reader, we give the basic definitions and results in measure theory that are required for the purposes of this paper in Appendix A. In Appendix B we provide the proofs of some technical lemmas.

2. Preliminaries and Problem Description. In this section we give a precise description of the dynamic shortest path problem in the continuous-time model. To motivate our treatment, we first describe the static shortest path problem.

We consider a directed graph $G := (V, E)$ with finite *node set* V and finite *arc set* $E \subseteq V \times V$. An arc e from a node v to a node w is denoted by $e := (v, w)$ to emphasize that e leaves v and enters w . In this case, we say that node v is the *tail* of e and w is the *head* of e and write $\text{tail}(e) := v$ and $\text{head}(e) := w$. Further, we use $\delta^+(v) := \{e \in E \mid \text{tail}(e) = v\}$ and $\delta^-(v) := \{e \in E \mid \text{head}(e) = v\}$ to denote the set of arcs leaving node v and entering node v , respectively.

A *walk* P from node v to node w is an alternating sequence of nodes and arcs of the form $P := (v_1, e_1, v_2, \dots, v_n, e_n, v_{n+1})$ such that $v_1 = v$, $e_i = (v_i, v_{i+1})$ for $i = 1, \dots, n$, and $v_{n+1} = w$. Throughout the paper we denote the walk P by the arc sequence (e_1, \dots, e_n) , assuming that $\text{head}(e_i) = \text{tail}(e_{i+1})$ for $i = 1, \dots, n$. Further, we denote by $E(P) := \{e_1, \dots, e_n\}$ and $V(P) := \{v_1, \dots, v_{n+1}\}$ the set of arcs and nodes, respectively, involved in P . The walk P is said to be a *path* from v to w (or simply *v-w-path*) if v_1, v_2, \dots, v_{n+1} are pairwise distinct, except v_1 and v_{n+1} . If in addition $v_1 = v_{n+1}$, the path P is called a *cycle*. A node w is said to be *reachable* from node v if there exists a *v-w-path*.

Each arc $e \in E$ has an associated *cost* c_e . The *cost* of a path $P = (e_1, \dots, e_n)$ is defined as the sum of the costs of all arcs in the path, that is, $c_P := \sum_{i=1}^n c_{e_i}$. Let $s \in V$ be a given *source* and $t \in V$ be a given *sink*. We assume without loss of generality that every node of G is reachable from s and that t is reachable from every node. The (static) *shortest path problem* is to determine a path from source s to sink t with minimal cost. This problem can be seen as the problem of sending one unit of flow from s to t at minimal cost and hence, can be formulated as follows:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = \begin{cases} 1 & \text{if } v = s \\ 0 & \text{if } v \neq s, t \\ -1 & \text{if } v = t \end{cases} \quad \forall v \in V, \quad (\text{LP}) \\ & x_e \geq 0 \quad \forall e \in E. \end{aligned}$$

Here the decision variable x_e gives the amount of flow on arc e . It is well-known that the underlying constraint matrix is total unimodular. This implies that any extreme point of (LP) corresponds one-to-one to a path from s to t with the same cost. In particular, in any extreme point, x_e is either 1 or 0 for each $e \in E$ which indicates whether or not the corresponding path involves the arc e , respectively. Thus, an optimal extreme point of (LP) yields a shortest *s-t-path*.

So far we have considered the setting of the *static* shortest path problem in which time does not enter the model. We now turn to the dynamic case in which each arc $e \in E$ has an associated *transit time* τ_e , specifying the required amount of time to travel from the tail to the head of e . More precisely, if we leave node v at time θ along an arc $e = (v, w)$, we arrive at w at time $\theta + \tau_e$. Further, waiting is allowed at

the nodes of the network for later departure. In the following we extend the definition of (static) walk, path, and cycle to the dynamic case.

A *dynamic walk* is a pair of a walk $P = (e_1, \dots, e_n)$ together with a family of *waiting times* $(\lambda_1, \dots, \lambda_{n+1})$. For $i = 1, \dots, n + 1$ after arriving at node $v_i \in V(P)$ we wait λ_i time units before we leave v_i . Given a *starting time* θ , let α_i be the time when we arrive at node v_i and β_i be the time when we departure from node v_i . For $i = 1, \dots, n + 1$, the *arrival time* α_i and the *departure time* β_i can be computed recursively as follows:

$$\alpha_i := \begin{cases} \theta & \text{for } i = 1 \\ \beta_{i-1} + \tau_{e_{i-1}} & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_i := \alpha_i + \lambda_i .$$

A walk P is called a *dynamic path* if P does not revisit any node (except the endpoints) at the same point in time, i.e., $[\alpha_i, \beta_i] \cap [\alpha_j, \beta_j] = \emptyset$ for each $1 \leq i < j \leq n + 1$ with $v_i = v_j$ and $(i, j) \neq (1, n + 1)$. Note that the underlying (static) walk of a dynamic path need not to be a (static) path since it is allowed that a node can be revisited at different points in time. Moreover, P is said to be a *dynamic cycle* if P is a dynamic path, and in addition $v_1 = v_{n+1}$ and $\alpha_1 = \beta_{n+1}$. Further, we say that the path P has *time horizon* Θ if $\beta_{n+1} = \Theta$.

Each arc e and each node v has a *cost function* $c_e : \mathbb{R} \rightarrow \mathbb{R}$ and $c_v : \mathbb{R} \rightarrow \mathbb{R}$, respectively. For a certain point in time $\theta \in \mathbb{R}$, the cost for leaving the tail of e at time θ and traveling along e is $c_e(\theta)$ and the cost per time unit for the waiting at v at time θ is $c_v(\theta)$. The *cost* of a dynamic walk $P = (e_1, \dots, e_n)$ with arrival times $\alpha_1, \dots, \alpha_{n+1}$ and departure times $\beta_1, \dots, \beta_{n+1}$ is thus given by

$$\text{cost}(P) := \sum_{i=1}^n c_{e_i}(\beta_i) + \sum_{i=1}^{n+1} \int_{\alpha_i}^{\beta_i} c_{v_i}(\theta) d\theta .$$

Here the first sum gives the cost for traveling along arcs in the path and the second one gives the cost for waiting at nodes of the path. A dynamic path P from v to w is called a *dynamic shortest path* if $\text{cost}(P) \leq \text{cost}(P')$ for all dynamic v - w -paths P' with the same starting time and the same time horizon as P .

Given a *source* $s \in V$, a *sink* $t \in V$, and a *time horizon* Θ , the *continuous-time dynamic shortest path problem* is to determine a dynamic shortest path from s to t with starting time 0 and time horizon Θ :

CONTINUOUS-TIME DYNAMIC SHORTEST PATH PROBLEM (CDSP)

Input: A network consisting of a directed graph $G := (V, E)$, cost functions $(c_e)_{e \in E}$ and $(c_v)_{v \in V}$, a source $s \in V$, a sink $t \in V$, and a time horizon Θ ,

Task: Find a dynamic shortest s - t -path with starting time 0 and time horizon Θ .

For the case that transit times as well as the time horizon are integral and further, waiting times have to be integral, the problem is a *discrete-time model*. Actually, in the discrete-time model, one is only allowed to leave each node at integral points in time. Hence, the resulting problem can be solved by the time-expanded network technique (see, e.g., [10]). In this paper we focus on the more challenging continuous-time model in which we can leave each node at any point in time and further, transit times as well as the time horizon can be any real value.

Throughout the paper, if not mentioned otherwise, the starting time and the time horizon of a dynamic path from source s to sink t are assumed to be 0 and Θ , respectively.

3. An Infinite-dimensional Linear Program for CDSP. We observed that the static shortest path problem has an equivalent LP formulation as (LP). More precisely, there is a one-to-one correspondence between static s - t -paths and extreme points of (LP) which preserves costs. Hence, a natural question is “Does CDSP have an equivalent LP formulation?” In order to answer this question we go along the same lines as in the static case. In fact, CDSP can be seen as the problem of sending one unit of flow from source s at time 0 to sink t at time Θ at minimal cost which can be modeled as a minimum cost flow over time problem. Unlike the static case the flow over time on each arc $e \in E$ is given by a *measure* x_e on the real line \mathbb{R} (time axis) which assigns to each suitable subset B a nonnegative real value $x_e(B)$. Intuitively, $x_e(B)$ is interpreted as the amount of flow entering arc e within the times in the subset B . This idea is due to Philpott [19], who formulates CDSP as a linear program in space of measures. In what follows, we give a detailed description of the LP formulation for CDSP, which differs slightly from that of Philpott [19] and is easier to recognize as a generalization of (LP). But first we motivate the use of measures.

Let \mathcal{B} be the collection of all intervals of \mathbb{R} , whose elements can be seen as time intervals. In order to distribute flow over time on an arc $e \in E$ we assign a value $x_e(I)$ to each time interval I , determining the amount of flow entering arc e within the time interval I . Thus, intuitively, the function $x_e : \mathcal{B} \rightarrow \mathbb{R}$ has to satisfy the following properties:

- (i) The flow assigned to the empty set is 0, i.e., $x_e(\emptyset) = 0$.
- (ii) An amount of flow is always nonnegative, i.e., $x_e(I) \geq 0$ for all $I \in \mathcal{B}$.
- (iii) For any countable collection $(I_i)_{i \in \mathbb{N}}$ of pairwise disjoint intervals in \mathcal{B} , it holds:

$$x_e \left(\bigcup_{i \in \mathbb{N}} I_i \right) = \sum_{i \in \mathbb{N}} x_e(I_i) .$$

On closer inspection of property (iii) we observe that \mathcal{B} must be closed under countable unions of its members. Otherwise property (iii) is not well-defined. In addition, we require that \mathcal{B} is closed under complementation. Henceforth, we extend \mathcal{B} to the smallest set containing all (open) intervals which is closed under countable union and complementation. We call \mathcal{B} the *Borel σ -algebra on \mathbb{R}* and a set $B \in \mathcal{B}$ a *Borel set* or a *measurable set*. In this manner properties (i)–(iii) make x_e a *Borel measure*. Thus, measures provide an appropriate tool for defining flow distributions over time.

