

Strong duality for the maximum Borel flow problem*

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Abstract Research on flows over time has been conducted mainly in two separate and mainly independent approaches, namely *discrete* and *continuous* models, depending on whether a discrete or continuous representation of time is used. Recently, Borel flows have been introduced to build a bridge between these two models. In this paper, we consider the maximum Borel flow problem formulated in a network where capacities on arcs are given as Borel measures and storage might be allowed at the nodes of the network. This problem is formulated as a linear program in a space of measures. We define a dual problem and prove a strong duality result. We show that strong duality is closely related to a MaxFlow-MinCut Theorem.

1 Introduction

Network flows over time (also called *dynamic network flows* in the literature) form a fascinating area of study. In contrast to classical *static* flows, transit times are introduced on the arcs to describe how long it takes to traverse an arc. This would imply that flows on arcs are not constant but may change over time. Ford and Fulkerson [6, 7] study the *maximum flow over time* problem where the aim is to find the maximum amount of flow that can be sent from a source node to a sink node within a given time horizon. They show that this problem can be solved efficiently by one minimum cost flow computation on the given network.

In the model studied by Ford and Fulkerson [6, 7], time is represented in discrete time steps and arc capacities are constant over time. In contrast to this, Philpott [10] and Anderson, Nash, and Philpott [2] study the maximum flow over time problem

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in a network with zero transit times and time-varying transit and storage capacities for the case where time is modeled as a continuum. They extend the concept of cuts to the continuous-time setting and establish a MaxFlow-MinCut theorem (see also [1]) for the case that transit times are zero and the transit capacities are bounded measurable. This result was later extended to arbitrary transit times by Philpott [11].

Since the seminal research of Ford and Fulkerson in the 1950s, many authors have extensively studied flows over time from different viewpoints (see, e.g., [14] for a recent survey on the topic), but in two separate ways with respect to time-modeling, leading to discrete and continuous models. For the case of constant network parameters (e.g., costs, capacities, supplies, and demands) and integral transit times, Fleischer and Tardos [5] point out a close correspondence between discrete and continuous models. In particular, they show how a number of results and algorithms for the discrete model can be carried over to the analogous continuous model.

Recently, Koch, Nasrabadi, and Skutella [9] introduced the notion of Borel flows to unify discrete and continuous flows over time into a single model. Here the flow on each arc is modeled as a *measure* on the real line (time axis) which assigns to each suitable subset a real value, interpreted as the amount of flow entering the arc over the subset. Koch, Nasrabadi, and Skutella [9] establish a MaxFlow-MinCut Theorem for the maximum Borel flow problem.

Developing a duality theory for various classes of optimization problems has received a great deal of attention because of its importance in designing solution algorithms. In the classical static network flows, it is known that the dual problem for the maximum flow problem corresponds to the cuts and strong duality is equivalent to the MaxFlow-MinCut Theorem. Anderson and Philpott [3] explore this relationship for the maximum flow over time in the continuous model for the special case where transit times are zero and the arc and node capacities are piecewise analytic. In the case that all transit times are zero, the network flow over time problem can be treated as a special case of the so-called *Separated Continuous Linear Program (SCLP)*. Pullan [12] establishes strong duality for SCLP given piecewise analytic problem data. Moreover, Pullan [13] examines a larger class of SCLP to include time-delays and derive a strong duality result for the problem with rational transit times and piecewise constant/linear input functions. Hashemi, Koch, and Nasrabadi [8] make use of ideas from the area of static network flows to establish a strong duality result for network flow over time problems with piecewise analytic input functions and rational transit times.

The aim of this paper is to establish a strong duality result for the maximum Borel flow problem and examine its relationship to the MaxFlow-MinCut Theorem. In Section 2 we give an overview of the maximum Borel flow problem as well as the MaxFlow-MinCut Theorem that are required for the purposes of this paper. In Section 3 we formulate a dual problem for the maximum Borel flow problem. In Section 4 we derive a strong duality result between the maximum Borel flow problem and its dual.

2 The Maximum Borel flow Problem

We consider a directed graph $G = (V, E)$ with finite *node set* V and finite *arc set* E . Let $s \in V$ be a *source* and $t \in V$ be a *sink* in G . Each arc $e \in E$ has an associated *transit time* $\tau_e \in \mathbb{R}$ specifying the required amount of time for traveling from the tail to the head of e . More precisely, if flow leaves node v at time θ along an arc $e = (v, w)$, it arrives at w at time $\theta + \tau_e$.

