# Optimal elliptic Sobolev regularity near three-dimensional, multi-material Neumann vertices

Robert Haller-Dintelmann,[*] Wolfgang Höppner[†]
Hans-Christoph Kaiser,[‡] Joachim Rehberg,[§] Günter M. Ziegler[¶]

April 19, 2011

*2000 Mathematics Subject Classification:* 35D10, 35J25, 35R05, 57Q25

*Keywords:* Elliptic div–grad operators, piecewise linear 3D flattening, anisotropic ellipticity in three dimensions, transmission at material interfaces, mixed Dirichlet–Neumann boundary conditions, optimal Sobolev regularity.

*Abstract:* We investigate optimal elliptic regularity (within the scale of Sobolev spaces) of anisotropic div–grad operators in three dimensions at a multi-material vertex on the Neumann boundary part of a polyhedral spatial domain. The gradient of a solution to the corresponding elliptic PDE (in a neighbourhood of the vertex) is integrable to an index greater than three.

*Résumé :* On considère des opérateurs div-grad dans un polyèdre en trois dimensions avec des conditions aux bords de Neumann. Dans l'échelle des espaces de Sobolev on s'intéresse à la régularité optimale elliptique proche d'un sommet dans lequelle des matériaux différents sont présent. On montre que (dans un voisinage du sommet) le gradient de la solution de l'EDP correspondante est intégrable avec un exposant supérieur à trois.

[*]Technische Universität Darmstadt, Institut für Mathematik, Schlossgartenstraße 7, 64289 Darmstadt, Germany. Email: haller@mathematik.tu-darmstadt.de

[†]Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany. Email: Wolfgang.Hoeppner@wias-berlin.de

[‡]Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany. Email: Hans-Christoph.Kaiser@wias-berlin.de

[§]Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany. Email: Joachim.Rehberg@wias-berlin.de

[¶]Freie Universität Berlin, Department of Mathematics and Computer Science, Arnimallee 2, 14195 Berlin, Germany. Email: ziegler@math.fu-berlin.de

# 1 Introduction

In recent years several elliptic regularity results were established in the $W^{1,q}$ scale, see [22] and [39] for the pure Dirichlet/Neumann Laplacian on domains with Lipschitz boundary, [31] and [9] for Neumann and mixed problems on polyhedra, and [33, 15, 6, 7, 27, 30, 10, 11, 16, 8] in case of discontinuous coefficients. Meanwhile, this covers quite a nice zoo of geometries and coefficient functions, even including mixed boundary conditions. What has not been treated, is optimal $W^{1,p} \leftrightarrow W^{-1,p}$ regularity (for a $p > 3$) in a neighbourhood of a Neumann vertex when additionally heterogeneous materials are involved. The referee of [16] had asked us about this situation. It turns out that one can establish integrability of the gradient of the solution to the corresponding elliptic equation (in a neighbourhood of the vertex) to an index greater than three and we present this result here.

The interest in such problems comes from the natural sciences. Here, many phenomena are described by elliptic or parabolic equations, and the influence of heterogeneous materials often is an important issue. For a quite detailed list of such problems we refer to the introduction of [16]. In particular, in the investigations of quite a few nonlinear problems (see [20], [19]) it is relevant that $W^{1,p}$ admits nice multiplier properties if $p$ is larger than the space dimension. Alternatively, the reader may think of a parabolic equation with a quadratic gradient term on the right hand side. Then it is important to have, on one hand, $L^{p/2} \hookrightarrow W^{-1,p}$ (what implies, by Sobolev embedding, that $p$ must be larger than the space dimension) and, on the other hand, an embedding for the domain of the elliptic operator into $W^{1,p}$, see [19], [20], [21].

Let $\Pi \subseteq \mathbb{R}^3$ be a domain whose closure $\overline{\Pi}$ is simultaneously a polyhedron and a manifold with boundary. For a bounded, measurable coefficient function $\mu : \Pi \to \mathbb{R}^{3\times 3}$ we define the operator $-\nabla \cdot \mu\nabla : W^{1,2}(\Pi) \to (W^{1,2}(\Pi))'$ as usual by

$$\langle -\nabla \cdot \mu\nabla v, w\rangle := \int_\Omega \mu\nabla v \cdot \nabla \overline{w} \,\mathrm{dx} \;, \quad v, w \in W^{1,2}(\Pi),$$

in order to have (homogeneous) Neumann boundary conditions for the restriction of this operator to $L^2(\Pi)$.

In our main result, Theorem 6.1, we show that there is a $p > 3$, such that, for any $f \in (W^{1,p'}(\Pi))'$, around any vertex $\mathrm{a}_\blacktriangle$ of $\Pi$, every solution $v$ of $-\nabla \cdot \mu\nabla v = f$ is in $W^{1,p}$ locally , provided the following assumptions are verified:

– $\mu$ is elliptic and takes symmetric matrices as values.
– $\Pi = |K|$ for some finite, euclidean complex $K$ and $\mu$ is constant on the interior of every 3-cell belonging to $K$, i.e. $\mu$ is piecewise constant on a cellular subpartition of the polyhedron $\Pi$.
– Any edge from the boundary of $\Pi$ that has one endpoint in $\mathrm{a}_\blacktriangle$ is a monomaterial edge or a bimaterial outer edge (see Definition 5.9 below).
– Every interior edge with endpoint $\mathrm{a}_\blacktriangle$ is well-behaved, i.e. the singularity exponent associated to this edge (cf. Definition 5.5), is larger than 1/3.

In fact, even more is true. In this situation the divergence form operator $-\nabla \cdot \mu\nabla$ turns out to be a topological isomorphism between some appropriately chosen Sobolev spaces of order 1 and $-1$, respectively; see Theorem 6.1 below.

In any case, our aim was not maximal generality – then necessarily including implicit conditions which are hard to control in examples – but to present a broad class of geometric constellations and coefficients which naturally comes up in real world problems. See Section 9 for further details.

Let us emphasise that the matrices which constitute the coefficient function $\mu$ may be not diagonal and, in particular, not multiples of the identity, see [2] and [26, Ch. IV/V]. This is motivated by the applications. Moreover, anisotropic coefficients are unavoidable in view of (local) deformation and transformation of the domain in the localisation procedure, see Proposition 4.1. It should be noted that in case of an essentially anisotropic coefficient matrix $\mu$ the generic properties of the elliptic operator differ dramatically from the case of a scalar coefficient function, see [10, Remark 5.1], [11, §4], and [37, Ch. 5].

As the referee of [16] suggested, we solve the problem by a reflection argument. This aim in mind, one has to flatten a part of the boundary in a way that this piece then becomes part of a plane. In order to preserve the cellular structure of the constancy domains for the transformed coefficient function, one must, additionally, take a piecewise linear homeomorphism for the transformation. That this can be achieved is not hard to see for convex vertices. But, since our aim was the treatment of general vertices, nontrivial – but classical – instruments from geometric topology in dimensions 2 and 3 are required. What comes out as a by-product, is the affirmative answer (Theorem 4.15) to the problem: "Does every TOP manifold have a LIP structure?" in case of 3-dimensional polyhedra, see [28, Ch. 9].

We do not know whether instruments from modern geometric topology allow to carry over this result to the case of higher dimensions – but this is beyond the scope of this work.

# 2 Notation

Throughout the text we will employ the following notation. By $\mathcal{C} := \,]-1,1[^3$ we denote the open cube in $\mathbb{R}^3$, centered at 0, while $\mathcal{C}_\pm := \mathcal{C} \cap \{\mathrm{x} = (x,y,z) \in \mathbb{R}^3 : x, z \in \mathbb{R}, y \gtrless 0\}$ and $\Sigma := \mathcal{C} \cap \{\mathrm{x} = (x,0,z) : x,z \in \mathbb{R}\}$.

By a *bi-Lipschitz map* between two metric spaces we mean a Lipschitz continuous, bijective map whose inverse is again Lipschitz continuous.

Following [14], we denote by $\Omega \subseteq \mathbb{R}^3$ a bounded Lipschitz domain in the sense of the following definition:

**Definition 2.1.** A bounded domain $\Omega \subset \mathbb{R}^3$ is a *Lipschitz domain* if for every $\mathrm{x} \in \partial\Omega$ there is an open neighbourhood $\mathcal{U}_\mathrm{x}$ in $\mathbb{R}^3$ and a bi-Lipschitz mapping $\phi_\mathrm{x} : \mathcal{U}_\mathrm{x} \to \mathbb{R}^3$, such that $\phi_\mathrm{x}(\mathcal{U}_\mathrm{x} \cap \Omega) = \mathcal{C}_+$, $\phi_\mathrm{x}(\mathcal{U}_\mathrm{x} \cap \partial\Omega) = \Sigma$, and $\phi_\mathrm{x}(\mathrm{x}) = 0 \in \mathbb{R}^3$.

**Remark 2.2.** If $\Omega$ is a Lipschitz domain, $\mathcal{U}$ an open neighbourhood of $\overline{\Omega}$ and $\phi : \mathcal{U} \to \mathbb{R}^3$ is bi-Lipschitz, then $\phi(\Omega)$ is again a Lipschitz domain.

In all what follows, $\Gamma \subseteq \partial\Omega$ is always a relatively open part of the boundary $\partial\Omega$.

$W^{1,p}(\Omega)$ denotes the (complex) *Sobolev space* on $\Omega$, consisting of the $L^p(\Omega)$ functions whose first order distributional derivatives also belong to $L^p(\Omega)$ (see [12] or [29]). This $\Omega$ enjoys the extension property for $W^{1,p}(\Omega)$ in view of being a bounded Lipschitz domain, see [12, Thm. 7.25] or [13, Thm. 3.10]. Thus, $W^{1,p}(\Omega)$ is identical to the completion of the set $\{v|_\Omega : v \in C^\infty(\mathbb{R}^3)\}$ with respect to the norm $\|v\|_{W^{1,p}} := \left(\int_\Omega |\nabla v|^p + |v|^p \,\mathrm{dx}\right)^{1/p}$. We use the symbol $W^{1,p}_\Gamma(\Omega)$ for the closure of

$$\left\{v|_\Omega : v \in C^\infty(\mathbb{R}^3),\ \mathrm{supp}(v) \cap (\partial\Omega\backslash\Gamma) = \emptyset\right\}$$

in $W^{1,p}(\Omega)$. If $\Gamma = \emptyset$ we write as usual $W^{1,p}_0(\Omega)$ instead of $W^{1,p}_\emptyset(\Omega)$. $W^{-1,p'}_\Gamma(\Omega)$ denotes the space of continuous antilinear forms on $W^{1,p}_\Gamma(\Omega)$ and $W^{-1,p'}(\Omega)$ denotes the space of continuous antilinear forms on $W^{1,p}_0(\Omega)$, when $\frac{1}{p} + \frac{1}{p'} = 1$ holds.

The expression $\langle \cdot, \cdot \rangle_X$ always indicates the pairing between a Banach space $X$ and its (anti-)dual; in case of $X = \mathbb{C}^d$ we mostly write $\langle \cdot, \cdot \rangle$. If $\omega$ is a Lebesgue measurable, essentially bounded function on $\Omega$ taking its values in the set of $d \times d$ matrices, then we define $-\nabla \cdot \omega\nabla : W^{1,2}_\Gamma(\Omega) \to W^{-1,2}_\Gamma(\Omega)$ by

$$\langle -\nabla \cdot \omega\nabla v, w\rangle_{W^{1,2}_\Gamma} := \int_\Omega \omega\nabla v \cdot \nabla\overline{w} \,\mathrm{dx}\ , \quad v, w \in W^{1,2}_\Gamma(\Omega).$$

The maximal restriction of $-\nabla \cdot \omega\nabla$ to any of the spaces $W^{-1,p}_\Gamma(\Omega)$, $p > 2$, we will denote by the same symbol.

**Remark 2.3.** In the context of divergence operators on complex spaces it is necessary to work with the anti-dual spaces (instead of the dual ones) in order to obtain on the complex Hilbert space $L^2$ (equipped with the canonic scalar product) the usual elliptic, self-adjoint operators, compare [4] or [35]. Clearly the antilinear forms and linear forms correspond to each other by the isomorphism $\langle \mathcal{I}f, \psi\rangle := \langle f, \overline{\psi}\rangle$ and in this spirit have the same functional analytic quality.

# 3 Strategy of proof

Since the proof of Theorem 6.1 is rather technical, we give here an exposition for the convenience of the reader, which at the same time serves as an outline of the rest of the paper.

The principal strategy is to first deform the problem in a small neighborhood of the vertex by a piecewise linear homeomorphism, such that a (small) boundary part at the vertex becomes part of a plane. This is obtained by means from classical geometric topology. Since we could not find this result of piecewise linear (PL) local flattening of the boundary in the literature, we give a comprehensive proof in Section 4; see Theorem 4.15. The crucial point is that for our purposes the transformation has to be piecewise linear in order to maintain the cellular structure of the constancy domains of the coefficient function.