Based on the above observations, we define a *Borel flow* x by Borel measures

$$x_e : \mathcal{B} \rightarrow \mathbb{R} \qquad e \in E .$$

For a Borel set $B \in \mathcal{B}$ the value $x_e(B)$ gives the amount of flow entering arc e within the times in B . For the purposes of the paper, we require x_e to be a *finite* Borel measure on \mathbb{R} , i.e., $|x_e| := x_e(\mathbb{R}) < \infty$ for all $e \in E$. This means that the total amount of flow traversing an arc is finite. Through the paper, we focus on Borel flows and therefore the term *flow* is used to refer to Borel flows.

Recall that the problem we want to model is to send one unit of flow from s at time 0 to t at time Θ so that the cost is minimized. This means that there is a supply of one flow unit at the source at time 0, and a demand of one flow unit at the

sink at time Θ . Hence, the supply or demand at a node v is given by a signed Borel measure b_v defined as

$$b_v(B) := \begin{cases} 1 & v = s, 0 \in B \\ -1 & v = t, \Theta \in B \\ 0 & \text{otherwise} \end{cases} \quad \forall B \in \mathcal{B} .$$

The value $|b_v(B)|$ is interpreted as the amount of supply or demand at node v over the Borel set B depending on whether $b_v(B) > 0$ or $b_v(B) < 0$, respectively.

Flow has to be stored at a node $v \in V$ if more flow enters v than leaves that node at certain points in time. Given a flow x , the *storage* at node v is determined by a signed measure y_v defined as

$$y_v(B) := b_v(B) - \sum_{e \in \delta^+(v)} x_e(B) + \sum_{e \in \delta^-(v)} x_e(B - \tau_e) \quad \forall B \in \mathcal{B} , \quad (3.1)$$

where $B - \tau_e := \{\theta - \tau_e \mid \theta \in B\}$. For a Borel set B the value $y_v(B)$ shows the amount of flow which is in total additionally stored at v over B if $y_v(B) \geq 0$. Note that flow can also leave v , even if $y_v(B) \geq 0$. Further, if $y_v(B) \leq 0$ the value $-y_v(B)$ can be interpreted as the total amount of stored flow leaving v over B . Since the space of signed measures is a vector space (see Appendix A), (3.1) can be rewritten as follows:

$$\sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} (x_e - \tau_e) + y_v = b_v .$$

Here $x_e - \tau_e$ is understood to be a Borel measure defined by $(x_e - \tau_e)(B) := x_e(B - \tau_e)$ for every Borel set $B \in \mathcal{B}$.

We know that every signed Borel measure can be uniquely decomposed into a sum of a discrete and a continuous measure (see for more details Appendix A). This implies that for each arc e the flow x_e is the sum of a continuous flow x_e^c and a discrete flow x_e^d . Similarly, for each node $v \in V$ the storage y is the sum of a continuous storage y^c and a discrete storage y^d . Since the supply or demand b_v is a discrete measure for each node v , we get the following equation for $x^c = (x_e^c)_{e \in E}$ and $y^c = (y_v^c)_{v \in V}$:

$$\sum_{e \in \delta^+(v)} x_e^c - \sum_{e \in \delta^-(v)} (x_e^c - \tau_e) + y_v^c = 0 \quad \forall v \in V . \quad (3.2)$$

A flow x is called *discrete* (continuous) if $x_e^c = 0$ ($x_e^d = 0$) for all arcs $e \in E$. So it follows from (3.2) that $y^c = 0$ whenever x is discrete.

For a node $v \in V$ we let Y_v be the distribution function of the storage y_v , that is, $Y_v(\theta) := y_v((-\infty, \theta])$, for all $\theta \in \mathbb{R}$. So we have

$$Y_v(\theta) = b_v((-\infty, \theta]) - \sum_{e \in \delta^+(v)} x_e((-\infty, \theta]) + \sum_{e \in \delta^-(v)} x_e((-\infty, \theta - \tau_e]) \quad \forall \theta \in \mathbb{R} .$$

Here the first sum denotes the total amount of flow arriving at node v up to time θ and the second one represents the total amount of flow leaving node v up to time θ . Further, $|b_v((-\infty, \theta])|$ gives the total amount of supply or demand at node v up to time θ , depending on whether $b_v((-\infty, \theta])$ is positive or negative, respectively. Thus, the value $Y_v(\theta)$ gives us the amount of flow stored at node v at the point in time $\theta \in \mathbb{R}$. It is assumed that there is no initial storage at any node and flow must not remain at

any node. This means $Y_v(-\infty) := \lim_{\theta \rightarrow -\infty} Y_v(\theta)$ and $Y_v(\infty) := \lim_{\theta \rightarrow \infty} Y_v(\theta)$ must be zero for each node $v \in V$. Notice that both limits exist since Y_v is of bounded variation.

A flow x with corresponding storage y fulfills the *flow conservation constraint* at node v if

$$Y_v(\theta) \geq 0 \quad \forall \theta \in \mathbb{R} .$$

The flow x fulfills the *strict flow conservation constraints* at node v if the equality holds in the above inequality. This implies that storage is not allowed at v .

We suppose that the cost functions $(c_e)_{e \in E}$ and $(c_v)_{v \in V}$ are measurable. The value $c_e(\theta)$ can be interpreted as the cost per flow unit for sending flow into arc e at time θ and $c_v(\theta)$ as the cost per time unit for storing one unit of flow at node v at time θ . The *cost* of a flow x with corresponding storage y is thus given by

$$\text{cost}(x) := \int_{-\infty}^{\infty} \sum_{e \in E} c_e(\theta) dx_e + \int_{-\infty}^{\infty} \sum_{v \in V} c_v(\theta) Y_v(\theta) d\theta . \quad (3.3)$$

Summarizing the above discussion, the problem of sending one unit of flow from a source s at time 0 to a sink t at time Θ at minimal cost can be expressed as the following linear program in the space of signed Borel measures:

$$\begin{aligned} \min \quad & \int_{-\infty}^{\infty} \sum_{e \in E} c_e(\theta) dx_e + \int_{-\infty}^{\infty} \sum_{v \in V} c_v(\theta) Y_v(\theta) d\theta \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} (x_e - \tau_e) + y_v = b_v \quad \forall v \in V , \\ & x_e \geq 0 \quad \forall e \in E , \\ & Y_v \geq 0 \quad \forall v \in V . \end{aligned} \quad (\text{LPM})$$

This formulation is quite similar to the formulation of (LP) and can be seen as its extension. In fact if waiting is not allowed and the transit times as well as the time horizon are zero, then (LPM) reduces to (LP).

A flow x with corresponding storage y satisfying the constraints of (LPM) is called a feasible solution or feasible flow. Similarly as in the finite-dimensional linear programming, a feasible solution of (LPM) is called an extreme point if it cannot be derived from a convex combination of any two other feasible solutions. In the next section we want to show that extreme points of (LPM) correspond to dynamic s - t -paths with the same cost and vice versa. Hence, we have to encode a dynamic path with measures, whereas in the static case a path is identified with its incidence vector whose elements are 0 or 1. Let $P := (e_1, \dots, e_n)$ be a dynamic path with arrival times α_i and departure times β_i for $i = 1, \dots, n+1$. The *incidence vector* χ^P of P is a family $(\chi_e^P)_{e \in E}$ of discrete measures defined by

$$\chi_e^P := \begin{cases} \sum_{i | e_i = e} \chi_i^P & \text{if } e \in E(P) \\ 0 & \text{otherwise} \end{cases} \quad \forall e \in E , \quad (3.4)$$

where

$$\chi_i^P(B) := \begin{cases} 1 & \text{if } \alpha_i \in B \\ 0 & \text{otherwise} \end{cases} \quad \forall B \in \mathcal{B} \quad i = 1, \dots, n .$$

The corresponding storage $\psi^P := (\psi_v^P)_{v \in V}$ is defined by:

$$\psi_v^P := b_v - \sum_{e \in \delta^+(v)} \chi_e^P + \sum_{e \in \delta^-(v)} (\chi_e^P - \tau_e) \quad \forall v \in V. \quad (3.5)$$

For each $v \in V$ let Ψ_v^P denote the distribution function of the measure ψ_v^P , i.e., $\Psi_v^P(\theta) := \psi_v^P((-\infty, \theta])$ for all $\theta \in \mathbb{R}$. It is not hard to observe that

$$\Psi_v^P(\theta) = \begin{cases} \sum_{i|v_i=v} \Psi_i^P(\theta) & \text{if } v \in V(P) \\ 0 & \text{otherwise} \end{cases} \quad \forall v \in V,$$

where

$$\Psi_i^P(\theta) := \begin{cases} 1 & \text{for } \theta \in [\alpha_i, \beta_i) \\ 0 & \text{otherwise} \end{cases} \quad \forall i = 1, \dots, n+1.$$

Therefore, $\Psi_v^P \geq 0$ and the incidence vector χ^P is a feasible solution of (LPM). In the following section we show that χ^P is not only a feasible solution, but also an extreme point of (LPM).

4. Continuous Part of Extreme Points. In this section, we show that the continuous part of any extreme point is 0. We begin our discussion with an important result concerning the characterization of extreme points. Roughly speaking, it deals with the following situation: whenever there exists a walk P carrying a continuous measure of flow and in addition there is waiting at the beginning and at the end but not at intermediate nodes of P , then the corresponding feasible solution is not an extreme point. For the proof, we require the following technical lemma, proven in Appendix B.