Consider the real line \mathbb{R} as the time domain. A *Borel flow* x is defined by a family of Borel measures

$$x_e : \mathcal{B} \longrightarrow \mathbb{R}_{\geq 0} \quad \forall e \in E ,$$

where \mathcal{B} is the *Borel σ -algebra* on \mathbb{R} . We refer readers to [9] for the motivation of Borel flows. A member $B \in \mathcal{B}$ is called a *Borel set* or *measurable set*. The value $x_e(B)$ gives the amount of flow entering arc e over the Borel set B .

Each arc $e \in E$ has an associated Borel measure $u_e : \mathcal{B} \rightarrow \mathbb{R}_{\geq 0}$ which denotes its capacity. We assume that $u_e(\mathbb{R}) < \infty$ for each arc $e \in E$. The value $u_e(B)$ is an upper bound on the amount of flow that is able to enter arc e over the Borel set B . We require that a Borel flow x fulfills *arc capacity constraints*

$$x_e(B) \leq u_e(B) \quad \forall e \in E, B \in \mathcal{B} .$$

The flow particles arriving at a node may wait for some time before they leave again that node. Thus the Borel flow x induces a *storage function* Y_v on \mathbb{R} at each node v by

$$Y_v(\theta) := \sum_{e \in \delta^-(v)} x_e((-\infty, \theta - \tau_e]) - \sum_{e \in \delta^+(v)} x_e((-\infty, \theta]) \quad \forall \theta \in \mathbb{R} , \quad (1)$$

where $\delta^+(v)$ and $\delta^-(v)$ denote the sets of arcs leaving node v and entering node v , respectively. The value $Y_v(\theta)$ gives the amount of flow stored at node v at the point in time $\theta \in \mathbb{R}$. The function Y_v is the difference between two right-continuous, monotonic increasing functions and thus is a right-continuous function of bounded variation.

We suppose that the storage of flow at a node $v \in V$ is bounded from above by a function $U_v : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$. The value $U_v(\theta)$ is an upper bound on the amount of flow that can be stored at node v at time θ . We assume that there is no initial storage at any node and flow must not remain at any node except s and t . This means $Y_v(-\infty) := \lim_{\theta \rightarrow -\infty} Y_v(\theta)$ and $Y_v(\infty) := \lim_{\theta \rightarrow \infty} Y_v(\theta)$ must be zero for each node $v \in V \setminus \{s, t\}$. Notice that both limits exist since Y_v is of bounded variation. Therefore we require $U_v(-\infty) = U_v(\infty) = 0$ for each $v \in V \setminus \{s, t\}$.

A Borel flow x with corresponding storage Y fulfills the *node capacity constraint* at node $v \in V \setminus \{s, t\}$ if

$$0 \leq Y_v(\theta) \leq U_v(\theta) \quad (2)$$

for all $\theta \in \mathbb{R}$. We assume that U_v is of bounded variation and continuous from the right for each node v . This imposes no restriction since Y_v is a right-continuous function of bounded variation.

The Borel flow x is called an *s-t Borel flow* if it satisfies the node capacity constraint at all nodes $v \in V \setminus \{s, t\}$. The *value* $\text{val}(x)$ of an *s-t Borel flow* x is defined as the total net flow out of node s , that is, $\text{val}(x) := \sum_{e \in \delta^+(s)} |x_e| - \sum_{e \in \delta^-(s)} |x_e|$ where $|x_e|$ denotes the total amount of flow entering arc e , i.e., $|x_e| := x_e(\mathbb{R})$. An *s-t Borel flow* is called *maximum* if it has maximum value among all *s-t Borel flows*. The *maximum Borel flow problem* is to find a maximum flow. This problem can be formulated as follows:

$$\begin{aligned} \max \quad & \sum_{e \in \delta^+(s)} |x_e| - \sum_{e \in \delta^-(s)} |x_e| \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} (x_e - \tau_e) + y_v = 0 \quad \forall v \in V \setminus \{s, t\}, \\ & 0 \leq x_e \leq u_e \quad \forall e \in E, \\ & 0 \leq Y_v \leq U_v \quad \forall v \in V \setminus \{s, t\}. \end{aligned} \quad (\text{MBFP})$$

In this formulation, for each $v \in V \setminus \{s, t\}$, y_v *signed Borel measure* derived from the formula $y_v((-\infty, \theta]) = Y_v(\theta)$. For a Borel set B , the value $y_v(B)$ can be interpreted as the overall change in storage at v over the Borel set B . Moreover, for each $e \in E$, $x_e - \tau_e$ is a *shifted measure* defined by $(x_e - \tau_e)(B) = x_e(B - \tau_e)$ for all $B \in \mathcal{B}$, where $B - \tau_e := \{\theta - \tau_e \mid \theta \in B\}$.

