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The classical solvability of the contact angle problem for generalized equations of mean curvature type

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Abstract

In this paper, mean curvature type equations with general potentials and contact angle boundary conditions are considered. We extend the ideas of Ural'tseva, formulating sharper hypotheses for the existence of a classical solution. Corner stone for these results is a method to estimate quantities on the boundary of the free surface. We moreover provide alternative proofs for the higher-order estimates, and for the existence result.

1 Introduction

We consider the problem to determine the shape and position of a free surface S parameterized by a function $\psi : G \subset \mathbb{R}^n \to \mathbb{R}$ ($n =$ space dimension), obeying on S the relation

$$\operatorname{div}_S \sigma_q(\cdot,\, -\nabla\psi,\, 1) = \phi(\cdot,\, \psi)\,, \tag{1}$$

where div_S is the surface divergence operator, $\sigma : G \times \mathbb{R}^{n+1} \to \mathbb{R}$, $(x, q) \mapsto \sigma(x, q)$ is a potential, and $\sigma_q := \nabla_q \sigma$. In the case of constant isotropic potential $\sigma^0(q) = |q|$, the equation (1) reduces to the problem of surfaces with prescribed mean curvature. The right-hand side $\phi :\ G \times \mathbb{R} \to \mathbb{R}$ is a given function.

Since the free surface S is represented as the graph of the function ψ, the equation (1) on the manifold S can be reduced to a problem posed in the domain G of parameterization. We define for $x \in G$, $p \in \mathbb{R}^n$

$$F(x,\, p) := \sigma(x,\, -p,\, 1), \quad F_p := \nabla_p F\,. \tag{2}$$

Under the assumption that σ is twice continuously differentiable, we can define

$$R(x,\, p) := \sum_{i,j=1}^{n+1} \sigma^0_{q_i}(x,\, -p,\, 1)\, \sigma^0_{q_j}(x,\, -p,\, 1)\, \sigma_{q_i,x_j}(x,\, -p,\, 1)\,, \tag{3}$$

and introduce a function $\Phi :\ G \times \mathbb{R} \times \mathbb{R}^n$ via

$$\Phi(x,\, x_{n+1},\, p) := \phi(x,\, x_{n+1}) + R(x,\, p)\,. \tag{4}$$

In the domain G, we consider the contact angle problem

$$-\operatorname{div} F_p(\cdot,\, \nabla\psi) = \Phi(\cdot,\, \psi,\, \nabla\psi) \quad \text{in } G, \tag{5}$$

$$-F_p(\cdot,\, \nabla\psi) \cdot n_G = \kappa \qquad \text{on } \partial G, \tag{6}$$

with the (generalized) angle of contact κ. Under the assumption that the potential σ is positively homogeneous of degree one in the $q-$variable, the equation (5) is equivalent to (1) (cf. Lemma B.1).

Physical applications of the model (1), respectively (5), (6) are to find for instance in thermodynamical context. The equation (1) is known as generalized Gibbs-Thomson relation. The surface S typically represents a phase transition, and σ is the surface tension on S. The right-hand side ϕ in (1) may involve quantities such as chemical potential, temperature and mechanical stresses on S: see the book [Vis96], Ch. IV for models in crystallization, or [LM89], [GK10] for related mathematical developments. Technical applications for the model (1) are for instance processes in industrial crystal growth, where curvature effects on the crystallization interface are assumed to be responsible for the formation of defects (cf. [DDEN08]).

The problem (5) has been thouroughly studied in the seventies, in connection with either the Dirichlet or the contact angle problem: see [Gia74], [Ger74], [Giu76] among others for the BV approach, see [Fin65], [Ser69], [Ura73], [Ura75], [SS76] a. o. for the classical approach, which is going to retain our attention in this paper.

To our knowledge a *local* L^∞ estimate on $\nabla\psi$ was first obtained in the papers [Mir67], [BDM69] for the problem of minimal surfaces (that is $\phi = 0$, $\sigma(q) = |q|$). The local boundedness of the gradient was then proved for general singular quasilinear equations in [LU70] on the basis of profound results of geometric measure theory. Local estimates employing other methods were also derived early (cf. [Tru73]) by the authors of [GT01] (see Chapter 16). It is to note that the *a priori* estimate derived in these papers for C^2 solutions being local, they did not lead to the solvability of (5), (6).

The global estimate on the gradient for the contact angle problem (5), (6) was first obtained in the papers [Ura71], [Ura73], [Ura75] for general $\sigma = \sigma(q)$, mainly via extension of the methods of [LU70]. In connection with at that time advanced continuation methods in Banach-spaces, the estimate also allowed to prove the classical solvability. In [Ura71] the validity of these results was restricted to (strictly) convex $\mathcal{C}^{2,\alpha}-$domains G, and a vanishing angle of contact. The theory for convex domains and a constant nonvanishing angle of contact κ was introduced in [Ura73]; Finally, the results were extended in [Ura75] to variable κ and nonconvex $\mathcal{C}^3-$domains, but only for the case $\sigma = |q|$ (mean curvature equation).

With respect to the $\phi-$data, it was in these papers necessary to assume that $\phi \in C^{1,\alpha}(\overline{G} \times \mathbb{R})$ satisfies the strict inequality

$$-\gamma_0 := \operatorname*{esssup}_{G\times\mathbb{R}} \phi_{x_{n+1}} < 0\,. \tag{7}$$

Other approaches to the results of [Ura75] for the mean curvature equation were later discussed in the papers [SS76], Th. 3 or in [Ger79], that state the gradient estimate for (nonconvex) $\mathcal{C}^4$ domains. The case of singular contact angles is discussed in [KS96]. More recent studies are more often related to the Dirichlet problem for the mean curvature equation: see [Mar10] and references. Note, also in connection to the Dirichlet problem, the interesting generalizations of (5) considered in [SW87].

In this paper, we generalize the ideas of Ural'tseva on classical solvability of (5), (6) to potentials $\sigma = \sigma(x, q)$. Note that in this case, the function R defined in (3) in general does not vanish. We moreover significantly weaken the required hypotheses regarding the regularity of the domain G (allowed to be $C^{2,\alpha}$ and nonconvex) and of the function ϕ. In particular, we allow for discontinuities of $\nabla\phi$ in $G\times\mathbb{R}$, which might be important in case that the function ϕ involves quantities that satisfy transmission conditions on the surface S. The key-point for these new results is a method (lacking in the references) to estimate quantities on the boundary of S.

Our hope is that these improvements will in particular allow to tackle coupled problems involving (5). In this context, we also propose a new method to estimate $\max|\psi|$, a quite important point if the quantity ϕ is defined only in a bounded cylinder $G\times]-M,\, M[$. Finally, we provide new proofs under sharper hypotheses for the higher-order estimates, and for the existence result.

2 Notations and statement of the main result

In this section, we introduce the basic notation and state the main hypotheses and results of our paper.

Let $n\geq 2$ denote the space dimension, and let the parameterization domain $G\subset\mathbb{R}^n$ be simply connected and belong to the class $\mathcal{C}^2$. To fix the notation, we recall that

$$\begin{aligned}\overline{G}\times\mathbb{R}^{n+1} &:= \{(x,\, q)\,:\, x\in\overline{G},\, q\in\mathbb{R}^{n+1}\}\\ \overline{G}\times\mathbb{R}^{n} &:= \{(x,\, p)\,:\, x\in\overline{G},\, p\in\mathbb{R}^{n}\}\,.\end{aligned} \tag{8}$$

For the potential $\sigma:\ \overline{G}\times\mathbb{R}^{n+1}\to\mathbb{R}$ appearing in (1), and for the function $F:\ \overline{G}\times\mathbb{R}^n\to\mathbb{R}$ defined in (2), it is convenient to introduce the abbreviations

$$\sigma_{q_i} := \partial_{q_i}\sigma \text{ for } i=1,\ldots,n+1,\quad F_{p_i}=\partial_{p_i}F \text{ for } i=1,\ldots,n\,, \tag{9}$$

Moreover, if $p:G\to\mathbb{R}^n$, we use the symbol $\frac{d}{dx}$ with the convention

$$\frac{d}{dx}F(x,\,p(x)) = F_x(x,p)+F_p(x,p)\cdot\nabla_x p\,. \tag{10}$$

Throughout the paper, the function σ is assumed to satisfy

$$\sigma\in C^{4,\alpha}(\overline{G}\times\mathbb{R}^{n+1}\setminus\{0\})\,(\alpha>0)\,. \tag{11}$$

We assume that there exist positive constants ν_j ($j=0,2$) and μ_i ($i=0,\ldots,4$) such that for all $(x,\,q)\in\overline{G}\times\mathbb{R}^{n+1}$

$$\nu_0\,|q|\leq\sigma(x,\,q)\leq\mu_0\,|q| \tag{12a}$$

$$|\sigma_q(x,\,q)|\leq\mu_1 \tag{12b}$$

$$\frac{\nu_2}{|q|}\,|\xi|^2\leq\sum_{i,j=1}^{n+1}\sigma_{q_i,q_j}(x,\,q)\,\xi_i\,\xi_j\leq\frac{\mu_2}{|q|}\,|\xi|^2 \quad \text{for all } \xi\in\mathbb{R}^{n+1} \text{ such that } \xi\cdot q=0 \tag{12c}$$

$$\sum_{i=j}^{n+1}\sigma_{q_i,q_j}(x,\,q)\,q_j=0\quad \text{for } i=1,\ldots,n+1 \tag{12d}$$

$$|\sigma_{q,x}(x,\, q)| \leq \mu_3\,, \quad |\sigma_{q,x,x}(x,\, q)| \leq \mu_4\,. \tag{12e}$$

Note that the hypotheses (12a), (12b), (12c) and (12d) are in particular satisfied if σ is positively homogeneous of degree one in the q variable (cf. [LU70], [Ura71] for a proof). For the function $\phi : \overline{G} \times \mathbb{R} \to \mathbb{R}$, the regularity

$$\phi \in W^{1,\infty}(G \times \mathbb{R})\,, \tag{13}$$

is at least required. Moreover, the gradient of Φ should have traces on manifolds. However, it is too restrictive to assume continuous differentiability, if, for instance in a coupled problem, the function ϕ involves quantities satisfying some transmission conditions on a surface $S' \subset \overline{G}\times\mathbb{R}$. Therefore, in addition to (13), we assume that there is $\psi' \in C^{0,1}(\overline{G})$ such that

$$\phi \in V_{\psi'} \Leftrightarrow \phi \in W^{2,1}(G \times \{x_{n+1} < \psi'\}) \text{ and } \phi \in W^{2,1}(G \times \{x_{n+1} > \psi'\})\,. \tag{14}$$

An important property of the class $V_{\psi'}$ is stated in Lemma A.2. We at last assume that

$$\kappa \in C^{1,\alpha}(\partial G)\ (\alpha > 0)\,. \tag{15}$$

The main result on existence, uniqueness and regularity for the problem (5), (6) is formulated in the following theorem, that extends the Theorems 1 and 2 of [Ura73].

Theorem 2.1. *Assume that $G \subset \mathbb{R}^n$ is a domain of class $\mathcal{C}^{2,\alpha}$ ($\alpha > 0$). Let σ satisfy the assumptions* (11)*,* (12)*, and assume that $\phi \in W^{1,\infty}(G \times \mathbb{R}) \cap V_{\psi'}$ (cp.* (14)*) satisfies* (7)*. Let Φ be given by* (4)*. Let $\kappa \in C^{1,\alpha}(\partial G)$ satisfy $\|\kappa\|_{L^\infty(\partial G)} < \nu_0$.*

Then, there exists a solution $\psi \in C^{2,\alpha}(\overline{G})$ to the problem (5), (6)*. If $\mu_3 < 2\,\gamma_0\,\nu_2$, the function ψ is the unique (weak) solution to* (5), (6) *in the class $W^{1,1}(G)$.*

Our second main result states an exact dependence on $\|\psi\|_{L^\infty(G)}$ with regard to the ϕ data.

Theorem 2.2. *Same assumptions as in 2.1. Let ψ denote the unique classical solution to* (5)*,* (6)*. Define $\gamma_0 > 0$ as in* (7)*. Then, there exist constants $M,\, c > 0$ independent on ϕ such that*

$$\|\psi\|_{L^\infty(G)} \leq M + c\,\gamma_0^{-1}\,(1 + \|\phi\|_{L^\infty(G\times[-M,M])})\,.$$

In our last main result, we prove that the condition (7) can be relaxed, and that the strong monotonicity is actually needed only for large arguments.

Theorem 2.3. *Same assumptions as in 2.1 except for* (7)*. Assume instead that there is $M > 0$ such that*

$$\operatorname*{esssup}_{G\times\mathbb{R}}\ \phi_{x_{n+1}} \leq 0\,, \tag{16}$$

$$-\gamma_0 := \operatorname*{esssup}_{G\times\{|x_{n+1}|>M\}}\ \phi_{x_{n+1}} < 0\,. \tag{17}$$

Then there exists a solution $\psi \in C^{2,\alpha}(\overline{G})$ to the problem (5), (6)*. If $\mu_3 = 0$, the solution is unique in this class provided that there is $G' \subseteq G$ of positive measure, such that*

$$\phi_{x_{n+1}} < 0 \quad \textit{almost everywhere in } G' \times \mathbb{R}\,. \tag{18}$$

3 Preliminary propositions

We begin this section by stating a few elementary consequences of the hypotheses formulated in the preceding section; They are are already to find in [LU70] or [Ura73], and are recalled for convenience. Afterwards we state a boundary inequality on the manifold S.

Note first for $p \in \mathbb{R}^n$ that $q := (-p, 1) \in \mathbb{R}^{n+1} \setminus B_1(0)$, and therefore, the definitions (2), (3) and the assumption (11) imply that

$$F \in C^{4,\alpha}(\overline{G} \times \mathbb{R}^n), \quad R \in C^{2,\alpha}(\overline{G} \times \mathbb{R}^n)\,. \tag{19}$$

Moreover, using (12e), the definitions (3) and (4) immediately yield

$$|R(x,\, p)| \leq \mu_3\,, \quad |\Phi(x,\, x_{n+1},\, p)| \leq |\phi(x,\, x_{n+1})| + \mu_3\,. \tag{20}$$

Due to (12a) and the Taylor formula, there is for all $(x, q) \in \overline{G} \times \mathbb{R}^{n+1} \setminus \{0\}$ a $\lambda \in]0, 1[$ such that[1]

$$0 = \sigma(x,\, 0) = \sigma(x,\, q) - \sigma_q(x\, q)\, q + \frac{1}{2}\, \sigma_{q_i, q_j}(x,\, \lambda\, q)\, q_i\, q_j\,.$$

The property (12d) therefore implies for all $q \in \mathbb{R}^{n+1} \setminus \{0\}$ that $\sigma(x,\, q) = \sigma_q(x,\, q) \cdot q$, and it follows from (12a) that

$$\sigma_q(x,\, q) \cdot q \geq \nu_0\, |q| \quad \text{for all } q \in \mathbb{R}^{n+1}\,. \tag{21}$$

For $p \in \mathbb{R}^n$, $q := (-p,\, 1)$, it follows from (21) and the definition (2) that

$$\begin{aligned} F_p(x,\, p) \cdot p &= \sigma_q(x,\, q) \cdot q - \sigma_{q_{n+1}}(x,\, q) \\ &= \sigma(x,\, q) - \sigma_{q_{n+1}}(x,\, q)\,. \end{aligned}$$

Using (12a) and (12b), one therefore obtains from the previous assumptions on the growth of σ that

$$F_p(x,\, p) \cdot p \geq \nu_0\, \sqrt{1 + |p|^2} - \mu_1 \quad \text{for all } (x, p) \in \overline{G} \times \mathbb{R}^n\,. \tag{22}$$

Since $\sigma_q(x,\, q) \cdot q = \sigma(x, q)$, the assumption (12e) also implies that

$$|\sigma_x(x, q)| \leq \mu_3\, |q| \quad \text{for all } q \in \mathbb{R}^{n+1}\,. \tag{23}$$

For $\xi \in \mathbb{R}^n$, let $\hat{\xi} := (\xi_1, \ldots, \xi_n, 0)$ be the projection onto $\mathbb{R}^{n+1}$. For $(x, x_{n+1}) \in \overline{G} \times \mathbb{R}$, define a vector field $\tilde{\xi} := \hat{\xi} - n_S\, (\hat{\xi} \cdot n_S) \in \mathbb{R}^{n+1}$. Denoting by $(\cdot)_n$, $|\cdot|_n$ the Euclidean scalar product and the Euclidean norm in $\mathbb{R}^n$, one has

$$|\xi|_n^2 = |\hat{\xi}|_{n+1}^2 = |\tilde{\xi}|_{n+1}^2 + (\hat{\xi} \cdot n_S)_{n+1}^2\,. \tag{24}$$

[1] Whenever confusion is impossible, we use the convention that repeated indices imply summation.

Therefore, if $\xi \in \mathbb{R}^n$

$$|\tilde{\xi}|_{n+1}^2 = |\hat{\xi}|_{n+1}^2 - (\hat{\xi} \cdot n_S)_{n+1}^2 \geq \frac{|\hat{\xi}|_{n+1}^2}{1 + |\nabla\psi|^2} = \frac{|\xi|_n^2}{1 + |\nabla\psi|^2} \quad \text{in } \overline{G} \times \mathbb{R}\,. \tag{25}$$

The condition (12d) implies for $x \in G$ that

$$\begin{aligned} \sum_{i,j=1}^{n} F_{p_i,p_j}(x,\, \nabla\psi)\, \xi_i\, \xi_j &= \sum_{i,j=1}^{n+1} \sigma_{q_i,q_j}(x,\, -\nabla\psi,\, 1)\, \hat{\xi}_i\, \hat{\xi}_j \\ &= \sum_{i,j=1}^{n+1} \sigma_{q_i,q_j}(x,\, -\nabla\psi,\, 1)\, \tilde{\xi}_i\, \tilde{\xi}_j\,, \end{aligned}$$

and it follows from (12c) that

$$\frac{\nu_2\, |\tilde{\xi}|_{n+1}^2}{(1 + |\nabla\psi|^2)^{1/2}} \leq F_{p_i,p_j}(x,\, \nabla\psi)\, \xi_i\, \xi_j \leq \frac{\mu_2\, |\tilde{\xi}|_{n+1}^2}{(1 + |\nabla\psi|^2)^{1/2}}\,, \tag{26}$$

In particular, the relation (26) and (25) imply that

$$\frac{\nu_2\, |\xi|^2}{(1 + |\nabla\psi|^2)^{3/2}} \leq F_{p_i,p_j}(x,\, \nabla\psi)\, \xi_i\, \xi_j \quad \text{for all } \xi \in \mathbb{R}^n\,. \tag{27}$$

Using the hat-projection operator for $n-$vectors, we associate with functions v and vector fields w defined in the domain G extensions $\hat{v}$ and $\hat{w}$ defined in $G \times \mathbb{R}$ setting

$$\hat{v}(x, x_{n+1}) := v(x), \quad \hat{w}(x,\, x_{n+1}) = \widehat{w(x)}\,. \tag{28}$$

Observe that, $\widehat{\nabla v} = \nabla \hat{v}$ for all $v \in C^1(\overline{G})$. We also need extensions into G of the data n_G and κ given on ∂G. Note the following Remark.