LEMMA 4.1. *Let μ be a finite signed measure on \mathbb{R} with a nonnegative distribution function F . Let $S := \{\theta \in \mathbb{R} \mid F|_{(a,b)} > 0 \text{ for some interval } (a,b) \ni \theta\}$ be the set of points with a strictly positive neighborhood regarding F . Then $\mathbb{R} \setminus S$ is a strict μ^c -null set, i.e., $\mu^c|_{\mathbb{R} \setminus S} = 0$.*

For a signed measure μ the positive and negative part is denoted by μ^+ and μ^- , respectively (see Appendix A). This is used in the following lemma.

LEMMA 4.2. *Let x with corresponding storage y be a feasible solution for (LPM) and $P = (e_1, \dots, e_n)$ be a walk from node v to node w . Further, assume that there exists a positive measure f such that*

$$f - (\tau_{e_1} + \dots + \tau_{e_n}) \leq y_v^{c-}, \quad (4.1)$$

$$\sum_{i|e_i=e} f - (\tau_{e_1} + \dots + \tau_{e_i}) \leq x_e^c \quad \forall e \in E, \quad (4.2)$$

$$f \leq y_w^{c+}. \quad (4.3)$$

Then x is not an extreme point.

Proof. We show that x can be written as the convex combination of two feasible solutions x^1 and x^2 . The basic idea is to increase and decrease x along P to construct x^1 and x^2 . This will change the flow x on arcs e_1, \dots, e_n and affect the storage y at endpoints v and w . To maintain the feasibility of x_1 and x_2 , we first find a closed interval satisfying certain properties. We then send at the beginning of the interval less flow and at the end more flow as compared with x . But in total

we send the same amount of flow as x over the interval along P . This gives us x^1 . The solution x^2 is constructed the other way around, i.e., we send more flow at the beginning of the interval and less flow at the end.

Let us first look for an interval which ensures that we are able to increase and decrease the flow x slightly along P without violating feasibility. Let $\tau := \sum_{k=1}^n \tau_{e_i}$ be the transit time of the walk P . Notice that P can be seen as a dynamic walk with zero waiting times at nodes. We show that there exists a closed interval I satisfying the following properties:

1. $f(I) > 0$ implying that the flow can be reduced along P over $I - \tau$ with a nonzero measure;
2. $Y_v|_{I-\tau} > \epsilon$ and $Y_w|_I > \epsilon$ for some $\epsilon > 0$ implying that the storage can be reduced at v and w .

As in Lemma 4.1, we define

$$\begin{aligned} S_v &:= \{\theta \in \mathbb{R} \mid Y_v|_{I'} > 0 \text{ for some open interval } I' \ni \theta\}, \\ S_w &:= \{\theta \in \mathbb{R} \mid Y_w|_{I'} > 0 \text{ for some open interval } I' \ni \theta\}. \end{aligned}$$

Because of (4.3) we know that f is absolutely continuous with respect to y_w^{c+} and, as a consequence, with respect to $|y_w^c| := y_w^{c+} + y_w^{c-}$. On the other hand, it follows from Lemma 4.1 that $y_w^c|_{\mathbb{R} \setminus S_w} = 0$. Hence, we can conclude that $|y_w^c|(\mathbb{R} \setminus S_w) = 0$ and further, $f = f|_{S_w}$ since f is absolutely continuous with respect to $|y_w^c|$. Similarly, (4.1) and Lemma 4.1 imply $f = (f - \tau)|_{S_v}$. Consequently, we get from the definitions of S_v and S_w that there exists an $\theta \in \text{supp}(f)$ and an open interval I' containing θ such that $Y_v|_{I'-\tau} > 0$ and $Y_w|_{I'} > 0$. Since θ is contained in the support of f , every closed interval $I := [\alpha, \beta] \subseteq I'$ with $\theta \in (\alpha, \beta)$ satisfies the properties above.

In the following let $I = [\alpha, \beta]$ be a closed interval and $\epsilon > 0$ such that $f(I) > 0$, $Y_v|_{I-\tau} > \epsilon$, and $Y_w|_I > \epsilon$. Without loss of generality, assume that $f(\mathbb{R} \setminus I) = 0$ (i.e., $\text{supp}(f) \subseteq I$). This can be done by letting $f := f|_I$. Further, let $\alpha_1, \beta_1 \in I$ with $\alpha_1 \leq \beta_1$ be chosen in such a way that $f([\alpha, \alpha_1]) = f([\beta_1, \beta]) = \min\{\epsilon, \frac{f(I)}{2}\}$. Note that α_1 and β_1 exist since f is a continuous measure. Then we define

$$\begin{aligned} f_1 &:= f|_{(\alpha, \alpha_1)} - f|_{[\beta_1, \beta]}, \\ f_2 &:= -f|_{(\alpha, \alpha_1)} + f|_{[\beta_1, \beta]} = -f_1. \end{aligned}$$

It is easy to see that $f_1 + f_2 = 0$ and that $|f_1(B)| < \epsilon$ and $|f_2(B)| < \epsilon$ for all measurable set B . We now define x^q for $q = 1, 2$ by

$$x_e^q = x_e^q + \sum_{i|e_i=e} (f_q|_{[\alpha, \beta]} - (\tau_1 + \dots + \tau_i)) \quad \forall e \in E.$$

The equation $f_1 + f_2 = 0$ implies $\frac{1}{2}x^1 + \frac{1}{2}x^2 = x$. Thus it remains to check the feasibility of x^1 and x^2 . The flows x^1 and x^2 are nonnegative because of (4.2) and the fact that $|f_1| \leq f$ and $|f_2| \leq f$. Let y^1 and y^2 be the corresponding storage of x^1 and x^2 , respectively. It is not hard to see that for $q = 1, 2$ y^q is equal to y except for nodes v and w where we have $y_v^q = y_v - (f_q - \tau)$ and $y_w^q = y_w - f_q$. It then follows from the definition of f_1 and f_2 that the distribution functions Y^1, Y^2 , and Y coincide everywhere except on $I - \tau$ and I at nodes v and w , respectively. Within the points in time in $I - \tau$ and I at nodes v and w , respectively, we get

$$Y_v^q|_{I-\tau} \geq 0, \quad Y_w^q|_I \geq 0, \quad q = 1, 2,$$

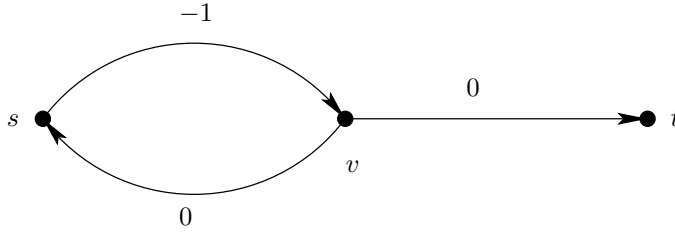


FIG. 4.1. Network for Example 4.3. The transit times are shown on the arcs.

due to the fact that $|f_1(B)| < \epsilon$, $|f_2(B)| < \epsilon$ and the definition of I . This yields the desired result. \square

As already mentioned, our first goal in this section is to show that the continuous part of an extreme point is 0. Thus, if we find a walk and a nonzero measure satisfying the assumptions of Lemma 4.2 whenever the continuous part is positive, we are done. The next example shows that an algorithm constructing such a path could cycle and, hence, must be designed carefully.

EXAMPLE 4.3. Consider the network depicted in Figure 4.1 where the transit times are shown on the arcs and suppose that f is some continuous Borel measure such that $f([0, 1]) = 1$ and $f(\mathbb{R} \setminus [0, 1]) = 0$. Let $\Theta := 0$ be the time horizon and consider the following feasible solution of (LPM): The flow f circulates on the cycle C induced by s and v . Every time the flow reaches the node v half of the remaining flow is sent in the arc (v, t) . Thus we get the following solution

$$x_{(s,v)} = \sum_{i=0}^{\infty} \frac{1}{2^i} (f + i), \quad x_{(v,s)} = \frac{1}{2} (x_{(s,v)} + 1), \quad x_{(v,t)} = \frac{1}{2} (x_{(s,v)} + 1).$$

Every finite s - t -walk P satisfies the assumptions of Lemma 4.2. The corresponding measure is $\frac{f}{2^k}$ assuming that the cycle C is used $k - 1$ times by P . On the other hand an uncarefully designed algorithm could be caught within the cycle C .

If we want to apply Lemma 4.2 we have to ensure that there exists a node whose continuous part of storage is nonzero. The next lemma shows that whenever we have an extreme point x such that the continuous part of the corresponding storage is 0, then the continuous part of x has to be 0 as well.

LEMMA 4.4. Let x be an extreme point of (LPM) with corresponding storage y . If $y^c = 0$, then $x^c = 0$.