The MaxFlow-MinCut Theorem, which is due to Ford and Fulkerson [7], is one of the most elaborated results in network flow theory. Koch, Nasrabadi and Skutella [9] extend this theorem for Borel flows. To state this result, we require to give some definitions. An *s-t Borel cut* $S := (S_v)_{v \in V}$ is defined by measurable sets S_v , one for each $v \in V$ so that $S_s = \mathbb{R}$, $S_t = \emptyset$ and for every node $v \in V \setminus \{s, t\}$ the set

$$\Gamma_v := S_v \cap \{\theta \in \mathbb{R} : U_v(\theta-) > 0 \text{ or } U_v(\theta) > 0\} \quad (3)$$

is a countable union of pairwise disjoint intervals. Here, $U_v(\theta-)$ denotes the limit of U_v at θ from the left, i.e., $U_v(\theta-) := \lim_{\vartheta \nearrow \theta} U_v(\vartheta)$.

For a right-continuous function $M : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ of bounded variation, we shall use $M^{\succ 0}$ to denote the set of all points $\theta \in \mathbb{R}$ where M or its left limit is positive at θ . More precisely:

$$M^{\succ 0} := \{\theta \in \mathbb{R} \mid M(\theta-) > 0 \text{ or } M(\theta) > 0\}. \quad (4)$$

By this notation, Γ_v can be simplified as $S_v \cap U_v^{\succ 0}$.

We say that node v belongs to the *s-side* of S for the points in time $\theta \in S_v$ and to the *t-side* of S for the points in time $\theta \in S_v^{c2}$. Thus, an arc $e = (v, w)$ connects the *s-side* to the *t-side* for all times in $S_v \cap (S_w - \tau_e)^c$.

² For a set $A \in \mathbb{R}$ we denote by $A^c := \mathbb{R} \setminus A$ the complement of A .

Let $S = (S_v)_{v \in V}$ be an s - t Borel cut and consider a node v . By definition, Γ_v is expressible as $\bigcup_{i \in J_v} I_{v,i}$, where J_v is a countable set of indices and $I_{v,i}$, $i \in J_v$, are pairwise disjoint intervals. Each interval $I_{v,i}$ is supposed to be inclusion-wise maximal, i.e., there is no interval $I \subseteq \Gamma_v$ strictly containing $I_{v,i}$. We keep this assumption throughout the paper; whenever we consider a countable union of intervals we suppose that the intervals are inclusion-wise maximal. Let $\alpha_{v,i}$ and $\beta_{v,i}$ be the left and right boundary of the interval $I_{v,i}$, respectively. An interval $I_{v,i}$ can be of the form $(\alpha_{v,i}, \beta_{v,i})$, $[\alpha_{v,i}, \beta_{v,i})$, $(\alpha_{v,i}, \beta_{v,i}]$, or $[\alpha_{v,i}, \beta_{v,i}]$. Therefore we partition the set J_v of indices into four subsets. Let J_v^1 (J_v^2 , J_v^3 , and J_v^4) be the set of indices i for which $I_{v,i}$ is open (left-closed & right-open, right-closed & left-open, and closed, respectively). With these constructions, the *capacity* $\text{cap}(S)$ of S is defined by

$$\begin{aligned} \text{cap}(S) := & \sum_{e=(v,w) \in E} u_e (S_v \cap (S_w - \tau_e)^c) + \\ & \sum_{v \in V \setminus \{s,t\}} \left(\sum_{i \in J_v^1 \cup J_v^2} U_v(\beta_{v,i}-) + \sum_{i \in J_v^3 \cup J_v^4} U_v(\beta_{v,i}) \right). \end{aligned} \quad (5)$$

We set the capacity $\text{cap}(S)$ to ∞ if any infinite sum does not converge. We refer to an s - t Borel cut whose capacity is minimum among all s - t Borel cuts as a *minimum Borel cut*. The following theorem summarizes the main results of [9]