After the application of a suitable cut-off function, one is then confronted with a mixed boundary value problem on the half cube $\mathcal{C}_+$ whose Neumann boundary part is $\Sigma$, the mid plane of the cube $\mathcal{C}$. This allows to reflect the problem symmetrically across $\Sigma$. One ends up with a Dirichlet problem on the cube $\mathcal{C}$. If the resulting Dirichlet problem provides a topological isomorphism between the corresponding spaces $W_0^{1,p}$ and $W^{-1,p}$, then this also the case for the corresponding spaces associated with the mixed boundary value problem, cf. Proposition 4.20. Thus, it remains to treat the Dirichlet problem. Here, the essential advantage is that one can make use of a deep idea of Maz'ya [30] that allows to avoid the complicated discussion of vertex singularities and, hence, to restrict the investigation to the edge singularities as far as the integrability of the gradient of the solution up to an index $p > 3$ is concerned; see Proposition 5.6 below.

Thus, one has to identify the resulting edges of the Dirichlet problem on $\mathcal{C}$ and to show that the kernel of an associated generalized Sturm-Liouville operator is trivial for all $\lambda \in \mathbb{C}$ with $0 < \Re(\lambda) \leq 1/3$. For some of the edges this follows directly from the construction of the geometric setting, while for the interior edges in the cube and those from $\Sigma$ one proceeds as follows: assuming that the Sturm-Liouville operator admits a nontrivial kernel even in case of $\Re\lambda \in \,]0, 1/3]$ one constructs first a corresponding function on $\mathbb{R}^2$ and afterwards by extension and cut off a function on $\mathbb{R}^3$ which fulfills an elliptic equation in a neighbourhood of this edge, where the right hand side is from $W^{-1,6}$, but the solution does not belong to $W^{1,3}$, see Lemma 5.12. Revoking the transformation of the problem, one ends up with an elliptic equation on a prismatic domain whose boundary includes the pre-image of the edge and where the coefficient constellation around this edge is the original one. Due to Proposition 4.1 and the cut-off, the resulting right hand side is of type $W^{-1,6}$ and the gradient of the solution lacks integrability up to index 3. But for all occurring prismatic constellations we already know that this must be false.

The last prismatic model problem is a rather interesting one. It possesses one opening angle between the boundary plate and a material interface of exactly $\pi$, so it is, for instance, the local model problem of a buried quantum well structure in an edge emitting semiconductor qw–laser, see [3]. An alternative proof, resting on new and nontrivial insights into singularities at edges where three different materials meet, is given in [18].

# 4 Transformation of the problem

All our transformation techniques heavily rely on the fact that the regularity of solutions is not altered by bi-Lipschitz transformations. Thus we first quote from [16] the essential lemma that allows to transform elliptic divergence operators under bi-Lipschitz mappings maintaining optimal regularity.

**Proposition 4.1** ([16, Prop. 16]). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\Gamma$ be an open subset of its boundary. Assume that $\phi$ is a mapping from a neighbourhood of $\overline{\Omega}$ into $\mathbb{R}^d$, which is bi-Lipschitz and denote $\phi(\Omega) = \Omega_\star$ and $\phi(\Gamma) = \Gamma_\star$. Then the following assertions are true.*

1. *For any $p \in ]1,\infty[$ the mapping $\phi$ induces a linear, topological isomorphism*

$$\Psi_p : W^{1,p}_{\Gamma_\star}(\Omega_\star) \to W^{1,p}_{\Gamma}(\Omega)$$

*that is given by $(\Psi_p f)(x) = f(\phi(x)) = (f \circ \phi)(x)$.*

2. *$\Psi^*_{p'}$ is a linear, topological isomorphism between $W^{-1,p}_{\Gamma}(\Omega)$ and $W^{-1,p}_{\Gamma_\star}(\Omega_\star)$.*
3. *If $\omega$ is a bounded measurable function on $\Omega$, taking its values in the set of $d \times d$ matrices, then*

$$\Psi^*_{p'} \nabla \cdot \omega \nabla \Psi_p = \nabla \cdot \underline{\omega} \nabla$$

*with*

$$\underline{\omega}(\mathrm{y}) = (D\phi)(\phi^{-1}(\mathrm{y}))\omega(\phi^{-1}(\mathrm{y}))\big(D\phi\big)^T(\phi^{-1}(\mathrm{y}))\frac{1}{\big|\det(D\phi)(\phi^{-1}\mathrm{y})\big|},$$

*where $D\phi$ denotes the Jacobian of $\phi$ and $\det(D\phi)$ the corresponding determinant. Furthermore, if $-\nabla \cdot \omega \nabla : W^{1,p}_{\Gamma}(\Omega) \to W^{-1,p}_{\Gamma}(\Omega)$ is a topological isomorphism, then $-\nabla \cdot \underline{\omega} \nabla : W^{1,p}_{\Gamma_\star}(\Omega_\star) \to W^{-1,p}_{\Gamma_\star}(\Omega_\star)$ also is (and vice versa).*

Later we will simply write $\Psi$ and $\Psi^*$ instead of $\Psi_p$ and $\Psi^*_{p'}$. This causes no difficulties, since the families $\Psi_p$ and $\Psi^*_p$, $1 < p < \infty$, are consistent for different values of $p$.

## 4.1 Local flattening of the boundary by piecewise linear maps

In this section we will prove – under very general conditions – that the boundary of a polyhedron in $\mathbb{R}^3$ may be locally flattened around any boundary point by means of a piecewise linear homeomorphism. This means in particular that if the polyhedron is subdivided into cells, on each of which the coefficient function of an elliptic operator is constant, then one can find a mapping which locally flattens the boundary and, additionally, does not destroy this configuration.

In order to do so, we will need some notions and results from geometric topology, which we will introduce briefly. All this material is principally in the spirit of the books by Moise [34] and Alexandroff & Hopf [1], see also Bing [5].

### 4.1.1 Some notions and results from geometric topology

If $v_0, \ldots, v_m \in \mathbb{R}^3$, and the convex hull of these points contains a $d$-dimensional ball and no $(d+1)$-dimensional ball, then this convex hull is called the *$d$-cell* generated by $v_0, \ldots, v_m$. The sides, edges and vertices from the boundary of the $d$-cell are called *faces*. If $m \le 3$ and the points $v_0, \ldots, v_m \in \mathbb{R}^3$ lie in general position, then we call the cell a simplex. Simplices in $\mathbb{R}^3$ are either tetrahedra, triangles, edges, or vertices.

A *euclidean complex* $K$ is a locally finite collection of cells in $\mathbb{R}^3$ such that $K$ contains all faces of all elements of $K$ and if $\sigma$ and $\tau$ are two cells in $K$ with $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ must be a face both of $\sigma$ and $\tau$ (see [1, Ch. III]). We call a complex a *simplicial complex*, if all cells involved are simplices. For a complex $K$ in $\mathbb{R}^3$ we denote by $|K| := \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^3$ the *polyhedron* given by the complex $K$. If v is a vertex of the euclidean complex $K$, then the set of all cells from $K$ which contain v, together with all their faces, form the *star* $K_{\mathrm{v}}^{\bigstar}$ of v in $K$. All those faces in the star that do not have v as a vertex form the *link* of v. The *open star* is the set-theoretic difference of star minus link. (The open star is an open set in $|K|$ and not carried by a subcomplex.)

If $K$ is a finite simplicial complex such that $|K|$ is additionally an $m$-dimensional topological manifold (with boundary), then the complex $K$ is a *triangulated $m$-manifold (with boundary)* and the topological space $|K|$ a *polyhedral $m$-manifold (with boundary).* We denote by $K_\partial$ its *boundary complex*, which is generated by all the $(m-1)$-cells of the complex that lie in exactly one $m$-cell. All the manifolds $M$ considered in the following are compact.

If $K$ and $K'$ are euclidean complexes in $\mathbb{R}^d$ with $|K| = |K'|$ and every $\sigma \in K'$ is contained in some element from $K$, then $K'$ is called a *subdivision* of $K$. Below we will repeatedly need the following important relation between euclidean complexes and simplicial complexes:

**Proposition 4.2** ([1, Ch. III.2])**.** *Every euclidean complex $K$ admits a simplicial subdivision $K'$ without new vertices. Moreover, $K'$ can be constructed such that all edges of $K'$ that come to lie in the open star of one specified vertex* v $\in K$ *have* v *as an endpoint.*

Let $K$ be a complex in $\mathbb{R}^d$. A continuous mapping $f$ from $|K|$ onto a subset of $\mathbb{R}^m$ is *piecewise linear* if there is a subdivision $K'$ of $K$ such that the restricted function $f|_\sigma$ is linear for every $\sigma \in K'$.

**Remark 4.3.** By Proposition 4.2 we may always assume that the cells on which a piecewise linear mapping is linear in fact are simplices.

If $K$ is a finite complex and $f$ is injective, then on $f(|K|)$ one has the structure of a complex, induced from $K$ by $f$. In the case of simplicial complexes this definition coincides with that in [34]; compare also [5, Ch. II]. See [34, Chap. 5] also for basics about *piecewise linear homeomorphisms.*

Finally, we need different notions of "boundary" for manifolds and complexes. Let $M$ be an $m$-dimensional topological manifold in $\mathbb{R}^3$, with or without boundary. Then all points in $M$ having an open neighbourhood in $M$ that is homeomorphic to $\mathbb{R}^m$ form the *interior* $\mathrm{Int}(M)$ of $M$ and the rest $\mathrm{Bd}(M) := M\backslash\mathrm{Int}(M)$ is the *manifold-theoretic boundary* of $M$.

The expression $\partial A$ stands for the *topological frontier* of a set $A \subset \mathbb{R}^d$, i.e. $\partial A = \overline{A} \cap \overline{\mathbb{R}^d\backslash A}$, where the closure has to be taken in $\mathbb{R}^d$.

**Remark 4.4.** Let $K$ be a triangulated 3-manifold with boundary. Assume that $\mu$ is a coefficient function on $|K|$ that is constant on the interiors of all 3-cells from $K$, and that $\phi : |K| \to \mathbb{R}^3$ is a piecewise linear mapping which establishes a homeomorphism from $|K|$ onto its image. Then the resulting coefficient function on $\phi(|K|)$ (see Proposition 4.1) is constant on the interiors of 3-cells whose pre-images are contained in the 3-cells of $K$.

In the sequel, we will exploit the following results:

**Proposition 4.5** ([5, Thm. I.2.A])**.** *If $K$ is a finite simplicial complex in $\mathbb{R}^3$, then there is a triangulation $K_{\mathbb{R}^3}$ of $\mathbb{R}^3$ which contains $K$ as a subcomplex.*

**Proposition 4.6** ([28, Thm. 2.18] [38, p. 504]). *Let $K$ be a finite euclidean complex in $\mathbb{R}^d$ and let $\phi : |K| \to \mathbb{R}^m$ be piecewise linear and continuous. Then $\phi$ is Lipschitz continuous.*

Furthermore, we need the following results, all taken from Moise [34].

**Proposition 4.7** ([34, Thm. 17.1]). *Let $M$ be a 3-manifold with boundary in $\mathbb{R}^3$. If $M$ is a closed subset of $\mathbb{R}^3$, then* $\mathrm{Bd}(M)$ *equals the topological frontier $\partial M$.*

**Proposition 4.8** ([34, Thm. 23.3]). *If $M$ is a 3-manifold with boundary, then* $\mathrm{Bd}(M)$ *is a 2-manifold.*

**Proposition 4.9** ([34, Thm. 23.7]). *Let $K$ be a complex whose polyhedron $|K|$ is a 3-manifold with boundary. Then* $\mathrm{Bd}(|K|)$ *is identical with the polyhedron whose underlying complex is the boundary complex $K_\partial$ of $K$.*

**Proposition 4.10** ([34, Thm. 4.8]). *Let $K$ be a complex such that $M = |K|$ is a 2-manifold. Then, for every vertex* $\mathrm{v}$ *the star $K^\star_{\mathrm{v}}$ of* $\mathrm{v}$ *within $K$ may be mapped by a PL homeomorphism onto a 2-cell.*

**Proposition 4.11** (The 2D Schoenflies Theorem [34, Thm 10.2]). *Let $J$ be a topological 1-sphere in a topological 2-sphere $S^2$. Then $S^2$ is the union of two topological discs with $J$ as their common frontier in $S^2$.*

**Proposition 4.12** (The 3D PL Schoenflies Theorem [34, Thm 17.12] [5, Thm. XIV.I]). *Let $S$ be a polyhedron in $\mathbb{R}^3$, whose boundary is topologically a 2-sphere, and let $\mathcal{W}$ be an open convex set that contains $S$. Then there is a piecewise linear homeomorphism*

$$\phi_S : \mathbb{R}^3 \to \mathbb{R}^3, \quad S \to \partial\sigma^3,$$

*where $\sigma^3$ is a tetrahedron, such that $\phi_S|_{\mathbb{R}^3\setminus\mathcal{W}}$ is the identity.*

The interior of $\partial\sigma^3 \subset \mathbb{R}^3$ is contractible, the exterior is not — thus the homeomorphism $\phi_S$ of Theorem 4.12 maps the interior of $S$ to the interior of $\sigma^3$.