Proof. Since $y^c = 0$, from (3.2) we get

$$\sum_{e \in \delta^+(v)} x_e^c - \sum_{e \in \delta^-(v)} (x_e^c - \tau_e) = 0 \quad \forall v \in V.$$

Thus we can add and subtract this equation from equation (3.1) without changing its the right hand side. This yields two feasible solutions $x^1 := x + x^c$ and $x^2 := x - x^c$ both with corresponding storage y if $x^c \geq 0$. Since $x = \frac{1}{2}(x^1 + x^2)$ is a convex combination of two feasible solutions, x is not an extreme point. \square

Having established Lemma 4.4, we consider the case where $y_v^c \neq 0$ for some node v , which requires a more complicated treatment. In this case, we prove the existence of a walk and a nonzero measure satisfying the assumptions (4.1)-(4.3) of Lemma 4.2. The approach is based on an algorithm, which is a kind of the well-known breadth-first

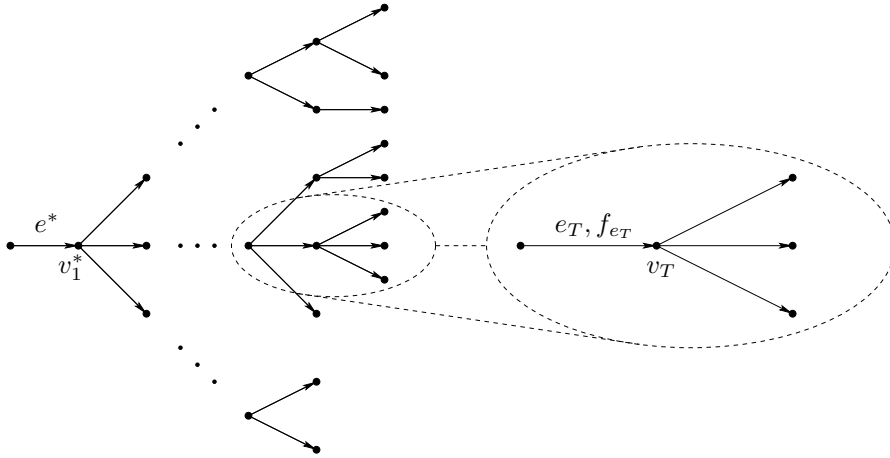


FIG. 4.2. Construction of the BFS-tree.

search (BFS). The node set and arc set of the BFS-tree are denoted by V_T and E_T , respectively. Each node in V_T corresponds to one node in V and each arc in E_T to one arc in E . Actually, the BFS-tree contains in general multiple copies of a node $v \in V$ and multiple copies of an arc $e \in E$. An arc in E_T whose head is a leaf is called a *leaf arc*. The BFS-tree is an out-tree and constructed in a way that each path starting at the root node corresponds to a walk satisfying the assumptions (4.1) and (4.2). The termination condition ensures that also the assumption (4.3) is satisfied. An illustration of the algorithm is shown in Figure 4.2.

Before giving a detailed description of the algorithm, we present the following lemma that will help us to show correctness. For the proof, see Appendix B.

LEMMA 4.5. *Let μ, μ_1, \dots, μ_n be finite Borel measures on \mathbb{R} with $\mu \leq \sum_{i=1}^n \mu_i$. Then there exists Borel measures $\bar{\mu}_1, \dots, \bar{\mu}_n$ such that $\bar{\mu}_i \leq \mu_i$, for $i = 1, \dots, n$ and $\mu = \sum_{i=1}^n \bar{\mu}_i$.*

BFS ALGORITHM

Input: A feasible solution x of (LPM) with corresponding storage y and a node v_1 with $y_{v_1}^{c^-} > 0$.

Output: A walk and a measure satisfying the assumptions of Lemma 4.2.

- (1) Init $\bar{x}_e := x_e^c$ for all $e \in E$, $V_T := \emptyset$, and $E_T := \emptyset$.
- (2) Add an (artificial) arc e^* to E_T and let the head of e^* be the copy v_1^* of v_1 . Assign the flow $f_{e^*} := y_{v_1}^{c^-}$ to e^* .
- (3) For each leaf arc $e_T \in E_T$ with head v_T do:
 - (a) Let $v \in V$ be the original node of v_T and τ_{e_T} be transit time of the original arc of e_T (In the case of $e_T = e^*$ we set $\tau_{e_T} := 0$).
 - (b) If $f_{e_T} - \tau_{e_T}$ and y_v^{c+} are not mutually singular, then go to (5).
 - (c) For each arc $e \in \delta^+(v)$ compute a (continuous) measure f_e such that $f_e \leq \bar{x}_e$ for all $e \in \delta^+(v)$ and $f_{e_T} - \tau_{e_T} = \sum_{e \in \delta^+(v)} f_e$.
 - (d) For each arc $e \in \delta^+(v)$ with $f_e \gtrsim 0$ add a copy e' to E_T , connect e' to e_T via v_T and set $f_{e'} := f_e$ and $\bar{x}_e := \bar{x}_e - f_e$.
- (4) Go to (3).
- (5) Return the walk consisting of the original arcs of the unique path from v_1^* to v_T

in the BFS-Tree and the measure $f := \min\{f_{e_T} - \tau_{e_T}, y_v^{c+}\}$.

In what follows, we analyse the correctness of the BFS ALGORITHM in details. One complete execution of Step (3) is called *phase*. Thus, in each phase every leaf arc is treated and the depth of the tree is increased by 1. Note that the first phase is not interrupted since $e_T = e^*$, $v = v_1$, $f_{e^*} = y_{v_1}^{c-}$, and we know that $y_{v_1}^{c-}$ and $y_{v_1}^{c+}$ are mutually singular. Further, \bar{x}_e denotes the remaining continuous flow on arc e since \bar{x}_e is initialized with x_e^c and after assigning the flow f_e to a tree arc e' in Step (3d) we reduce \bar{x}_e by the same flow. The next lemma shows that the BFS ALGORITHM works properly.

LEMMA 4.6. *The BFS ALGORITHM is well-defined and correct. In particular, the algorithm is able to execute Step (3c), terminates in a finite time, and produces the desired output.*

Proof. Assume that we are at Step (3c) and let e^* , e_T and v be as defined by the algorithm. With each arc $e \in E$ we associate two measures g_e and g_e^l , which denote the total flow assigned to e within the BFS-tree and the total flow assigned to e within the leaf arcs of the BFS-tree, respectively. More precisely, measures g_e and g_e^l are defined by

$$g_e := \sum_{e' \in E_T | e' = e} f_{e'}, \quad g_e^l := \sum_{\text{leaf arc } e' | e' = e} f_{e'}.$$

Note that the artificial arc e^* does not appear in any of the two sums above. Steps (3a)-(3d) ensure that flow $(g_e)_{e \in E_T}$ fulfills the strict flow conservation constraints at intermediate nodes of the BFS-tree. Hence, for each intermediate node $v \in V_T$ we get:

$$\begin{aligned} \sum_{e \in \delta^-(v)} (g_e - \tau_e) - \sum_{e \in \delta^+(v)} g_e &= \sum_{e \in \delta^-(v)} (g_e^l - \tau_e) - \begin{cases} f_{e^*} & \text{if } v = v_1 \\ 0 & \text{otherwise} \end{cases} \\ &\geq \sum_{e \in \delta^-(v)} (g_e^l - \tau_e) - y_v^{c-}. \end{aligned} \quad (4.4)$$

Further, because of Step (3d) we see that the sum $\bar{x}_e + g_e$ remains constant during the execution of the algorithm. In fact, we always have

$$x_e^c = \bar{x}_e + g_e \quad \forall e \in E. \quad (4.5)$$

Therefore, by substituting $\bar{x}_e + g_e$ instead of x^c in (3.2), we get

$$y_v^c = \sum_{e \in \delta^-(v)} ((\bar{x}_e - \tau_e) + (g_e - \tau_e)) - \sum_{e \in \delta^+(v)} (\bar{x}_e + g_e).$$

On the other hand, $\bar{x}_e \geq 0$ for each arc e because of earlier executions of (3c). Now it follows from the above equation and inequality (4.4) that

$$\sum_{e \in \delta^-(v)} (g_e^l - \tau_e) \leq y_v^{c+} + \sum_{e \in \delta^+(v)} \bar{x}_e.$$

Because of Step (3b), we know that $f_{e_T} - \tau_{e_T}$ and y_v^{c+} are mutually singular. Hence, from the above inequality we obtain

$$f_{e_T} - \tau_{e_T} \leq \sum_{e \in \delta^+(v)} \bar{x}_e.$$

Now we can construct a decomposition of $f_{e_T} - \tau_{e_T}$ into $\sum_{e \in \delta^+(v)} f_e$ so that $f_e \leq \bar{x}_e$ for each arc $e \in \delta^+(v)$ (see for more details Lemma 4.5). This establishes the validity of Step (3c).

Next we prove the termination of the algorithm. We first observe that the number of tackled leafs in one phase is finite. This can be seen by induction and the fact that the number of outgoing arcs of an original node is finite. Thus it suffices to show that the number of phases is finite. In each phase the flow in the leaf arcs is completely routed to the outgoing arcs of the corresponding head nodes. Thus, by induction the total amount of flow in the leaf arcs is always equal to $f_{e_1^*}(\mathbb{R}) =: \epsilon$. Hence, in each phase the total amount of flow $(\sum_{e \in E} g_e)(\mathbb{R})$ is increased by ϵ . On the other hand, because of (4.5) the total amount of flow is bounded by $M := (\sum_{e \in E} x_e^c)(\mathbb{R})$ which is finite since $(x_e)_{e \in E}$ are assumed to be finite measures. Thus, the number of iterations is bounded by $\lceil \frac{M}{\epsilon} \rceil$ and the algorithm terminates in a finite time.

For the correctness of the algorithm we show that the output of the algorithm satisfies the assumptions (4.1)-(4.3) of Lemma 4.2. Consider Step (3c). Since this step is well-defined the flow which is assigned to the outgoing arcs of v is smaller than the flow f_{e_T} . Therefore we obtain the following invariance from Steps (2) and (4.5): For each tree arc $e_T \in E_T$ with head node v_T the walk consisting of the original arcs of the unique path from v_1^* to v_T and the measure f_{e_T} satisfies (4.1) and (4.2). Hence, the correctness of the algorithm follows directly from the termination condition in Step (3b) and the definition of f in the final Step (5). Note that f is nonzero since f_{e_T} and y_v^{c+} are not mutually singular when reaching Step (5). This completes the proof of the lemma. \square

The following lemma concludes the first part of this section.