Remark 3.1 (Data extension)**.** *Since G has a $\mathcal{C}^{2,\alpha}$ boundary, the unit normal has an extension $n_G := \nabla \operatorname{dist}(\cdot,\, \partial G)$ into G such that $n_G \in [C^{1,\alpha}(\overline{G})]^n$. Under the assumption* (15)*, it is possible to assume that $\kappa \in C^{1,\alpha}(\overline{G})$. See the Lemma A.1 in the appendix. With the help of the hat-operator* (28)*, we then obtain extensions $n_G \in [C^{1,\alpha}(\overline{G} \times \mathbb{R})]^{n+1}$ and $\kappa \in C^{1,\alpha}(\overline{G} \times \mathbb{R})$*

We now want to prove an inequality associated with the surface S. For $\psi \in C^{0,1}(\overline{G})$, the graph $S \subset \mathbb{R}^{n+1}$ of ψ is the set

$$S := \{(x,\, x_{n+1}) \in \overline{G} \times \mathbb{R} \,:\, x_{n+1} = \psi(x)\}\,. \tag{29}$$

A unit normal on the surface S is given almost everywhere by

$$n_S(x,\, \psi(x)) := \frac{1}{\sqrt{1 + |\nabla\psi(x)|^2}} \begin{pmatrix} -\nabla\psi(x) \\ 1 \end{pmatrix}\,. \tag{30}$$

The natural surface measure on the surface S is given by

$$dH_n := \sqrt{1 + |\nabla\psi|^2}\, d\lambda_n\,. \tag{31}$$

For $f \in C^1(\mathbb{R}^{n+1})$, the differential operator

$$\delta f := \nabla f - (\nabla f \cdot n_S)\, n_S\,, \tag{32}$$

plays a crucial role in the problem (5), (6). On S, it is identical with the surface gradient. For $v \in C^1(\overline{G})$, we also introduce

$$\delta v := \nabla v - \frac{\nabla v \cdot \nabla \psi}{1 + |\nabla \psi|^2}\, \nabla \psi \quad \text{in } G\,. \tag{33}$$

The definitions (28), (32) and (33) imply for $v \in C^1(\overline{G})$ that (cp. (24))

$$|\nabla v|_n^2 = |\nabla \hat{v}|_{n+1}^2 = |\,\delta \hat{v}|_{n+1}^2 + (\nabla \hat{v} \cdot n_S)_{n+1}^2\,. \tag{34}$$

The identity (34) thus implies that

$$|\,\delta \hat{v}|_{n+1}^2 = |\nabla v|_n^2 - (\nabla \hat{v} \cdot n_S)_{n+1}^2 = |\,\delta v|_n^2\,. \tag{35}$$

Moreover, the definition (33) yields the important property (cp. (25))

$$|\,\delta v|^2 \geq (1 + |\nabla \psi|^2)^{-1}\, |\nabla v|^2 \quad \text{in } G\,. \tag{36}$$

Throughout the paper, we denote

$$\partial S := \{(x,\, x_{n+1}) \in \mathbb{R}^{n+1} \,:\, x \in \partial G,\; x_{n+1} = \psi(x)\}\,. \tag{37}$$

An important quantity related to ∂S is the tangential gradient of ψ on ∂G given by

$$\psi_t := \nabla \psi - (\nabla \psi \cdot n_G)\, n_G \quad \text{on } \partial G\,. \tag{38}$$

If α denotes the angle of contact between S and ∂G (that is, $\cos\alpha := -\nabla\psi \cdot n_G / \sqrt{1 + |\nabla \psi|^2}$ on ∂G), then

$$\sin\alpha = \left(\frac{1 + |\psi_t|^2}{1 + |\nabla \psi|^2}\right)^{1/2} \quad \text{on } \partial G\,. \tag{39}$$

Denote dH_{n-1} the standard surface measure on ∂G. Then, a natural surface measure on ∂S is defined by

$$d\mathcal{H}_{n-1} = \sqrt{1 + |\psi_t|^2}\, dH_{n-1}\,. \tag{40}$$

With these preliminaries, we can state the following inequality, which is a corner stone of the proof of Theorem 2.1.

Lemma 3.2. *Assume that G is a bounded domain of class $\mathcal{C}^2$ in $\mathbb{R}^n$. Let $S \subset \mathbb{R}^{n+1}$ denote the graph of the function $\psi \in C^2(\overline{G})$ that satisfies* (5), (6). *Assume moreover that $\|\kappa\|_{L^\infty(\partial G)} < \nu_0$. Taking into account the assumptions* (12b) *and* (12e) *and the Remark 3.1, introduce the positive constant*

$$c_0 = c_0(G,\, \Phi) := \|\Phi\|_{L^\infty(S)} + 2\, \mu_1\, \|\nabla n_G\|_{L^\infty(G)} + \mu_3\,. \tag{41}$$

Then, for every nonnegative $f \in C^1(\mathbb{R}^{n+1})$

$$\int_{\partial S} \sin\alpha^{-1}\, f\, d\mathcal{H}_{n-1} \leq (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^{-1} \left(\int_S |\delta\, f|\, dH_n + c_0 \int_S f\, dH_n\right)\,,$$

where δ is defined by (32). *The function $\sin\alpha$ and the measure $d\mathcal{H}_{n-1}$ are defined in* (39) *and* (40)*, and satisfy in particular $\sin\alpha^{-1} d\mathcal{H}_{n-1} = \sqrt{1 + |\nabla \psi|^2}\, dH_{n-1}$ on ∂S.*

Proof. Throughout the proof, we abbreviate $\sigma_q = \sigma_q(x, -\nabla\psi, 1)$. We draw the attention of the reader on the definition (28) of the hat-extension operator, and on the Remark 3.1. On the surface S, define vector fields (cf. also (39))

$$T := -(\hat{n}_G \cdot n_S)\, \sigma_q + (\sigma_q \cdot n_S)\, \hat{n}_G, \quad n_{\partial S} := \sin\alpha^{-1}\, (\hat{n}_G - (n_S \cdot \hat{n}_G)\, n_S)\,. \tag{42}$$

Note that T and $n_{\partial S}$ are tangent vectors to the surface S. Due to the divergence theorem in Lemma B.2, the identity

$$\int_S T \cdot \delta\, f\, dH_n + \int_S \operatorname{div}_S T\, f\, dH_n = \int_{\partial S} (T \cdot n_{\partial S})\, f\, d\mathcal{H}_{n-1}\,, \tag{43}$$

is valid. Using the definitions (42) and the property (21), one easily verifies that

$$\begin{aligned} T \cdot n_{\partial S} &= \sin\alpha^{-1}\, (\sigma_q - \kappa\, \hat{n}_G) \cdot n_S \\ &\geq (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})\, \sin\alpha^{-1} \quad \text{on } \partial S\,. \end{aligned} \tag{44}$$

Denote $q := (\nabla\psi, -1)$. Since all quantities appearing in the definition of the vector field T only vary in the domain G, the equivalent expression (129) for the operator div_S implies that

$$\begin{aligned} \sqrt{1 + |\nabla\psi|^2}\, \operatorname{div}_S T &= \operatorname{div}((n_G \cdot \nabla\psi)\, \sigma_q - (\sigma_q \cdot q)\, n_G) \\ &= \sum_{i,j=1}^{n} \{\partial_{x_i} n_{G,j}\, \partial_j\psi\, \sigma_{q_i} + n_{G,j}\, \partial^2_{j,i}\psi\, \sigma_{q_i} + n_{G,j}\, \partial_j\psi\, \frac{d\sigma_{q_i}}{dx_i}\} \\ &\quad - \sum_{i,j=1}^{n} \sigma_{q_j}\, \partial_{x_i} q_j\, n_{G,i} - \sum_{i=1}^{n}\sum_{j=1}^{n+1} \{\frac{d\sigma_{q_j}}{dx_i}\, q_j\, n_{G,i} + \sigma_{q_j}\, q_j\, \partial_{x_i} n_{G_i}\}\,. \end{aligned} \tag{45}$$

For $i \in \{1, \ldots, n\}$, the property (12d) yields

$$\sum_{j=1}^{n+1} \frac{d\sigma_{q_j}}{dx_i}\, q_j = \sum_{j=1}^{n+1} \{\sigma_{q_j,x_i}\, q_j - \sigma_{q_j,q_k}\, \partial^2_{k,i}\psi\, q_j\} = \sum_{j=1}^{n+1} \sigma_{q_j,x_i}\, q_j\,.$$

Using on the other hand that $\partial_{x_i} q_j = \partial^2_{i,j}\psi$, the relation (45) is equivalent to

$$\begin{aligned} \sqrt{1 + |\nabla\psi|^2}\, \operatorname{div}_S T &= (n_G \cdot \nabla\psi)\, \operatorname{div}(\sigma_q) - \operatorname{div} n_G\, (\sigma_q \cdot q) \\ &\quad + \sum_{i,j=1}^{n} \sigma_{q_i}\, \partial_{x_i} n_{G,j}\, \partial_{x_j}\psi - \sum_{i=1}^{n}\sum_{j=1}^{n+1} n_{G,i}\, q_j\, \sigma_{q_j,x_i}\,. \end{aligned} \tag{46}$$

Therefore, using also the equation (5), it follows from (46) that

$$\operatorname{div}_S T = -(\hat{n}_G \cdot n_S)\, \Phi - (\sigma_q \cdot \nabla)\hat{n}_G \cdot n_S + \operatorname{div} n_G\, (\sigma_q \cdot n_S) - \frac{n_{G,i}\, q_j\, \sigma_{q_j,x_i}}{\sqrt{1 + |\nabla\psi|^2}}\,,$$

and, with the help of (12b) and (12e), that

$$|\operatorname{div}_S T| \leq \|\Phi\|_{L^\infty(S)} + 2\,\mu_1\, \|\nabla n_G\|_{L^\infty(G)} + \mu_3\,. \tag{47}$$

As a consequence of (43), (44) and (47), the claim follows. $\square$

4 *A priori* estimates on ψ in L^p

The natural $W^{1,1}$ estimate, and the global boundedness of weak solutions to (5) have been discussed in different papers. For the contact angle problem (6), the monotonicity condition (7), the $C^{1,\alpha}$ regularity of the data $n_G,\, \kappa$ and the inequality $\|\kappa\|_{L^\infty(\partial G)} < \nu_0$ are sufficient to obtain such bounds (this was proved in [Ura71]). In the two next Lemma, we show that it is possible to relax these conditions, assuming only that there are γ_0 and $M > 0$ such that (17) is valid.

Lemma 4.1. *Assume that G is of class $\mathcal{C}^{0,1}$. Let $\psi \in W^{1,1}(G)$ satisfy* (5), (6) *in the weak sense, where Φ is given by* (4)*, with $\phi \in C^{0,1}(G \times \mathbb{R})$ satisfying* (14) *and* (17)*. Then, there is a constant c depending on G, n and on $\|\kappa\|_{L^\infty(\partial G)}$, ν_0 such that*

$$\nu_0 \, \|\nabla \psi\|_{L^1(G)} + \gamma_0 \, \|\psi\|^2_{L^2(G)} \leq c \, (M^2 \, \gamma_0 + \gamma_0^{-1} \, (1 + \|\phi\|^2_{L^\infty(G \times [-M, M])})) \, .$$

Proof. We denote $F_p = F_p(x, \, \nabla \psi)$. By assumption, the integral relation

$$\int_G F_p \cdot \nabla \xi + \int_{\partial G} \kappa \, \xi = \int_G (\phi(\cdot, \, \psi) + R(\cdot, \, \nabla \psi)) \, \xi \, , \quad \forall \, \xi \in W^{1,1}(G) \tag{48}$$

is valid. Denote $T_M(\psi)$ the truncature of ψ at the levels M and $-M$. For all $x \in G$, the fundamental theorem of integration implies that

$$\phi(x, \, \psi) = \phi(x, \, T_M(\psi)) + \int_0^1 \phi_{x_{n+1}}(x, \, t \, \psi + (1-t) \, T_M(\psi)) \, dt \, (\psi - T_M(\psi)) \, . \tag{49}$$

The assumption (17) and Young's inequality then imply that

$$\begin{aligned} \phi(x, \, \psi) \, (\psi - T_M(\psi)) &\leq \phi(x, \, T_M(\psi)) \, (\psi - T_M(\psi)) - \gamma_0 \, (\psi - T_M(\psi))^2 \\ &\leq -\frac{\gamma_0}{2} \, (\psi - T_M(\psi))^2 + \frac{1}{2\gamma_0} \, |\phi(x, \, T_M(\psi))|^2 \, . \end{aligned} \tag{50}$$

It follows that

$$\begin{aligned} \phi(x, \, \psi) \, \psi &\leq -\frac{\gamma_0}{2} \, (\psi - T_M(\psi))^2 + \frac{1}{2\gamma_0} \, |\phi(x, \, T_M(\psi))|^2 + \phi(x, \psi) \, T_M(\psi) \\ &\leq -\frac{\gamma_0}{2} \, (\psi - T_M(\psi))^2 + \frac{1}{2\gamma_0} \, |\phi(x, \, T_M(\psi))|^2 + \phi(x, T_M(\psi)) \, T_M(\psi) \, . \end{aligned} \tag{51}$$

The estimate (20) and Young's inequality yield

$$R(x, \, \nabla \psi) \, \psi \leq \frac{\gamma_0}{4} \, (\psi - T_M(\psi))^2 + \gamma_0^{-1} \, \mu_3^2 + \mu_3 \, M \, . \tag{52}$$

On the other hand, the formula (22) implies that

$$F_p \cdot \nabla \psi \geq \nu_0 \, \sqrt{1 + |\nabla \psi|^2} - \mu_1 \, . \tag{53}$$

Inserting $\xi = \psi$ into (48), and using (51), (52) and (53) yield

$$
\begin{aligned}
&\nu_0 \int_G \sqrt{1+|\nabla\psi|^2} + \frac{\gamma_0}{4} \int_G (\psi - T_M(\psi))^2 \\
&\quad \le (\mu_1 + \gamma_0^{-1}\,\mu_3^2 + M\,\mu_3)\ \mathrm{meas}(G) + \int_{\partial G} |\kappa|\,|\psi| \\
&\qquad + \frac{1}{2\gamma_0} \int_G |\phi(x,\,T_M(\psi))|^2 + \int_G \phi(x, T_M(\psi))\, T_M(\psi)\,.
\end{aligned}
\tag{54}
$$

Since $W^{1,1}(G) \hookrightarrow L^1(\partial G)$ compactly, it follows for arbitrary $\delta > 0$ that there is $c = c(\delta)$ such that

$$
\int_{\partial G} |\kappa|\,|\psi| \le \|\kappa\|_{L^\infty(\partial G)}\, \|\psi\|_{L^1(\partial G)} \le \|\kappa\|_{L^\infty(\partial G)}\, (\delta\, \|\nabla\psi\|_{L^1(G)} + c(\delta)\, \|\psi\|_{L^1(G)})\,.
$$

Therefore, if $\kappa \neq 0$, we choose $\delta = \nu_0/2\|\kappa\|_{L^\infty(\partial G)}$, $\tilde{c} := c(\delta)$, and using Young's inequality again, it follows that

$$
\int_{\partial G} |\kappa|\,|\psi| \le \frac{\nu_0}{2}\, \|\nabla\psi\|_{L^1(G)} + \frac{\gamma_0}{8}\, \|\psi - T_M(\psi)\|^2_{L^2(G)} + (\gamma_0^{-1}\,\tilde{c}^2 + \tilde{c}\,M)\ \mathrm{meas}(G)\,,
$$

and, due to (54), that

$$
\begin{aligned}
\frac{\nu_0}{2} \int_G \sqrt{1+|\nabla\psi|^2} + \frac{\gamma_0}{8} \int_G (\psi - T_M(\psi))^2 &\le (\mu_1 + \gamma_0^{-1}\,\bar{c}^2 + \bar{c}\,M)\ \mathrm{meas}(G) \\
&+ \frac{1}{2\gamma_0} \int_G |\phi(x,\,T_M(\psi))|^2 + M \int_G |\phi(x,\,T_M(\psi))|\,,
\end{aligned}
$$

with $\bar{c} = \tilde{c} + \mu_3$, yielding the claim. □

For the following L^∞ estimate, weaker assumptions are needed than in the classical references [Ura71], [Ura73]. A restriction on $\|\kappa\|_{L^\infty(\partial G)}$ is neither needed.

Lemma 4.2. *Same assumptions as in Lemma 4.1. Define $m_0 := \min\{\gamma_0,\, \nu_0\}$. Then, there is a constant $c = c(G,\, n)$ depending continuously on the constants $\nu_i,\ \mu_i$ in* (12) *such that*

$$
\|\psi\|_{L^\infty(G)} \le M + c\, m_0^{-1}\, (1 + \gamma_0^{-1}\, (1 + \|\phi\|^2_{L^\infty(G\times[-M,\,M])}))\ \mathrm{meas}(G)^{1/(n-1)}\,.
$$

Proof. The proof is to carry out with arguments very similar to the natural estimate of Lemma 4.1, and the well-known technique of Lemma C.2. □

Remark 4.3. *The estimates in Lemma 4.1 and 4.2 depend on the ϕ data via the quantity $\|\phi\|_{L^\infty(G\times[-M,M[)}/\gamma_0^{1/2}$. The bounds are therefore not invariant on proportional increment of ϕ and its gradient (choose for instance a sequence $\{\phi_k\}$ where $\phi_k(x,\, x_{n+1}) = -\gamma_0\, k\, x_{n+1}$ with $k \in \mathbb{N}$).*

In the following Lemma, we prove for more regular data another type of L^∞ estimate, that proves the invariance mentioned in Remark 4.3. Note that the assumptions in Lemma 4.4 are exactly the usual ones for proving the boundedness of ψ in this problem ([Ura71], Th. 3).

Lemma 4.4. *Assumptions of the Theorem 2.1. If $\psi \in W^{1,1}(G) \cap C(\overline{G})$ is a weak solution to* (5), (6)*, then, there is a function $\psi_0 \in C^{2,\alpha}(\overline{G})$ determined only by σ, G and κ, such that $\psi_0 = 0$ on ∂G and such that*

$$\|\psi - \psi_0\|_{C(\overline{G})} \le \gamma_0^{-1} \left(\|\phi(\cdot, \psi_0) - K_0\|_{L^\infty(G)} + \mu_3\right),$$

with $K_0 := \operatorname{div} F_p(x,\, \nabla\psi_0) \in C^\alpha(\overline{G})$, and μ_3 from (12e).