**Remark 4.13.** In the following, we will apply the 3D PL Schoenflies Theorem to get a homeomorphism not to the boundary of a tetrahedron, but instead to the boundary of the brick $\overline{\mathcal{C}}_+ = [-1, 1] \times [0, 1] \times [-1, 1] = \{(x, y, z) \in [-1, 1]^3 : y \geq 0\}$. Moreover, we need that a specified point of $\mathrm{a} \in S$ is mapped to the point $0 \in \partial\mathcal{C}_+$. Both these properties may be achieved by constructing a piecewise linear homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ that maps $\sigma^3$ onto $\overline{\mathcal{C}}_+$ and carries a given point $\mathrm{a} \in \partial\sigma^3$ to 0. This is easily achieved by explicit geometric construction, where we distinguish the three cases when a lies in the relative interior of a triangle face, lies on an edge, or is a vertex of $\sigma^3$.

Next, we prove the key lemma needed for the flattening theorem in the next subsection.

**Lemma 4.14.** *Let $K$ be a triangulated 3-manifold with boundary in $\mathbb{R}^3$ and* $\mathrm{v}$ *be a vertex in the boundary of $|K|$. Then the polyhedron $|K^\star_{\mathrm{v}}|$ of the star of* $\mathrm{v}$ *in $K$ is homeomorphic to the closed unit ball in $\mathbb{R}^3$. In particular, the boundary of $K^\star_{\mathrm{v}}$ is a polyhedral 2-sphere.*

*Proof.* By Propositions 4.7 and 4.8, the boundary complex $K_\partial$ of $K$ is a 2-manifold, and hence the star of the vertex $\mathrm{v} \in K_\partial$, denoted $K^\bigstar_{\partial,\mathrm{v}}$, is a disk (see Proposition 4.10). Consider a sufficiently small ball $B$ centered at v, small enough so that it intersects only those faces of $K$ that lie in $K^\bigstar_{\mathrm{v}}$, i.e., that contain v. One first observes that $K^\bigstar_{\partial,\mathrm{v}} \cap B$ is homeomorphic to $K^\bigstar_{\partial,\mathrm{v}}$. Hence, the intersection of the disk $K^\bigstar_{\partial,\mathrm{v}}$ with the boundary of $B$ yields a Jordan curve in the 2-sphere $\partial B$, which by the 2D Schoenflies theorem, Prop. 4.11, divides the sphere into two disks, one of them being $\partial B \cap K^\bigstar_{\mathrm{v}}$. Thus radial projection from v shows that the star $K^\bigstar_{\mathrm{v}}$ is a cone (with apex v) over a disk, hence it is a ball; its boundary, which is the union of two discs, namely the link of v in the boundary of $K$, $K^\bigstar_{\partial,\mathrm{v}}$ and the link of v in $K$, is a 2-sphere. ◻

### 4.1.2 The PL flattening theorem

The aim of this section is to establish the following result.

**Theorem 4.15.** *Let $\Pi \subseteq \mathbb{R}^3$ be a domain such that $\overline{\Pi}$ is a polyhedral 3-manifold with boundary with $\Pi = \mathrm{Int}(\overline{\Pi})$. Then $\Pi$ is a Lipschitz domain and the local bi-Lipschitz charts around the boundary points may be taken as piecewise linear homeomorphisms, as follows.*

*Let $K_\Pi$ be a euclidean complex with $|K_\Pi| = \overline{\Pi}$ and extend this to a triangulation $K_{\mathbb{R}^3} \supset K_\Pi$ of $\mathbb{R}^3$. Then for every $\mathrm{a} \in \partial\Pi$ there is a piecewise linear map $\varphi_\mathrm{a}$ of the cube $\overline{\mathcal{C}} = [-1,1]^3 \subset \mathbb{R}^3$ onto a closed neighborhood $\mathcal{W}_\mathrm{a}$ of a in $\mathbb{R}^3$ with the following properties:*

(1) $0$ *maps to* a,

(2) $\Sigma = ]-1,1[ \times \{0\} \times ]-1,1[$ *maps to the boundary $\partial\Pi$ of the domain $\Pi$,*

(3) $\mathcal{C}_+ = ]-1,1[ \times ]0,1[ \times ]-1,1[$ *maps to the domain $\Pi$,*

(4) $\mathcal{C}_- = ]-1,1[ \times ]-1,0[ \times ]-1,1[$ *maps to the exterior $\mathbb{R}^3 \backslash \overline{\Pi}$ of the domain $\Pi$.*

*Moreover, the complex $L_{\overline{\mathcal{C}}}$ which supports the mapping $\varphi_\mathrm{a}$ has the following properties:*

(5) *For every $A \in L_{\overline{\mathcal{C}}}$, $\varphi_\mathrm{a}(A)$ is a subset of an element from the star of* a *in $K_{\mathbb{R}^3}$. In particlular, if $A \in L_{\overline{\mathcal{C}}}$ is a subset of $\overline{\mathcal{C}_+}$, then $\varphi_\mathrm{a}(A)$ is a subset of an element from the star of* a *in $K_\Pi$.*

(6) *Every 3-cell of $L_{\overline{\mathcal{C}}}$ has one of its vertices in $0$.*

(7) *Every edge from $L_{\overline{\mathcal{C}}}$ which intersects $\mathcal{C}$ connects $0$ with a vertex on $\partial\mathcal{C}$.*

**Remark 4.16.** The crucial condition is that $\overline{\Pi}$ has to be a manifold with boundary. This excludes e.g. a pyramid minus another pyramid in its interior with the same apex. Likewise, the union of a plate and a tetrahedron sharing one vertex or one edge is also not admitted.

*Proof.* All the claims of this theorem are easy to establish if a is a relative interior point of a boundary triangle of $K_\Pi$, and quite easy to prove in the case where a lies on an edge of $K_\Pi$, since then a projection parallel to the edge reduces the situation to a 2-dimensional problem. The hard case is if a is a vertex of $K_\Pi$.

Lemma 4.14 in conjunction with Proposition 4.12 – adapted according to Remark 4.13 – provides a piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ that carries the star $K_{\mathrm{a}}^{\bigstar}$ onto the brick $\overline{\mathcal{C}_+} = [-1,1] \times [0,1] \times [-1,1]$, while mapping a to the boundary point $0 \in \partial\mathcal{C}_+$. Let $L_{\mathrm{ref}}$ be a simultaneous refinement of both the complexes $K_{\mathbb{R}^3}$ and a triangulation of $\mathbb{R}^3$ that supports the piecewise linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ (with a subcomplex triangulating $K_{\mathrm{a}}^{\bigstar}$). Modulo another refinement (which does not reduce the polyhedron of the star of a) we may assume that the 3-cells of $L_{\mathrm{ref}}$ are tetrahedra; we perform this refinement, but maintain the notation $L_{\mathrm{ref}}$. Let $L_{\mathrm{a}}^{\bigstar}$ denote the star of a in $L_{\mathrm{ref}}$. Now we consider the complex $h(L_{\mathrm{a}}^{\bigstar})$. The polyhedron $|h(L_{\mathrm{a}}^{\bigstar})|$ is a neighbourhood of 0. Obviously, all cells of $h(L_{\mathrm{a}}^{\bigstar})$ are again tetrahedra – this time with one vertex in 0. Moreover, all edges which intersect the interior of the polyhedron $|h(L_{\mathrm{a}}^{\bigstar})|$ originate in 0. Let $\varepsilon\overline{\mathcal{C}}$ be a cube that is situated in the interior of $|h(L_{\mathrm{a}}^{\bigstar})|$. Thus, the intersection with $|h(L_{\mathrm{a}}^{\bigstar})|$ induces on $\varepsilon\overline{\mathcal{C}}$ the structure of a complex the cells of which have one vertex in 0. Furthermore, the mapping $\varphi_{\mathrm{a}} := h^{-1}|_{\varepsilon\overline{\mathcal{C}}}$ is linear on each of these cells. Thus, finally, rescaling $\mathrm{x} \mapsto \varepsilon\mathrm{x}$ and defining $\mathcal{W}_{\mathrm{a}} := \varphi_{\mathrm{a}}(\overline{\mathcal{C}})$, one obtains the asserted neighbourhood. The stated bi-Lipschitz property of $\varphi_{\mathrm{a}}$ follows from Proposition 4.6. □

## 4.2 Localization and reflection

Having in mind our aim to show optimal elliptic regularity in a neighbourhood of a vertex a of the polyhedral domain $\Pi$, the next task will be to define a suitable neighbourhood $\mathcal{U}$ of the point a under consideration, where we investigate the regularity of the solution.

**Definition 4.17.** From now on we always suppose that a is a vertex of $\Pi$ and that $\varphi_{\mathrm{a}}$ is chosen as above. We define $\mathcal{U} := \varphi_{\mathrm{a}}(\mathcal{C})$. Moreover, we define the euclidean complex $K_{\overline{\mathcal{C}}}^{+}$ as the complex $L_{\overline{\mathcal{C}}} \cap \overline{\mathcal{C}_+}$.

Now, let a cut-off function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\eta) \subseteq \mathcal{U}$ and a right hand side $f \in (W^{1,p'}(\Pi))'$ for some $p \in \,]3,6]$ be given. The following lemma establishes an equation for the truncated function $\eta v$. A proof of this is given in [17, Lemma 4.7].

**Lemma 4.18** ([17, Lemma 4.7])**.** *Let $p \in \,]3,6]$ and assume that $\mathcal{V} \subset \mathbb{R}^3$ is open such that $\Pi_\bullet := \Pi \cap \mathcal{V}$ is again a Lipschitz domain. Define $\Gamma := \partial\Pi \cap \mathcal{V}$ and let $v \in W^{1,2}(\Pi)$ be the solution to*

$$-\nabla \cdot \mu\nabla v + v = f \in \big(W^{1,p'}(\Pi)\big)'. \tag{4.1}$$

*Then $\eta v \in W^{1,2}_\Gamma(\Pi_\bullet)$, and it holds*

$$-\nabla \cdot \mu|_{\Pi_\bullet}\nabla(\eta v) = g \tag{4.2}$$

*for some $g \in W^{-1,p}_\Gamma(\Pi_\bullet)$, provided that $-\nabla \cdot \mu|_{\Pi_\bullet}\nabla : W^{1,2}_\Gamma(\Pi_\bullet) \to W^{-1,2}_\Gamma(\Pi_\bullet)$ is defined by*

$$\langle -\nabla \cdot \mu|_{\Pi_\bullet}\nabla\psi, w\rangle_{W^{1,2}_\Gamma(\Pi_\bullet)} := \int_{\Pi_\bullet} \mu|_{\Pi_\bullet}\nabla\psi \cdot \nabla\overline{w}\,\mathrm{dx}\,, \quad \psi, w \in W^{1,2}_\Gamma(\Pi_\bullet).$$

This Lemma states that, given $f \in (W^{1,p'}(\Pi))'$ and the variational solution $v$ of (4.1), the function $\eta v$ is the variational solution of problem (4.2) with mixed boundary conditions (in the sense that $g \in L^2(\Pi_\bullet)$ implies Neumann condition on $\Gamma$ and Dirichlet condition on $\partial\Pi_\bullet \backslash \Gamma$). So, we may rephrase our original regularity problem about the behaviour in the neighbourhood of a Neumann edge (given $f$, how good is $v$ near the point a) into a question about optimal regularity for a mixed boundary value problem (given $g$, how good is the solution to (4.2)).