Theorem 1. *The MaxFlow-MinCut Theorem holds for Borel flows, i.e., there exists an s - t -flow over time x and an s - t -cut over time S for which $\text{val}(x) = \text{cap}(S)$.*

3 Dual Formulation

In the context of static network flows, the MaxFlow-MinCut Theorem is equivalent to strong duality. Here we wish to establish a similar result for Borel flows. To do this, we need a dual problem for (MBFP). In order to state a dual formulation, we require the concept of a function of σ -bounded variation. Let f be a real-valued function on \mathbb{R} . The *total variation* of f within a bounded interval $[a, b]$ is defined by

$$V(f; [a, b]) := \sup \left\{ \sum_{i=1}^n (f(a_i) - f(a_{i-1})) \mid \{a_1, \dots, a_n\} \text{ is a partition of } [a, b] \right\}.$$

The function f is called of *bounded variation* on $[a, b]$ if $V(f, [a, b]) < \infty$. The function f is said to be of *bounded variation* on \mathbb{R} if there exists a constant $K < \infty$ such that $V(f; [a, b]) < K$ for any (bounded) interval $[a, b] \subset \mathbb{R}$.

The function f is said to be of σ -bounded variation if it can be decomposed into a countable sum of functions of bounded variation on \mathbb{R} . Similarly, f is said to be σ -monotonic increasing if it can be decomposed into a countable sum of monotonic increasing and bounded functions on \mathbb{R} . These two definitions can be regarded as the extension of finite measures to σ -finite measures and because of that we have used the symbol σ .

It is a well-known result that a function is of bounded variation if and only if it is the difference between two monotonic increasing and bounded functions (see, e.g., Chapter 6 in [4]). This implies that a function is of σ -bounded variation if and only if it is the difference between two σ -monotonic increasing functions. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of σ -bounded variation with $f(\infty) = 0$ where $f(\infty) := \lim_{\theta \rightarrow \infty} f(\theta)$. Notice that this limit exists since f is of σ -bounded variation. Then there exist functions $f^{(+)}$ and $f^{(-)}$ (subsequently referred to as the *Jordan decomposition* of f) that are σ -monotonic increasing on f with $f^{(+)}(\infty) = f^{(-)}(\infty) = 0$ and $f(\theta) = f^{(+)}(\theta) - f^{(-)}(\theta)$ for $\theta \in \mathbb{R}$. The functions $f^{(+)}$ and $f^{(-)}$ are called the positive and negative parts of f , respectively.

We now consider a dual problem for (MBFP) as follows:

$$\begin{aligned}
\min \quad & \sum_{e \in E} \int_{\mathbb{R}} \rho_e du_e + \sum_{v \in V \setminus \{s, t\}} \int_{\mathbb{R}} U_v d\pi_v^{(-)} \\
\text{s.t.} \quad & \rho_e(\theta) - \lambda_v(\theta) + \lambda_w(\theta + \tau_e) \geq 0 \quad \forall e = (v, w) \in E, \theta \in \mathbb{R}, \\
& \rho_e(\theta) \geq 0 \quad \forall e \in E, \theta \in \mathbb{R}, \\
& \lambda_s(\theta) = 1 \quad \forall \theta \in \mathbb{R}, \\
& \lambda_t(\theta) = 0 \quad \forall \theta \in \mathbb{R}, \\
& \pi_v := \lambda_v|_{U_v^{>0}} \text{ of } \sigma\text{-BV on } \mathbb{R} \quad \forall v \in V \setminus \{s, t\},
\end{aligned} \tag{MBFP*}$$

where $\rho_e, e \in E$ and $\lambda_v, v \in V \setminus \{s, t\}$ are measurable functions. We should mention that this problem generalizes the dual problem studied by Anderson and Philpoot [3] for the continuous-time maximum flow problem. Note that $\pi_v^{(-)}$ denotes the negative part of π_v and $\pi_v := \lambda_v|_{U_v^{>0}}$ is given for each $v \in V \setminus \{s, t\}$ as follows:

$$\pi_v(\theta) := \begin{cases} \lambda_v(\theta) & \text{if } \theta \in U_v^{>0}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from this definition that $\pi_v(\infty) = 0$ since $Y_v(\infty) = 0$ due to our assumption.