Proof. According to the Remark 3.1, the data n_G and κ can be extended into the domain G. For $s \in \mathbb{R}$ and $x \in \overline{G}$, define

$$H(x,\, s) := F_p(x,\, s\, n_G(x)) \cdot n_G(x) + \kappa(x)\,. \tag{55}$$

One computes

$$\begin{aligned}\partial_{x_i} H(x,\, s) =& \partial_{x_i} n_{G,j}\, F_{p_j}(x,\, s\, n_G) + n_{G,j}\, F_{p_j, x_i}(x,\, s\, n_G)\\ &+ n_{G,j}\, F_{p_j,p_k}(x,\, s\, n_G)\, s\, \partial_{x_i} n_{G,k} + \partial_{x_i}\kappa\,.\end{aligned}$$

Let $q \in \mathbb{R}^{n+1}$ be the vector $q := (-s\, n_G, 1)$. The assumption (12d) implies for $k = 1, \ldots, n$ that

$$n_{G,j}\, F_{p_j,p_k}(x,\, s\, n_G(x))\, s = -\sum_{j=1}^{n+1} \sigma_{q_j,q_k}(x,\, q)\, q_j + \sigma_{q_{n+1},q_k}(x,\, q) = \sigma_{q_{n+1},q_k}(x,\, q)\,,$$

and therefore, if $i \in \{1, \ldots, n\}$

$$\begin{aligned}\partial_{x_i} H(x,\, s) =& \partial_{x_i} n_{G,j}\, F_{p_j}(x,\, s\, n_G) + n_{G,j}\, F_{p_j, x_i}(x,\, s\, n_G)\\ &+ \sigma_{q_{n+1},q_k}(x,\, -s\, n_G,\, 1)\, \partial_{x_i} n_{G,k} + \partial_{x_i}\kappa\,.\end{aligned} \tag{56}$$

On the other hand, the assumption (12c) implies that

$$\begin{aligned}\partial_s H(x, s) &= n_{G,j}\, F_{p_j,p_k}(x,\, s\, n_G)\, n_{G,k}\\ &= \sum_{j,k=1}^{n+1} \sigma_{q_j,q_k}(x,\, -s\, n_G,\, 1)\, \hat{n}_{G,j}\, \hat{n}_{G,k} \ge \frac{\nu_2}{(1+s^2)^{3/2}}\,.\end{aligned} \tag{57}$$

Due to (22), one moreover has for $|s| > 0$

$$F_p(x,\, s\, n_G(x)) \cdot n_G(x) = s^{-1}\, F_p(x,\, s\, n_G(x)) \cdot s\, n_G(x) \ge s^{-1} \left(\nu_0 \sqrt{1+s^2} - \mu_1\right).$$

It follows that

$$\lim_{s \to +\infty} F_p(x,\, s\, n_G(x)) \cdot n_G(x) \ge \nu_0, \quad \lim_{s \to -\infty} F_p(x,\, s\, n_G(x)) \cdot n_G(x) \le -\nu_0\,.$$

Due to the property $\|\kappa\|_{L^\infty(G)} < \nu_0$ (cf. Lemma A.1) there is for each $x \in G$ some $s(x) \in \mathbb{R}$ such that $H(x,\, s(x)) = 0$, and

$$\partial_{x_i} s = (n_{G,j}\, F_{p_j,p_k}(x,\, s\, n_G)\, n_{G,k})^{-1}\, \partial_{x_i} H(x,\, s)\,. \tag{58}$$

From (56), (57), we at first conclude that $s \in C^1(\partial G)$, with estimate

$$|\nabla s| \leq \nu_2^{-1} \, (1+s^2)^{3/2} \, (\mu_3 + (\mu_1+\mu_2) \, \|\nabla n_G\|_{L^\infty(G)} + \|\nabla \kappa\|_{L^\infty(G)}) \, .$$

The regularity assumption on n_G, F and κ then imply together with (58) that $s \in C^{1,\alpha}(\partial G)$. Denote by $\tilde{s} \in C^{1,\alpha}(\overline{G})$ the solution to

$$-\triangle \tilde{s} = 0 \text{ in } G, \quad \tilde{s} = s \text{ on } \partial G \, . \tag{59}$$

Let $\mu_0 > 0$ and $d(x) = \operatorname{dist}(x, \partial G)$ according to Lemma A.1. Let $f \in C_c^\infty([0, \mu_0[)$ satisfy $f(0) = 0$, $f'(0) = 1$. The function $\psi_0 := \tilde{s} \, f(d)$ then satisfies

$$-\triangle \psi_0 = 2 \, \nabla \tilde{s} \cdot \nabla d + \tilde{s} \, \triangle d \in C^{0,\alpha}(\overline{G}) \quad \text{in } G \, , \quad \psi_0 = 0 \quad \text{on } \partial G \, . \tag{60}$$

Therefore, classical results for linear second order elliptic equations (cf. the case (4) of Theorem C.1) imply that $\psi_0 \in C^{2,\alpha}(\overline{G})$. Obviously, $\nabla \psi_0 = f(d) \, \nabla s + f'(d) \, \nabla d \, s$. Therefore, $\nabla \psi_0(x) = s(x) \, n_G(x)$ for all $x \in \partial G$, and it follows that

$$-F_p(x, \, \nabla \psi_0(x)) \cdot n_G(x) = -F_p(x, \, s(x) \, n_G(x)) \cdot n_G(x) = \kappa(x) \, .$$

Define $K_0 := \operatorname{div} F_p(x, \, \nabla \psi_0)$. If $\psi \in W^{1,1}(G)$ is a solution to the problem (5), (6), then for all $\xi \in W^{1,1}(G)$

$$\int_G (F_p(x, \, \nabla \psi) - F_p(x, \, \nabla \psi_0)) \cdot \nabla \xi = \int_G (\Phi(x, \, \psi, \, \nabla \psi) - K_0) \, \xi \, . \tag{61}$$

For $1 \leq q < \infty$, choose in (61) $\xi = |\psi - \psi_0|^{q-1} \, (\psi - \psi_0)$, and use the definition (4) and the assumption (12c) to obtain that

$$-\int_G |\phi(x, \, \psi) - \phi(x, \, \psi_0)| \, |\psi - \psi_0|^q \leq \int_G |\phi(x, \, \psi_0) - K_0 + R| \, |\psi - \psi_0|^q \, .$$

Due to (7) and Hölder's inequality, it obviously follows that

$$\gamma_0 \, \|\psi - \psi_0\|_{L^{q+1}(G)}^{q+1} \leq \|\phi(x, \, \psi_0) - K_0 + R\|_{L^{q+1}(G)} \, \|\psi - \psi_0\|_{L^{q+1}(G)}^{q} \, .$$

The claim follows letting $q \to \infty$ and using (20). ☐

5 Global $L^\infty-$ estimate on $\nabla \psi$

In this section, we are concerned with *a priori* estimates satisfied by $\nabla \psi$ in L^∞, where the function ψ satisfies (5), (6) and is assumed to belong to $C^2(\overline{G})$. With respect to ϕ, we only assume that ϕ is given in $C^{0,1}(\overline{G} \times \mathbb{R})$, and in the class $V_{\psi'}$ (cp. (14)). Moreover, we are going to prove a gradient estimate for the cases (7) as well as (16). The results of this section have been proved in [Ura73] for convex $\mathcal{C}^{2,\alpha}$ domains $G \subset \mathbb{R}^n$, $\sigma = \sigma(q)$ (that is in particular $R = 0$) and $\kappa =$ const. The proof was extended in [Ura75] to arbitrary $\mathcal{C}^3$ domain and a variable angle of contact, but only for the particular case $\sigma(q) = |q|$. Slightly weaker estimates are to find in [SS76], [Ger79] for the case $\sigma(q) = \sigma^0(q) = |q|$: A $\mathcal{C}^4$ boundary is required, and the weakening (16) of the condition (7) is not discussed.

We show that the method of [Ura73] yields the global gradient estimate also for general σ, variable angle of contact and domains of class $\mathcal{C}^2$. Our contribution is to apply the Lemma 3.2 in this context. The remaining part of the estimate is obtained following the lines of [Ura73], Lemma 3 and 4, even though we consider a more general situation.

We denote by n_G and κ the extensions constructed in Remark 3.1. It was noticed for the first time in [Ura73] that under the condition $\|\kappa\|_{L^\infty(\partial G)} < \nu_0$, it is sufficient to estimate the quantity

$$v := F(x,\, \nabla\psi) + \kappa\, \nabla\psi \cdot n_G\,, \tag{62}$$

in view of the inequalities

$$(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})\, \sqrt{1+|\nabla\psi|^2} \leq v \leq (\mu_0 + \|\kappa\|_{L^\infty(\partial G)})\, \sqrt{1+|\nabla\psi|^2} \text{ in } G\,, \tag{63}$$

which are clear consequences of the condition (12a).

Throughout this section, we apply the convention that integration on the surfaces S and ∂G is always performed with respect to the standard surface measures dH_n and dH_{n-1}. The integrals on ∂S are performed with respect to the measure $d\mathcal{H}_{n-1}$ (cf. (40)) occurring in the Lemma 3.2 and B.2.

Lemma 5.1. *Assume that $G \in \mathcal{C}^2$ and that ψ satisfies* (5), (6) *and belongs to $C^2(\overline{G})$. Assume that ϕ is given in $C^{0,1}(\overline{G}\times\mathbb{R})$ and in $V_{\psi'}$, and that Φ is defined by* (4). *Let v be the function* (62). *Then, there are nonnegative numbers c_i ($i = 1,\ldots,4$) such that for all $\eta \in C^1(\overline{G})$*

$$\begin{aligned}\int_G F_{p_i,p_j}\, \partial_j v\, \partial_i \eta - (\nu_0 - \|\kappa\|_{L^\infty(\partial G)}) \int_G \phi_{x_{n+1}}\, \sqrt{1+|\nabla\psi|^2}\, \eta \\ \leq c_1 \int_G |\,\delta\,\eta| + c_2 \int_G |\eta| + c_3 \int_G \frac{|\,\delta\, v|}{\sqrt{1+|\nabla\psi|^2}} + c_4 \int_{\partial G} |\eta|\,.\end{aligned}$$

Here, c_i are continuous functions of the constants appearing in the conditions (12), *and of $\|\nabla(\kappa\, n_G)\|_{L^\infty(G)}$. The constant c_2 is depends linearly on $\|\nabla\phi^{+,-}\|_{L^\infty(S)}$ (cf. the Lemma A.2).*

Proof. For $k = 1,\ldots,n$, the relation (5) implies that

$$\int_G \frac{dF_p}{dx_k}\cdot\nabla\xi = \int_G \frac{d\Phi}{dx_k}\,\xi + \int_{\partial G} n_G \cdot \frac{dF_p}{dx_k}\,\xi\,, \tag{64}$$

for all $\xi \in W^{1,1}(G)$. For $i, k \in \{1,\ldots,n\}$, observe that

$$\frac{dF_{p_i}}{dx_k} = F_{p_i,x_k} + F_{p_i,p_j}\, \partial^2_{j,k}\psi\,. \tag{65}$$

On the other hand, if v is defined by (62), the relation

$$\partial_j v = (F_{p_k} + \kappa\, n_{G,k})\, \partial^2_{j,k}\psi + F_{x_j} + \partial_k\psi\, \partial_j(n_{G,k}\,\kappa)\,, \tag{66}$$

is valid. In (64), consider a testfunction of the form $\xi = [F_{p_k}(\cdot,\, \nabla\psi) + \kappa\, n_{G,k}]\,\eta$, with $k \in \{1,\dots,n\}$ and $\eta \in C^1(\overline{G})$ arbitrary. Due to (65) and (66), summation over $k = 1,\dots,n$ implies for each $i \in \{1,\dots,n\}$ that

$$
\begin{aligned}
[F_{p_k} + \kappa\, n_{G,k}]\,\frac{dF_{p_i}}{dx_k} =& F_{p_i,p_j}\,(\partial_j v - F_{x_j} - \partial_k\psi\,\partial_j(n_{G,k}\,\kappa)) \\
&+ [F_{p_k} + \kappa\, n_{G,k}]\,F_{p_i,x_k}\,,
\end{aligned}
$$

and it follows from (64) that

$$
\begin{aligned}
&\int_G \{F_{p_i,p_j}\,(\partial_j v - F_{x_j} - \partial_k\psi\,\partial_j(n_{G,k}\,\kappa)) + [F_{p_k} + \kappa\, n_{G,k}]\,F_{p_i,x_k}\}\,\partial_i\eta \\
&\quad + \int_G \{(F_{p_i,x_k} + F_{p_i,p_j}\,\partial^2_{j,k}\psi)\,(F_{p_k,p_l}\,\partial^2_{l,i}\psi + F_{p_k,x_i} + \partial_i(\kappa\, n_{G,k}))\}\,\eta \\
&= \int_G \frac{d\Phi}{dx_k}\,[F_{p_k} + \kappa\, n_{G,k}]\,\eta + \int_{\partial G}[F_{p_k} + \kappa\, n_{G,k}]\,\frac{dF_{p_i}}{dx_k}\cdot n_G\,\eta\,.
\end{aligned}
\tag{67}
$$

With the abbreviations

$$
\begin{aligned}
b_i^{(1)} &:= -F_{p_i,p_j}\,(F_{x_j} + \partial_j(\kappa\, n_{G,k})\,\partial_k\psi), \quad \text{for } i = 1,\dots,n\,, \\
b^{(2)} &:= F_{p_i,p_j}\,F_{p_k,p_l}\,\partial^2_{j,k}\psi\,\partial^2_{l,i}\psi \,+ F_{p_i,p_j}\,\partial^2_{j,k}\psi\,(F_{p_k,x_i} + \partial_i(\kappa\, n_{G,k})) \\
&\quad + F_{p_k,p_l}\,F_{p_i,x_k}\,\partial^2_{l,i}\psi\,, \\
b^{(3)} &:= F_{p_i,x_k}\,(F_{p_k,x_i} + \partial_i(\kappa\, n_{G,k}))\,,
\end{aligned}
\tag{68}
$$

the relation (67) reads

$$
\begin{aligned}
&\int_G F_{p_i,p_j}\,\partial_j v\,\partial_i\eta + \int_G b_i^{(1)}\,\partial_i\eta + \int_G \{b^{(2)} + b^{(3)}\}\,\eta \\
&\quad = \int_G \frac{d\Phi}{dx_k}\,[F_{p_k} + \kappa\, n_{G,k}]\,\eta + \int_{\partial G}[F_{p_k} + \kappa\, n_{G,k}]\,\frac{dF_p}{dx_k}\cdot n_G\,\eta\,.
\end{aligned}
\tag{69}
$$

The claim will follow from appropriate estimates on the functions $b^{(i)}$. Using for $j = 1,\dots,n$ the abbreviation $\xi_j := F_{x_j} + \partial_j(\kappa\, n_{G,k})\,\partial_k\psi$, the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
|b_i^{(1)}\,\partial_i\eta| &= |F_{p_i,p_j}\,\partial_i\eta\,(F_{x_j} + \partial_j(\kappa\, n_{G,k})\,\partial_k\psi)| \\
&\le \left(F_{p_i,p_j}\,\partial_i\eta\,\partial_j\eta\right)^{1/2}\,(F_{p_i,p_j}\,\xi_i\,\xi_j)^{1/2}\,.
\end{aligned}
$$

Thanks to (12c), the property (23) and (26), it follows that

$$
|b_i^{(1)}\,\partial_i\eta| \le \mu_2\,|\,\delta\,\eta|\,(\mu_3 + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)}\,\frac{|\nabla\psi|}{(1+|\nabla\psi|^2)^{1/2}}) \le c_1\,|\,\delta\,\eta|\,. \tag{70}
$$

On the other hand, Lemma 3 of [Ura73] (cp. the Lemma D.1 in the appendix) yields

$$
b^{(2)} \ge -\hat{c}_1\,\frac{|\,\delta\, v|}{\sqrt{1+|\nabla\psi|^2}} - \hat{c}_2\,. \tag{71}
$$

The assumptions (12e) moreover imply that

$$|b^{(3)}| \leq \mu_3\,(\mu_3 + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)})\,. \tag{72}$$

Due to Lemma D.2 (see the appendix for the proof)

$$\begin{aligned}\frac{d\Phi}{dx_k}\,[F_{p_k} + \kappa\, n_{G,k}] \leq & \hat{c}_3 + \hat{c}_4\, \frac{|\,\delta\, v|}{\sqrt{1+|\nabla\psi|^2}} \\ & + (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})\, \phi_{x_{n+1}}\, \sqrt{1+|\nabla\psi|^2}\,.\end{aligned} \tag{73}$$

Define $c_4 := \hat{c}_1 + \hat{c}_4$, and $c_2 := \hat{c}_3 + \hat{c}_2 + \mu_3\,(\mu_3 + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)})$.

We at last consider in (69) the surface integral, which was estimated in [Ura71], [Ura73], [Ura75] only for special cases of F, κ and G. Due to the condition (6), the operator $[F_{p_k} + \kappa\, n_{G,k}]\frac{d}{dx_k}$ is a tangential differential operator. If $\{\tau^{(k)}\}_{k=1,\ldots,n-1}$ is a system of orthonormal tangential vectors on ∂G, then, the relation (6) implies that

$$\begin{aligned}\int_{\partial G} [F_{p_k} + \kappa\, n_{G,k}]\, \frac{dF_p}{dx_k}\cdot n_G\, \eta = & \int_{\partial G} (F_p\cdot\tau^{(l)})\,(\tau^{(l)}\cdot\nabla)F_p\cdot n_G\,\eta \\ = \int_{\partial G} (F_p\cdot\tau^{(l)})\,(\tau^{(l)}\cdot\nabla\kappa)\,\eta - & \int_{\partial G} (F_p\cdot\tau^{(l)})\,(\tau^{(l)}\cdot\nabla)n_G\cdot F_p\,\eta \\ = \int_{\partial G} (F_p\cdot\tau^{(l)})\,(\tau^{(l)}\cdot\nabla\kappa)\,\eta - & \int_{\partial G} (F_p\cdot\tau^{(l)})\,(F_p\cdot\tau^{(j)})\,(\tau^{(l)}\cdot\nabla)n_G\cdot\tau^{(j)}\,\eta \\ - \int_{\partial G} (F_p\cdot\tau^{(l)})\,(F_p\cdot n_G)\,(\tau^{(l)}\cdot\nabla)n_G\cdot n_G\,\eta\,. &\end{aligned} \tag{74}$$

Note that $(\tau^{(l)}\cdot\nabla)n_G\cdot n_G = 0$, and that the eigenvalues of the symmetric matrix $a_{l,j} := (\tau^{(l)}\cdot\nabla)n_G\cdot\tau^{(j)}$ $(j,l = 1,\ldots,n-1)$ are nothing else but the principal curvatures k_i of ∂G. In the paper [Ura73], Lemma 2 these facts and the assumption $\kappa =$ const. were used to prove that the surface integral has a sign for all nonnegative η. In the general case, it follows that

$$\int_{\partial G} [F_{p_k} + \kappa\, n_k]\, \frac{dF_p}{dx_k}\cdot n_G\,\eta \leq (\|\nabla\kappa\|_{L^\infty(\partial G)} + \mu_1^2 \sup_{i=1,\ldots,n-1} |k_i|)\int_{\partial G} |\eta|\,. \tag{75}$$

Using the estimates (70), (71), (72), (73), (75) and the relation (69), the claim follows. ☐

5.1 The gradient estimate I

In this subsection, we show on the base of Lemma 5.1 that under the strong monotonicity condition (7), the estimate on $\nabla\psi$ is only polynomial in the norm of the data.