In a next step, we transform our problem by the piecewise linear homeomorphism $\phi = \varphi_{\mathrm{a}}$ constructed before. According to Proposition 4.1, Equation (4.2) then transforms into

$$-\nabla \cdot \underline{\mu} \nabla u = \Psi^* g =: h \in W_\Sigma^{-1,p}(\mathcal{C}_+) \tag{4.3}$$

with $u := \Psi^{-1}(\eta v)$ and we additionally know by this proposition that the regularity behaviour of (4.2) and (4.3) is the same. So, our final aim will be to prove

**Lemma 4.19.**

$$-\nabla \cdot \underline{\mu} \nabla : W_\Sigma^{1,p}(\mathcal{C}_+) \to W_\Sigma^{-1,p}(\mathcal{C}_+)$$

*is a topological isomorphism for some $p > 3$.*

This is again a problem with mixed boundary conditions, but now the Neumann boundary part $\Sigma$ is planar. Thus, we may apply a reflection argument across the $x$-$z$-plane. After this we end up with a Dirichlet problem on the cube $\mathcal{C}$, which will be treated afterwards. The precise reflection argument is contained in the next

**Proposition 4.20** ([16, Proposition 17])**.** *Let $\Omega \subseteq \{(x,y,z) : y > 0\}$ be a bounded, polyhedral Lipschitz domain and $\Gamma$ be an open subset of $\partial\Omega$ with $\overline{\Omega} \cap \{(x,0,z) : x, z \in \mathbb{R}\} = \overline{\Gamma}$. Let for any $\mathrm{x} = (x,y,z)$ the symbol $\mathrm{x}_-$ denote the element $(x,-y,z)$ and define $\hat{\Omega}$ as the interior of*

$$\Omega \cup \{\mathrm{x} : \mathrm{x}_- \in \Omega\} \cup \overline{\Gamma}.$$

*Furthermore, for a bounded, measurable function $\omega$ on $\Omega$, taking its values in the set of $3 \times 3$ matrices, we define*

$$\hat{\omega}(\mathrm{x}) := \begin{cases} \omega(\mathrm{x}), & \text{if } \mathrm{x} \in \Omega, \\ \begin{pmatrix} \omega_{11}(\mathrm{x}_-) & -\omega_{12}(\mathrm{x}_-) & \omega_{13}(\mathrm{x}_-) \\ -\omega_{12}(\mathrm{x}_-) & \omega_{22}(\mathrm{x}_-) & -\omega_{23}(\mathrm{x}_-) \\ \omega_{13}(\mathrm{x}_-) & -\omega_{23}(\mathrm{x}_-) & \omega_{33}(\mathrm{x}_-) \end{pmatrix}, & \text{if } \mathrm{x}_- \in \Omega. \end{cases}$$

*If $-\nabla \cdot \hat{\omega} \nabla : W_0^{1,p}(\hat{\Omega}) \to W^{-1,p}(\hat{\Omega})$ is a topological isomorphism, then the same is true for $-\nabla \cdot \omega \nabla : W_\Gamma^{1,p}(\Omega) \to W_\Gamma^{-1,p}(\Omega)$.*

Thus, in order to prove Lemma 4.19, it suffices to show

**Lemma 4.21.**

$$-\nabla \cdot \hat{\mu} \nabla : W_0^{1,p}(\mathcal{C}) \to W^{-1,p}(\mathcal{C}) \tag{4.4}$$

*is a topological isomorphism for some $p > 3$.*

Obviously, (4.4) is a Dirichlet problem on a polyhedral domain. In order to identify a triangulation of $\overline{\mathcal{C}}$, such that the coefficient function $\hat{\mu}$ is constant on its 3-cells, we introduce the following

**Definition 4.22.** We define the complex $K_{\overline{\mathcal{C}}}$ as the union of all cells from the complex $K_{\overline{\mathcal{C}}}^{+}$ together with all cells which are images of cells from $K_{\overline{\mathcal{C}}}^{+}$ under the mapping $(x, y, z) \mapsto (x, -y, z)$.

**Remark 4.23.** The triangulation $K_{\overline{\mathcal{C}}}$ of $\overline{\mathcal{C}}$ is different from $L_{\overline{\mathcal{C}}}$ in general, but it coincides with $L_{\overline{\mathcal{C}}}$ on $\overline{\mathcal{C}_+}$. Furthermore, the triangulation $K_{\overline{\mathcal{C}}}$ is of such kind that the coefficient function $\hat{\mu}$ is constant on the interior of all its 3-cells. Thus, this complex determines the edges which are then the relevant ones for the Dirichlet problem in Lemma 4.21. For this it is essential that the whole coefficient configuration on $\mathcal{C}$ relates to the original coefficient configuration on $\Pi$ in a way that lateron will allow us to restrict the elliptic edge singularities via the invested assumptions on the coefficient function $\mu$ on $\Pi$.

As announced above, for the proof of Lemma 4.21 it suffices to delimitate the edge singularities. We present the details in the next section.

# 5 Edge singularities

In this section, after some preparations, we recall the optimal regularity result from [30] for heterogeneous Dirichlet problems on polyhedral domains and explain how to identify the occurring edge singularities. We first introduce some notions and notation corresponding to our geometric situation of a polyhedral domain $\Pi$ and the piecewise constant coefficient function $\mu$.

**Definition 5.1.** For $\iota, \vartheta \in \,]-\pi, \pi]$ with $\iota < \vartheta$ we define the *sector*

$$S_\iota^\vartheta := \{(r\cos\theta, r\sin\theta) : \; r > 0, \theta \in \,]\iota, \vartheta[\}.$$

**Definition 5.2.** Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain, such that $\overline{\Omega}$ is a polyhedral 3-manifold with boundary, associated to the finite complex $K$. If $\{\sigma_k\}_k$ is the collection of all 3-cells from $K$, then let $\mu$ be a matrix function on $\Omega$ which is constant on the interior of each cell $\sigma_k$ and takes real, symmetric, positive definite $3 \times 3$ matrices as values. Take an edge $E$ of any of the $\sigma_k$'s and consider an arbitrary point $P$ of this edge that is not an endpoint of it.

Choose a new orthogonal coordinate system $(x, y, z)$ with origin at the point $P$ such that the direction of $E$ coincides with the $z$-axis. We denote by $\mathcal{O}_E$ the corresponding orthogonal transformation matrix and by $\mu_{E,P}$ the piecewise constant matrix function, which coincides in a neighbourhood of $P$ with $\mathcal{O}_E \mu(\mathcal{O}_E^{-1}(x + P))\mathcal{O}_E^{-1}$ and which satisfies

$$\mu_{E,P}(tx, ty, z) = \mu_{E,P}(x, y, 0), \quad \text{for all } (x, y, z) \in \mathbb{R}^3, \; t > 0.$$

By $\mu_E(\cdot, \cdot)$ we denote the upper left $2 \times 2$ block of $\mu_{E,P}(\cdot, \cdot, 0)$.

**Remark 5.3.** It is essential that – as the above notation suggests – the coefficient function $\mu_E$ is the same for every point $P$ from the (relative) interior of the edge $E$.

There exist angles $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$, such that $\mu_E$ is constant on each of the sectors $S_{\theta_j}^{\theta_{j+1}}$ and takes real, symmetric, positive definite matrices as values. Note that $\theta_n = \theta_0 + 2\pi$ if $\mu_E$ corresponds to an interior edge $E$, otherwise $\mu_E$ is given on an infinite sector $S_{\theta_0}^{\theta_n}$ which coincides near $P$ with the intersection of the transformed $\Omega$ with the $x$-$y$-plane.

In order to cite the optimal regularity result from [30], we now introduce the Sturm-Liouville operator associated to an edge and to the coefficient function $\mu$.

**Definition 5.4.** Let numbers $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$ be given and, additionally, real, symmetric, positive definite $2 \times 2$ matrices $\rho^1, \ldots, \rho^n$, which are asscociated to the sectors $S_{\theta_0}^{\theta_1}, \ldots, S_{\theta_{n-1}}^{\theta_n}$. We introduce on $]\theta_0, \theta_n[ \setminus \{\theta_1, \ldots, \theta_{n-1}\}$ coefficient functions $b_0$, $b_1$, $b_2$, whose restrictions to the interval $]\theta_j, \theta_{j+1}[$, $j = 0, \ldots, n-1$, are given by

$$\begin{aligned}
b_0(\theta) &= \rho_{11}^j \cos^2\theta + 2\rho_{12}^j \sin\theta\cos\theta + \rho_{22}^j \sin^2\theta\,, \\
b_1(\theta) &= (\rho_{22}^j - \rho_{11}^j)\sin\theta\cos\theta + \rho_{12}^j(\cos^2\theta - \sin^2\theta)\,, \\
b_2(\theta) &= \rho_{11}^j \sin^2\theta - 2\rho_{12}^j \sin\theta\cos\theta + \rho_{22}^j \cos^2\theta\,.
\end{aligned}$$

If $\theta_n \neq \theta_0 + 2\pi$, then we define the space $\mathcal{H}$ as $W^{1,2}(]\theta_0, \theta_n[)$ in case of a Neumann condition and as $W_0^{1,2}(]\theta_0, \theta_n[)$ in case of a Dirichlet condition. If $\theta_n = \theta_0 + 2\pi$ – the case of an interior edge – we define $\mathcal{H}$ as the periodic Sobolev space $H^1(]\theta_0, \theta_n[) \cap \{v : v(\theta_0) = v(\theta_n)\}$, which clearly may be identified with the Sobolev space $H^1(S^1)$ on the unit circle $S^1$. For every $\lambda \in \mathbb{C}$ we define the quadratic form $\mathfrak{t}_\lambda$ on $\mathcal{H}$ by

$$\mathfrak{t}_\lambda[v] := \int_{\theta_0}^{\theta_n} \big(b_2\, v'\, \overline{v'} + \lambda b_1 v\, \overline{v'} - \lambda b_1 v'\, \overline{v} - \lambda^2 b_0 v \overline{v}\big)\, \mathrm{d}\theta \tag{5.1}$$

and $\mathcal{A}_\lambda$ as the operator which is induced by $\mathfrak{t}_\lambda$ on $L^2(]\theta_0, \theta_n[)$.

**Definition 5.5.** Let $E$ be any edge of the triangulation of $\overline{\Omega}$ and $\mathcal{A}_\lambda$ as defined in Definition 5.4 with $\rho = \mu_E$. Then we call the number

$$\inf\{\Re\lambda > 0 : \ker(\mathcal{A}_\lambda) \neq \{0\}\}$$

the *singularity exponent* associated to $E$, compare Lemma 5.12 below.

We proceed by quoting the central linear regularity result [30, Thm. 2.3], by means of which the crucial Lemma 4.21 will be deduced.

**Proposition 5.6.** *Let $\Omega$, $\{\sigma_k\}_k$ and $\mu$ be as in Definition 5.2 and for any edge $E$ of $\overline{\Omega}$ let $\mu_E$ be the $2 \times 2$ matrix valued function on $S_{\theta_0}^{\theta_n}$ in the sense of this definition. If for every such edge the associated singularity exponent is larger than $\frac{1}{3}$, then there is a $p > 3$, such that*

$$-\nabla \cdot \mu \nabla : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$$

*is a topological isomorphism.*

**Remark 5.7.** Unfortunately, there are some errors in the paper [30]. First, the assertion of [30, Thm. 2.3] that the exponent $p$ can be taken from the interval $[2, 2/(1 - \widehat{\lambda}_\Upsilon)[$ is erroneous, since the assumptions of [30, Thm. 2.4] have to be taken into account. The correct formulation of the linear regularity result proved in [30] is given in Proposition 5.6 above. Furthermore, the signs in formulas for the coefficients of certain generalized Sturm-Liouville equations are not correct, in detail: in [30, p. 240] there is a wrong sign in the formula for the Mellin transform $\widetilde{r\partial_r u} = -\lambda\tilde{u}$, which has to be replaced by $\widetilde{r\partial_r u} = \lambda\tilde{u}$. Therefore the formulas [30, (3.33)] for the sesquilinear form $a(u, v; \lambda)$ and [30, (3.32)] for the corresponding differential problem differ in sign from the correct formulas (5.1) and

$$(b_2 u')' + \lambda(b_1 u)' + \lambda b_1 u' + \lambda^2 b_0 u = 0, \quad [u]_\theta = 0, \quad [b_2 u' + \lambda b_1 u]_\theta = 0.$$

The correctness of the other considerations in [30] is not affected by this.

**Remark 5.8.** It is known that in every strip $\{\lambda = \lambda_1 + i\lambda_2 : |\lambda_1| \le c\}$ there is only a finite number of values $\lambda$, for which the kernel of $\mathcal{A}_\lambda$ is not trivial, see Corollary 3.11 from [30]. Therefore, it suffices to show that no $\mathcal{A}_\lambda$ with $0 < \Re\lambda \le \frac{1}{3}$ does admit a nontrivial kernel.

In [30] and [16] it is pointed out in great detail how to find parameters $\lambda$ for which the operator $\mathcal{A}_\lambda$ has only a trivial kernel – we will not repeat this here. Basing on this, in the next sections all edges resulting from our problem are inspected, concerning the triviality of $\ker(\mathcal{A}_\lambda)$. Here, two essential types are monomaterial edges and bimaterial outer edges:

**Definition 5.9.** Let $E$ be an edge in $\overline{\Omega}$ that lies in $\partial\Omega$. Then we define in the terminology of Definition 5.2:

1. $E$ is a *monomaterial edge*, if all relative interior points of $E$ possess a neighbourhood in $\overline{\Omega}$ on which $\mu$ is constant a.e. with respect to 3-dimensional Lebesgue measure.
2. $E$ is a *bimaterial outer edge*, if the function $[\theta_0, \theta_n] \ni \theta \mapsto \mu_E(\cos(\theta), \sin(\theta))$, has exactly 1 jump, and the lengths of both the constancy intervals of this function do not exceed $\pi$. Here $\theta_0$ and $\theta_n$ are again the angles associated to $E$ in Remark 5.3.