The integrals in the objective function of (MBFP*) should be explained. For each $e \in E$ the first integral involves the function ρ_e as the integrand and the measure u_e as the integrator and is understood in the sense of Lebesgue-Stieltjes. Since ρ_e is supposed to be measurable and u_e is a Borel measure, the first integral is well defined and exists. The second integral for each $v \in V \setminus \{s, t\}$ involves two functions U_v and $\pi_v^{(-)}$ (U_v as the integrand and $\pi_v^{(-)}$ as the integrator) and is regarded as a generalization of the Riemann-Stieltjes integral. More precisely, if $\pi_v^{(-)} = \sum_{i \in \mathbb{N}} \pi_{v,i}^{(-)}$ where $\pi_{v,i}^{(-)}, i \in \mathbb{N}$ are monotonic increasing and bounded, then

$$\int_{\mathbb{R}} U_v d\pi_v^{(-)} := \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} U_v d\pi_{v,i}^{(-)}.$$

Each left integral of the above equation is treated as the Riemann-Stieltjes integral as developed in [4, Chapter 7]. However, although U_v is supposed to be of bounded

variation and right-continuous and $\pi_{v,i}^{(-)}$ is monotonic increasing and bounded, the Riemann-Stieltjes integral $\int_{\mathbb{R}} U_v d\pi_{v,i}^{(-)}$ need not exist for some $i \in \mathbb{N}$ as U_v and $\pi_{v,i}^{(-)}$ may have common discontinuities from the left or from the right at some points. In such a case, the integral does not exist (see [4, Theorem 7.29]) and we replace U_v by another function, say \bar{U}_v , defined by

$$\bar{U}_v(\theta) := \begin{cases} U_v(\theta-) & \text{if } U_v \text{ is discontinuous at } \theta \text{ and } \pi_{v,i} \text{ is right-continuous at } \theta, \\ U_v(\theta) & \text{otherwise.} \end{cases}$$

Note that the function \bar{U}_v differs from U_v only at those points θ for which U_v and $\pi_{v,i}^{(-)}$ share a common discontinuity from the left or from the right at θ . Then the integral $\int_{\mathbb{R}} U_v d\pi_{v,i}^{(-)}$ is defined by

$$\int_{\mathbb{R}} U_v d\pi_{v,i}^{(-)} := \int_{\mathbb{R}} \bar{U}_v d\pi_{v,i}^{(-)}$$

Note that the left integral of the above equation is guaranteed to exist (see [4, Theorem 7.29]). In what follows, each integral with a measure as the integrator is regarded in the sense of Lebesgue-Stieltjes and each one with a function as the integrator is regarded in the sense of Riemann-Stieltjes as defined above.

For each $e \in E$, the dual variable ρ_e can be eliminated from (MBFP*) since it appears in the objective function integrated with respect to measure u_e which is nonnegative, and hence at an optimum solution ρ_e should be made as small as possible. This observation implies that if we know optimal values for the dual variables λ_v , $v \in V$, we can compute the optimal values for ρ_e , $e \in E$ by

$$\rho_e(\theta) = \max \{0, \lambda_v(\theta) - \lambda_w(\theta + \tau_e)\} \quad \forall e = (v, w) \in E, \theta \in \mathbb{R}. \quad (6)$$

4 Strong duality

The first result that we would like to establish between (MBFP) and (MBFP*) is weak duality. Before we prove this, we require the following technical result.

Lemma 1. *Let μ be a finite signed Borel measure on \mathbb{R} with a nonnegative distribution function M . Suppose that $h : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that $h(\theta) = 0$ for all $\theta \in M^{\succ 0}$. Then $\int_{\mathbb{R}} h(\theta) d\mu = 0$.*

Proof. Let $A := \mathbb{R} \setminus M^{\succ 0}$ be the set of points $\theta \in \mathbb{R}$ for which M is continuous and zero at θ . We can write

$$\int_{\mathbb{R}} h(\theta) d\mu = \int_{M^{\succ 0}} h(\theta) d\mu + \int_A h(\theta) d\mu$$

The first integral on the right-hand side of the equality sign is zero because $h(\theta) = 0$ for all $\theta \in M^{\succ 0}$. Moreover, we can conclude from Lemma 1 in [9] that the set A is

a strict μ -null set, i.e., $\mu(B) = 0$ for all measurable sets $B \subseteq A$. This implies that the second integral is zero as well. \square

We introduce some notations that are used throughout the rest of the paper. For a given optimization problem (OP), we use the notation $V[OP]$ to denote its optimal value and use the notation $V[OP, x]$ to denote the objective function value for a given feasible solution x .