Lemma 5.2. *Same assumptions as in Lemma 5.1. Assume moreover that $\|\kappa\|_{L^\infty(\partial G)} < \nu_0$. Then, there is $K > 0$ such that for all $1 \leq q \leq \infty$*

$$\int_G v^{q-4}\,|\nabla v|^2 - \frac{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2}{\nu_2\, q}\int_G \phi_{x_{n+1}}\, v^{q+1} \leq K\int_G v^q\,. \tag{76}$$

The number K is a continuous function of the constants in the conditions (12)*, of the constants c_i of Lemma 5.1, and on $\nu_0 - \|\kappa\|_{L^\infty(\partial G)}$. Moreover, K depends linearly on $\|\nabla\phi^{+,-}\|_{L^\infty(S)}$ and on $\|\phi\|_{L^\infty(S)}$.*

Proof. For $f \in C^1(]0,\infty[)$ nonnegative and increasing, choose in Lemma 5.1 $\eta = f(v)$. The estimates (63) imply that

$$\begin{aligned} &\int_G f'(v)\, F_{p_i,p_j}\, \partial_j v\, \partial_i v - (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2 \int_G \phi_{x_{n+1}}\, v\, f(v) \\ &\quad \le c_1 \int_G |\,\delta\, v|\, f'(v) + c_2 \int_G f(v) + c_3 \int_G \frac{|\,\delta\, v|\, f(v)}{\sqrt{1+|\nabla\psi|^2}} + c_4 \int_{\partial G} f(v)\,. \end{aligned} \tag{77}$$

Due to the identity (cp. (38), (39))

$$\int_{\partial G} f(v) = \int_{\partial G} \sin\alpha^{-1}\, \frac{f(v)}{\sqrt{1+|\nabla\psi|^2}}\, \sqrt{1+|\psi_t|^2}\,, \tag{78}$$

the inequality (63) implies that (cf. (40) for the definition of the measure $d\mathcal{H}_{n-1}$)

$$\begin{aligned} \int_{\partial G} f(v) &\le (\mu_0 + \|\kappa\|_{L^\infty(\partial G)}) \int_{\partial G} \sin\alpha^{-1}\, f(v)/v\, \sqrt{1+|\psi_t|^2} \\ &= (\mu_0 + \|\kappa\|_{L^\infty(\partial G)}) \int_{\partial S} \sin\alpha^{-1}\, f(v)/v\, d\mathcal{H}_{n-1}\,. \end{aligned}$$

Applying Lemma 3.2, it follows that

$$\begin{aligned} \int_{\partial G} f(v) &\le \frac{\mu_0 + \|\kappa\|_{L^\infty(\partial G)}}{\nu_0 - \|\kappa\|_{L^\infty(\partial G)}} \left(\int_S |\,\delta(f(v)/v)| + c_0 \int_S f(v)/v \right) \\ &\le \frac{\mu_0 + \|\kappa\|_{L^\infty(\partial G)}}{\nu_0 - \|\kappa\|_{L^\infty(\partial G)}} \left(\int_S \frac{f'(v)\, v + f(v)}{v^2}\, |\,\delta\, v| + c_0 \int_S f(v)/v \right)\,. \end{aligned}$$

Writing the surface integrals over G again, the property (35) and the inequality (63) yield

$$\int_{\partial G} f(v) \le \frac{\mu_0 + \|\kappa\|_{L^\infty(\partial G)}}{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2} \left(\int_G (f'(v) + f(v)/v)\, |\,\delta\, v| + c_0 \int_G f(v) \right)\,. \tag{79}$$

Define

$$\begin{aligned} \tilde{c_1} &:= c_1 + c_4\, \frac{\mu_0 + \|\kappa\|_{L^\infty(\partial G)}}{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2} \qquad \tilde{c}_2 := c_2 + c_4\, c_0 \frac{\mu_0 + \|\kappa\|_{L^\infty(\partial G)}}{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2}\,, \\ \tilde{c_3} &:= \frac{c_3}{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})} + c_4\, \frac{\mu_0 + \|\kappa\|_{L^\infty(\partial G)}}{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2}\,. \end{aligned}$$

The estimates (77) and (79) together imply that

$$\begin{aligned} &\int_G f'(v)\, F_{p_i,p_j}\, \partial_j v\, \partial_i v - (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2 \int_G \phi_{x_{n+1}}\, v\, f(v) \\ &\quad \le \tilde{c}_1 \int_G |\,\delta\, v|\, f'(v) + \tilde{c}_2 \int_G f(v) + \tilde{c}_3 \int_G |\,\delta\, v|\, v^{-1}\, f(v)\,. \end{aligned} \tag{80}$$

From Young's inequality and (63), it follows that

$$\begin{aligned} \int_G |\,\delta\, v|\, v^{-1}\, f(v) &\le \frac{\nu_2}{4\,\tilde{c}_3} \int_G \frac{|\,\delta\, v|^2\, f'(v)}{\sqrt{1+|\nabla\psi|^2}} + \frac{\tilde{c}_3}{\nu_2\,(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})} \int_G \frac{f^2(v)}{f'(v)\, v}\,, \\ \int_G |\,\delta\, v|\, f'(v) &\le \frac{\nu_2}{4\,\tilde{c}_1} \int_G \frac{|\,\delta\, v|^2\, f'(v)}{\sqrt{1+|\nabla\psi|^2}} + \frac{\tilde{c}_1}{\nu_2\,(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})} \int_G f'(v)\, v\,. \end{aligned} \tag{81}$$

Due to (26), it follows from (80) and (81) that

$$\begin{aligned}
&\frac{\nu_2}{2} \int_G f'(v) \frac{|\,\delta\, v|^2}{\sqrt{1+|\nabla\psi|^2}} - (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2 \int_G \phi_{x_{n+1}}\, v\, f(v)\\
&\quad \leq \frac{\tilde{c}_3}{\nu_2\,(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})} \int_G \frac{f^2(v)}{f'(v)\, v} + \frac{\tilde{c}_1}{\nu_2\,(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})} \int_G f'(v)\, v\\
&\quad + \tilde{c}_2 \int_G f(v)\,.
\end{aligned} \tag{82}$$

Choose now $f(v) = v^q$ with $q \geq 1$ in (82). Then

$$\begin{aligned}
&q\, \nu_2\, (\mu_0 + \|\kappa\|_{L^\infty(\partial G)})^{-1} \int_G v^{q-2}\, |\,\delta\, v|^2 - (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2 \int_G \phi_{x_{n+1}}\, v^{q+1}\\
&\quad \leq (\tilde{c}_2 + (\nu_2\,(\nu_0 - \|\kappa\|_{L^\infty(\partial G)}))^{-1}\, (\tilde{c}_1\, q + \tilde{c}_3/q)) \int_G v^q\,.
\end{aligned}$$

Using the property (36), the claim follows. ☐

Proposition 5.3. *Same assumptions as in Lemma 5.2. Assume moreover that* (16) *is valid. Let* $q_0 > n$ *arbitrary. Then, there exist a constant* $c_5 = c_5(q_0)$ *and positive numbers* $\zeta_i = \zeta_i(q_0)$ *(*$i = 1, 2, 3$*) such that*

$$\max_G v \leq c_5\, (K^{\zeta_1} + \operatorname{meas}(G)^{\zeta_2}))\, \|v\|^{\zeta_0}_{L^{q_0}(G)}\,. \tag{83}$$

Proof. Due to the condition (16), Lemma 5.2 implies that

$$\int_G |\nabla v^{(q-2)/2}|^2 \leq \frac{K\,(q-2)^2}{4} \int_G v^q\,. \tag{84}$$

We add $\|v^{(q-2)/2}\|^2_{L^2(G)}$ on both sides of (84). Thanks to Hölder's inequality, it follows that

$$\int_G \{|\nabla v^{(q-2)/2}|^2 + |v|^{q-2}\} \leq \frac{K\,(q-2)^2}{4} \int_G v^q + \operatorname{meas}(G)^{2/q}\, \|v\|^{q-2}_{L^q(G)}\,. \tag{85}$$

Define $\chi := \frac{q_0-2}{n-2}\, \frac{n}{q_0}$ if $n > 2$, and $\chi := 2$ if $n = 2$. The choice of q_0 implies that $\chi > 1$ and that

$$\frac{2\,\chi\, q}{q-2} \leq \frac{2\, n}{n-2} \text{ for all } q_0 \leq q < \infty\,.$$

It follows that the embedding $W^{1,2}(G) \hookrightarrow L^r(G)$ for $r := 2\,\chi q/(q-2)$ is continuous, and that the embedding constant are uniformely bounded. The relation (85) implies that

$$\begin{aligned}
\|v\|^{q-2}_{L^{\chi\, q}(G)} &= \|v^{(q-2)/2}\|^2_{L^r(G)} \leq c\,((q-2)^2\, K\, \|v\|^q_{L^q(G)} + \operatorname{meas}(G)^{2/q}\, \|v\|^{q-2}_{L^q(G)})\\
&\leq c\, \max\{(q-2)^2\, K,\, \operatorname{meas}(G)^{2/q}\}\, \max\{\|v\|^q_{L^q(G)},\, \|v\|^{q-2}_{L^q(G)}\}\,.
\end{aligned} \tag{86}$$

For $m \in \mathbb{N}$, set $q_m := \chi\, q_{m-1}$, $A_m := \|v\|_{L^{q_m}(G)}$, $\lambda_m := q_m/(q_m-2)$, and define

$$\xi_m := \begin{cases} \lambda_m & \text{if } A_m \geq 1 \\ 1 & \text{otherwise} \end{cases}, \quad c_m := c\,[(q_m-2)^2\,K + \operatorname{meas}(G)^{2/q_m}]\,. \tag{87}$$

As a consequence of (86), one finds the recursive inequalities

$$\begin{aligned} & A_{m+1} \leq c_m^{1/(q_m-2)}\,A_m^{\xi_m}\,, \\ \Longrightarrow\ & A_{m+1} \leq c_m^{1/(q_m-2)}\,\{\prod_{i=0}^{m-1}[c_i]^{\xi_{i+1}/(q_i-2)}\}\,A_0^{\prod_{i=0}^m \xi_i}\,. \end{aligned} \tag{88}$$

As usual for Moser-iteration-type arguments, we now provide (rough) bounds for the products appearing in (88). Note first that

$$\log(\prod_{i=0}^m \xi_i) \leq \sum_{i=0}^m \log(q_i/(q_i-2)) \leq 2\sum_{i=0}^m \frac{1}{q_i-2} \leq \frac{2}{q_0-2}\sum_{i=0}^\infty \chi^{-i}\,,$$

Thus $\zeta_0 := \prod_{i=0}^\infty \xi_i$ satisfies the estimate $\zeta_0 \leq \exp(2\chi/(q_0-2)(\chi-1))$. Observe also that

$$\begin{aligned} \log(\prod_{i=0}^{m-1}[c_i]^{\xi_{i+1}/(q_i-2)}) &= \sum_{i=0}^{m-1}\frac{\xi_{i+1}}{q_i-2}\,\log(c_i), \\ \log(c_i) \leq \log c + \log K + 2\,\log(q_i-2) &+ 2/q_i\,\log \operatorname{meas}(G)\,. \end{aligned} \tag{89}$$

Using the estimate

$$\xi_{i+1} \leq \lambda_{i+1} = \frac{\chi^{i+1}\,q_0}{\chi^{i+1}\,q_0-2} \leq \frac{q_0}{q_0-2} \text{ for } i \in \mathbb{N}\,,$$

we can bound

$$\sum_{i=0}^{m-1}\frac{\xi_{i+1}}{q_i-2}\,\log(q_i-2) \leq \frac{q_0}{q_0-2}\sum_{i=0}^{m-1}\frac{i\,\log\chi + \log q_0}{\chi^i\,q_0-2} \leq \frac{q_0\,\log q_0}{(q_0-2)^2}\,(\sum_{i=0}^{m-1}\frac{i+1}{\chi^i})\,,$$

and

$$\begin{aligned} \zeta_1 &:= \sum_{i=0}^\infty \frac{\xi_{i+1}}{q_i-2} \leq \frac{q_0}{q_0-2}\sum_{i=0}^\infty \frac{1}{\chi^i q_0-2} \leq \frac{2\,q_0\,\chi}{(q_0-2)^2\,(\chi-1)}\,, \\ \zeta_2 &:= \sum_{i=0}^\infty \frac{2\,\xi_{i+1}}{q_i\,(q_i-2)} \leq 2\,\frac{q_0}{q_0-2}\sum_{i=0}^\infty \frac{1}{\chi^i\,(\chi^i\,q_0-2)} \leq \frac{\chi^2\,q_0}{(\chi^2-1)\,(q_0-2)^2}\,. \end{aligned}$$

Therefore, (89) implies that $\prod_{i=0}^{m-1}[c_i]^{\xi_{i+1}/(q_i-2)} \leq c_1(q_0)\,(K^{\zeta_1} + \operatorname{meas}(G)^{\zeta_2})$, and the claim follows from (88). □

Everything is therefore reduced to estimating the $L^{q_0}-$norm of v for a $q_0 > n$. We directly obtain this bound, if we require the strong monotonicity condition (7).

Proposition 5.4. *Same assumptions as in Lemma 5.2 and* (7)*. Then, there is a positive number* $c = c(G)$ *such that for all* $2 < t < \infty$*,*

$$\|v\|_{L^t(G)} \le \frac{\nu_2\,(1+t)}{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2}\ \operatorname{meas}(G)^{1/t}\,\frac{K}{\gamma_0}\,, \tag{90}$$

where K *is the constant of Lemma 5.2.*

Proof. Use Lemma 5.2 and Hölder's inequality, to obtain for $q \ge 1$ that

$$\frac{(\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^2\,\gamma_0}{q\,\nu_2}\int_G v^{q+1} \le K\int_G v^q \le K\,\|v\|^q_{L^{q+1}(G)}\ \operatorname{meas}(G)^{1/(q+1)}\,.$$

□

5.2 Gradient estimate II. The limiting case $\gamma_0 = 0$

Ural'tseva has proved in [Ura73], [Ura75] for the gradient estimate that the assumption (7) can be replaced by (16) either if the domain G is convex and κ is a constant, or if $\sigma = \sigma^0(q) = |q|$. Here, we prove the validity of these results in a more general situation. The key-point in the proof is the Sobolev embedding theorem on the manifold S recalled in the appendix under Lemma B.3.

An important feature is that the gradient estimate here depends on lower-order norm of ψ, in particular on $\|\psi\|_{L^\infty(G)}$, whereas in the preceding section, the condition (7) guaranteed an independent estimate on $\nabla\psi$ (cp. Proposition 5.4). Note also that the estimate in this section is exponential in the data.

Lemma 5.5. *Let* v *be defined by* (62)*, and* $w := \log v$*. Under the assumptions of Lemma 5.2, there is a constant* c *such that for all* $\zeta \in C^1(\overline{G})$ *and all* $q \in [1,\infty[$

$$\int_S \zeta^2\,|\,\delta[w^+]^{(q+1)/2}|^2 \le c\,(q+1)^2\int_S ([w^+]^{q-1} + [w^+]^q/q)\,(\zeta + |\,\delta\,\zeta|)^2\,. \tag{91}$$

The constant c *is a continuous function of the constants* ν_i*,* μ_i *in the conditions* (12)*, of the constants* c_i *in Lemma 5.1, and depends linearly on* $\|\nabla\phi^{+,-}\|_{L^\infty(S)}$ *and on* $\|\phi\|_{L^\infty(S)}$*.*

Proof. We start from the Lemma 5.2, choosing $\eta := \zeta^2\,v\,f(w)$, with $f \in C^1(\mathbb{R})$ positive and nondecreasing. The assumption (16) yields

$$\begin{aligned}
&\int_G f(w)\,\zeta^2\,F_{p_i,p_j}\,\partial_j v\,\partial_i v + \int_G v^2\,f'(w)\,\zeta^2\,F_{p_i,p_j}\,\partial_j w\,\partial_i w \le c_2\int_G f(w)\,v\,\zeta^2 \\
&- 2\int_G v\,f(w)\,\zeta\,F_{p_i,p_j}\,\partial_j v\,\partial_i\zeta + c_3\int_G \zeta^2\,\frac{|f(w)|\,|\,\delta\,v|\,v}{\sqrt{1+|\nabla\psi|^2}} + c_4\int_{\partial G} v\,f(w)\,\zeta^2 \\
&+ c_1\int_G \{\zeta^2\,|f(w)|\,|\,\delta\,v| + \zeta^2\,f'(w)\,v\,|\,\delta\,w| + 2\zeta\,v\,f(w)\,|\,\delta\,\zeta|\}\,.
\end{aligned} \tag{92}$$

Using Young's and Cauchy-Schwarz's inequalities

$$
\begin{aligned}
&\left|\int_G v\, f(w)\, \zeta\, F_{p_i,p_j}\, \partial_j v\, \partial_i \zeta\right| \\
&\quad \le \frac{1}{4} \int_G f(w)\, \zeta^2\, F_{p_i,p_j}\, \partial_j v\, \partial_i v + \int_G f(w)\, v^2\, F_{p_i,p_j}\, \partial_j \zeta\, \partial_i \zeta \\
&\quad \le \frac{1}{4} \int_G f(w)\, \zeta^2\, F_{p_i,p_j}\, \partial_j v\, \partial_i v + \mu_2\, (\mu_0 + \|\kappa\|_{L^\infty(\partial G)}) \int_G f(w)\, v\, |\, \delta\, \zeta|^2\,,
\end{aligned}
$$

where we used the assumption (12c) in the second line. Using also (63), it follows from (92) that

$$
\begin{aligned}
\int_G f(w)\, \zeta^2\, F_{p_i,p_j}\, \partial_j v\, \partial_i v + \int_G v^2\, f'(w)\, \zeta^2\, F_{p_i,p_j}\, \partial_j w\, \partial_i w \le c_4 \int_{\partial G} v\, f(w)\, \zeta^2 \\
+ c \int_G \{\zeta^2\, |f(w)|\, |\, \delta\, v| + \zeta^2\, f'(w)\, v\, |\, \delta\, w| + v\, f(w)\, (|\, \delta\, \zeta| + \zeta)^2\}\,. \qquad (93)
\end{aligned}
$$

The inequality (63) and Lemma 3.2 imply that (cp. (78))

$$
\begin{aligned}
&\int_{\partial G} v\, f(w)\, \zeta^2 \le (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^{-1} \int_{\partial S} \sin\alpha^{-1}\, f(w)\, \zeta^2\, d\mathcal{H}_{n-1} \\
&\quad \le (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^{-2}\, (\int_S \{f'(w)\, \zeta^2\, |\, \delta\, w| + 2\, \zeta\, f(w)\, |\, \delta\, \zeta|\} + c_0 \int_S f(w)\, \zeta^2)\,.
\end{aligned}
$$