LEMMA 4.7. *Let x with corresponding storage y be an extreme point of (LPM). Then $x^c = 0$.*

Proof. We assume the opposite and derive a contradiction. Suppose that x_e^c is nonzero for some arc e . Then, it follows from Lemma 4.4 that y_v^c is nonzero for some node v . On the other hand, we can conclude from (3.2) that

$$\sum_{v \in V} y_v^c(\mathbb{R}) = \sum_{v \in V} \left(\sum_{e \in \delta^-(v)} x_e^c(\mathbb{R}) - \sum_{e \in \delta^+(v)} x_e^c(\mathbb{R}) \right) = 0.$$

since, for each edge $e = vw \in E$, the term $x_e^c(\mathbb{R})$ occurs once with a positive sign if w is considered in the sum above and once with a negative sign if v is considered in the sum above. Hence, we assume without loss of generality that $y_v^{c-} \geq 0$. Then, the BFS ALGORITHM which gets as input the feasible solution x and the node v , gives as output a walk and a nonzero measure satisfying the assumptions of Lemma 4.2. Then Lemma 4.2 implies that x is not an extreme point, which is a contradiction. This yields the result. \square

5. Discrete Part of Extreme Points. Up to this point, we have shown that x^c must be zero whenever x is an extreme point of (LPM). In this section, we restrict our attention to discrete feasible solutions, when identifying extreme points and show that an extreme point of (LPM) corresponds to a dynamic s - t -path. We first give some definitions.

Suppose that x is a discrete feasible solution of (LPM) with corresponding storage y and that $P = (e_1, \dots, e_n)$ is a dynamic walk with arrival times $\alpha_1, \dots, \alpha_{n+1}$ and departure times $\beta_1, \dots, \beta_{n+1}$. Let δ be a positive real number. The walk P carries a

flow of value δ (with respect to x) if

$$\begin{aligned} x_{e_k}(\{\beta_k\}) &\geq \delta & \forall i = 1, \dots, n, \\ Y_{v_k}(\theta) &\geq \delta & \forall \theta \in [\alpha_i, \beta_i], i = 1, \dots, n+1. \end{aligned}$$

The *flow value* of P is defined as the maximum amount of flow that can be carried by P . The walk P is called a *flow-carrying walk* if its flow value is positive. For the case where P is a dynamic path or dynamic cycle, it is called a *flow-carrying path* or *cycle*, respectively.

For a dynamic s - t -path let χ^P be the corresponding incidence vector and ψ^P be the corresponding storage, respectively, given by (3.4) and (3.5). Then it is not hard to observe that P carries a flow of value δ if and only if $\delta \cdot \chi_e^P \leq x_e$ for all $e \in E$ and $\delta \cdot \Psi_v^P \leq Y_v$ for all $v \in V$.

Next we show that an extreme point provides a flow-carrying s - t -path. We do this along the same lines as showing that the continuous part of an extreme point is 0. Here a BFS-tree is constructed in a way that each path starting at the root node corresponds to a flow-carrying walk. To do so, we assign to each tree arc $e_T = (v_T, w_T)$ a measure f_{e_T} whose support consists only of one point in time θ . This is interpreted as follows: Let $e = (v, w)$ be the original arc of e_T . Then the arc e is entered at time θ in the corresponding walk, i.e., the departure time from node v is θ and consequently the arrival time at node w is $\theta + \tau_e$. If we consider the unique path from the root node to w_T , then the corresponding s - w -walk carries a flow of value $f_{e_T}(\{\theta\})$. The termination condition ensures that in the end we obtain a flow carrying s - t -walk.

The construction of the BFS-tree follows the same lines as the CONTINUOUS BFS ALGORITHM. But in contrast, we have to take care that the constructed BFS-Tree has finite width since incoming flow could be split into an infinite but countable number of different parts. Therefore in Step (3c) of the algorithm, we do not propagate the entire flow of an arc. In particular, we use the following result, the proof of which can be found in Appendix B.

LEMMA 5.1. *Let μ_1 and μ_2 be two finite discrete measures. Furthermore, let γ be a signed measure with $\gamma(\mathbb{R}) = 0$ and nonnegative distribution function F_γ such that $\mu_2 + \gamma \geq \mu_1$. Consider a point $\theta \in \mathbb{R}$ and let $\nu_1 \leq \mu_1$ be a measure with $\text{supp}(\nu_1) = \{\theta\}$. Then for every $\rho \in [0, 1)$ there exists a (discrete) measure $\nu_2 \leq \mu_2$ with finite support and a signed (discrete) measure η with distribution function F_η such that:*

$$\begin{aligned} \rho \cdot \nu_1 &= \eta + \nu_2, \\ 0 &\leq F_\eta \leq F_\gamma, \\ \eta(\mathbb{R}) &= 0. \end{aligned}$$

We are now in a position to give a detailed description of the algorithm.

DISCRETE BFS ALGORITHM

Input: A discrete feasible solution x of (LPM) with corresponding storage y .

Output: An s - t -walk carrying a flow of value of f .

- (1) Init $\bar{x}_e := x_e$ for all $e \in E$, $V_T := \emptyset$, $E_T := \emptyset$, $i := 1$, and $\rho := \frac{3}{4}$.
- (2) Add an (artificial) arc e^* to E_T and let the head of e^* be the copy s^* of s . Assign the measure $f_{e^*} := b_s$ to e^* .

- (3) For each leaf arc $e_T \in E_T$ with head v_T do:
- Let $v \in V$ be the original node of v_T and τ_{e_T} be transit time of the original arc of e_T (In the case of $e_T = e^*$ we set $\tau_{e_T} := 0$). Further, let $\theta \in \mathbb{R}$ be such that $\{\theta - \tau_{e_T}\} = \text{supp}(f_{e_T})$.
 - If $v = t$, $\theta \leq T$ and $\bar{y}_t|_{(\theta, T]} > \epsilon$ for a some positive $\epsilon \in \mathbb{R}$ then go to (5).
 - Compute a signed measure y_{v_T} and for each arc $e \in \delta^+(v)$ a measure f_e with finite support such that $f_e \leq \bar{x}_e$, $0 \leq Y_{v_T} \leq \bar{Y}_v$, $y_{v_T}(\mathbb{R}) = 0$ and $\rho \cdot (f_{e_T} - \tau_{e_T}) = y_{v_T} + \sum_{e \in \delta^+(v)} f_e$
 - For each arc $e \in \delta^+(v)$ and each time $\vartheta \in \text{supp}(f_e)$ add a copy e' to E_T , connect e' to e_T via v_T and set $f_{e'} := f_e|_{\{\vartheta\}}$, $\bar{x}_e := \bar{x}_e - f_{e'}$, and $\bar{y}_v := \bar{y}_v - y_{v_T}$.
- (4) Set $i := i + 1$ and then $\rho := \frac{2^i + 1}{2^{i+2}}$. Go to (3).
- (5) Return the dynamic walk corresponding to the unique path from s^* to v_T in the BFS-Tree and the positive real number $\delta := \min\{f_{e_T}(\{\theta - \tau_{e_T}\}), \epsilon\}$.

It is worth mentioning that the continuous and the discrete BFS algorithm are quite similar. Regardless of the kinds of measures participating in these algorithms, the discrete BFS algorithm can be seen as a generalization of the continuous version as follows: We obtain the notion of the continuous BFS algorithm if we set ρ always equal to 1 and assume that y_{v_T} computed in Step (3c) is always zero. As in the continuous BFS algorithm, \bar{x}_e is the remaining flow on an arc e . In addition \bar{y}_v is the remaining storage of a node v .

LEMMA 5.2. *The DISCRETE BFS ALGORITHM works correctly, i.e., Step (3c) is always valid executable, the algorithm terminates, and the output is a flow carrying s-t-path.*

Proof. Assume that we reach Step (3c) and let e^* , e_T and v be as defined by the algorithm. We define for each arc $e \in E$ two measures: The measure g_e is the total flow assigned to e within the BFS-tree and g_e^l is the total flow assigned to e within the leaf arcs of the BFS-tree. In addition, we define a measure z_v for each node $v \in V$ determining the stored flow which is already propagated. Thus, we have:

$$g_e := \sum_{e' \in E_T | e' = e} f_{e'}, \quad g_e^l := \sum_{\text{leaf arc } e' | e' = e} f_{e'}, \quad z_v := \sum_{v' \in V_T | v' = v} y_{v'}.$$

Note that the artificial arc e^* does not appear in any of the first two sums above and that v_T does not appear in the last sum. Because of the above definitions and the flow conservation equation in Step (3c) we get:

$$\begin{aligned} \sum_{e \in \delta^-(v)} (g_e - \tau_e) - \sum_{e \in \delta^+(v)} g_e &\geq z_v + \sum_{e \in \delta^-(v)} (g_e^l - \tau_e) - \begin{cases} f_{e^*} & \text{if } v = v_1 \\ 0 & \text{otherwise} \end{cases} \\ &= z_v + \sum_{e \in \delta^-(v)} (g_e^l - \tau_e) - b_v^+. \end{aligned} \quad (5.1)$$

Further, because of Step (3d) we see that the two sums $\bar{x}_e + g_e$ and $\bar{y}_v + z_v$ remain constant during the execution of the algorithm. Thus we have

$$\bar{x}_e + g_e = x_e \quad \forall e \in E \quad \text{and} \quad \bar{y}_v + z_v = y_v \quad v \in V. \quad (5.2)$$

Further, by induction we have $\bar{y}_v(\mathbb{R}) = 0$ and $\bar{Y}_v \geq 0$ for each $v \in V$. Inserting the first equation of (5.2) in (3.1) we obtain:

$$y_v = b_v + \sum_{e \in \delta^-(v)} ((\bar{x}_e - \tau_e) + (g_e - \tau_e)) - \sum_{e \in \delta^+(v)} (\bar{x}_e + g_e).$$