Lemma 2. *Suppose that x is feasible for (MBFP) and λ is feasible for (MBFP*). Then $V[(MBFP), x] \leq V[(MBFP^*), \lambda]$.*

Proof. Assume that y is the storage with respect to x derived from (??) and ρ is the function corresponding to λ given by (6). Since $\lambda_v(\theta) = 1$ for each $\theta \in \mathbb{R}$, we can express the objective function value of (MBFP) for x as

$$V[(MBFP), x] = \sum_{e=(v,w) \in \delta^+(s)} \int_{\mathbb{R}} \lambda_v(\theta) dx_e - \sum_{e=(v,w) \in \delta^-(s)} \int_{\mathbb{R}} \lambda_w(\theta + \tau_e) dx_e. \quad (7)$$

Moreover, as $\lambda_t(\theta) = 0$ for all $\theta \in \mathbb{R}$, we have

$$\sum_{e=(v,w) \in \delta^+(t)} \int_{\mathbb{R}} \lambda_v(\theta) dx_e - \sum_{e=(v,w) \in \delta^-(t)} \int_{\mathbb{R}} \lambda_w(\theta + \tau_e) dx_e = 0. \quad (8)$$

On the other hand, the flow conservation constraint (1) at node $v \in V \setminus \{s, t\}$ holds for all $\theta \in \mathbb{R}$. Therefore,

$$\int_{\mathbb{R}} \left(\sum_{e \in \delta^+(v)} \int_{-\infty}^{\theta} dx_e - \sum_{e \in \delta^-(v)} \int_{\infty}^{\theta - \tau_e} dx_e + Y_v(\theta) \right) d\pi_v(\theta) = 0 \quad v \in V \setminus \{s, t\}.$$

By summing up these equations over $v \in V \setminus \{s, t\}$, we get

$$\sum_{v \in V \setminus \{s, t\}} \int_{\mathbb{R}} \left(\sum_{e \in \delta^+(v)} \int_{-\infty}^{\theta} dx_e - \sum_{e \in \delta^-(v)} \int_{\infty}^{\theta - \tau_e} dx_e + Y_v(\theta) \right) d\pi_v(\theta) = 0. \quad (9)$$

By integration by parts and the fact that $\pi_v(\infty) = 0$ for each $v \in V \setminus \{s, t\}$, we have for each arc $e = (v, w) \in E$

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_{-\infty}^{\theta} dx_e \right) d\pi_v(\theta) &= \int_{\mathbb{R}} \pi_v(\theta) dx_e, \\ \int_{\mathbb{R}} \left(\int_{-\infty}^{\theta - \tau_e} dx_e \right) d\pi_w(\theta) &= \int_{\mathbb{R}} \pi_w(\theta + \tau_e) dx_e. \end{aligned}$$

Thus, (9) can be rewritten as follows:

$$\begin{aligned}
& \sum_{e=(v,w) \in \delta^-(\{s,t\})} \int_{\mathbb{R}} \pi_v(\theta) dx_e - \sum_{e=(v,w) \in \delta^+(\{s,t\})} \int_{\mathbb{R}} \pi_w(\theta + \tau_e) dx_e \\
& \sum_{e=(v,w) \in E \setminus (\delta^+(\{s,t\}) \cup \delta^-(\{s,t\}))} \int_{\mathbb{R}} (\pi_v(\theta) - \pi_w(\theta + \tau_e)) dx_e + \\
& \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} Y_v(\theta) d\pi_v(\theta) = 0. \tag{10}
\end{aligned}$$