Therefore, there is a constant c such that the inequality (93) is preserved dropping the boundary term. Estimating

$$
\begin{aligned}
\int_G \zeta^2\, |f(w)|\, |\, \delta\, v| \le& \frac{\nu_2}{4\, c} \int_G f(w)\, \zeta^2\, \frac{|\, \delta\, v|^2}{\sqrt{1 + |\nabla\psi|^2}} \\
&+ c\, \nu_2^{-1}\, (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^{-1} \int_G \zeta^2\, f(w)\, v \\
\int_G \zeta^2\, f'(w)\, v\, |\, \delta\, w| \le& \frac{\nu_2}{4\, c} \int_G \zeta^2\, |f'(w)|\, v^2\, \frac{|\, \delta\, w|^2}{\sqrt{1 + |\nabla\psi|^2}} \\
&+ c\, \nu_2^{-1}\, (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})^{-1} \int_G \zeta^2\, |f'(w)|\, v\,,
\end{aligned}
$$

the property (12c) yields

$$
\begin{aligned}
&\frac{\nu_2}{2} \left(\int_G f(w)\, \zeta^2\, \frac{|\, \delta\, v|^2}{\sqrt{1 + |\nabla\psi|^2}} + \int_G v^2\, f'(w)\, \zeta^2\, \frac{|\, \delta\, w|^2}{\sqrt{1 + |\nabla\psi|^2}}\right) \\
&\qquad \le c \int_G v\, \{f'(w) + f(w)\}\, (|\, \delta\, \zeta| + \zeta)^2\}\,.
\end{aligned}
$$

It follows from (63) that

$$
\frac{\nu_2}{2} \int_G v\, f'(w)\, \zeta^2\, |\, \delta\, w|^2 \le c \int_G v\, (f'(w) + f(w))\, (|\, \delta\, \zeta| + \zeta)^2\,. \qquad (94)
$$

For $q \in [1, \infty[$ and $s \in \mathbb{R}$, define $f(s) = \max\{s,\, 0\}^q$. Then, writing the integrals in (94) over S, we obtain that

$$q \int_S \zeta^2\, [w^+]^{q-1}\, |\, \delta\, w^+|^2 \leq c \int_S (q\, [w^+]^{q-1} + [w^+]^q)\, (|\, \delta\, \zeta| + \zeta)^2\,, \tag{95}$$

and the claim follows. ☐

Lemma 5.6. *Same assumptions as in Lemma 5.5. Then, there is $\zeta \in]0, 1]$ such that $w^+ \leq c\, \|w^+\|^{\zeta}_{L^2(S)}$. The constant c depends on the constants ν_i, μ_i and c_i. Additionally c depends on $H_n(S)$ and on the constant β of Lemma B.3.*

Proof. Observe at first that

$$\int_S \zeta^2\, |\, \delta [w^+]^{(q+1)/2}|^2 \geq \int_S |\, \delta(\zeta\, [w^+]^{(q+1)/2})|^2 - \int_S [w^+]^{q+1}\, |\, \delta\, \zeta|^2\,. \tag{96}$$

For $s = 2n/(n-2)$ if $n = 2$, $s < \infty$ arbitrary if $n = 2$, Theorem B.3 yields

$$\|\zeta\, [w^+]^{(q+1)/2})\|^2_{L^s(S)} \leq c \int_S |\, \delta(\zeta\, [w^+]^{(q+1)/2})|^2\,,$$

and it follows from Lemma B.3 that

$$\begin{aligned} &\|\zeta\, [w^+]^{(q+1)/2})\|^2_{L^s(S)} \\ &\quad \leq c\,(q+1)^2 \int_S \{[w^+]^{q-1} + [w^+]^q/q + [w^+]^{q+1}/(q+1)^2\}\, (\zeta + |\, \delta\, \zeta|)^2\,. \end{aligned} \tag{97}$$

Setting $\chi := n/(n-2)$ for $n > 2$, $\chi > 1$ arbitrary if $n = 2$, it follows that

$$\begin{aligned} &\left(\int_S \zeta^{2\chi}\, [w^+]^{\chi\,(q+1)}\right)^{1/2} \\ &\quad \leq c\,(q+1)^2 \int_S \{[w^+]^{q-1} + [w^+]^q/q + [w^+]^{q+1}/(q+1)^2\}\, (\zeta + |\, \delta\, \zeta|)^2\,. \end{aligned} \tag{98}$$

From (98), the claim follows from fairly standard arguments. For $x_0 \in G$ and $0 < r < R$, we choose $\eta \in C^\infty(\overline{G})$ so that $\eta = 1$ on $B_r(x_0)$, $\eta = 0$ in $G \setminus B_R(x_0)$, and $|\nabla \eta| \leq c_0\,(R-r)^{-1}$. We then define $\zeta := \hat{\eta}$, $S_r := \{(x,\, \psi(x)) \,:\, x \in G \cap B_r\}$. The relation (98) and Hölder's inequality yield

$$\left(\int_{S_r} [w^+]^{\chi\,(q+1)}\right)^{1/2} \leq c\, \frac{(q+1)^2}{(R-r)^2}\, N_q(S) \left(\int_{S_R} [w^+]^{q+1}\right)^{\lambda_q}\,, \tag{99}$$

where

$$\begin{aligned} &\lambda_q \in \{(q-1)/(q+1),\, q/(q+1),\, 1\}, \\ &N_q(S) := \begin{cases} H_n(S)^{2/(q+1)} & \text{if } \lambda_q = (q-1)/(q+1) \\ q^{-1}\, H_n(S)^{1/(q+1)} & \text{if } \lambda_q = q/(q+1) \\ (q+1)^{-2} & \text{if } \lambda_q = 1\,. \end{cases} \end{aligned} \tag{100}$$

For $m \in \mathbb{N}$, choose $\alpha_m = \chi^m$, $r_m = r\,(1+2^{-m})$. Putting in (99) the choice $q+1 = \alpha_m$, $r = r_{m+1}$ and $R = r_m$ yields

$$\left(\int_{S_{r_{m+1}}} [w^+]^{\alpha_{m+1}}\right)^{1/\alpha_{m+1}} \leq [c\,\frac{\chi^{2m}\,4^{m+1}}{r^2}\,N_m(S)]^{1/\chi^m}\,\left(\int_{S_{r_m}} [w^+]^{\alpha_m}\right)^{\lambda_m/\alpha_m}\,, \tag{101}$$

with the iterative structure

$$A_{m+1} \leq [c\,\frac{\chi^{2m}\,4^{m+1}}{r^2}\,N_m(S)]^{1/2^m}\,A_m^{\lambda_m}\,. \tag{102}$$

We obtain

$$A_{m+1} \leq [c\,\frac{\chi^{2m}\,4^{m+1}}{r^2}\,N_m(S)]^{1/2^m}\,\prod_{i=0}^{m-1}[c\,\frac{\chi^{2i}\,4^{i+1}}{r^2}\,N_i(S)]^{\lambda_{i+1}/2^i}\,A_0^{\prod_{i=0}^m \lambda_i}\,. \tag{103}$$

Here $\lambda_m \leq 1$. The claim follows with similar methods as for Proposition 5.3. □

Thanks to Lemma 5.6, the problem of estimating w^+ is reduced to obtaining bounds on $H_n(S)$ and on $\|w^+\|_{L^2(S)}$. This can be done exactly as in the paper [Ura73], Lemma 5.

Remark 5.7. *Since the estimate of $\|w^+\|_{L^2(S)}$ in [Ura73], Lemma 5 depends on the number $\|\psi\|_{L^\infty(G)}$, it seems that a truly* a priori *bound can be obtained only via a strong monotonicity such as* (7) *or* (17).

6 Higher-order estimates

The gradient bound is crucial in the problem (5), (6). Higher-order estimates can be derived whenever a $L^\infty-$bound on the derivatives of ψ has been proved, since the equation (5) is then a *uniformely elliptic* equation of quasilinear type, due to (cp. (27))

$$F_{p_i,p_j}\,\xi_i\,\xi_j \geq \nu_2\,(1+\sup_G |\nabla\psi|^2)^{-3/2}\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n\,.$$

Define $a_0 := \sup_G |\nabla\psi|$. Here, and also in the next section about existence, we provide alternative proofs to the ones of the literature. We at first state an auxiliary proposition, whose proof, technical, is to find in the appendix, section D.

Lemma 6.1. *Let ∂G be a $n-1$ dimensional manifold of class $\mathcal{C}^{2,\alpha}$ ($\alpha \in [0,1]$). Then there exist $k \in \mathbb{N}$ and a family $\{T^{(1)},\ldots,T^{(k)}\}$ of vector fields, such that $T^{(l)} \in [C^{1,\alpha}(\overline{G})]^n$ and such that*

$$\operatorname{div} T^{(l)} = 0 \text{ in } G, \quad T^{(l)}\cdot n_G = 0 \text{ on } \partial G\,, \tag{104}$$

for $l = 1,\ldots,k$. Moreover, if $\tau \in [C^{1,\alpha}(\partial G)]^n$ is such that $\tau\cdot n_G = 0$ on ∂G, there exist $\lambda_1,\ldots\lambda_k \in [C^{1,\alpha}(\overline{G})]^n$ such that

$$\tau = \sum_{l=1}^k \lambda_l\,T^{(l)} \quad \text{on } \partial G\,. \tag{105}$$

The following Lemma states the Hoelder continuity estimate on $\nabla\psi$.

Lemma 6.2. *Assume that G is a domain of class $\mathcal{C}^{2,\alpha}$, ($0<\alpha\leq 1$). Let $\psi\in\mathcal{C}^2(\overline{G})$ be a solution to* (5)*,* (6)*. Then, for all $\beta\in[0,1[$, there is $c=c(G,\,a_0,\,\beta)$ such that*

$$\|\nabla\psi\|_{C^{0,\beta}(\overline{G})}\leq c\,(1+\|\nabla(n_G\,\kappa)\|_{L^\infty(G)}+\|\phi\|_{L^\infty(S)})\,,$$

Proof. Due to Remark 3.1 and Gauss' theorem, the relation (48) is equivalent to

$$\int_G (F_p+\kappa\,n_G)\cdot\nabla\xi=\int_G(\Phi-\operatorname{div}(\kappa\,n_G))\,\xi\quad\forall\,\xi\in W^{1,1}(G)\,. \tag{106}$$

For $l=1,\ldots,n$, we insert the testfunction $\partial_{x_l}\xi$ for $\xi\in C^1(\overline{G})$ in (106). Using integration by parts, it follows that

$$\begin{aligned}-\int_G\{F_{p_i,p_j}\,\partial^2_{x_j,x_l}\psi+\partial_{x_l}(\kappa\,n_{G,i})\}\,\partial_{x_i}\xi+\int_{\partial G}(F_{p_i}+\kappa\,n_{G,i})\,\partial_{x_i}\xi\,n_{G,l}&\\ =\int_G(\Phi-\operatorname{div}(\kappa\,n_G))\,\partial_{x_l}\xi\,.&\end{aligned} \tag{107}$$

Let $\tau\in\{T^{(1)},\ldots,T^{(k)}\}$, where k and the family $\{T^{(i)}\}$ are given by Lemma 6.1.

Define $\psi_\tau:=\tau\cdot\nabla\psi$. In (107), choose $\xi=\tau_l\,\eta$, with $\eta\in C^1(\overline{G})$. Summation over $l=1,\ldots,n$ yields

$$\begin{aligned}&-\int_G F_{p_i,p_j}\,\partial_{x_j}\psi_\tau\,\partial_{x_i}\eta+\int_G F_{p_i,p_j}\,\partial_{x_j}\tau_l\,\partial_{x_l}\psi\,\partial_{x_i}\eta-\int_G F_{p_i,p_j}\,\partial^2_{x_j,x_l}\psi\,\partial_{x_i}\tau_l\,\eta\\ &=-\int_{\partial G}(F_{p_i}+\kappa\,n_{G,i})\,\partial_{x_i}\tau_l\,n_{G,l}\,\eta+\int_G\partial_{x_l}(n_{G,i}\,\kappa)\,\partial_{x_i}(\tau_l\,\eta)\\ &\quad+\int_G(\Phi-\operatorname{div}(\kappa\,n_G))\,\partial_{x_l}(\tau_l\,\eta)\,.\end{aligned}$$

Since τ is divergence-free in G according to the Lemma 6.1, it follows that

$$\begin{aligned}\int_G F_{p_i,p_j}\,\partial^2_{x_j,x_l}\psi\,\partial_{x_i}\tau_l\,\eta&=\int_G\frac{dF_{p_i}}{dx_l}\,\partial_{x_i}\tau_l\,\eta-\int_G F_{p_i,x_l}\,\partial_{x_i}\tau_l\,\eta\\ &=\int_{\partial G}F_{p_i}\,\partial_{x_i}\tau_l\,\eta\,n_{G,l}-\int_G F_{p_i}\,\partial_{x_i}\tau_l\,\partial_{x_l}\eta-\int_G F_{p_i,x_l}\,\partial_{x_i}\tau_l\,\eta\,,\\ -\int_{\partial G}\kappa\,n_{G,i}\,\partial_{x_i}\tau_l\,n_{G,l}\,\eta&=-\int_G\partial_{x_l}(\kappa\,n_{G,i})\,\partial_{x_i}\tau_l\,\eta-\int_G\kappa\,n_{G,i}\,\partial_{x_i}\tau_l\,\partial_{x_l}\eta\,,\end{aligned}$$

and therefore

$$\begin{aligned}&-\int_G F_{p_i,p_j}\,\partial_{x_j}\psi_\tau\,\partial_{x_i}\eta+\int_G F_{p_i,p_j}\,\partial_{x_j}\tau_l\,\partial_{x_l}\psi\,\partial_{x_i}\eta+\int_G F_{p_i}\,\partial_{x_i}\tau_l\,\partial_{x_l}\eta\\ &\quad+\int_G F_{p_i,x_l}\,\partial_{x_i}\tau_l\,\eta=-\int_G\kappa\,n_{G,i}\,\partial_{x_i}\tau_l\,\partial_{x_l}\eta+\int_G\partial_{x_l}(n_{G,i}\,\kappa)\,\tau_l\,\partial_{x_i}\eta\\ &\quad+\int_G(\Phi-\operatorname{div}(\kappa\,n_G))\,\tau_l\,\partial_{x_l}\eta\,.\end{aligned}$$

Equivalently

$$\int_G F_{p_i,p_j}\,\partial_{x_j}\psi_\tau\,\partial_{x_i}\eta = \int_G \{w\,\eta + V\cdot\nabla\eta\}\,, \tag{108}$$

with $w := F_{p_i,x_l}\,\partial_{x_i}\tau_l$, and with a vector field V given by

$$\begin{aligned} V_k :=& F_{p_k,p_j}\,\partial_{x_j}\tau_l\,\partial_{x_l}\psi + F_{p_i}\,\partial_{x_i}\tau_k + \kappa\,n_{G,i}\,\partial_{x_i}\tau_k - \partial_{x_l}(n_{G,k}\,\kappa)\,\tau_l \\ & - (\Phi - \operatorname{div}(\kappa\,n_G))\,\tau_k \quad \text{for } k = 1,\ldots,n\,. \end{aligned} \tag{109}$$

Using in particular the growth assumption (12b), it follows that

$$\begin{aligned} |V_k| \le& (\mu_2 + \mu_1 + \|\kappa\|_{L^\infty(\partial G)})\,\|\nabla\tau\|_{L^\infty(G)} + \|\nabla\kappa\|_{L^\infty(G)} \\ & + \|\kappa\|_{L^\infty(\partial G)}\,\|\nabla n_G\|_{L^\infty(G)} + \|\Phi\|_{L^\infty(S)} \quad \text{for } k \in \{1,\ldots,n\}\,. \end{aligned}$$

According to the case (1) of Theorem C.1, there is for $0 \le \beta < 1$ arbitrary a constant c depending only on β, G, the ellipticity constant of the matrix $\{F_{p_i,p_j}\}$ and its norm in L^∞ such that

$$\|\psi_\tau\|_{C^{0,\beta}(\overline{G})} \le c\,(\|V\|_{[L^\infty(G)]^n} + \|w\|_{L^\infty(G)})\,. \tag{110}$$

Let $\{\tau^{(1)},\ldots,\tau^{(n-1)}\}$ be an orthonormal basis of the tangential space on ∂G. Using Lemma 6.1, a representation

$$\nabla\psi - (n_G\cdot\nabla\psi)\,n_G = \sum_{i=1}^{n-1}(\tau^{(i)}\cdot\nabla\psi)\,\tau^{(i)} = \sum_{i-1}^{n-1}\left(\sum_{l=1}^{k}\lambda_l^{(i)}\,(T^{(l)}\cdot\nabla\psi)\right)\,\tau^{(i)}\,,$$

is valid, with $\lambda_l^{(i)} \in C^{1,\alpha}(\overline{G})$. With the help of the result (110), it follows that $\psi_t := \nabla\psi - (n_G\cdot\nabla\psi)\,n_G \in C^{0,\beta}(\overline{G})$.

We finally show that also $\psi_n := n_G\cdot\nabla\psi$ satisfies a Hölder condition. For $x \in G$, $y \in \mathbb{R}$, define (cp. (55))

$$H(x,\,y) := F_p(x,\,\psi_t(x) + n_G(x)\,y)\cdot n_G(x) + \kappa(x)\,.$$

Using the growth condition (12b),

$$|H(x,\,y)| \le \mu_1 + \|\kappa\|_{L^\infty(\partial G)} \quad \forall (x,y) \in \overline{G}\times\mathbb{R}\,.$$

Moreover, for $x_1,\,x_2 \in \overline{G}$, $y \in \mathbb{R}$

$$\begin{aligned} |H(x_1,\,y) - H(x_2,\,y)| \le& \|\nabla^2_{p,x}F\|_{L^\infty}\,|x_1 - x_2| \\ & + \|\nabla^2_{p,p}F\|\,(|\psi_t(x_1) - \psi_t(x_2)| + |y|\,|n_G(x_1) - n_G(x_2)|)\,, \end{aligned}$$

so that the following estimate holds:

$$\frac{|H(x_1,\,y) - H(x_2,\,y)|}{|x_1 - x_2|^\beta} \le c\,(1 + \|\psi_t\|_{C^{0,\beta}(\overline{G})} + |y|\,\|n_G\|_{C^{0,\beta}(\overline{G})})\,. \tag{111}$$

By virtue of the condition (12c), note that

$$\partial_y H(x,\,y) = F_{p_i,p_j}\, n_{G,j}(x)\, n_{G,i}(x) \geq \nu_2\,(1+|\nabla\psi|^2)^{-1/2}\,(1-(n_S\cdot n_G)^2)$$
$$\geq \nu_2\,(1+a_0)^{-3/2}\,. \tag{112}$$

On the other hand, the boundary condition (6) implies that $H(x,\,\psi_n(x)) = 0$ on ∂G. For $x,\,x' \in \partial G$ arbitrary, it follows that

$$\nu_2\,(1+a_0)^{-3/2}\,(\psi_n(x)-\psi_n(x')) \leq \int_{\psi_n(x)}^{\psi_n(x')} \partial_y H(x,\,s)\,ds$$
$$= H(x,\,\psi_n(x')) - H(x,\,\psi_n(x)) = H(x,\,\psi_n(x')) - H(x',\,\psi_n(x'))\,.$$

The latest yields

$$\frac{|\psi_n(x)-\psi_n(x')|}{|x-x'|^\beta} \leq c\,\frac{|H(x,\,\psi_n(x')) - H(x',\,\psi_n(x'))|}{|x-x'|^\beta}\,. \tag{113}$$

Therefore, taking (111) into account

$$\|\psi_n\|_{C^{0,\beta}(\partial G)} \leq c\,(1+\|\psi_t\|_{C^{0,\beta}(\overline{G})} + a_0\,\|n_G\|_{C^{0,\beta}(\overline{G})})$$

which finally implies that $\nabla\psi \in C^{0,\beta}(\partial G)$. Return to (107) for $\xi \in C^1_0(\overline{G})$

$$-\int_G \{F_{p_i,p_j}\,\partial^2_{x_j,x_l}\psi + \partial_{x_l}(\kappa\, n_{G,i})\}\,\partial_{x_i}\xi = \int_G (\Phi - \operatorname{div}(\kappa\, n_G))\,\partial_{x_l}\xi\,. \tag{114}$$

With the help of the case (1) of Theorem C.1, it now follow that $\partial_{x_l}\psi \in C^{0,\beta}(\overline{G})$ for $l = 1,\ldots n$ with corresponding norm estimate. □

The estimate in $C^{2,\alpha}$ is obtained with similar ideas.