For the treatment of the corresponding singularities, we have the following two theorems, see [10, Lemma 2.3] or [16, Thm. 24/25].

**Proposition 5.10.** *For any monomaterial edge $E$ the kernels of the associated operators $\mathcal{A}_\lambda$ are trivial, if $\Re\lambda \in ]0, 1/2]$.*

**Proposition 5.11.** *Let $S_{\theta_0}^{\theta_1}, S_{\theta_1}^{\theta_2}$ be two neighbouring sectors in $\mathbb{R}^2$ with $\theta_1 - \theta_0$, $\theta_2 - \theta_1 \le \pi$ and $\theta_2 - \theta_0 < 2\pi$. Let $\rho^1$, $\rho^2$ be two real, symmetric, positive definite $2 \times 2$ matrices corresponding to the sectors $S_{\theta_0}^{\theta_1}$ and $S_{\theta_1}^{\theta_2}$, respectively. Let $\mathfrak{t}_\lambda$ be the form defined in (5.1) either on $H_0^1(]\theta_0, \theta_2[)$ or on $H^1(]\theta_0, \theta_2[)$. Then there is an $\varepsilon > 0$, such that the kernel of the corresponding operator $\mathcal{A}_\lambda$ (see Definition 5.4) is trivial for $\Re\lambda \in ]0, 1/2 + \varepsilon]$.*

In the rest of this section we collect some technical lemmata on edge singularities that will be needed in the sequel.

**Lemma 5.12** ([16, Lemma 14]). *Let $E$ be some edge of $\overline{\Omega}$, let $\mu_E$ be the $2 \times 2$ matrix associated to $E$ in Definition 5.2 and denote by $\theta_0, \ldots, \theta_n$ the corresponding angles from Remark 5.3. Assume that the operator $\mathcal{A}_\lambda$ belonging to $\mu_E$ (cf. Definition 5.4) has a non-trivial kernel for some $\lambda \in \mathbb{C}$ with $\Re\lambda \in ]0,1[$. Furthermore, let $\mathcal{H}$ be as in Definition 5.4 and let $v_\lambda \in \mathcal{H}$ be a non-trivial function from $\ker(\mathcal{A}_\lambda)$. Set $\mathcal{S}_{\theta_0}^{\theta_n} := S_{\theta_0}^{\theta_n} \times \mathbb{R}$ if $\theta_n \neq \theta_0 + 2\pi$ and $\mathcal{S}_{\theta_0}^{\theta_n} := \mathbb{R}^3$ else. Then the following assertions are true.*

1. *The function $\psi_0$ given by*

$$\psi_0(x,y) = (x^2+y^2)^{\lambda/2} v_\lambda\big(\arg(x+iy)\big), \qquad (x,y) \in S_{\theta_0}^{\theta_n},$$

*belongs to $W^{1,p}_{\mathrm{loc}}(S_{\theta_0}^{\theta_n})$ for $p \in \big[2, \frac{2}{1-\Re\lambda}\big[$, but not to $W^{1,\frac{2}{1-\Re\lambda}}_{\mathrm{loc}}(S_{\theta_0}^{\theta_n} \cap ]-r,r[^2)$ for any $r>0$.*

2. *The function $\psi$, given by $\psi(x,y,z) := \psi_0(x,y)$, belongs to $W^{1,p}_{\mathrm{loc}}(\mathcal{S}_{\theta_0}^{\theta_n})$ for $p \in \big[2, \frac{2}{1-\Re\lambda}\big[$, but not to $W^{1,\frac{2}{1-\Re\lambda}}_{\mathrm{loc}}(\mathcal{S}_{\theta_0}^{\theta_n} \cap r\mathcal{C})$ for any $r>0$.*

3. *The function $\psi$ satisfies*

$$-\nabla \cdot \mu_{E,P} \nabla \psi = 0 \quad \text{on } \mathcal{S}_{\theta_0}^{\theta_n},$$

*precisely: one has*

$$\int_{\mathcal{S}_{\theta_0}^{\theta_n}} \mu_{E,P} \nabla\psi \cdot \nabla\overline{\varphi} \,\mathrm{dx} = 0 \tag{5.2}$$

*for all compactly supported $\varphi \in W^{1,2}(\mathbb{R}^3)$ in the Neumann case or case of an interior edge and for all compactly supported $\varphi \in W^{1,2}(\mathbb{R}^3)$, vanishing on $\partial\mathcal{S}_{\theta_0}^{\theta_n}$, in the Dirichlet case.*

**Corollary 5.13.** *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain, such that $\Upsilon := \Omega \cap \mathcal{S}_{\theta_0}^{\theta_n}$ is again a Lipschitz domain. Let $\xi \in W^{1,\infty}(\mathbb{R}^3)$ be a real function with $\operatorname{supp}(\xi) \subset \Omega$. Put $\Gamma := \emptyset$ in the Dirichlet case or in the case of an interior edge and $\Gamma := \Omega \cap \partial\mathcal{S}_{\theta_0}^{\theta_n}$ in the Neumann case. Then*

$$-\nabla \cdot \mu_{E,P}|_\Upsilon \nabla(\xi\psi)|_\Upsilon = g \text{ on } \Upsilon, \tag{5.3}$$

*where $\psi$ is the function from Lemma 5.12 and $g \in W^{-1,6}_\Gamma(\Upsilon)$ is the antilinear form*

$$W^{1,\frac{6}{5}}_\Gamma \ni \varphi \mapsto \int_\Upsilon \psi\, \mu_{E,P}\, \nabla\xi \cdot \nabla\overline{\varphi} \,\mathrm{dx} - \int_\Upsilon \overline{\varphi}\, \mu_{E,P}\, \nabla\psi \cdot \nabla\xi \,\mathrm{dx}.$$

*Proof.* For any function $\varphi \in W^{1,2}_\Gamma(\Upsilon)$ one calculates

$$\begin{aligned} &\int_\Upsilon \mu_{E,P} \nabla(\xi\psi)|_\Upsilon \cdot \nabla\overline{\varphi} \,\mathrm{dx} \qquad (5.4)\\ &= \int_\Upsilon \mu_{E,P}\, \nabla\psi \cdot \nabla(\overline{\xi\varphi}) \,\mathrm{dx} + \int_\Upsilon \psi\, \mu_{E,P}\, \nabla\xi \cdot \nabla\overline{\varphi} \,\mathrm{dx} - \int_\Upsilon \overline{\varphi}\, \mu_{E,P}\, \nabla\psi \cdot \nabla\xi \,\mathrm{dx}. \end{aligned}$$

The extension of the function $\xi\varphi$ by zero to the set $\mathcal{S}_{\theta_0}^{\theta_n}$ leads to an admissable test function, thus the first term of the right hand side in (5.4) vanishes, according to (5.2). Due to $\nabla\xi \in L^\infty(\Upsilon)$ and $\psi|_\Upsilon \in W^{1,2}(\Upsilon) \hookrightarrow L^6$, the second term in (5.4) defines a continous antilinear form on $W_\Gamma^{1,\frac{6}{5}}(\Upsilon)$. The same is true for the third term, due to $\nabla\psi \in L^2(\Upsilon)$ and $W^{1,\frac{6}{5}}(\Upsilon) \hookrightarrow L^2(\Upsilon)$. □

The final result in this section allows to delimitate the singularities at edges in $\Sigma$ which result from the even reflection across the former Neumann boundary part.

**Lemma 5.14.** *Assume that the half space $\{(x,y) : y > 0\}$ splits up into the sectors $S_0^{\theta_1}, \ldots, S_{\theta_{n-1}}^{\pi}$ with the associated matrices*

$$\begin{pmatrix} \rho_{11}^1 & \rho_{12}^1 \\ \rho_{12}^1 & \rho_{22}^1 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} \rho_{11}^n & \rho_{12}^n \\ \rho_{12}^n & \rho_{22}^n \end{pmatrix}. \tag{5.5}$$

*Assume that to the sectors $S_0^{-\theta_1}, \ldots, S_{-\theta_{n-1}}^{-\pi}$ the matrices*

$$\begin{pmatrix} \rho_{11}^1 & -\rho_{12}^1 \\ -\rho_{12}^1 & \rho_{22}^1 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} \rho_{11}^n & -\rho_{12}^n \\ -\rho_{12}^n & \rho_{22}^n \end{pmatrix} \tag{5.6}$$

*are assigned. Let $\mathcal{A}_\lambda$ denote the operator that corresponds to the matrices* (5.5), (5.6) *within the given sector partition of $\mathbb{R}^2$. Let further $\mathcal{A}_\lambda^D$ and $\mathcal{A}_\lambda^N$ denote the operators, which correspond to the matrices* (5.5) *within the sectors in the half space $\{(x,y) : y > 0\}$, once combined with Dirichlet and once with Neumann boundary conditions. Then for any number $\lambda$ with $\Re\lambda > 0$ the kernel of the operator $\mathcal{A}_\lambda$ is trivial, if for this same $\lambda$ the kernels of $\mathcal{A}_\lambda^D$ and $\mathcal{A}_\lambda^N$ are trivial.*

*Proof.* A proof is given in [16, Lemma 22] in the case of two sectors in the half space. Mutatis mutandis, the proof can be carried over to the case of many sectors. □

# 6 The optimal regularity result

We are now in the position to state our main result on optimal regularity of the elliptic problem near a heterogeneous Neumann vertex.

**Theorem 6.1.** *Let $\Pi \subseteq \mathbb{R}^3$ be a bounded domain, such that $\Pi$ and $\overline{\Pi}$ have the same boundary and its closure $\overline{\Pi}$ is a polyhedral 3-manifold with boundary. Let $\mathrm{a}_\blacktriangle$ be a vertex of $\overline{\Pi}$ and suppose:*

1. *The coefficient function $\mu$ on $\Pi$ is elliptic and takes its values in the set of real, symmetric, positive definite $3 \times 3$ matrices.*
2. *There is a triangulation of $\overline{\Pi}$ by a (finite) euclidean complex $K$, such that $\mu$ is constant on the interior of each 3-cell belonging to $K$.*
3. *Any edge in $K$, belonging to the boundary of $\Pi$ and having one endpoint in $\mathrm{a}_\blacktriangle$, is a monomaterial edge or a bimaterial outer edge.*

4. *For every interior edge with endpoint* $\mathrm{a}_\blacktriangle$*, resulting from the triangulation* $K$*, the singularity exponent, associated to this edge, is larger than* $\frac{1}{3}$.

*Then there is an open neighbourhood* $\mathcal{U}$ *of* $\mathrm{a}_\blacktriangle \in \mathbb{R}^3$ *and a* $p > 3$ *such that, for any* $f \in (W^{1,p'}(\Pi))'$ *and any* $\eta \in C^\infty(\mathbb{R}^3)$ *with support in* $\mathcal{U}$*, every solution* $v$ *of* $-\nabla\cdot\mu\nabla v = f$ *satisfies* $\eta v \in W^{1,p}(\Pi \cap \mathcal{U})$.

*Even more, setting* $\Pi_\bullet := \Pi \cap \mathcal{U}$ *and* $\Gamma := \partial\Pi \cap \mathcal{U}$*, the operator*

$$-\nabla \cdot \mu|_{\Pi_\bullet} \nabla : W^{1,p}_\Gamma(\Pi_\bullet) \to W^{-1,p}_\Gamma(\Pi_\bullet)$$

*is a topological isomorphism for this* $p > 3$.

**Remark 6.2.** Condition 2 in Theorem 6.1 says by no means that the coefficient function $\mu$ has to take different values on different cells.

Some comment on Condition 4 in this theorem is in order. Naturally, it is an unsatisfactory assumption, postulating a technical need rather than giving a descriptive condition on $\mu$. Unfortunately, for interior edges it seems to be extremely difficult to decide, whether the kernels of $\mathcal{A}_\lambda$ are trivial or not in generality, see the detailed discussion in [10]. However, there are several important constellations, where this assumption is known to be true. In order to formulate some of them, we need the following definition.

**Definition 6.3.** Let angles $-\pi = \theta_0 < \theta_1 < \cdots < \theta_n = \pi$ be given and let $\hat{\mu}$ be a constant matrix on every sector $S_j := S^{\theta_j}_{\theta_{j-1}}$, $j = 1, \ldots, n$, in $\mathbb{R}^2$. Then $\hat{\mu}$ is distributed *quasi-monotonely*, if there exist indices $j_{\min}, j_{\max} \in \{1, \ldots, n\}$, such that

$$\hat{\mu}|_{S_{j_{\max}}} \geq \hat{\mu}|_{S_{j_{\max}+1}} \geq \cdots \geq \hat{\mu}|_{S_{j_{\min}-1}} \geq \hat{\mu}|_{S_{j_{\min}}} \leq \hat{\mu}|_{S_{j_{\min}+1}} \leq \cdots \leq \hat{\mu}|_{S_{j_{\max}-1}} \leq \hat{\mu}|_{S_{j_{\max}}}$$

and there exists a point $\mathrm{x} \in \mathbb{R}^2$, such that $\mathrm{x} \in S_{j_{\max}}$ and $-\mathrm{x} \in S_{j_{\min}}$. Here the indices are to be understood modulo $n$, i.e. $S_{n+1}$ is again $S_1$.