Moreover, for each $v \in V \setminus \{s,t\}$ we have $\int_{\mathbb{R}} (\lambda_v(\theta) - \pi_v(\theta)) dy_v = 0$ by means of Lemma 1 and $y_v = \sum_{e \in \delta^-(v)} (x_e - \tau_e) - \sum_{e \in \delta^+(v)} x_e$ because of the flow conservation constraint at node v . Thus, we can write down

$$\begin{aligned}
0 &= \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} (\lambda_v(\theta) - \pi_v(\theta)) dy_v \\
&= \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} (\lambda_v(\theta) - \pi_v(\theta)) d\left(\sum_{e \in \delta^-(v)} (x_e - \tau_e) - \sum_{e \in \delta^+(v)} x_e \right) \\
&= \sum_{e=(v,w) \in \delta^+(\{s,t\})} \int_{\mathbb{R}} (\lambda_v(\theta) - \pi_v(\theta)) dx_e \\
&\quad - \sum_{e=(v,w) \in \delta^+(\{s,t\})} \int_{\mathbb{R}} (\lambda_w(\theta + \tau_e) - \pi_w(\theta + \tau_e)) dx_e \\
&\quad + \sum_{e=(v,w) \in E \setminus (\delta^+(\{s,t\}) \cup \delta^-(\{s,t\}))} \int_{\mathbb{R}} ((\lambda_v(\theta) - \pi_v(\theta)) - (\lambda_w(\theta + \tau_e) - \pi_w(\theta + \tau_e))) dx_e. \tag{11}
\end{aligned}$$

Then, adding (10) and (11) leads to

$$\begin{aligned}
& \sum_{e=(v,w) \in \delta^-(\{s,t\})} \int_{\mathbb{R}} \lambda_v(\theta) dx_e - \sum_{e=(v,w) \in \delta^+(\{s,t\})} \int_{\mathbb{R}} \lambda_w(\theta + \tau_e) dx_e \\
& \sum_{e=(v,w) \in E \setminus (\delta^+(\{s,t\}) \cup \delta^-(\{s,t\}))} \int_{\mathbb{R}} (\lambda_v(\theta) - \lambda_w(\theta + \tau_e)) dx_e + \\
& \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} Y_v(\theta) d\lambda_v(\theta) = 0. \tag{12}
\end{aligned}$$

Now by summing up (7), (8), and (12), we obtain

$$V[(\text{MBFP}), x] = \sum_{e=(v,w) \in E} \int_{\mathbb{R}} (\lambda_v(\theta) - \lambda_w(\theta + \tau_e)) dx_e + \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} Y_v(\theta) d\pi_v(\theta).$$

Then by comparing the objective function values of (MBFP) and (MBFP*) for x and λ , respectively, we have

$$\begin{aligned}
V[(\text{MBFP}^*), \lambda] - V[(\text{MBFP}), x] &= \sum_{e \in E} \int_{\mathbb{R}} \rho_e(\theta) d(u_e - x_e) + \\
&\quad \sum_{e=(v,w) \in E} \int_{\mathbb{R}} (\rho_e(\theta) - \lambda_v(\theta) + \lambda_w(\theta + \tau_e)) dx_e + \\
&\quad \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} Y_v(\theta) d\pi^{(+)}(\theta) + \\
&\quad \sum_{v \in V \setminus \{s,t\}} \int_{\mathbb{R}} (U_v(\theta) - Y_v(\theta)) d\pi^{(-)}(\theta).
\end{aligned}$$

Each of the above integrals is nonnegative due to feasibility of x, y for (MBFP) and feasibility of λ, ρ for (MBFP*). Thus, $V[(\text{MBFP}), x] \leq V[(\text{MBFP}^*), \lambda]$, which establishes the desired result. \square

Corollary 1. *Suppose that x, y is feasible for (MBFP) and λ, ρ is feasible for (MBFP*). If*

$$\sum_{e \in E} \int_{\mathbb{R}} \rho_e(\theta) d(u_e - x_e) = 0, \quad (13)$$

$$\sum_{e=(v,w) \in E} \int_{\mathbb{R}} (\rho_e(\theta) - \lambda_v(\theta) + \lambda_w(\theta + \tau_e)) dx_e = 0, \quad (14)$$

$$\sum_{v \in V \setminus \{s,t\}} \int_0 Y_v(\theta) d\pi^{(+)}(\theta) = 0, \quad (15)$$

$$\sum_{v \in V \setminus \{s,t\}} \int_0 (U_v(\theta) - Y_v(\theta)) d\pi^{(-)}(\theta) = 0, \quad (16)$$

then x, y and π, ρ are optimal for (MBFP) and (MBFP*), respectively. Moreover, strong duality holds between (MBFP) and (MBFP*).