Lemma 6.3. *Same assumptions as in Lemma 6.2. Then, the norm of $\nabla\psi$ in $C^{1,\alpha}(\overline{G})$ is bounded by a constant that depends on the constants in the conditions* (12)*, on $\|\phi\|_{C^{0,\alpha}(S)}$ and a_0.*

Proof. Consider the relation (108). Lemma 6.2 implies that $F_{p_i,p_j} \in C^{0,\beta}(\overline{G})$ for all $\beta \in [0,1[$. Analogously, $\Phi \in C^{0,\beta}(\overline{G})$ for all $\beta \in [0,1[$ (cf. (4) and (19)).

The definition (109) together with Lemma 6.2 now implies that $V \in [C^{0,\alpha}(\overline{G})]^n$ (cp. (109)). Thus, the case (2) of Theorem C.1 now yields

$$\|\psi_\tau\|_{[C^{1,\alpha}(\overline{G})]^n} \leq c\,(\|V\|_{[C^{0,\alpha}(\overline{G})]^n} + \|w\|_{L^\infty(G)})\,,$$

and it follows from Lemma 6.1 that

$$\|\psi_t\|_{C^{1,\alpha}(\overline{G})} = \|\nabla\psi - (n_G\cdot\nabla\psi)\,n_G\|_{C^{1,\alpha}(\overline{G})} \leq c\,\|V\|_{[C^{0,\alpha}(\overline{G})]^n}\,. \tag{115}$$

We are now allowed to differentiate the relation $H(x,\, \psi_n(x)) = 0$ in any tangential direction τ over ∂G, which yields

$$\partial_y H(x,\, \psi_n(x))\, (\tau \cdot \nabla \psi_n) = \tau \cdot \nabla_x H(x,\, \psi_n(x)) \quad \text{for } x \in \partial G\,.$$

Due to (115), the regularity of ∂G and the preceding Proposition 6.2, the function $\tau \cdot \nabla_x H(x,\, \psi_n(x))$ belongs to $C^{0,\alpha}(\partial G)$. For the same reasons, and (112), the function $[\partial_y H(x,\, \psi_n(x))]^{-1}$ belongs to $C^{0,\alpha}(\partial G)$ as well. Thus, the mixed-derivatives $\psi_{t,n}$ belongs to $C^{0,\alpha}(\partial G)$, with corresponding continuity estimates.

We now want to show that also $\psi_{n,n} \in C^{0,\alpha}(\partial G)$. Using integration by parts, the relation (107) in particular implies for $l = 1, \ldots, n$ that

$$\begin{aligned} - \int_{\partial G} n_{G,i} \, \{F_{p_i,p_j}\, \partial^2_{x_j,x_l} \psi - \partial_{x_l}(\kappa\, n_{G,i})\}\, \xi + \int_{\partial G} (F_{p_i} + \kappa\, n_{G,i})\, \partial_{x_i} \xi \, n_{G,l} & \\ = \int_{\partial G} n_{G,l}\, (\Phi - \mathrm{div}(\kappa\, n_G))\, \xi\,. & \end{aligned}$$

Choosing $\xi = \eta\, n_{G,l}$ and summing up for $l = 1, \ldots, n$, it follows after straightforward simplifications that

$$\int_{\partial G} n_{G,i}\, n_{G,l}\, F_{p_i,p_j}\, \partial^2_{x_j,x_l} \psi\, \eta = \int_{\partial G} (F_{p_i} + \kappa\, n_{G,i})\, \partial_{x_i} \eta - \int_{\partial G} (\Phi - \kappa\, \mathrm{div}(n_G))\, \eta\,.$$

Denoting $\mathrm{div}_{\partial G}$ the surface divergence on ∂G, the latest yields

$$n_{G,i}\, n_{G,l}\, F_{p_i,p_j}\, \partial^2_{x_j,x_l} \psi = - \mathrm{div}_{\partial G}(F_p + \kappa\, n_G) - \Phi + \kappa\, \mathrm{div}(n_G) \text{ on } \partial G\,. \tag{116}$$

If τ is a tangential vector on ∂G, then for $i = 1, \ldots, n$

$$\begin{aligned} \tau \cdot \nabla (F_{p_i} - \kappa\, n_{G,i}) &= \tau_l \, (F_{p_i,x_l} + F_{p_i,p_j}\, \partial^2_{j,l} \psi - \partial_l(\kappa\, n_{G,i})) \\ &= \tau_l \, F_{p_i,x_l} + F_{p_i,p_j}\, \tau \cdot \nabla \psi_{x_j} \, + \tau \cdot \nabla (\kappa\, n_{G_i})\,. \end{aligned}$$

Using the preceding results, $\mathrm{div}_{\partial G}(F_p + \kappa\, n_G)$ involves only mixed tangential normal second derivatives of ψ and therefore belongs to $C^{0,\alpha}(\partial G)$ with estimate. Note also that

$$\begin{aligned} & n_{G,i}\, n_{G,l}\, F_{p_i,p_j}\, \partial^2_{x_j,x_l} \psi = n_{G,i}\, F_{p_i,p_j}\, \partial_{x_j} \psi_n - n_{G,i}\, F_{p_i,p_j}\, \partial_{x_j} n_{G,l}\, \partial_{x_l} \psi \\ & \quad = (n_{G,i}\, F_{p_i,p_j}\, n_{G,j})\, (n_G \cdot \nabla \psi_n) + n_{G,i}\, F_{p_i,p_j}\, \psi_{t_j,n} - n_{G,i}\, F_{p_i,p_j}\, \partial_{x_j} n_{G,l}\, \partial_{x_l} \psi\,. \end{aligned}$$

The relation (116) implies that

$$(n_{G,i}\, F_{p_i,p_j}\, n_{G,j})\, (n_G \cdot \nabla \psi_n) \in C^{0,\alpha}(\partial G)\,.$$

Since $n_{G,i}\, F_{p_i,p_j}\, n_{G,j} \geq \nu_2\, (1 + a_0)^{-3/2}$, then the function $(n_{G,i}\, F_{p_i,p_j}\, n_{G,j})^{-1}$ belongs also to $C^{0,\alpha}(\partial G)$. We finally can conclude that $(n_G \cdot \nabla \psi_n) \in C^{0,\alpha}(\partial G)$. Since $D^2 \psi \in C^{0,\alpha}(\partial G)$, the relation (114) yields the claim (case (2) of Theorem C.1). $\square$

7 Existence

It was shown for the first time in [Ura71] that *a priori* estimates on the gradient of C^2 solutions to (5), (6) joined to the Hoelder estimate of Lemma 6.2 leads to an existence theorem via continuation methods in Banach-spaces exposed in [LU68], Ch. 10.

We are able to generalize these ideas and to significantly improve the hypotheses on ϕ, required to satisfy only (13) and (14) instead of being Hoelder continuously differentiable. Moreover, once the *a priori* estimate of Lemma 6.3 is given, existence is obtained more directly via the following simple continuation Lemma.

Proposition 7.1. *Let $X,\, Y$ be two Banach spaces such that $Y \hookrightarrow X$ with compact embedding. Let $S\,:\, X \to X$ be a Fréchet differentiable mapping. Assume that there are $a,\, K > 0$ such that for all $\lambda \in]-a,\, 1[$, all solutions $x \in X$ to the equation*

$$\lambda\, S(x) = x\,, \tag{117}$$

belong to $B_K(0;Y)$. Then S has a fixed-point in $B_K(0;Y)$.

Proof. Define a mapping $\mathcal{G}\,:\, X\times]-a,\, 1[\to X$ by $\mathcal{G}(x,\, \lambda) = \lambda\, S(x) - x$. Since S is Fréchet differentiable on X, $\mathcal{G}$ is Fréchet differentiable on $X\times] - a,\, 1[$.

Define $M \subseteq [-a,\, 1]$ by

$$M := \{\lambda \in [-a,\, 1]\, : \exists x \in X,\, \mathcal{G}(x,\, \lambda) = 0\}\,.$$

The set M is nonvoid since $\mathcal{G}(0, 0) = 0$. Moreover $\lambda^* := \sup M$ belongs to M. To see this, choose $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq M$, $\lambda_k \to \lambda^*$. By definition, there is $x_k \in X$ such that $\mathcal{G}(x_k,\, \lambda_k) = 0$. By assumption $x_k \in B_K(0;Y)$ for all $k \in \mathbb{N}$, and therefore, there is a subsequence x_{k_j} that strongly converges in X to some x^*. Obviously, $\mathcal{G}(x^*,\, \lambda^*) = 0$, implying $\lambda^* \in M$.

Seeking a contradiction, assume that $\lambda^* < 1$. Then, due to the implicit function theorem (see [GT01], Th. 17.6), there is an open neighborhood $]\lambda^* - \epsilon,\, \lambda^* + \epsilon[$ in $] - a, 1[$ such that the equation $\mathcal{G}(x,\, \lambda) = 0$ defines a unique implicit vector-valued function $\lambda \mapsto x(\lambda) \in X$. Therefore $\lambda^* \neq \sup M$, the contradiction. ☐

Theorem 7.2. *Assumptions of the Theorem 2.1. Then, there is $\psi \in C^{2,\alpha}(\overline{G})$ that solves* (5), (6).

Proof. At first, the fundamental theorem of integration implies that

$$\begin{aligned} F_{p_i}(x,\, p) &= \int_0^1 F_{p_i,p_j}(x,\, t\, p)\, dt\, p_j + F_{p_i}(x,\, 0) \\ &=: a_{i,j}(x,\, p)\, p_j + b_i(x)\,. \end{aligned} \tag{118}$$

Due to (118), the problem (5), (6) is equivalent to

$$\begin{aligned} -\partial_{x_i}(a_{i,j}(x,\, \nabla\psi)\, \partial_{x_j}\psi) &= \Phi(x,\, \psi,\, \nabla\psi) + \partial_{x_i} b_i(x) && \text{in } G \\ -n_{G,i}(x)\, a_{i,j}(x,\, \nabla\psi)\, \partial_{x_j}\psi &= \kappa(x) + n_{G,i}(x)\, b_i(x) && \text{on } \partial G\,. \end{aligned}$$

Let $w \in C^{2,\beta}(\overline{G})$, with $\beta < \alpha$ be given. Consider the following semilinear Neumann-problem for the unknown u:

$$\begin{aligned} -\partial_{x_i}(a_{i,j}(x,\, \nabla w)\, \partial_{x_j} u) &= \phi(x,\, u) + R(x,\, \nabla w) + \partial_{x_i} b_i(x) \quad && \text{in } G \\ -n_{G,i}(x)\, a_{i,j}(x,\, \nabla w)\, \partial_{x_j} u &= \kappa(x) + n_{G,i}(x)\, b_i(x) && \text{on } \partial G\,. \end{aligned} \tag{119}$$

Since ∇w is in $C^{1,\beta}$, the coefficient matrix $\{a_{i,j}(x\, \nabla w)\}$ is uniformely elliptic, and it moreover belongs $[C^{1,\beta}(\overline{G})]^{n\times n}$. In Lemma D.4 hereafter, we prove that the problem (119) is uniquely solvable in the space $C^{2,\beta}(\overline{G})$, and that the nonlinear mapping $S\, :\, C^{2,\beta}(\overline{G}) \rightarrow C^{2,\beta}(\overline{G})$ given by $Sw := u$ is Fréchet differentiable in $C^{2,\beta}(\overline{G})$.

Let $a > 0$ arbitrary, let $\lambda \in]-a, 1] \setminus \{0\}$, and assume that $w \in C^{2,\beta}(\overline{G})$ satisfies $\lambda\, S(w) = w$. The latest means that

$$\begin{aligned} -\partial_{x_i}(a_{i,j}(x,\, \nabla w)\, \partial_{x_j} w) &= \lambda\, \Phi(x,\, w/\lambda,\, \nabla w) + \lambda\, \partial_{x_i} b_i(x) \quad && \text{in } G \\ -n_{G,i}(x)\, a_{i,j}(x,\, \nabla w)\, \partial_{x_j} w &= \lambda\, (\kappa(x) + n_{G,i}(x)\, b_i(x)) && \text{on } \partial G\,. \end{aligned} \tag{120}$$

Introduce

$$\begin{aligned} \bar{\Phi}_\lambda(x,\, x_{n+1},\, p) &:= \lambda\, \Phi(x,\, x_{n+1}/\lambda,\, p) + (\lambda - 1)\, \partial_{x_i} b_i(x)\,, \\ \bar{\kappa}_\lambda &:= \lambda\, \kappa(x) + (\lambda - 1)\, n_{G,i}(x)\, b_i(x)\,. \end{aligned}$$

Taking into account the definitions (118), (120) is nothing else but

$$\begin{aligned} -\partial_{x_i} F_{p_i}(x,\, \nabla w) &= \bar{\Phi}_\lambda(x,\, w,\, \nabla w) \quad && \text{in } G \\ -n_{G,i}(x)\, F_{p_i}(x,\, \nabla w) &= \bar{\kappa}_\lambda(x) && \text{on } \partial G\,. \end{aligned}$$

The new functions $\bar{\Phi}_\lambda$, $\bar{\kappa}_\lambda$ satisfy the requirements of the statements 4.4, 5.3, 6.2 and 6.3: In particular, (7) is valid, and all derivatives of $\bar{\Phi}_\lambda$ are bounded independently of the parameter λ. Therefore, w is unique, and there is a continuous function K of the data such that $\|w\|_{C^{2,\alpha}(\overline{G})} \leq K$. The claim follows from Proposition 7.1, with $X = C^{2,\beta}$ and $Y = C^{2,\alpha}$. □

It remains to prove the existence claim of Theorem 2.3 with the weakening of the condition (7).

Proposition 7.3. *Assumptions of the Theorem 2.3. Then, there exists a solution to* (5), (6) *in* $C^{2,\alpha}(\overline{G})$*. Under* (18) *is the solution unique.*

Proof. Assume that ϕ satisfies (16). For $\epsilon > 0$, the function $\phi_\epsilon(x,\, x_{n+1}) := \phi(x,\, x_{n+1}) - \epsilon\, x_{n+1}$ satisfies (7), with $\gamma_0 = \epsilon$. Applying the result of Theorem 7.2, there is a unique $\psi_\epsilon \in C^{2,\alpha}(\overline{G})$ that solves the problem

$$\begin{aligned} -\operatorname{div} F_p(x, \nabla\psi_\epsilon) &= \Phi_\epsilon(x,\, \psi_\epsilon,\, \nabla\psi_\epsilon) \quad && \text{in } G\,, \\ -F_p(x, \nabla\psi_\epsilon) &= \kappa(x) && \text{on } \partial G\,. \end{aligned}$$

On the other hand, $\phi_\epsilon(x,\, x_{n+1})$ satisfies (17) independently of ϵ, and therefore $\|\nabla\psi_\epsilon\|_{L^1(G)} + \|\psi_\epsilon\|_{L^\infty(G)} \leq C$ according to Lemma 4.1 and Lemma 4.2. Then, the statements 5.6 and 5.7 guaranty that also $\|\nabla\psi_\epsilon\|_{L^\infty(G)} \leq C$ with C independent of ϵ. Finally, one obtains uniform bounds for $\{\nabla\psi_\epsilon\}$ in Hoelder spaces applying the propositions 6.2 and 6.3. The existence claim follows letting $\epsilon \rightarrow 0$. □

8 Uniqueness

We recall the definition (3) and that $\sigma^0(q) := |q|$. We abbreviate $R = R(x,p)$ and for $q = (-p,1)$, $\sigma = \sigma(x,q) = \sigma(x,-p,1)$, $\sigma^0 = \sigma^0(x,q) = \sigma^0(x,-p,1)$. Then, for $k = 1,\dots,n$, the assumption (12d) implies that

$$\frac{\partial R}{\partial p_k} = -\sum_{i,j=1}^{n+1}\Big(2\,\sigma_{q_i,x_j}\,\sigma^0_{q_j}\,\sigma^0_{q_i,q_k} + \sigma_{x_j,q_k,q_i}\,\sigma^0_{q_i}\,\sigma^0_{q_j}\Big) = -2\sum_{i,j=1}^{n+1}\sigma_{q_i,x_j}\,\sigma^0_{q_j}\,\sigma^0_{q_i,q_k}\,.$$

For $i = 1,\dots,n+1$, denote $\eta_i := 2\,\sigma_{q_i,x_j}\,\sigma^0_{q_j}$. Due to (12e), $|\eta| \le 2\,\mu_3$. Let $\xi \in \mathbb{R}^n$ arbitrary. Using the latest, and the projection operator (28) it follows that $\frac{\partial R}{\partial p}\cdot\xi = \sum_{i,k=1}^{n+1}\sigma^0_{q_i,q_k}\eta_i\,\hat{\xi}_k$. Denote $\tilde{\xi} = \hat{\xi} - (\hat{\xi}\cdot q/|q|^2)\,q$. Since σ^0 satisfies (12c) with $\mu_2 = 1$, the Cauchy-Schwarz inequality yields

$$\left|\frac{\partial R}{\partial p}\cdot\xi\right| \le \frac{|\tilde{\xi}|\,|\tilde{\eta}|}{|q|} \le 2\,\mu_3\,\frac{|\tilde{\xi}|}{|q|}\,. \tag{121}$$

Let $p_1,\,p_2 \in \mathbb{R}^n$, and $x \in G$ arbitrary. Due to the fundamental theorem of integration

$$R(x,\,p_1) - R(x,\,p_2) = \int_0^1 \frac{\partial R}{\partial p}(x,\,tp_1 + (1-t)p_2)\cdot(p_1 - p_2)\,dt\,.$$

Using (121), with $q = q_t = (tp_1 + (1-t)p_2,\,-1)$ and $\xi = (p_1 - p_2)$ it follows that

$$|R(x,\,p_1) - R(x,\,p_2)| \le 2\,\mu_3\int_0^1 \frac{|\tilde{\xi}_t|}{(1 + |tp_1 + (1-t)p_2|^2)^{1/2}}\,dt\,, \tag{122}$$

where $\tilde{\xi}_t = \hat{\xi} - (\hat{\xi}\cdot q_t/|q_t|^2)\,q_t$. On the other hand, we can write

$$\begin{aligned}&(F_p(x,\,p_1) - F_p(x,\,p_2))\cdot(p_1 - p_2)\\ &= \int_0^1 F_{p_i,p_j}(x,\,tp_1 + (1-t)p_2)\,(p_1 - p_2)_i\,(p_1 - p_2)_j\,dt\,,\end{aligned}$$

and, employing the same notations as in (122), the assumptions (12c), (12d) yield

$$(F_p(x,\,p_1) - F_p(x,\,p_2))\cdot(p_1 - p_2) \ge \nu_2\int_0^1 \frac{|\tilde{\xi}_t|^2}{(1 + |tp_1 + (1-t)p_2|^2)^{1/2}}\,dt\,. \tag{123}$$

It ψ_1 and ψ_2 are two weak solutions to (5), (6), then

$$\begin{aligned}\int_G (F_p(x,\,\nabla\psi_1) - F_p(x,\,\nabla\psi_2))\cdot\nabla(\psi_1 - \psi_2) - \int_G (\phi(x,\psi_1) - \phi(x,\psi_2))\,(\psi_1 - \psi_2)\\ = \int_G (R(x,\,\nabla\psi_1) - R(x,\,\nabla\psi_2))\,(\psi_1 - \psi_2)\,.\end{aligned} \tag{124}$$

For $\rho > 0$, and $\gamma_0 > 0$ Youngs inequality yields

$$\begin{aligned}|R(x,\,\nabla\psi_1) - R(x,\,\nabla\psi_2)|\,|\psi_1 - \psi_2| &\le \rho\,\gamma_0\,|\psi_1 - \psi_2|^2\\ &+ (4\,\rho\,\gamma_0)^{-1}\,|R(x,\,\nabla\psi_1) - R(x,\,\nabla\psi_2)|^2\,.\end{aligned}$$

Due to the assumption $\mu_3 < 2\,\gamma_0\,\nu_2$, the estimates (122) and (123), there is $\rho < 1$ such that

$$(F_p(x,\,\nabla\psi_1) - F_p(x,\,\nabla\psi_2))\cdot\nabla(\psi_1-\psi_2) - (4\,\rho\,\gamma_0)^{-1}\,|R(x,\,\nabla\psi_1) - R(x,\,\nabla\psi_2)|^2$$
$$\geq (\nu_2 - (2\,\rho\,\gamma_0)^{-1}\,\mu_3)\int_0^1 \frac{|\tilde{\xi}_t|^2}{(1+|t\nabla\psi_1+(1-t)\nabla\psi_2|^2)^{1/2}}\,dt \geq 0\,.$$

Therefore, the assumption (7) implies that $(1-\rho)\,\gamma_0\,\int_G |\psi_1-\psi_2|^2 \leq 0$, proving the uniqueness.