**Example 6.4.** Condition 4 of Theorem 6.1 is satisfied in each of the following cases.

1. There is exactly one plane containing $\mathrm{a}_\blacktriangle$, which splits up $\Pi$ in a neighbourhood of $\mathrm{a}_\blacktriangle$ into two pieces, such that the coefficient function is locally constant on both intersections of $\Pi$ with the half spaces, induced by the plane. Of course, then no interior edges appear around $\mathrm{a}_\blacktriangle$.
2. There are exactly two planes containing $\mathrm{a}_\blacktriangle$ and intersecting $\Pi$. The coefficient function is constant in the induced four quarter spaces, scalar valued and strictly monotone if running from sector to sector, see [36, Thm. 6.4].
3. The matrices are distributed quasi-monotonely, see [10, Lemma 2.1], see also [25]. This is in particular the case, if $\mu$ is scalar valued, and $\mathbb{R}^2$ splits up into three sectors, each of which has an opening angle less than $\pi$.

**Remark 6.5.** Elschner's striking counterexample (see [11, Ch. 4]) in case of only two sectors and anisotropic coefficient matrices shows that there is no hope to find general conditions, beside quasimonotonicity, that assure for interior edges the triviality of $\ker(\mathcal{A}_\lambda)$ in case of $\Re\lambda \in ]0, 1/3]$.

**Remark 6.6.** 1. Condition 3 of Theorem 6.1 in particular forbids the constellation in Figure 1, despite the fact that only two different materials are involved.

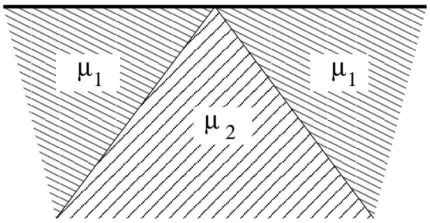


**Figure 1:** An outer edge with two materials that is *not* a bimaterial outer edge

# 7 Proof of Theorem 6.1

Part of the proof of Theorem 6.1 is already contained in the preceding sections. In fact, our aim will be to show Lemma 4.21, which by the considerations in Section 4.2 implies Theorem 6.1. This lemma in turn will be deduced by an application of Proposition 5.6. Thus we have to assure that the singularity exponent of all edges appearing in $\overline{\mathcal{C}}$, relative to the piecewise constant coefficient $\hat{\mu}$, in Lemma 4.21 are larger than 1/3.

## 7.1 Identification of edges

In order to identify all these edges, we consider for a given vertex $\mathrm{a}_\blacktriangle$ the coefficient function $\hat{\mu}$ from Lemma 4.21. Recall in this context Definition 4.22 and Remark 4.23, as well as the piecewise linear map $\varphi_{\mathrm{a}_\blacktriangle}$ and the complex $K_{\overline{\mathcal{C}}}$ from the definitions 4.17 and 4.22.

**Definition 7.1.**

1. For all $t_0 > 0$ and any vectors $\mathrm{a} \in \mathbb{R}^3$ and $\mathrm{b} \in \mathbb{R}^3 \setminus \{0\}$ we call the set $\{\mathrm{a}+t\mathrm{b}, t \in ]0, t_0[\}$ *segment* with *starting point* a.
2. A segment in $\Pi$ is *monomaterial* if it is contained in the interior of one 3-cell from $K_\Pi$. A segment in $\Pi$ is a *bi-cellular segment* if it lies in the interior of a 2-face of $K_\Pi$.

**Remark 7.2.** Recall that the coefficient matrix $\mu$ need not have different values on the two 3-cells which are adjacent to the 2-face which includes a bi-cellular segment.

The following facts about the edges from $K_{\overline{\mathcal{C}}}$ are essential in what follows.

**Lemma 7.3.** *Let for the edge $E$ the symbol $\mathring{E}$ denote its relative interior.*

1. *Any edge from the complex $K_{\overline{\mathcal{C}}}$, lying in $\partial\mathcal{C}$, is either a monomaterial edge or a bimaterial outer edge.*
2. *Any edge $E$ from $K_{\overline{\mathcal{C}}}$, which intersects $\mathcal{C}_+$ has one endpoint in* 0. *Consequently, the segment $\varphi_{\mathrm{a}_\blacktriangle}(\mathring{E})$ has starting point $\mathrm{a}_\blacktriangle$. Thus, $\varphi_{\mathrm{a}_\blacktriangle}(\mathring{E})$ is either a monomaterial or a bi-cellular segment in $\Pi$ or it is contained in an edge of $K_\Pi$, i.e. in the starting triangulation of $\overline{\Pi}$.*
3. *For every edge $E \subset \Sigma$, $\varphi_{\mathrm{a}_\blacktriangle}(\mathring{E}) \subset \partial\Pi$ is a segment with starting point $\mathrm{a}_\blacktriangle$ and is, hence, a monomaterial segment or part of a monomaterial edge or part of a bimaterial outer edge which was already present in the starting triangulation $K_\Pi$.*

*Proof.* All three assertions follow from the definition of the triangulation of the polyhedron $\varepsilon\overline{\mathcal{C}}$ which is induced by the complex $h(L_{\mathrm{a}}^{\bigstar})$, see the proof of Theorem 4.15.

1. Since the edges from $\partial\mathcal{C}_-\setminus\overline{\Sigma}$ are the reflected edges from $\partial\mathcal{C}_+\setminus\overline{\Sigma}$, it suffices to show the assertion only for the edges within the latter set and for edges from $\partial\Sigma$. Concerning the edges from $\partial\mathcal{C}_+\setminus\overline{\Sigma}$, it is important to remember that the value of $\varepsilon$ was chosen to assure that the whole link of $h(L_{\mathrm{a}}^{\bigstar})$ is disjoint to $\varepsilon\overline{\mathcal{C}}$. Thus, any edge, occurring on $\partial\mathcal{C}_+\setminus\overline{\Sigma}$, is either a "natural" edge of $\partial\mathcal{C}$ or results from the intersection of a 2-cell from $h(L_{\mathrm{a}}^{\bigstar})$, having one of its vertices in 0, with $\partial(\varepsilon\mathcal{C})$. Since such 2-cells are adjacent to exactly 2 tetrahedra, the resulting edge can be only monomaterial or bimaterial. Finally, each edge $E$ from $\partial\Sigma$ is, by construction, an edge of exactly one element from $K_{\overline{\mathcal{C}}}^{+}$ (see Definition 4.17). Recalling the Definition 4.22 of the complex $K_{\overline{\mathcal{C}}}$ it is then clear that $E$ is a monomaterial or a bimaterial edge within $K_{\overline{\mathcal{C}}}$ – depending whether the coefficient function on the reflected cell is different from that on the coresponding cell in $K_{\overline{\mathcal{C}}}^{+}$ or not.
2. By construction, all edges which intersect the interior of the polyhedron $|h(L_{\mathrm{a}}^{\bigstar})|$ originate in 0. Clearly, this property does not get lost when intersecting with $\varepsilon\overline{\mathcal{C}}$. Hence, if $E$ intersects $\mathcal{C}_+$, then $\varphi_{\mathrm{a}_\blacktriangle}(\mathring{E})$ must be a segment with $\mathrm{a}_\blacktriangle$ as one of its endpoints. But all such segments can only be monomaterial or bi-cellular or have to be part of an edge of the original triangulation $K_\Pi$ of $\overline{\Pi}$.
3. Every edge in $\Sigma$ is already present in $h(L_{\mathrm{a}}^{\bigstar})$ and must, hence, have its starting point in $0\in\mathbb{R}^3$. Thus, the assertion follows from (5) in Theorem 4.15. ☐

The edges whose investigation requires most efforts are those from $\Sigma$. The required regularity properties for the crucial model problem in view of these edges we formulate as Proposition 7.4. This altogether will then lead to the proof of Theorem 6.1 in Section 7.2.

**Proposition 7.4** ([23, Thm. 1.2]). *Let $\triangle_1\subseteq\mathbb{R}^2$ be an open triangle with vertices $P_1$, $P_2$, $P_3$ and $\triangle_2$ another open triangle with vertices $P_1$, $Q_2$, $Q_3$, such that $P_2$ is contained in the edge joining $P_1$ and $Q_2$. Furthermore, let $\mathcal{P}\subseteq\mathbb{R}^3$ be an open right prism with basis $\triangle_1\cup\triangle_2\cup\overline{P_1P_2}$ and height $h$ and let $\Gamma$ be the closure of the part of the nappe of $\mathcal{P}$ with basis $\overline{Q_2P_2}\cup\{P_2\}\cup\overline{P_2P_3}$ if $P_2\neq Q_2$ and with basis $\overline{Q_3P_2}\cup\{P_2\}\cup\overline{P_2P_3}$ in the case $P_2=Q_2$, cf. Figure 2.*

*Let $\varrho_1$ and $\varrho_2$ be two real, symmetric and positive definite $3\times 3$ matrices and set $\varrho$ to be the coefficient function on $\mathcal{P}$, which equals $\varrho_1$ on the right prism with basis $\triangle_1$ and $\varrho_2$ on the right prism with basis $\triangle_2$. Then there is a $p>3$, such that*

$$-\nabla\cdot\varrho\nabla : W_\Gamma^{1,p}(\mathcal{P})\to W_\Gamma^{-1,p}(\mathcal{P})$$

*provides a topological isomorphism.*

## 7.2 Estimates for the edge singularities

In this section we want to finish the proof of Theorem 6.1. Basing on our considerations in Section 5, we are done, if for all edges from the triangulation $K_{\overline{\mathcal{C}}}$ of $\overline{\mathcal{C}}$ the induced

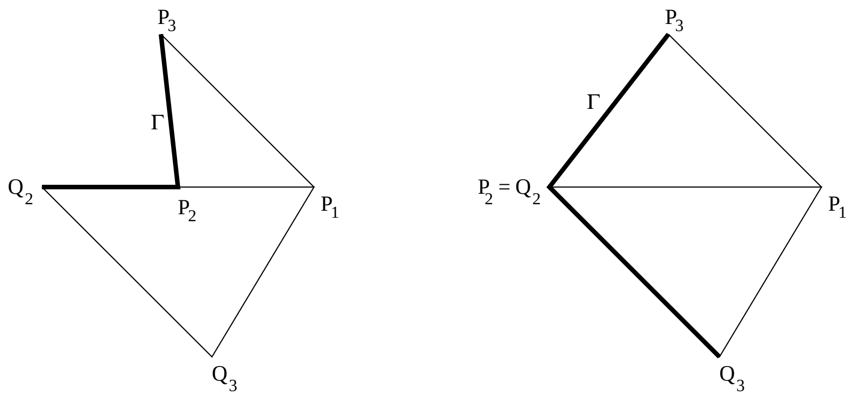


**Figure 2:** The basis of the prism $\mathcal{P}$ and of the Neumann boundary part $\Gamma$ in the two cases $P_2 \neq Q_2$ and $P_2 = Q_2$

operators $\mathcal{A}_\lambda$ have a trivial kernel for all $\lambda$ with $\Re\lambda \in \,]0, 1/3]$, see also Remark 5.8. The occurring edges are the following:

| | | | |
|---|---|---|---|
| I | edges from $\partial\mathcal{C}$, | III | edges from $\Sigma$, |
| II | edges from $\mathcal{C}_+$, | IV | edges from $\mathcal{C}_-$. |

It is not hard to see that the edges in IV may be treated analogously to the edges in II. Thus, we will discuss the cases I – III in the following.

**I Edges from $\partial\mathcal{C}$** According to 1. of Lemma 7.3, any edge contained in $\partial\mathcal{C}$ is a monomaterial edge or a bimaterial outer edge. Thus the corresponding operators $\mathcal{A}_\lambda$ have a trivial kernel if $\Re\lambda \in \,]0, 1/2]$ by Proposition 5.10 and Proposition 5.11.