So far we have seen that weak duality holds between (MBFP) and (MBFP*). A stronger result is to prove the existence of a feasible solution x for (MBFP) and a feasible solution λ for (MBFP*) in which $V[(\text{MBFP}), x] = V[(\text{MBFP}^*), \lambda]$. As noted previously, the arc capacities $u_e, e \in E$ are finite and this is sufficient to guarantee the existence of an optimal solution x , say, for (MBFP). Moreover, by Theorem 1, there exists an s - t Borel cut S for which $\text{val}(x) = \text{cap}(S)$. It thus enough to show that S corresponds to a feasible solution λ with $\text{cap}(S) = V[(\text{MBFP}^*), \lambda]$. This is the context of the next lemma.

Lemma 3. *Given an s - t Borel cut $S = (S_v)_{v \in V}$, let $\lambda_v : \mathbb{R} \rightarrow \mathbb{R}$ be the indicator function³ of S_v and $\rho_e : \mathbb{R} \rightarrow \mathbb{R}$ be the indicator function of $S_v \cap (S_w - \tau_e)^c$ for*

³ The indicator function χ_A of a set $A \subseteq \mathbb{R}$ is defined by

$$\chi_A(\theta) := \begin{cases} 1 & \text{if } \theta \in A, \\ 0 & \text{otherwise,} \end{cases} \quad \forall \theta \in \mathbb{R}.$$

each arc $e = (v, w) \in E$. Then $\lambda := (\lambda_v)_{v \in V}$, $\rho := (\rho_e)_{e \in E}$ is a feasible solution for (MBFP*) and moreover, $V[(\text{MBFP}^*), \lambda] = \text{cap}(S)$.

Proof. We first show that λ is feasible for (MBFP*). As $S_s = \mathbb{R}$ and $S_t = \emptyset$, we have $\lambda_s = 1$ and $\lambda_t = 1$. Moreover, it is easy to see that λ and ρ satisfy the equation (6). It thus suffices to prove for each $v \in V \setminus \{s, t\}$ that the function π_v defined by $\pi_v := \lambda_v|_{U_v^{>0}}$ is of σ -bounded variation. We easily observe that π_v is the indicator function of $\Gamma_v := S_v \cap U_v^{>0}$. By the definition of s - t Borel cuts, the set Γ_v is a countable union of pairwise disjoint intervals for each $v \in V \setminus \{s, t\}$ and, as an immediate result, the indicator function of Γ_v is a function of σ -bounded variation. This establishes the first part of the lemma.

It remains to show that $V[(\text{MBFP}^*), \lambda] = \text{cap}(S)$. To do so, we simplify the objective function value of (MBFP) with respect to solution λ , ρ . The first term in the objective function can be expressed as follows:

$$\sum_{e \in E} \int_{\mathbb{R}} \rho_e(\theta) d u_e = \sum_{e=(v,w) \in E} u_e (S_v \cap (S_w - \tau_e)^c),$$

since ρ_e is the indicator function of $S_v \cap (S_w - \tau_e)^c$ for each arc $e = (v, w) \in E$.

For each node $v \in V \setminus \{s, t\}$, the set $\Gamma_v := S_v \cap U_v^{>0}$ can be written as $\bigcup_{i \in J_v} I_{v,i}$, where J_v is a countable set and $I_{v,i}$, $i \in J_v$, are pairwise disjoint intervals. Then the second term in the objective function can be written as follows:

$$\sum_{v \in V \setminus \{s, t\}} \int_{\mathbb{R}} U_v(\theta) d \pi_v^{(-)}(\theta) = \sum_{v \in V \setminus \{s, t\}} \left(\sum_{i \in J_v^1 \cup J_v^2} U_v(\beta_{v,i}^-) + \sum_{i \in J_v^3 \cup J_v^4} U_v(\beta_{v,i}) \right).$$

Here $\alpha_{v,i}$ and $\beta_{v,i}$ are the left and right boundaries of the interval $I_{v,i}$ for each $v \in V$ and $i \in \mathbb{N}$. Further, J_v^1 , J_v^2 , J_v^3 , and J_v^4 are the sets of indices i for which $I_{v,i}$ is open, left-closed & right-open, right-closed & left-open, and closed, respectively. \square

Combining Theorem 1 and Lemma 3, we get the main result of this paper.

Theorem 2. *Strong duality holds between (MBFP) and (MBFP*), i.e., there exists a feasible solution x for (MBFP) and a feasible solution λ for (MBFP*) in which $V[(\text{MBFP}), x] = V[(\text{MBFP}^*), \lambda]$.*

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