A Useful properties

Let $G\subset\mathbb{R}^n$ be bounded, and define $d(x) := \operatorname{dist}(x,\partial G)$ for $x\in\overline{G}$. For $\mu>0$, denote $\Gamma_\mu := \{x\in G\,:\, d(x)<\mu\}$.

Lemma A.1. *If $G\in\mathcal{C}^{2,\alpha}$, there is $\mu_0=\mu_0(G)$, such that $d\in C^{2,\alpha}(\Gamma_{\mu_0})$. Moreover, if $\kappa\in C^{1,\alpha}(\partial G)$, there is $\tilde{\kappa}\in C^{1,\alpha}(\Gamma_{\mu_0}))$ such that $\tilde{\kappa}=\kappa$ on ∂G, and $\|\kappa\|_{L^\infty(G)}\leq\|\kappa\|_{L^\infty(\partial G)}$.*

Proof. A proof is given in [GT01], Lemma 14.16 for $d\in C^2(\Gamma_{\mu_0})$, but it obviously extends to the $C^{2,\alpha}$ case. It is moreover shown that for all $x\in\Gamma_{\mu_0}$, there is a unique $y=y(x)\in\partial G$ such that $d(x)=|y-x|$, and the mapping $x\mapsto y$ is of class $C^1(\Gamma_{\mu_0})$ (resp. of class $C^{1,\alpha}(\Gamma_{\mu_0})$). Given $\kappa\in C^{1,\alpha}(\partial G)$, define $\tilde{\kappa}(x):=\kappa(y(x))$ for $x\in\Gamma_{\mu_0}$. ☐

In the following Lemma, we show that the right-hand side Φ of (5) is allowed to have less regularity than $C^1(G\times\mathbb{R})$. If $\psi'\in C^{0,1}(G)$, we introduce

$$\Omega^- := \{(x,x_{n+1})\,:\,x_{n+1}\leq\psi'(x)\},\quad \Omega^+ := \{(x,x_{n+1})\,:\,x_{n+1}\geq\psi'(x)\}\,. \tag{125}$$

Lemma A.2. *Let $\phi\in V_{\psi'}$ (cp. (14)). Assume that $\psi\in C^{0,1}(\overline{G})$. Then, the function $\phi(\cdot,\,\psi)$ belongs to $W^{1,1}(G)$, and moreover*

$$\frac{d}{dx_i}\phi(\cdot,\,\psi) = \frac{\partial\phi^-}{\partial x_i}(\cdot,\,\psi) + \frac{\partial\phi^-}{\partial x_{n+1}}(\cdot,\,\psi)\,\frac{\partial\psi}{\partial x_i}\,, \tag{126}$$

where $\nabla\phi^{+,-}\in L^1(S)$ are the traces of $\nabla\phi$ on the surface S taken from the side of $\Omega^{+,-}$.

Proof. By assumption, $\phi\in W^{2,1}(\Omega^-)$. Therefore, the function ϕ and its first weak derivatives ϕ_{x_i} ($i=1,\ldots,n+1$) have well defined traces (same denotation) in $L^1(S\cap\Omega^-)$. Moreover, there is a sequence $\{\phi_m\}\subset C^\infty(\overline{\Omega^-})$ such that $\phi_m\to\phi$ in $W^{2,1}(\Omega^-)$. For the traces is in particular valid that $\phi_m\to\phi$ in $L^1(S\cap\Omega^-)$, and that $\phi_{x_i,m}\to\phi_{x_i}$ in $L^1(S\cap\Omega^-)$.

The chain rule implies in $G^- := \{x\in G\,:\,\psi(x)\leq\psi'(x)\}$ that

$$\frac{d}{dx_i}\phi_m(\cdot,\,\psi) = \frac{\partial\phi_m}{\partial x_i}(\cdot,\psi) + \frac{\partial\phi_m}{\partial x_{n+1}}(\cdot,\psi)\,\frac{\partial\psi}{\partial x_i}\,. \tag{127}$$

The right-hand of (127) is strongly convergent in $L^1(G^-)$, and therefore, $\phi_m(\cdot,\,\psi) \to w \in W^{1,1}(G)$, and it follows that $w = \phi(\cdot,\,\psi)$. This proves (126) in G^-. The claim follows from a similar consideration for the set Ω^+.

In the interior of the set $G_0 := \{x \in G \,:\, \psi(x) = \psi'(x)\}$, the traces $\phi^+_{x_i}$ from the side of Ω^+ and $\phi^-_{x_i}$ from the side Ω^- are in general different. $\square$

B Integration by parts and Sobolev embedding on manifolds

In this section, we assume that $G \in \mathcal{C}^1$, and we consider $\psi \in C^1(\overline{G})$. Denote by S the graph of ψ (cp. (29)) and by dH_n the surface measure $\sqrt{1+|\nabla\psi|^2}\,d\lambda_n$ on S. For vector fields $\eta \in [C^1(\mathbb{R}^{n+1})]^{n+1}$, the surface divergence $\operatorname{div}_S \eta$ is defined using the operator δ of (32) by

$$\operatorname{div}_S \eta := \sum_{i=1}^{n} \delta_i\ \eta_i \quad \text{on } S\,. \tag{128}$$

Lemma B.1. *Assume that σ satisfies the assumptions* (11) *and* (12d)*, that F and R are respectively defined by* (2) *and* (3)*, and that Φ is given by* (4)*. Then,* (1) *is valid for $\psi \in C^2(G)$ if and only if* (5) *is satisfied.*

Proof. Throughout the proof, $\sigma_q = \sigma_q(x,\,-\nabla\psi,\,1)$. According to (128), the relation (1) is equivalent to

$$\sum_{i=1}^{n}\left(\frac{d}{dx_i}\sigma_{q_i} - (n_S\cdot\nabla\sigma_{q_i})\,n_{S,i}\right) = \phi\,.$$

For $i = 1,\dots,n+1$, we compute

$$n_S\cdot\nabla\sigma_{q_i} = n_{S,j}\,\sigma_{q_i,x_j} - n_{S,j}\,\sigma_{q_i,q_k}\,\partial^2_{k,j}\psi\,,$$

and therefore, using that the assumption (12d) implies $\sigma_{q_i,q_k}\,n_{S,i} = 0$ for $k = 1,\dots,n+1$, it follows that

$$(n_S\cdot\nabla\sigma_{q_i})\,n_{S,i} = n_{S,j}\,\sigma_{q_i,x_j}\,n_{S,i} - n_{S,j}\,\partial^2_{k,j}\psi\,\sigma_{q_i,q_k}\,n_{S,i} = R\,.$$

Thus, (1) is equivalent to $\sum_{i=1}^n \frac{d}{dx_i}\sigma_{q_i} = \phi + R$, and the claim follows. $\square$

For vector fields $\eta \in [C^1(\overline{G}\times\mathbb{R})]^{n+1}$, and for $x \in G$, denote $\tilde\eta(x) := \eta(x,\,\psi(x))$, so that $\tilde\eta \in [C^1(\overline{G})]^{n+1}$. If $\eta\cdot n_S = 0$ on S, the surface divergence of η on S has the equivalent expression

$$(\operatorname{div}_S \eta)(x,\,\psi(x)) := (\sqrt{1+|\nabla\psi(x)|^2})^{-1}\ \operatorname{div}(\sqrt{1+|\nabla\psi|^2}\,\tilde\eta)(x) \quad \text{for } x \in G\,. \tag{129}$$

We recall the definition (32) of the operator δ, the notations (38) and (39). The following Lemma is well-known. It is proved here only for the convenience of the reader.

Lemma B.2. *Assume that $G \in \mathcal{C}^1$. Let H_{n-1} denote the surface measure on ∂G. Let $\mathcal{H}_{n-1}$ be the measure $\sqrt{1+|\psi_t|^2}\, dH_{n-1}$ on ∂S. Then, for all $\eta \in [C^1(\overline{G}\times\mathbb{R})]^{n+1}$ such that $\eta \cdot n_S = 0$ on S, and all $\xi \in C^1(\overline{G}\times\mathbb{R})$,*

$$\int_S \operatorname{div}_S \eta\, \xi\, dH_n = -\int_S \eta \cdot \delta\, \xi\, dH_n + \int_{\partial S} \eta \cdot n_{\partial S}\, \xi\, d\mathcal{H}_{n-1}\,, \tag{130}$$

with $n_{\partial S} = \sin\alpha^{-1}\,(\hat{n}_G - (n_S\cdot \hat{n}_G)\, n_S)$.

Proof. By definition

$$\begin{aligned}\int_S \operatorname{div}_S \eta\, \xi\, dH_n &= \int_G \operatorname{div}(\sqrt{1+|\nabla\psi|^2}\, \tilde{\eta})\, \tilde{\xi}\, dx \\ &= -\int_G \sqrt{1+|\nabla\psi|^2}\, \tilde{\eta}\cdot\nabla\tilde{\xi}\, dx + \int_{\partial G} \sqrt{1+|\nabla\psi|^2}\, \tilde{\eta}\cdot n_G\, \tilde{\xi}\, dH_{n-1}\,.\end{aligned}$$

We easily verify that $\int_G \sqrt{1+|\nabla\psi|^2}\, \tilde{\eta}\cdot\nabla\tilde{\xi}\, dx = \int_S \eta\cdot\delta\,\xi\, dH_n$. Moreover, since $\eta\cdot n_S = 0$ on S,

$$\tilde{\eta}\cdot n_G = \tilde{\eta}\cdot(n_G - (n_S\cdot n_G)\, n_S) = \sin\alpha\, \tilde{\eta}\cdot n_{\partial S} \tag{131}$$

Using (39) of $\sin\alpha$, the claim follows. □

The following Theorem was first proved in [LU70] to obtain local gradients estimates. It bases on the Federer/Fleming deformation theorem for (convex) currents and repeats arguments applied in [Mir67] for minimal surfaces. It was extended in [Ura71], Lemma 5 to nonconvex surfaces to prove a global estimate.

Theorem B.3. *Let $\psi \in C^2(\overline{G})$ solve* (5), (6)*. Let $G_1 \subset G$ and denote $\Gamma_1 = \partial G_1 \cap \partial G$. If Γ_1 is of class $\mathcal{C}^2$, there exists a constant β depending on the constants ν_0, μ_1, μ_3 in* (12)*, on $\|\kappa\|_{C^1(\partial G)}$ and on $\|\phi\|_{L^\infty(S)}$, such that for all $g \in C^1(\overline{G})$ such that $\operatorname{supp}(g) \subseteq G_1 \cup \Gamma_1$*

$$\left(\int_S |g|^{\frac{n}{n-1}}\, dH_n\right)^{(n-1)/n} \leq \beta \int_S |\,\delta\, g|\, dH_n\,. \tag{132}$$

A natural consequence of Theorem B.3 is the general embedding theorem on the manifold S.

Corollary B.4. *Let 2^* denote the usual Sobolev embedding exponent to $p = 2$ in $\mathbb{R}^n$. Let $S_1 := \{(x,\psi(x)) \,:\, x \in G_1\}$ with G_1 satisfying the assumptions of Theorem B.3. For all $g \in C^1(\overline{G})$ such that $\operatorname{supp}(g) \subseteq G_1 \cup \Gamma_1$*

$$\left(\int_S |g|^{2^*}\, dH_n\right)^{1/2^*} \leq c\, \left(\int_S |\,\delta\, g|^2\, dH_n\right)^{1/2}\,. \tag{133}$$

Proof. For $q \in [1,\, \infty[$, Theorem B.3 implies that

$$\|g\|^q_{L^{qn/(n-1)}(S_1)} \leq \beta\, q \int_S |g|^{q-1}\, |\,\delta\, g|\, dH_n\,.$$

Choose $q = 2(n-1)/(n-2)$ if $n > 2$, $q < \infty$ arbitrary if $n = 2$. □

C Auxiliary regularity theorems

For the higher-order estimates on $\nabla\psi$, we need auxiliary regularity. Throughout this section, we consider a function $u \in W^{1,2}_\Gamma(G)$, where $\Gamma \subseteq \partial G$ is a (relatively) closed subset of ∂G that might also be empty. We assume that u is a weak solution to the mixed boundary value problem

$$\begin{gathered} u \in W^{1,2}_\Gamma(G)\,, \\ \int_G a_{i,j}\,\partial_j u\,\partial_i v + \int_G c\,u\,v = \int_G (f\,v + f_i\,\partial_i v) \quad \text{for all } v \in W^{1,2}_\Gamma(G)\,, \end{gathered} \tag{134}$$

where $\{a_{i,j}\}_{i,j=1,\ldots,n}$ is uniformely elliptic.

Theorem C.1. *Let u satisfy* (134)*. Assume that $c \in L^\infty(G)$, and that $f \in L^\infty(G)$. Then, the following statements are valid:*

(1) If $G \in \mathcal{C}^1$, $a_{i,j} \in L^\infty(G)$ for $i,j = 1,\ldots,n$, and $f_i \in L^\infty(G)$ for $i = 1,\ldots,n$, then $u \in C^{0,\beta}(\overline{G})$ for all $\beta < 1$, and there is $C > 0$ depending on β and on the data $a_{i,j}$, c and G in their respective norm such that

$$\|u\|_{C^{0,\beta}(\overline{G})} \leq C\,(\|f\|_{L^\infty(G)} + \|f_i\|_{L^\infty(G)} + \|u\|_{W^{1,2}(G)})\,.$$

(2) If $G \in \mathcal{C}^{1,\alpha}$ ($\alpha > 0$), $a_{i,j} \in C^{0,\alpha}(\overline{G})$ for $i,j = 1,\ldots,n$, and $f_i \in C^{0,\alpha}(\overline{G})$ for $i = 1,\ldots,n$, then $u \in C^{1,\alpha}(\overline{G})$, and there is $C > 0$ depending on the data $a_{i,j}$, c and G in their respective norm such that

$$\|u\|_{C^{1,\alpha}(\overline{G})} \leq C\,(\|f\|_{L^\infty(G)} + \|f_i\|_{C^{0,\alpha}(\overline{G})} + \|u\|_{W^{1,2}(G)})\,.$$

(3) Same assumptions as in (2)*, but with $f_i \in L^p(G)$ for $i = 1,\ldots,n$ for some $p \in]2,\infty[$. Then, $u \in W^{1,p}(G)$, and there is $C > 0$ depending on the data $a_{i,j}$, c and G in their respective norm such that*

$$\|u\|_{W^{1,p}(G)} \leq C\,(\|f\|_{L^\infty(G)} + \|f_i\|_{L^p(G)} + \|u\|_{W^{1,2}(G)})\,.$$

(4) If $G \in \mathcal{C}^{2,\alpha}$ ($\alpha > 0$), $a_{i,j} \in C^{1,\alpha}(\overline{G})$ for $i,j = 1,\ldots,n$, $c \in C^{0,\alpha}(\overline{G})$, and $f \in C^{0,\alpha}(\overline{G})$, $f_i \in C^{1,\alpha}(\overline{G})$ for $i = 1,\ldots,n$, then $u \in C^{2,\alpha}(\overline{G})$, and there is $C > 0$ depending on the data $a_{i,j}$, c and G in their respective norm such that

$$\|u\|_{C^{2,\alpha}(\overline{G})} \leq C\,(\|f\|_{C^{0,\alpha}(\overline{G})} + \|f_i\|_{C^{1,\alpha}(\overline{G})} + \|u\|_{W^{1,2}(G)})\,.$$

Proof. The result (4) is already to find by Schauder. See the Ch. 3 of monograph [LU68]. A proof basing on Campanato space methods is to find in the more recent survey [Tro87], Theorem 3.16 and Theorem 3.17. ☐

The following Lemma, to find for instance in [Tro87], Theorem 2.7 (see also Lemma 2.9), is well-known.

Lemma C.2. *Let $u \in W^{1,2}(G)$. For $k \in \mathbb{N}$, let $A(k) := \{x \in G \,:\, |u(x)| \geq k\}$. Assume that there are $k_0 \in \mathbb{N}$, $p > 1$ and $C > 0$ such that for all $k \geq k_0$, the function u satisfies the inequality $\|u - T_k(u)\|_{L^p(G)} \leq C\,\operatorname{meas}(A(k))$. Then, for $p' := p/(p-1)$, $\|u\|_{L^\infty(G)} \leq k_0 + C\,2^{p'}\,\operatorname{meas}(G)^{p-1}$.*

D Proof of some technical results

The following Lemma was proved in [Ura73], Lemma 3 for $\sigma = \sigma(q)$ and $\kappa =$ const. The proof is elegant and relatively short. Because we consider a more general form of σ and κ, we present a sketch of the proof, but we emphasize that we introduce no new ideas.