**II Edges from $\mathcal{C}_+$** Let $E$ be an edge from $\mathcal{C}_+$. Then $E$ is an interior edge with one endpoint in $0 \in \mathbb{R}^3$, cf. 2. of Lemma 7.3. Let us assume that the corresponding operator $\mathcal{A}_\lambda$ has a nontrivial kernel for some $\lambda$ with $\Re\lambda \in \,]0, 1/3]$. Recall that the coefficient configuration for $\mathcal{A}_\lambda$ in $\mathbb{R}^2$ results from $\underline{\mu}$ after the affine transformation $a_{E,P}$ that rotates the edge into the $z$-axis and afterwards shifts an arbitrary point $P \in E$ to the origin, see Definition 5.2. Observe that the coefficient function $\underline{\mu}_{E,P}$ coincides on a sufficiently small neighbourhood $\mathcal{V}$ of 0 with the coefficient function, that results from $\underline{\mu}$ under the transformation $a_{E,P}$.

Now, choose a cut-off function $\xi \in C^\infty(\mathbb{R}^3)$ with $\xi \equiv 1$ in a neighbourhoof of 0 and with support in $\mathcal{V}$. Due to Lemma 5.12 and Corollary 5.13, arising from an element $v_\lambda \neq 0$ from $\ker(\mathcal{A}_\lambda)$, there is a function $\psi : \mathbb{R}^3 \to \mathbb{C}$ such that $\xi\psi$ fulfills the equation

$$-\nabla \cdot \underline{\mu}_{E,P} \nabla(\xi\psi) = f \quad \text{on } \mathcal{V}$$

with $f \in W^{-1,6}(\mathcal{V})$ and $\xi\psi \notin W^{1,3}(\mathcal{V})$.

Then we transform everything back to the original constellation in $\Pi$, i.e. we apply the map $\Phi := \varphi_{\mathrm{a}_\blacktriangle} a_{E,P}^{-1}$, where $\varphi_{\mathrm{a}_\blacktriangle}$ is again the piecewise linear homeomorphism associated to $a_\blacktriangle$ by Theorem 4.15. In view of Proposition 4.1, this leads to an equation

$$-\nabla \cdot \mu|_{\Pi_\bullet} \nabla \Psi(\xi\psi) = (\Psi^*)^{-1} f \quad \text{on} \quad \Phi(\mathcal{V}), \tag{7.1}$$

where $(\Psi^*)^{-1} f \in W^{-1,6}(\Phi(\mathcal{V}))$ and $\Psi(\xi\psi) \notin W^{1,3}(\Phi(\mathcal{V}))$. Due to 2. of Lemma 7.3, the edge $E$ transforms under the transformation $\Phi$ into either a monomaterial segment or a bi-cellular segment or part of an interior edge with endpoint in $\Phi(0) = a_\blacktriangle$ of the original setting in $\Pi$. Obviously, the point $P$ is mapped under $\Phi$ into an interior point of $\Phi(E)$. Since $\mathcal{V}$ may be chosen arbitrarily small and $\Phi$ is bi-Lipschitz, the supports of $(\Psi^*)^{-1} f$ and $\Psi(\xi\psi)$ are contained in an arbitrarily small cube $\mathcal{K}$ around $\Phi(P)$. In particular, one may assume that this cube contains no other interior edges of $K_\Pi$ except possibly $\Phi(E) \cap \mathcal{K}$. Furthermore, due to the support properties of $(\Psi^*)^{-1} f$ and $\Psi(\xi\psi)$, equation (7.1) may be read as a Dirichlet problem on $\mathcal{K}$.

If $\Phi(E)$ is a monomaterial or a bi-cellular segment, then in fact no interior edges appear in $\mathcal{K}$, while possibly occurring outer edges are either monomaterial edges or at worst bimaterial outer edges and in this second case all appearing angles of the sectors are less than $\pi$, since $\mathcal{K}$ is convex. Thus, $(\Psi^*)^{-1} f \in W^{-1,6}(\mathcal{K})$ and $\Psi(\xi\psi) \notin W_0^{1,3}(\mathcal{K})$ is contradictory due to Proposition 5.6, Proposition 5.10 and Proposition 5.11.

Finally, if $E$ is transformed into part of an interior edge of the original setting, then one is concerned with monomaterial edges, bimaterial outer edges and (part of) the original edge. Since Condition 4 of Theorem 6.1 guarantees also for the original edge, that $\ker(\mathcal{A}_\lambda)$ is trivial whenever $\Re\lambda \leq \frac{1}{3}$, we can argue as in the preceding case to obtain a contradiction.

**III Edges from $\Sigma$** Obviously, every edge from $\Sigma$ is an interior edge relative to $\mathcal{C}$ and by 2. of Lemma 7.3 one of its endpoints is $0 \in \mathbb{R}^3$. If, for a given edge $E$ from $\Sigma$, the coefficient matrices, belonging to the corresponding sectors in the half space $\{(x, y, z) \in \mathbb{R}^3 : y > 0\}$, are

$$M^1 = \begin{pmatrix} m^1_{11} & m^1_{12} & m^1_{13} \\ m^1_{12} & m^1_{22} & m^1_{23} \\ m^1_{13} & m^1_{23} & m^1_{33} \end{pmatrix}, \quad \ldots, \quad M^n = \begin{pmatrix} m^n_{11} & m^n_{12} & m^n_{13} \\ m^n_{12} & m^n_{22} & m^n_{23} \\ m^n_{13} & m^n_{23} & m^n_{33} \end{pmatrix},$$

then the corresponding matrices in the reflected sectors are

$$M^1_- = \begin{pmatrix} m^1_{11} & -m^1_{12} & m^1_{13} \\ -m^1_{12} & m^1_{22} & -m^1_{23} \\ m^1_{13} & -m^1_{23} & m^1_{33} \end{pmatrix}, \quad \ldots, \quad M^n_- = \begin{pmatrix} m^n_{11} & -m^n_{12} & m^n_{13} \\ -m^n_{12} & m^n_{22} & -m^n_{23} \\ m^n_{13} & -m^n_{23} & m^n_{33} \end{pmatrix},$$

see Proposition 4.20. According to Proposition 5.6, one has to perform a rotation in the $x$-$z$-plane, which moves the edge under consideration to the $z$-axis. This means, one has to consider the matrices

$$\begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} M \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

$M$ taken as $M^1, \dots, M^n, M^1_-, \dots, M^n_-$, respectively, and $\alpha$ being the angle between the corresponding edge and the $z$-axis. A straightforward calculation shows that the resulting upper $2 \times 2$ blocks look alike

$$\begin{pmatrix} \rho^1_{11} & \rho^1_{12} \\ \rho^1_{12} & \rho^1_{22} \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \rho^n_{11} & \rho^n_{12} \\ \rho^n_{12} & \rho^n_{22} \end{pmatrix}$$

in the sectors within the half space $\{(x, y, z) \in \mathbb{R}^3 : y > 0\}$ and

$$\begin{pmatrix} \rho^1_{11} & -\rho^1_{12} \\ -\rho^1_{12} & \rho^1_{22} \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \rho^n_{11} & -\rho^n_{12} \\ -\rho^n_{12} & \rho^n_{22} \end{pmatrix}$$

on the reflected sectors within the half space $\{(x, y, z) \in \mathbb{R}^3 : y < 0\}$. Concerning this constellation, we may apply Lemma 5.14, in order to reduce the problem, given on a sector partition of the whole of $\mathbb{R}^2$, to a Dirichlet and a Neumann problem on a sector partition of the half space.

Hence, it remains to prove the following

**Lemma 7.5.** *The kernels of $\mathcal{A}^D_\lambda$ and $\mathcal{A}^N_\lambda$ are trivial, if $\Re\lambda \in \left]0, \frac{1}{3}\right]$.*

*Proof.* Let us assume that the assertion is not true. We fix a (relative) interior point $P \in E$. Then, by Lemma 5.12 and Corollary 5.13, there is a function $\psi$ and a neighbourhood $\mathcal{V}$ of $P$ in $\mathbb{R}^3$ such that for every $\xi \in C^1(\mathbb{R}^3)$ with support in $\mathcal{V}$ and $\xi \equiv 1$ on a neighbourhood of $P$ we have (5.3) with $\Upsilon = \mathcal{C}_+$ and $\mu_{E,P} = \rho$, but $\xi\psi \notin W^{1,3}(\mathcal{V} \cap \mathcal{C}_+)$.

Now, we transform back the corresponding edge to the original setting in $\Pi$, i.e., using the notation of the foregoing point II, we apply the map $\Phi := \varphi_{a_\blacktriangle} a^{-1}_{E,P}$ in the sense of Proposition 4.1. The image of the edge $E$ under this map necessarily is either contained in a planar face of one cell from $K_\Pi$ or part of a monomaterial edge or part of a bimaterial outer edge which was already present in the starting triangulation $K_\Pi$, see 3. of Lemma 7.3. Thus, a cut trough $\Pi_\bullet$ perpendicular to $\Phi(E)$ in a neighbourhood of $\Phi(E)$ looks like indicated in Figure 3. Note that, due to the definition of a bimaterial

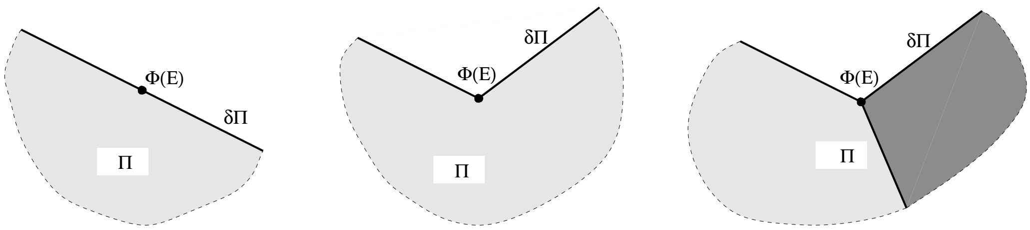


**Figure 3:** Cut through $\Pi_\bullet$ perpendicular to $\Phi(E)$ for a monomaterial segment, a monomaterial edge and a bimaterial edge, respectively

outer edge, the opening angles of the sectors in the third case all do not exceed $\pi$.

In order to generate a contradiction, we may in accordance with Figure 3 always choose a right prism $\mathcal{P} \subset \Pi$ with the following properties:

1. $\Phi(E)$ is a segment within the nappe of $\mathcal{P}$, and the basis and the upper plate of $\mathcal{P}$ are orthogonal to this segment.

2. $\mathcal{P}$ meets the conditions of Theorem 7.4.
3. The coefficient function $\mu|_{\mathcal{P}}$ satisfies the suppositions of Theorem 7.4; in particular it does not induce interior edges on $\mathcal{P}$.

Observe that in the case of a monomaterial segment the two triangles degenerate to a single one and that in the cases of a monomaterial segment and a monomaterial edge the coefficient function $\mu|_{\mathcal{P}}$ in fact is constant.

Having fixed $\mathcal{P}$ this way, we may assume, possibly diminishing $\mathcal{V}$, that $\Psi(\xi\psi)$ is zero on the boundary of $\mathcal{P}$, except for the side(s) of $\mathcal{P}$, which is (are) adjacent to $\Phi(E)$. Finally, elliptic regularity results for the arising prisms will show that the assumption $\xi\psi \notin W^{1,3}(\mathcal{K}_0^\pi)$ is contradictory in the case of $\mathcal{A}_\lambda^D$ and $\mathcal{A}_\lambda^N$. In the Dirichlet case, in view of Proposition 4.1, one ends up with a Dirichlet problem

$$-\nabla\cdot\mu|_{\mathcal{P}}\nabla\big(\Psi(\xi\psi)\big) = f \text{ with } f\in W^{-1,6}(\mathcal{P}), \quad \text{but } \Psi(\xi\psi)\notin W^{1,3}(\mathcal{P}).$$

This, however, cannot be true in view of 2, 3 and [10, Thm. 2.1]. In the Neumann case, due to Corollary 5.13 and Proposition 4.1, one ends up with the following mixed boundary value problem

$$-\nabla\cdot\mu|_{\mathcal{P}}\nabla\big(\Psi(\xi\psi)\big) = f \text{ with } f\in W_\Gamma^{-1,6}(\mathcal{P}), \quad \text{but } \Psi(\xi\psi)\notin W^{1,3}(\mathcal{P}). \tag{7.2}$$

But (7.2) cannot be correct, due to Theorem 7.4. □

# 8 Counterexamples

In this section we show that an edge with a critical singularity (a corresponding operator $\mathcal{A}_\lambda$ with $\Re\lambda \in \,]0, 1/3]$ and nontrivial kernel) gives rise to a vertex which does not fulfil the conclusion of Theorem 6.1. This shows that our suppositions are in essence also necessary. Remember that for a sector $S_\theta^\vartheta$ we set $\mathcal{S}_\theta^\vartheta := S_\theta^\vartheta \times \mathbb{R}$.