On the other hand, we know that $\bar{x}_e \geq 0$ for each arc e because of (3c). Hence the above equation and inequality (5.1) imply

$$\sum_{e \in \delta^-(v)} (g_e^l - \tau_e) \leq -b_v^- + (y_v - z_v) + \sum_{e \in \delta^+(v)} \bar{x}_e .$$

Further, Step (3b) implies that there exists a $\bar{\theta} \in \mathbb{R} \cup \{\infty\}$ with $\bar{y}_v(-\infty, \bar{\theta}) = 0$ such that the measures $(f_{e_T} - \tau_{e_T}) - \bar{y}_v|_{(-\infty, \bar{\theta})}$ and $b_v^-(-\infty, \bar{\theta}) + \bar{y}_v|_{[\bar{\theta}, \infty)}$ are mutually singular (note that in the case $v = t$ we can choose $\bar{\theta} \in [\theta, \Theta]$ and otherwise we choose $\bar{\theta} = \infty$). Then we can conclude:

$$f_{e_T} - \tau_{e_T} \leq \bar{y}_v|_{(-\infty, \bar{\theta})} + \sum_{e \in \delta^+(v)} \bar{x}_e .$$

By the application of Lemma 5.1 we obtain a discrete measure $\nu \leq \sum_{e \in \delta^+(v)} \bar{x}_e$ of finite support and the signed measure y_{v_T} satisfying $\rho \cdot (f_{e_T} - \tau_{e_T}) = y_{v_T} + \nu$. Now we apply Lemma 4.5 to ν in order to find the measure f_e for each $e \in \delta^+(v)$. From the conclusions of both Lemmas 4.5 and 5.1, we get the validity of Step (3c).

For proving the termination of the algorithm we first observe that the number of tackled leaves in one phase is finite. This is seen by induction and the fact that in Step (3d) only a finite number of (new) leaves are added to the tree. Thus it suffices to show that the number of phases is finite. At the end of each phase the amount of flow in the new leave arcs is ρ times the amount of flow in the old leaves. Let $\rho_i = \frac{2^i+1}{2^i+2}$ be the ρ used in phase i . Then at the end of phase i the amount of flow in the new leaves is equal to

$$f_{e^*}(\mathbb{R}) \cdot \prod_{j=1}^i \rho_j = b_s(\mathbb{R}) \cdot \prod_{j=1}^i \frac{2^j+1}{2^j+2} = 1 \cdot \frac{1}{2^i} \prod_{j=1}^i \frac{2^j+1}{2^{j-1}+1} = \frac{2^i+1}{2^i} \geq \frac{1}{2} . \quad (5.3)$$

Hence, in each phase the total amount of flow $(\sum_{e \in E} g_e)(\mathbb{R})$ is increased by at least $\frac{1}{2}$. Further, because of (4.5) the total amount of flow is bounded by $M := (\sum_{e \in E} x)(\mathbb{R})$ which is finite since we restrict to finite measures. Thus, the number of iteration is bounded by $2M$ and the algorithm terminates in finite time.

To prove the correctness of the algorithm it is enough to show that the output is a walk-carrying flow of amount f . Consider Step (3c) and a point in time $\vartheta \geq \theta + \tau_{e_T}$. It holds:

$$\begin{aligned} \sum_{e \in \delta^+(v)} f_e((\vartheta, \infty)) &= \sum_{e \in \delta^+(v)} f_e(\mathbb{R}) - \sum_{e \in \delta^+(v)} f_e((-\infty, \vartheta]) \\ &= \rho \cdot (f_{e_T} - \tau_{e_T})((-\infty, \vartheta]) - \sum_{e \in \delta^+(v)} f_e((-\infty, \vartheta]) \\ &= y_{v_T}((-\infty, \vartheta]) . \end{aligned}$$

For $e \in \delta^+(v)$ and $\vartheta \in \text{supp}(f_e)$ we know $Y_{v_T}|_{[\theta + \tau_{e_T}, \vartheta]} \geq f_e(\vartheta)$. Therefore we obtain the following: For tree arc $e_T \in E_T$ with head node v_T the dynamic walk corresponding to the unique path from s^* to v_T in the BFS-Tree carries a flow of value $f_{e_T}(\mathbb{R})$. Hence, the correctness of the algorithm follows directly from the termination condition in Step (3b) and the definition of δ in the final step (5). \square

As mentioned previously, the nonzero components of any extreme point of (LP) are one which indicate a static s - t -path. The next lemma shows that this result can be extended to CDSP.

LEMMA 5.3. *Suppose that x is an extreme point of (LPM). Then the network G contains no flow-carrying cycle. Moreover, there is a unique flow-carrying s - t -path P of flow value 1. In fact, we have $x = \chi^P$ where χ^P is the incidence vector of P .*

Proof. Let us first assume by contradiction that there is a flow-carrying cycle C and let χ^C be the incidence vector of C . If C carries a flow of value δ then $x^1 := x + \delta \cdot \chi^C$ and $x^2 := x - \delta \cdot \chi^C$ are both feasible solutions. Obviously $x = \frac{1}{2}(x^1 + x^2)$ is the convex combination of x^1 and x^2 . This implies that x is not an extreme point, which is a contradiction. Hence, there exists no flow-carrying cycles with respect to x .

Now suppose that there are two s - t -paths P_1 and P_2 with incidence vectors χ^{P_1} and χ^{P_2} carrying a flow of values δ_1 and δ_2 , respectively. Let $\delta := \min\{\delta_1, \delta_2\}$. Then $x^1 := x + \delta\chi^{P_1} - \delta\chi^{P_2}$ and $x^2 := x - \delta\chi^{P_1} + \delta\chi^{P_2}$ are both feasible solutions and we have $x = \frac{1}{2}(x^1 + x^2)$. Hence, x is not an extreme point, which is again a contradiction. This implies that there must be at most one flow-carrying s - t -path.

We are left to prove the existence of a flow-carrying s - t -path of flow value 1. Since x is an extreme point, it follows from Lemma 4.7 that the continuous part of x is 0. This means that x is discrete and thus applying DISCRETE BFS ALGORITHM yields a flow-carrying s - t -path P with respect to x . Define $x^* := x - \delta \cdot \chi^P$ where δ and χ^P are the flow value and incidence vector of P , respectively. We show that δ must be 1 and further, x^* must be zero. Note that $\delta \leq 1$ because of the definition of b_s . Now suppose that $\delta < 1$. Then it is not hard to see that $x^* \neq 0$ and $\frac{1}{1-\delta}x^*$ is also a discrete feasible solution of (LPM). Thus there exists a flow carrying s - t -path P^* with respect to x^* and hence, also with respect to x . Because of the maximality of x^* we have $P^* \neq P$ contradicting the uniqueness of P . Thus we must have $\delta = 1$ implying $x^* = x - \chi^P$. In this case $x^1 := x + x^*$ and $x^2 := x - x^*$ are both feasible solutions and we have $x = \frac{1}{2}(x^1 + x^2)$. This implies that $x^* = 0$ since x is an extreme point. Hence, $x = \chi^P$, which completes the proof of the lemma. \square

As a direct consequence of the above lemma we obtain the following:

COROLLARY 5.4. *For an extreme point x of (LPM) the flow x_e is concentrated on a finite set (that is, x_e is discrete and $\text{supp}(x_e)$ is finite) for each arc $e \in E$. Further, $x_e(\{\theta\}) = 1$ for each arc $e \in E$ and each point $\theta \in \text{supp}(x_e)$.*

We can now summarize the results of this paper in the following theorem.

THEOREM 5.5. *Any extreme point of (LPM) corresponds one-to-one to a dynamic s - t -path. If the cost functions are given, this one-to-one correspondence preserves also costs.*

Proof. From Lemma 5.3 we know that for any extreme point x there exists an s - t -path P with $x = \chi^P$. Thus, it remains to show that for any dynamic s - t -path P the incidence vector χ^P is an extreme point of (LPM).

We assume the opposite, that is, $x := \chi^P$ is not an extreme point for some dynamic s - t -path P . Then there is a signed measure x^* such that $x^1 := x + x^*$ and $x^2 := x - x^*$ are both feasible solutions. Further, assume that x^* is maximal in the following sense: for any $\rho > 1$ at least one of $x + \rho \cdot x^*$ and $x - \rho \cdot x^*$ is not feasible. Obviously x^* is a discrete measure and hence, x_1 and x_2 are. Then by Lemma 5.3, there are flow-carrying s - t -paths P_1 and P_2 with respect to x^1 and x^2 respectively. Because of the maximality of x^* at least one of P_1 and P_2 is not equal to P . Without loss of generality let P_1 be this path. We have $x^2 = 2x - x^1 \leq 2\chi^P - \delta \cdot \chi^{P_1}$ for some $\delta > 0$. But this contradicts the feasibility of x^2 since $2\chi^P - \delta \cdot \chi^{P_1} \not\leq 0$. Hence, x is an extreme point.

The definition of the incidence vector implies that the cost of a dynamic s - t -path

and its corresponding incidence vector are equal. This establishes the theorem. \square

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Appendix A. Preliminaries on measure theory. In this appendix we briefly present some basic definitions and notation that are used frequently throughout the paper. For a detailed treatment we refer to, e.g., [12].