**Lemma 8.1.** *Let $S_{\theta_0}^{\theta_n}$ be a sector in $\mathbb{R}^2$ and $\mu$ a coefficient function on $\mathcal{S}_{\theta_0}^{\theta_n}$ which is constant on wedges $\mathcal{S}_{\theta_0}^{\theta_1}, \dots, \mathcal{S}_{\theta_{n-1}}^{\theta_n} \subseteq \mathcal{S}_{\theta_0}^{\theta_n}$. Let $\mathcal{A}_\lambda$ be the operator associated to $\mu_E$ (cf. Definition 5.2), combined with Neumann conditions on $]\theta_0, \theta_n[$ in case of $\theta_n \neq \theta_0 + 2\pi$. If $\mathcal{A}_\lambda$ has a non-trivial kernel for a $\lambda$ with $\Re\lambda \in \,]0, 1[$, then there exists a pyramidal vertex $\Pi_0$ with vertex in $0 \in \mathbb{R}^3$ and basis $\Lambda$ and an element $g \in W_{\partial\Pi_0\setminus\Lambda}^{-1,6}(\Pi_0)$, such that the solution $u \in W_{\partial\Pi_0\setminus\Lambda}^{1,2}(\Pi_0)$ of*

$$-\nabla\cdot\mu\nabla u = g$$

*is not in $W^{1,p}(\Pi_0 \cap \kappa\mathcal{C})$ for every $\kappa > 0$ and every $p \geq 2/(1-\Re\lambda)$.*

*Proof.* Let $S := [-1, 1]^2$ denote the square in $\mathbb{R}^2$. By 1. of Lemma 5.12 there is a function $\psi_0 \in W_{\mathrm{loc}}^{1,2}(S_{\theta_0}^{\theta_n})$, whose gradient does not belong to $L^p(S_{\theta_0}^{\theta_n} \cap \kappa S)$ for any $\kappa > 0$ and $p \geq 2/(1-\Re\lambda)$ and which satisfies $\nabla\cdot\mu_E\nabla\psi_0 = 0$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function that is identically 1 on $[-\frac{1}{2}, \frac{1}{2}]$ and 0 outside $[-1, 1]$. Furthermore, let $\xi_0 : \mathbb{R} \to \mathbb{R}$ be a $C^2$ function with

$$\xi_0(z) = \begin{cases} 0, & \text{for } z \leq -2, \\ z^4, & \text{for } z \in [-1, 0], \\ 0, & \text{for } z > 0. \end{cases}$$

We define

$$\xi(x, y, z) := \xi_0(z)\chi\Big(\frac{x}{|z|}\Big)\chi\Big(\frac{y}{|z|}\Big).$$

Obviously, $\xi$ is from $C^1(\mathbb{R}^3)$ and has, additionally, a compact support that is contained in $\bigcup_{z\in[-2,0]}(|z| \cdot S \times \{z\})$. Now, we set $\Pi_1 := \bigcup_{z\in]-3,0[}(2|z| \cdot S \times \{z\})$ and define the pyramid $\Pi_0 := \mathcal{S}^{\theta_n}_{\theta_0} \cap \Pi_1$. Clearly, its basis is then $\Lambda := \big(S^{\theta_n}_{\theta_0} \cap 6S\big) \times \{3\}$. As in 2. of Lemma 5.12, we define $\psi : \mathbb{R}^3 \to \mathbb{C}$ by $\psi(x, y, z) = \psi_0(x, y)$. According to Lemma 5.12, the function $\xi\psi$ then satisfies (5.3), where $\Upsilon := \Pi_0$ and $\Gamma := \emptyset$ in case of an interior edge and $\Gamma := \partial\mathcal{S}^{\theta_n}_{\theta_0} \cap \Pi_0$, if $\mathcal{A}_\lambda$ is complemented by a Neumann condition.

It follows easily from Lemma 5.12, the definition of $\xi$ and Fubini's theorem that $\nabla(\xi\psi) \notin L^p(\Pi_0 \cap \kappa\mathcal{C})$ for every $p \geq 2/(1 - \Re\lambda)$ and every $\kappa > 0$.

Moreover, the function $\xi\psi$ vanishes in a neighbourhood of $\partial\Pi_0\backslash\{0\}$ and therefore satisfies Neumann conditions in the distributional sense. □

Having this lemma at hand, we will see that our suppositions in Theorem 6.1 cannot be relaxed in general, since there are always situations, where a critical $\lambda$ with real part smaller than 1/3 appears. In this case $2/(1 - \Re\lambda)$ is inferior to 3, so the assertion of Theorem 6.1 cannnot be true.

In detail: If there are more than two sectors meeting in a boundary edge, i.e. Condition 3 of Theorem 6.1 does not hold, then there are examples (see [32, Ch 2.3] and references there), where $\ker(\mathcal{A}_\lambda)$ is nontrivial for $\Re\lambda$ arbitrarily small, even in the case of only scalar coefficients. If one drops the assumption that the opening angles should not exceed $\pi$, cf. the notion of bimaterial outer edge in Definition 5.9, then, inspecting the proof of [10, Lemma 2.3], one finds situations where the minimal real part of the singular values is arbitrarily close to only $\frac{1}{4}$. Finally, in the case of interior edges, Elschner's counterexample (see [11, Ch. 4]) provides examples violating Condition 4 of Theorem 6.1.

# 9 Concluding remarks

**Remark 9.1.** The regularity result of this paper easily carries over to problems with Robin boundary conditions. Indeed, one can prove that if $\varpi$ is the surface measure on $\partial\Omega$ and $\varkappa \in L^\infty(\partial\Omega, \varpi)$, then the linear map $T : W^{1,p}(\Pi) \to \big(W^{-1,p'}(\Pi)\big)'$ given by

$$\langle T\psi, \varphi\rangle = \int_\Gamma \varkappa\, \psi\, \overline{\varphi}\, \mathrm{d}\varpi,$$

representing the Robin boundary condition, is infinitesimally small with respect to the operator $-\nabla \cdot \mu\nabla$. Thus, the domains of both operators are the same by classical perturbation theory, see [24, Ch. IV.1].

**Remark 9.2.** The assumption that at most two materials should meet at the nappe could be relaxed by supposing that the kernels of the corresponding operators $\mathcal{A}_\lambda$ also are trivial in either the Dirichlet and the Neumann case, provided that $\Re\lambda \leq \frac{1}{3}$. But, because we do not know a natural class of geometric and coefficient configurations, besides our supposition that only two materials meet, we abandon this.


## Acknowledgements

We are grateful to Hans-Günther Bothe for very useful discussions towards Section 4.1.

Research by GMZ is supported by DFG Research Center MATHEON and by ERC Advanced Grant agreement no. 247029-SDModels.


# References


[1] P. Alexandroff, H. Hopf, Topologie, vol. 45 of Grundlehren der math. Wissenschaften, Springer, Berlin, 1935; berichtigter Reprint, Springer, 1974.

[2] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: "Function spaces, differential operators and nonlinear analysis" (H.-J. Schmeisser et al., eds.), vol. 133 of Teubner-Texte Math., Teubner, Stuttgart, 1993, 9–126.

[3] U. Bandelow, H.-C. Kaiser, T. Koprucki, J. Rehberg, Modeling and simulation of strained quantum wells in semiconductor lasers, in: "Mathematics – Key Technology for the Future" (W. Jäger, H.-J. Krebs, eds.), Springer, Berlin, 2003, 377–390.

[4] Y. M. Berezanskij, Selfadjoint operators in spaces of infinitely many variables, vol. 63 of Translations of Mathematical Monographs, Amer. Math. Soc., Providence/RI, 1986.

[5] R. H. Bing, The Geometric Topology of 3-Manifolds, vol. 40 of AMS Colloquium Publications, Amer. Math. Soc., Providence/RI, 1983.

[6] L. A. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, Comm. Pure Appl. Math. 51 (1998) 1–21.

[7] M. Chipot, D. Kinderlehrer, G. V. Caffarelli, Smoothness of linear laminates, Arch. Rational Mech. Anal. 96 (1986) 81–96.

[8] A. Cianchi, V. Maz'ya, Neumann problems and isocapacitary inequalities, J. Math. Pures Appl. 89 (2008) 71–105.

[9] M. Dauge, Neumann and mixed problems on curvilinear polyhedra, Integral Equations Operator Theory 15 (1992) 227–261.

[10] J. Elschner, H.-C. Kaiser, J. Rehberg, G. Schmidt, $W^{1,q}$ regularity results for elliptic transmission problems on heterogeneous polyhedra, Math. Models Methods Appl. Sci. 17 (2007) 593–615.

[11] J. Elschner, J. Rehberg, G. Schmidt, Optimal regularity for elliptic transmission problems including $C^1$ interfaces, Interfaces Free Bound. 9, No. 2, (2007) 233-252.

[12] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer, 1983.

[13] E. Giusti, Metodi diretti nel calcolo delle variazioni, Unione Matematica Italiana, Bologna, 1994.

[14] P. Grisvard, Elliptic problems in nonsmooth domains, vol. 24 of Monographs and Studies in Math., Pitman, Boston, 1985.

[15] K. Gröger, A $W^{1,p}$ estimate for solutions to mixed boundary value problems for second order elliptic differential equations, Math. Ann. 283 (1989) 679–687.

[16] R. Haller-Dintelmann, H.-C. Kaiser, J. Rehberg, Elliptic model problems including mixed boundary conditions and material heterogeneities, J. Math. Pures Appl. 89 (2008) 25–48.

[17] R. Haller-Dintelmann, C. Meyer, J. Rehberg, A. Schiela, Hölder continuity and optimal control for nonsmooth elliptic problems, Appl. Math. Optim. 60 (2009) 397–428.

[18] R. Haller-Dintelmann, H.-C. Kaiser, J. Rehberg, Direct computation of elliptic singularities across anisotropic, multi-material edges, J. Math. Sci. (N.Y.) 172 (2011) 589–622.

[19] R. Haller-Dintelmann, J. Rehberg, Maximal parabolic regularity for divergence operators including mixed boundary conditions, J. Differential Equations 247 (2009) 1354–1396.

[20] M. Hieber, J. Rehberg, Quasilinear parabolic systems with mixed boundary conditions, SIAM J. Math. Anal. 40 (2008) 292–305.

[21] D. Hömberg, Ch. Meyer, J. Rehberg, W. Ring, Optimal control for the thermistor problem, SIAM J. Control Optim. 48 (2010) 3449–3481.

[22] D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.

[23] H.-C. Kaiser, J. Rehberg, Optimal elliptic regularity at the crossing of a material interface and a Neumann boundary edge, J. Math. Sci. (N.Y.) 169 (2010) 145–166.

[24] T. Kato, Perturbation Theory for Linear Operators, vol. 132 of Grundlehren der math. Wissenschaften, 2nd ed., Springer, Berlin, 1976.

[25] D. Knees, On the regularity of weak solutions of quasi-linear elliptic transmission problems on polyhedral domains, Z. Anal. Anwendungen 23 (2004) 509–546.

[26] D. Leguillon, E. Sanchez-Palenzia, Computation of Singular Solutions in Elliptic Problems and Elasticity, Wiley, Chichester, 1987.

[27] Y. Y. Li, M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, Arch. Ration. Mech. Anal. 153 (2000) 91–151.

[28] J. Lukkainen, J. Väisälä, Elements of Lipschitz topology, Ann. Acad. Sci. Fenn., Ser. A I 3 (1977) 85–122.

[29] V. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[30] V. Maz'ya, J. Elschner, J. Rehberg, G. Schmidt, Solutions for quasilinear evolution systems in $L^p$, Arch. Ration. Mech. Anal. 171 (2004) 219–262.

[31] V. G. Maz'ya and S. A. Nazarov, *Singularities of solutions of the Neumann problem at a conic point*, Sibirsk. Mat. Zh. **30** (1989) 52–63, 218.

[32] D. Mercier, Minimal regularity of the solutions of some transmission problems, Math. Methods. Appl. Sci. 26 (2003) 321–348.

[33] N. G. Meyers, An $L^p$-estimate for the gradient of solutions of second order elliptic divergence equations, Ann. Scuola Norm. Sup. Pisa 17 (1963) 189–206.

[34] E. E. Moise, Geometric Topology in Dimensions 2 and 3, vol. 47 of Graduate Texts in Math., Springer, New York, 1977.

[35] E. M. Ouhabaz, Analysis of heat equations on domains, vol. 31 of London Math. Soc. Monographs, Princeton University Press, Princeton/NJ, 2005.

[36] M. Petzoldt, Regularity results for Laplace interface problems in two dimensions, Z. Anal. Anwendungen 20 (2001) 431–455.

[37] G. Savaré, Regularity results for elliptic equations in Lipschitz domains, J. Funct. Anal. 152 (1998) 176–201.

[38] L. Siebenmann, D. Sullivan: On complexes that are Lipschitz manifolds, in: "Geometric Topology", Proc. 1977 Georgia Topology Conf., Athens/GA 1977 (J. C. Cantrell, ed.), Academic Press, New York, 1979, 503–525.

[39] D. Zanger, The inhomogeneous Neumann problem in Lipschitz domains, Comm. Partial Differential Equations 25 (2000) 1771–1808.