A σ -algebra on the real line \mathbb{R} is a nonempty collection of subsets of \mathbb{R} that is closed under countable unions and complements. The smallest σ -algebra on \mathbb{R} containing all open sets (or, equivalently, closed sets) is called the *Borel σ -algebra*. The elements of the Borel algebra are called *measurable sets* or *Borel sets*. Let \mathcal{B} denote the collection

of all Borel sets on \mathbb{R} . A function $\mu : \mathcal{B} \rightarrow \mathbb{R}_{\geq 0}$ is called a Borel measure on \mathbb{R} if

1. $\mu(B) \geq 0$ for any $B \in \mathcal{B}$ and $\mu(\emptyset) = 0$;
2. if $\{B_i\}_{i \in \mathbb{N}}$ is a countable collection of pairwise disjoint sets in \mathcal{B} , then

$$\mu\left(\bigcup_{i \in \mathbb{N}} B_i\right) = \sum_{i \in \mathbb{N}} \mu(B_i) .$$

A Borel measure μ on \mathbb{R} is called finite if $\mu(\mathbb{R}) < \infty$. Measures are by definition nonnegative, i.e., a nonnegative real number is assigned to each measurable set. However, it is sometimes convenient that also negative values can be assigned to some measurable sets. A measure which can take both positive and negative values is called a *signed* measure. The space of finite signed Borel measures becomes a vector space under the standard addition and scalar multiplication operations. In particular, for any two finite signed Borel measures μ_1 and μ_2 and any real value λ , the addition $\mu_1 + \mu_2$ and scalar multiplication $\lambda \cdot \mu_1$ are defined as

$$\begin{aligned} (\mu_1 + \mu_2)(B) &= \mu_1(B) + \mu_2(B) & \forall B \in \mathcal{B} , \\ (\lambda \cdot \mu_1)(B) &= \lambda \cdot \mu_1(B) & \forall B \in \mathcal{B} . \end{aligned}$$

Let μ_1 and μ_2 be two signed Borel measures. We write $\mu_1 = \mu_2$ ($\mu_1 \leq \mu_2$) if $\mu_1(B) = \mu_2(B)$ ($\mu_1(B) \leq \mu_2(B)$) for each $B \in \mathcal{B}$. Moreover, we write $\mu_1 \geq 0$ if $\mu_1(B) \geq 0$ for each Borel set B and $\mu \not\geq 0$ if $\mu(B) < 0$ for some Borel set B . For a measurable set A and a Borel measure μ , the *restriction* $\mu|_A$ of μ to the set A is a Borel measure defined by $\mu|_A(B) := \mu(B \cap A)$ for each $B \in \mathcal{B}$. If $\mu|_A = 0$, then A is called a *strict μ -null*. This implies $\mu(B) = 0$ for each Borel set $B \subseteq A$. A Borel set B for which $\mu(B) = 0$ is called a *μ -null set*. A (signed) measure μ is called *zero* if $\mu(B) = 0$ for each Borel set B . Otherwise μ is called nonzero.

A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is called *measurable* if the preimage of each measurable set is measurable, that is, $F^{-1}(B) := \{\theta \in \mathbb{R} \mid F(\theta) \in B\}$ is a measurable set for every Borel set B . It is well known that if F is measurable, then the integral of F on a measurable set B with respect to a Borel measure μ exists and is denoted by $\int_B F d\mu$. We refer to, e.g., [12] for more details.

A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is called a *distribution function* if it is of bounded variation and right-continuous, and moreover $\lim_{\theta \rightarrow -\infty} F(\theta) = 0$. Notice that the limit exists since F is of bounded variation. For every finite signed Borel measure μ , there is a unique distribution function F satisfying $F(b) - F(a) = \mu((a, b])$ for all $a, b \in \mathbb{R}$ with $a \leq b$. In fact F is given by the formula $F(\theta) = \mu((-\infty, \theta])$.

Given a measure μ on \mathbb{R} , the *support* of μ is defined to be the set of all points in \mathbb{R} with a neighborhood of positive measure, that is,

$$\text{supp}(\mu) := \{\theta \in \mathbb{R} : \mu(U) > 0 \text{ for every open neighborhood } U \text{ of } \theta\} .$$

A point $\theta \in \mathbb{R}$ is called an *atom* of μ if $\mu(\{\theta\}) > 0$. Obviously if μ is finite, the set of atoms of μ is countable. We define the *discrete part* μ^d and *continuous part* μ^c of a finite measure μ by

$$\mu^d(B) := \sum_{\text{atoms } \theta \in B} \mu(\{\theta\}) \quad \text{and} \quad \mu^c(B) := \mu(B) - \mu^d(B)$$

for every measurable set B . The measure μ is called a *discrete* (*continuous*)¹ if its continuous (discrete) part is zero. In fact, a finite Borel measure is continuous

¹A continuous measure is also called *nonatomic* in textbooks on measure theory.

(discrete) if and only if its corresponding distribution function is continuous (a jump function) (see, e.g., Section 9.3 in [12]). Moreover, the decomposition $\mu = \mu^c + \mu^d$ of a finite Borel measure μ into a sum of a discrete and a continuous measure is unique.

Two Borel measures μ_1 and μ_2 are called *singular* if there exist two disjoint measurable sets A and B whose union is \mathbb{R} such that μ_1 is zero on all measurable subsets of B while μ_2 is zero on all measurable subsets of A , i.e., $\mu_1(B) = 0$ and $\mu_2(A) = 0$. Moreover, μ_1 is said to be *absolutely continuous* with respect to μ_2 if $\mu_1(A) = 0$ for every measurable set A for which $\mu_2(A) = 0$.

The following theorem shows that any signed measure can be expressed as the difference of two singular (positive) measures.

THEOREM A.1 (Jordan Decomposition). *Every signed measure μ can be expressed as a difference of two nonnegative measures μ^+ and μ^- such that μ^+ and μ^- are singular and at least one of which is finite. Moreover if $\mu = \mu_1 - \mu_2$, then $\mu^+ \leq \mu_1^+$ and $\mu^- \leq \mu_2^-$. The measures μ^+ and μ^- are called the positive and negative part of μ , respectively. The pair (μ^+, μ^-) is called the Jordan decomposition (or sometimes Hahn–Jordan decomposition) of μ .*

Following this theorem, let μ be a signed measure with the Jordan decomposition (μ^+, μ^-) . The *absolute value* of μ is then defined by $|\mu| := \mu_1 + \mu_2$. Theorem A.1 helps us to define the minimum of two measures. Let μ_1 and μ_2 be two nonnegative measures on \mathbb{R} . The minimum of μ_1 and μ_2 is a nonnegative measure defined by $\min\{\mu_1, \mu_2\} := \mu^1 - \mu^+ = \mu^2 - \mu^-$, where (μ^+, μ^-) is the Jordan decomposition of the signed measure $\mu_1 - \mu_2$. It is not hard to see that $\min\{\mu_1, \mu_2\}$ is positive if μ_1 and μ_2 are positive and not singular.

Appendix B. Proofs of technical lemmas. In this Appendix, we provide the proofs of Lemmas 4.1, 4.5 and, 5.1 that were omitted from the main text. We start with the proof of Lemma 4.5.

Proof. [Proof of Lemma 4.5] The following algorithm computes $\bar{\mu}_1, \dots, \bar{\mu}_n$:

1. Set $\nu_1 := \mu$.
2. For $i := 1$ to n do the following:
 - (a) Let (z_i^+, z_i^-) be the Jordan decomposition of the signed measure $\mu_i - \nu_i$.
 - (b) Set $\bar{\mu}_i := \mu_i - z_i^+ = \nu_i - z_i^-$ and $\nu_{i+1} := \nu_i - \bar{\mu}_i = z_i^-$.

In order to complete the prove we have to show that μ is reduced to zero during the algorithm, i.e., $\nu_{n+1} = 0$. We assume the opposite and seek a contradiction. Let $B = \text{supp}(\nu_{n+1})$. It follows from the computations of the algorithm that

$$\begin{aligned} \sum_{i=1}^n \mu_i &= \sum_{i=1}^n (\nu_i + z_i^+ - z_i^-) = \mu + z_1^+ - z_1^- + \sum_{i=2}^k (z_{i-1}^- + z_i^+ - z_i^-) \\ &= \mu - \nu_{n+1} + \sum_{i=1}^n z_i^+ \end{aligned}$$

Since ν_{n+1} is mutually singular to z_i^+ for all $i = 1, \dots, n$, we get $(\sum_{i=1}^n \mu_i)(B) < \mu(B)$, which is a contradiction. \square

The proof of Lemma 5.1 is based on the following result.

LEMMA B.1. *Let μ be a finite discrete measure on \mathbb{R} , f be a positive real number, and θ be a real number such that: $f \leq \mu([\theta, \infty))$. Then for every $\rho \in [0, 1)$ there exists a (discrete) measure $\nu \leq \mu$ with finite support $\text{supp}(\nu) \subset [\theta, \infty)$ such that:*

$$\begin{aligned} \rho \cdot f &= \nu(\mathbb{R}), \\ f - \mu([\theta, \vartheta]) &\geq \nu([\vartheta, \infty)) \quad \forall \vartheta \in [\theta, \max(\text{supp}(\nu))]. \end{aligned}$$

Proof. Since $f \leq \mu([\theta, \infty))$, there is some point in time $\theta_{\max} \in \mathbb{R}$, such that:

$$\sqrt{\rho} \cdot f \leq \mu([\theta, \theta_{\max}]) .$$

In the following let θ_{\max} be the infimum over all such times. Then, θ_{\max} is in fact a minimum, because of the right continuity of distribution functions. Therefore:

$$0 \leq \underbrace{\sqrt{\rho} \cdot f - \mu([\theta, \theta_{\max}))}_{=: a} \leq \mu(\{\theta_{\max}\}) .$$

Since μ is discrete there exist a finite set $\Omega \subset [\theta, \theta_{\max})$ such that:

$$\sqrt{\rho} \cdot \mu([\theta, \theta_{\max})) \leq \mu(\Omega) .$$

We define the discrete measure ν as follows:

$$\nu(\{\vartheta\}) := \begin{cases} \mu(\{\vartheta\}) & \text{for } \vartheta \in \Omega \\ a & \text{for } \vartheta = \theta_{\max} \\ 0 & \text{otherwise} \end{cases}$$

Thus by definition we know that $\text{supp}(\nu) \subset [\theta, \infty]$. Further, we have

$$\nu(\mathbb{R}) = a + \mu(\Omega) \geq \mu(\{\theta_{\max}\}) + \sqrt{\rho} \cdot \mu([\theta, \theta_{\max})) \geq \sqrt{\rho} \cdot \mu([\theta, \theta_{\max}]) \geq \rho \cdot f .$$

For proving the second property let $\vartheta \in [\theta, \theta_{\max})$. We have:

$$\begin{aligned} f - \mu([\theta, \vartheta]) &= f - \mu([\theta, \theta_{\max})) + \mu((\vartheta, \theta_{\max})) \\ &\geq a + \mu((\vartheta, \theta_{\max})) \\ &\geq \nu((\vartheta, \infty)) . \end{aligned}$$

Scaling of ν such that equality is reached in the first property completes this proof. \square

Proof. [Proof of Lemma 5.1] The idea is to apply Lemma B.1 to $f := \nu_1(\{\theta\})$ and μ_2 . Therefore we have to show $f \leq \mu_2([\theta, \infty))$. Since $\gamma(\mathbb{R}) = 0$ and $F_\gamma \geq 0$ we know:

$$\gamma([\theta, \infty)) = \gamma(\mathbb{R}) - \gamma((-\infty, \theta)) \leq 0$$

Thus, from $\mu_2 + \gamma \geq \mu_1$ we obtain:

$$f \leq \mu_1([\theta, \infty)) \leq \gamma([\theta, \infty)) + \mu_2([\theta, \infty)) \leq \mu_2([\theta, \infty))$$

Hence, Lemma B.1 ensures the existence of a discrete measure $\nu_2 \leq \mu_2$ with finite support in $[\theta, \infty)$ such that:

$$\begin{aligned} \rho \cdot f &= \nu_2(\mathbb{R}), \\ f - \mu_2([\theta, \vartheta]) &\geq \nu_2((\vartheta, \infty)) , \quad \forall \vartheta \in [\theta, \max(\text{supp}(\nu_2))] . \end{aligned}$$

In order two satisfy the first statement we define $\eta := \rho \cdot \nu_1 - \nu_2$. Then from the first equation we get $\eta(\mathbb{R}) = \nu_2(\mathbb{R}) - \rho \cdot f = 0$. Further, the distribution function F_η is equal to 0 outside of $[\theta, \max(\text{supp}(\nu_2))$. For $\vartheta \in [\theta, \max(\text{supp}(\nu_2))$ we obtain:

$$\begin{aligned} F_\eta(\vartheta) &= \rho \cdot f - \nu_2((-\infty, \vartheta]) = \nu_2(\mathbb{R}) - \nu_2((-\infty, \vartheta]) = \nu_2((\vartheta, \infty)) \\ &\leq f - \mu_2([\theta, \vartheta]) \leq \mu_1([\theta, \vartheta]) - \mu_2([\theta, \vartheta]) \leq \gamma([\theta, \vartheta]) \leq F_\gamma(\vartheta) . \end{aligned}$$

This completes the proof. \square

It remains to prove Lemma 4.1. To this end, we first give some lemmas.

LEMMA B.2. *Suppose that $\mu_1, \mu_2 \geq 0$ are two finite continuous Borel measures on \mathbb{R} with distribution functions F_1 and F_2 , respectively. Let $F_1 \geq F_2$ on some interval $I := (-\infty, \theta]$, $\theta \in \mathbb{R}$, and $M := \{\vartheta \in I \mid F_1(\vartheta) = F_2(\vartheta)\}$ be the set of points in I where the two distribution functions are equal. Then $\mu_1(M) = \mu_2(M)$.*

Proof. For a given $\epsilon > 0$, we let $M_{<\epsilon} := \{\vartheta \in (-\infty, \theta) \mid F_1(\vartheta) - F_2(\vartheta) < \epsilon\}$ be the set of points in $(-\infty, \theta)$ where two distribution functions differ by less than ϵ . It is clear that $M_{<\epsilon}$ is an open set, so we can express it as a countable union of pairwise disjoint open intervals, unique up to order, as $M_\epsilon = \cup_{i \in J} (a_i, b_i)$, where J is a countable set of indices. Note that, for each $i \in J$, (a_i, b_i) is maximal in the following sense: There exists no open interval $(a', b') \subseteq M_\epsilon$ strictly containing (a_i, b_i) . We also know that the distribution functions F_1 and F_2 are continuous since μ_1 and μ_2 are continuous measures. Hence we can conclude that $F_1(a_i) - F_2(a_i) = \epsilon$ if $a_i \neq -\infty$, and $F_1(b_i) - F_2(b_i) = \epsilon$ if $b_i \neq \theta$. Then it follows that

$$\mu_1(M_\epsilon) - \mu_2(M_\epsilon) = \sum_{i \in J} \mu_1((a_i, b_i)) - \mu_2((a_i, b_i)) \leq \epsilon .$$

Now we let tend ϵ to 0 and get $\mu_1(M) = \mu_2(M)$. \square

The next corollary generalizes Lemma B.2 from $\mu_1(M) = \mu_2(M)$ to $\mu_1|_M = \mu_2|_M$, even for the more general case when the assumption of $F_1 \geq F_2$ is not met.

COROLLARY B.3. *Let $\mu_1, \mu_2 \geq 0$ be two finite continuous Borel measures on \mathbb{R} with distribution functions F_1 and F_2 , respectively. Moreover, let $M := \{\theta \in \mathbb{R} \mid F_1(\theta) = F_2(\theta)\}$ be the set of points where two distribution functions are equal. Then $\mu_1|_M = \mu_2|_M$.*

Proof. We first assume that $F_1 \geq F_2$. Then Lemma B.2 implies

$$\mu_1|_M((-\infty, \theta]) = \mu_1|_{(-\infty, \theta]}(M) = \mu_2|_{(-\infty, \theta]}(M) = \mu_2|_M((-\infty, \theta]) \quad \forall \theta \in \mathbb{R} .$$

It follows from this relation that the distribution functions with respect to $\mu_1|_M$ and $\mu_2|_M$ coincide on \mathbb{R} . This implies $\mu_1|_M = \mu_2|_M$.

For the general case, we define $F_{\max} : \mathbb{R} \rightarrow \mathbb{R}$ by $F_{\max}(\theta) := \max\{F_1(\theta), F_2(\theta)\}$. It is clear that F_{\max} is monotonic increasing and continuous on the right. So it is the distribution function of some measure μ_{\max} . Applying the previous result to F_{\max} and F_1 and also to F_{\max} and F_2 , we get

$$\mu_1|_M = \mu_{\max}|_M = \mu_2|_M .$$

\square

COROLLARY B.4. *Let μ be a finite signed Borel measure on \mathbb{R} with distribution function F and $Q \subset \mathbb{R}$ be a countable set of real numbers. If μ is continuous, then $M := \{\theta \mid F(\theta) \in Q\}$ is a strict μ -null set, i.e., $\mu|_M = 0$.*

Proof. For each $q \in Q$ define $M_q := \{\theta \mid F(\theta) = q\}$. Then M is the disjoint (countable) union of the M_q 's. Hence, in order to establish the lemma it is enough to show $\mu|_{M_q} = 0$ for each $q \in Q$.

Let $q \in Q$ be fixed and assume, without loss of generality, that $q \geq 0$. Further, let μ^+ and μ^- be the positive and negative part of μ with distribution functions F^+ and F^- , respectively. Since μ is continuous, $a := \min\{\theta \mid F^+(\theta) \geq q\}$ is well-defined and $F^+(a) = q$. We define $\bar{F} : \mathbb{R} \rightarrow \mathbb{R}_+$ by

$$\theta \mapsto \begin{cases} 0 & \text{if } \theta \leq a , \\ F^+(\theta) - q & \text{if } \theta > a . \end{cases}$$

Then \bar{F} is the distribution function of the restricted measure $\bar{\mu} := \mu^+|_{[a, \infty)}$ and we have $M_q = \{\theta \mid \bar{F}(\theta) = F^-(\theta)\}$. From Corollary B.3 and the fact that $M_q \cap (-\infty, a) = \emptyset$, it follows $\mu^+|_{M_q} = \bar{\mu}|_{M_q} = \mu^-|_{M_q}$, and as a direct consequence $\mu|_{M_q} = 0$. \square

We are now in a position to prove Lemma 4.1.

Proof. [Proof of Lemma 4.1] Let μ^d be the discrete part of μ . Then there exists a countable set Q of real numbers such that the distribution function of μ^d only takes its values in Q .

Let F^c be the distribution function of μ^c and define $\bar{M} := \{\theta \mid F^c(\theta) \in Q\}$. It now follows from Corollary B.4 that $\mu^c|_{\bar{M}} = 0$ since Q is countable. In order to prove the lemma, it suffices to show $\mathbb{R} \setminus M \subseteq \bar{M} \cup \text{supp}(x^d)$. Let $\theta \in \mathbb{R} \setminus M$ be fixed. Due to the definition of distribution functions, we have

$$\mu(\{\theta\}) = F(\theta) - \lim_{\vartheta \rightarrow \theta^-} F(\vartheta) .$$

Then exactly one of the following cases occurs:

1. $\mu(\{\theta\}) = 0$ and $F(\theta) = 0$,
2. $\mu(\{\theta\}) > 0$.

In the first case we have $\theta \in \bar{S}$ and in the second case $\theta \in \text{supp}(x^d)$. This completes the proof of the lemma. \square