Lemma D.1. *Same assumptions as in Lemma 5.1. Define $b^{(2)}$ as in* (68)*. Then, there are nonnegative $\hat{c}_i$ ($i = 1, 2$) depending continuously on the constants μ_i, ν_i in* (12)*, and on $\|\nabla(\kappa\, n_G)\|_{L^\infty(G)}$ such that*

$$b^{(2)} \geq -\hat{c}_1 \, \frac{|\,\delta\, v|}{\sqrt{1+|\nabla\psi|^2}} - \hat{c}_2 \quad \textit{for all } x \in G\,.$$

Proof. Throughout the proof, we denote $a_{i,j} = \sigma_{q_i,q_j}(x,\, -\nabla\psi,\, 1)$ for $i, j = 1, \ldots, n+1$, and $A := \{a_{i,j}\}$. It is convenient to use the notation $u := \hat{\psi} \in C^2(\overline{G} \times \mathbb{R})$ (cf. (28)) to avoid confusion with the superscripts.

First step. Obviously

$$\sum_{i,j,k,l=1}^{n} F_{p_i,p_j}\, F_{p_k,p_l}\, \partial^2_{j,k}\psi\, \partial^2_{l,i}\psi = \sum_{i,j,k,l=1}^{n+1} a_{i,j}\, a_{k,l}\, \partial^2_{j,k}u\, \partial^2_{l,i}u\,.$$

Denote $\lambda_1(x), \ldots, \lambda_n(x)$ the nonvanishing eigenvalues of the matrix $A(x)$, and $\lambda_{n+1} = 0$. Due to (12c), the eigenvalues satisfy the inequalities

$$\frac{\nu_2}{\sqrt{1+|\nabla\psi|^2}} \leq \lambda_i \leq \frac{\mu_2}{\sqrt{1+|\nabla\psi|^2}} \quad \text{for } i = 1, \ldots, n\,. \tag{135}$$

Due to the condition (12d), there is for each $x \in G \times \mathbb{R}$ an orthonormal system of vectors $\{\tau^m(x)\}_{m=1,\ldots,n}$ such that $\tau^m(x) \cdot n_S(x) = 0$, and such that

$$a_{i,j}\, \tau^r_i\, \tau^s_j = \delta_{r,s}\, \lambda_r \quad \text{for } r, s = 1, \ldots, n+1\,,$$

with the Kronecker δ. Using othonormal decomposition, and the fact that $A n_S = 0$, we see that

$$\Lambda := a_{i,j}\, a_{k,l}\, \partial^2_{j,k}u\, \partial^2_{l,i}u = \lambda_p\, \lambda_q\, [\tau^p_i\, \tau^q_j\, \frac{\partial^2 u}{\partial x_i \partial x_j}]^2\,. \tag{136}$$

Second step. For $l = 1, \ldots, n$, introduce $\xi_l := 2\, F_{p_l} + \kappa\, n_{G,l}$, and $g := \widehat{\xi}$ (cp. (28)). Due to the conditions (12b) and (12e), note that

$$\|g\|_{L^\infty(G\times\mathbb{R})} \leq 2\,\mu_1 + \|\kappa\|_{L^\infty(\partial G)}\,, \quad \|\partial_x g\|_{L^\infty(G\times\mathbb{R})} \leq 2\,\mu_3 + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)}\,. \tag{137}$$

Rearranging the indices in the definition (68) of $b^{(2)}$, it follows that

$$\begin{aligned} b^{(2)} &= \Lambda + F_{p_i,p_j}\, \partial^2_{j,k}\psi\, (F_{p_k,x_i} + \partial_i(n_{G,k}\,\kappa)) + F_{p_k,p_l}\, \partial^2_{l,i}\psi\, F_{p_k,x_i} \\ &= \Lambda + F_{p_i,p_j}\, \partial^2_{j,k}\psi\, (2\, F_{p_k,x_i} + \partial_i(n_{G,k}\,\kappa)) \\ &= \Lambda + a_{i,j}\, \partial^2_{j,k}u\, \partial_i g_k\,. \end{aligned}$$

Orthonormal decomposition implies the equivalent representation

$$
\begin{aligned}
&a_{i,j}\,\partial^2_{j,k}u\,\partial_i g_k = \lambda_p\,(\tau^p\cdot\nabla\partial_k u)\,(\tau^p\cdot\nabla g_k)\\
&\quad = \lambda_p\,\big((\tau^p\cdot\nabla)\nabla u\cdot\tau^q\big)\,\big((\tau^p\cdot\nabla)g\cdot\tau^q\big) + \lambda_p\,\big((\tau^p\cdot\nabla)\nabla u\cdot n_S\big)\,\big((\tau^p\cdot\nabla)g\cdot n_S\big)\,.
\end{aligned}
\tag{138}
$$

The Cauchy-Schwarz inequality, Young's inequality and (135) imply that

$$
\begin{aligned}
&\lambda_p\,\big((\tau^p\cdot\nabla)\nabla u\cdot\tau^q\big)\,\big((\tau^p\cdot\nabla)g\cdot\tau^q\big) = (\lambda_p\,\lambda_q)^{1/2}\,[(\tau^p_i\,\tau^q_j\,\frac{\partial^2 u}{\partial x_i\partial x_j}]\,\frac{\lambda_p}{\lambda_q}\,[(\tau^p_i\,\tau^q_j\,\frac{\partial g_i}{\partial x_j}]\\
&\leq \frac{1}{8}\,\lambda_p\,\lambda_q\,[(\tau^p_i\,\tau^q_j\,\frac{\partial^2 u}{\partial x_i\partial x_j}]^2 + 2\,\frac{\mu_2}{\nu_2}[(\tau^p_i\,\tau^q_j\,\frac{\partial g_i}{\partial x_j}]^2 \leq \frac{\Lambda}{8} + 2\,\frac{\mu_2}{\nu_2}\|\nabla g\|^2_{L^\infty}\,.
\end{aligned}
\tag{139}
$$

Third step. The second term on the right-hand of (138) contains mixed tangential-normal derivatives that are not contained in the expression Λ. To obtain a suitable estimate for the right-hand of (138), we have to state an auxiliary inequality, originating in the paper [Ura73] again. Consider for $y_0\in G\times\mathbb{R}$ and $p\in\{1,\ldots,n+1\}$ the vector $\tau^{p,0}:=\tau^p(y_0)$ and the function $w:=\tau^{p,0}\cdot\nabla u$. The inequality (36) yields

$$
|\nabla w| \leq |\,\delta\,w|\,\sqrt{1+|\nabla\psi|^2} = \sqrt{1+|\nabla\psi|^2}\,\sum_{k=1}^n|\tau^k\cdot\nabla w|\,.
$$

Using the latest, the definition of w implies that

$$
|\tau^{p,0}_i\,n_{S,j}\,\frac{\partial^2 u}{\partial x_i\partial x_j}| \leq |\nabla w| \leq \sqrt{1+|\nabla\psi|^2}\,|\tau^k_i\,\tau^{p,0}_j\,\frac{\partial^2 u}{\partial x_i\partial x_j}|\,.
$$

Writing the last expression in $(x,\,x_{n+1})=y_0$ arbitrary, it follows for all $(x,\,x_{n+1})\in G\times\mathbb{R}$ and $p\in\{1,\ldots,n+1\}$ that

$$
|\tau^p_i\,n_{S,j}\,\frac{\partial^2 u}{\partial x_i\partial x_j}| \leq \sqrt{1+|\nabla\psi|^2}\,|\tau^k_i\,\tau^p_j\,\frac{\partial^2 u}{\partial x_i\partial x_j}|\,.
\tag{140}
$$

On the other hand, the orthonormality of $\{\tau^m\}$, and the definition of the extension operator (28), yield for $k=1,\ldots,n$ the decomposition

$$
F_{p_k}+\kappa\,n_{G,k} = n_{S,k}\,n_S\cdot(F_p+\kappa\,n_G) + \sum_{q=1}^n\tau^q_k\,\tau^q\cdot(F_p+\kappa\,n_G)\,.
\tag{141}
$$

Consider now the formula (66) for the derivatives of the function v. Due to the latest formula (141) used in (66), we obtain for $p\in\{1,\ldots,n+1\}$ that

$$
\begin{aligned}
\tau^p_j\,\partial_j v &= \tau^p_j\,\partial^2_{j,k}\psi\,(F_{p_k}+\kappa\,n_{G,k}) + \tau^p_j\,F_{x_j} + \tau^p_j\,\partial_j(\kappa\,n_{G,k})\,\partial_k\psi\\
&= \tau^p_j\,\partial^2_{j,k}\psi\,n_{S,k}\,n_S\cdot(F_p+\kappa\,n_G) + \tau^p_j\,\partial^2_{j,k}\psi\,\tau^q_k\,\tau^q\cdot(F_p+\kappa\,n_G) + \tau^p_j\,F_{x_j}\\
&\quad + \tau^p_j\,\partial_j(\kappa\,n_{G,k})\,\partial_k\psi\,.
\end{aligned}
\tag{142}
$$

In view of (22), $-n_S \cdot (F_p + \kappa\, n_G) \geq \nu_0 - \mu_1/\sqrt{1+|\nabla\psi|^2}$. The latest yields, thanks also to the conditions (12e),

$$|\tau_j^p\, \partial^2_{j,k}\psi\, n_{S,k}|\,(\nu_0 - \frac{\mu_1}{\sqrt{1+|\nabla\psi|^2}}) \leq |\,\delta\, v| + |\tau_j^p\, \partial^2_{j,k}\psi\, \tau_k^q|\,(\mu_1 + \|\kappa\|_{L^\infty(\partial G)}) + |F_x| \\ + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)}\, |\nabla\psi|\,.$$

It follows from (140) that

$$\frac{\mu_1}{\sqrt{1+|\nabla\psi|^2}}\,|\tau_j^p\, \partial^2_{j,k}\psi\, n_{S,k}| \leq \mu_1\, |\tau_i^k\, \tau_j^p\, \frac{\partial^2 u}{\partial x_i \partial x_j}|\,,$$

and therefore, the last two relation yield

$$\nu_0\, |\tau_j^p\, \partial^2_{j,k}\psi\, n_{S,k}| \\ \leq |\,\delta\, v| + |\tau_j^p\, \partial^2_{j,k}\psi\, \tau_k^q|\,(2\,\mu_1 + \|\kappa\|_{L^\infty(\partial G)}) + |F_x| + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)}\, |\nabla\psi|\,, \tag{143}$$

which, in turn, thanks also to (23), implies that

$$\lambda_p\, \big((\tau^p\cdot\nabla)\nabla u\cdot n_S\big)\,\big((\tau^p\cdot\nabla) g\cdot n_S\big) \leq \nu_0^{-1}\, \|\nabla g\|_{L^\infty} \\ \left[\mu_2\,(\frac{|\,\delta\, v|}{\sqrt{1+|\nabla\psi|^2}} + \mu_3 + \|\nabla(\kappa\, n_G)\|_{L^\infty(G)}) + (2\,\mu_1 + \|\kappa\|_{L^\infty(\partial G)})\,\lambda_p\, |\tau_j^p\, \partial^2_{j,k} u\, \tau_k^q|\right]\,.$$

With arguments analogous to (139)

$$\nu_0^{-1}\, \|\nabla g\|_{L^\infty}\,(2\,\mu_1 + \|\kappa\|_{L^\infty(\partial G)})\,\lambda_p\, |\tau_j^p\, \partial^2_{j,k} u\, \tau_k^q| \\ \leq \frac{\Lambda}{8} + \frac{4\,\mu_2}{\nu_2\,\nu_0^2}\,(2\,\mu_1 + \|\kappa\|_{L^\infty(\partial G)})^2\, \|\nabla g\|^2_{L^\infty}\,.$$

Using the estimates (136), (139) and (137), the claim follows. ☐

Lemma D.2. *Same assumptions as in Lemma 5.1. Let* v *be the function* (62). *Then, there are nonnegative numbers* $\hat{c}_3$, $\hat{c}_4$ *depending continuously on the constants* μ_i, ν_i *in* (12), *and on* $\|\nabla(\kappa\, n_G)\|_{L^\infty(G)}$ *such that*

$$\sum_{k=1}^n [F_{p_k} + \kappa\, n_{G,k}]\, \frac{d}{dx_k}\Phi(x,\,\psi,\,\nabla\psi) \leq \hat{c}_3 + \hat{c}_4\, \frac{|\,\delta\, v|}{\sqrt{1+|\nabla\psi|^2}} \\ + (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})\, \phi_{x_{n+1}}\, \sqrt{1+|\nabla\psi|^2}\,.$$

The number $\hat{c}_3$ *depends moreover continuously on* $\|\nabla\phi^{+,-}\|_{L^\infty(S)}$.

Proof. Due to the definition (4), we have $\Phi = \phi + R$. We begin estimating the ϕ term. Lemma A.2, and the property (53) yield

$$\frac{d\phi}{dx_k}\,[F_{p_k} + \kappa\, n_{G,k}] = (\phi_{x_k} + \phi_{x_{n+1}}\, \partial_k\psi)\,[F_{p_k} + \kappa\, n_{G,k}]\,, \\ \partial_k\psi\,[F_{p_k} + \kappa\, n_{G,k}] \geq (\nu_0 - \|\kappa\|_{L^\infty(\partial G)})\,\sqrt{1+|\nabla\psi|^2} - \mu_1\,. \tag{144}$$

In the case that $\nabla\phi$ is discontinuous across S, Lemma A.2 moreover shows that the traces ϕ_{x_i} in (144) can be taken from arbitrary side of S. The latest implies the estimate

$$\begin{aligned}\frac{d\phi}{dx_k}\,[F_{p_k}+\kappa\,n_{G,k}] \leq & |\phi_{x_k}|+\mu_1\,|\phi_{x_{n+1}}| \\ & +(\nu_0-\|\kappa\|_{L^\infty(\partial G)})\,\phi_{x_{n+1}}\,\sqrt{1+|\nabla\psi|^2}\,.\end{aligned} \tag{145}$$

Now considering the R term (cf. (3)) with $\sigma^0(q)=|q|$, note for $k=1,\ldots,n$ that

$$\begin{aligned}\frac{dR}{dx_k}=&-\sum_{l=1}^{n}[\sigma_{q_i,x_j}\,(\sigma^0_{q_i,q_l}\,\sigma^0_{q_j}+\sigma^0_{q_j,q_l}\,\sigma^0_{q_i})+\sigma^0_{q_i}\,\sigma^0_{q_j}\,\sigma_{q_i,x_j,q_l}]\,\partial^2_{l,k}\psi \\ & +\sigma^0_{q_i}\,\sigma^0_{q_j}\,\sigma_{q_i,x_j,x_k}\,,\end{aligned}$$

where the summation over i,j occurs from $1,\ldots,n+1$. For all $(x,q)\in G\times\mathbb{R}^{n+1}$ and $l=1,\ldots n{+}1$, the assumption (12d) yields $\sigma_{q_l,q_i}(x,q)\,q_i=0$, and therefore also $\sigma_{x_j,q_l,q_i}(x,q)\,q_i=0$. It follows that $\sigma^0_{q_i}\,\sigma_{q_i,x_j,q_l}=0$, and that

$$\frac{dR}{dx_k}=-\,2\,\sigma_{q_i,x_j}\,\sigma^0_{q_j}\,\sum_{l=1}^{n}\sigma^0_{q_i,q_l}\;\partial^2_{l,k}\psi+\sigma^0_{q_i}\,\sigma^0_{q_j}\,\sigma_{q_i,x_j,x_k}\,.$$

It follows that

$$\begin{aligned}(F_{p_k}+\kappa\,n_{G,k})\,\frac{dR}{dx_k}=&-2\,\sigma_{q_i,x_j}\,\sigma^0_{q_j}\,\sum_{l=1}^{n}\sigma^0_{q_i,q_l}\,[(F_{p_k}+\kappa\,n_{G,k})\,\partial^2_{l,k}\psi] \\ & +\sigma^0_{q_i}\,\sigma^0_{q_j}\,\sigma_{q_i,x_j,x_k}\,(F_{p_k}+\kappa\,n_{G,k})\,.\end{aligned} \tag{146}$$

Due to (12a) and (12e), we obviously have

$$|\sigma^0_{q_i}\,\sigma^0_{q_j}\,\sigma_{q_i,x_j,x_k}\,(F_{p_k}+\kappa\,n_{G,k})\leq\mu_4\,(\mu_1+\|\kappa\|_{L^\infty(\partial G)}),\,.$$

On the other hand, using (66), we see that

$$\begin{aligned}[(F_{p_k}+\kappa\,n_{G,k})\,\partial^2_{l,k}\psi]&=\partial_{x_l}v-F_{x_l}-\partial_k\psi\,\partial_l(n_{G,k}\kappa) \\ \sum_{l=1}^{n}\sigma^0_{q_i,q_l}\,[(F_{p_k}+\kappa\,n_{G,k})\,\partial^2_{l,k}\psi]&=\sum_{l=1}^{n}\sigma^0_{q_i,q_l}\,(\partial_{x_l}v-F_{x_l}-\partial_k\psi\,\partial_l(n_{G,k}\kappa))\,.\end{aligned}$$

For $i=1,\ldots,n+1$, set $\eta_i:=\sigma_{q_i,x_j}\,\sigma^0_{q_j}$. The $|\eta|\leq\mu_3$ in view of (12e). Since $\sigma^0(q)=|q|$ satisfies the assumptions (12c), and (12d) with $\mu_2=1$, it follows that (cp. (70))

$$|\sigma^0_{q_i,q_l}\,\partial_{x_l}v\,\eta_i|\leq\sqrt{1+|\nabla\psi|^2}^{-1}\,|\,\delta\,v|\,|\eta|\,,$$

and analogously (cf. (23))

$$|\sigma^0_{q_i,q_l}\,(F_{x_l}+\partial_k\psi\,\partial_l(n_{G,k}\kappa))\,\eta_i|\leq(\mu_3+\|\nabla(\kappa\,n_G)\|_{L^\infty(G)})\,\mu_3\,.$$

The claim follows from (146). □

We now give the proof of Lemma 6.1.

Proof of Lemma 6.1. Let Z_1 be the cylinder $B_1(0;\mathbb{R}^{n-1})\times]-1,0]$ (here $B_1(0;\mathbb{R}^{n-1})$ is the unit ball centered at zero in the space $\mathbb{R}^{n-1}$). Due to the definition of a $\mathcal{C}^{2,\alpha}$ manifold, there are $m\in\mathbb{N}$, a family of smooth open sets $\{U_i\}_{i=0,\ldots,m}\subset\mathbb{R}^n$ such that $\partial G\subset\bigcup_{i=1}^m U_i$, and a family of $C^{2,\alpha}$ diffeormorphisms $\{F_i\}_{i=1,\ldots,m}$

$$\begin{aligned} F_i:\ Z_1 &\to U_i\cap\overline{G}\\ \hat{x} &\mapsto F_i(\hat{x})\,, \end{aligned}$$

such that $F_i(B_1(0;\mathbb{R}^{n-1})\times\{0\}) = U_i\cap\partial G$. Let $\{\zeta_0,\,\ldots,\zeta_m\}$ denote a partition of unity associated with the family $\{U_i\}_{i=0,\ldots,m}$.

We fix $i\in\{1,\ldots,m\}$ and let $\{e_j\}_{j=1,\ldots,n-1}$ denote the $n-1$ first standard basis vectors in $\mathbb{R}^n$. We define a vector field $T^{(i,j)}\in[C^{1,\alpha}(U_i\cap G)]^n$ by

$$T^{(i,j)}(F_i(\hat{x})) := \frac{1}{\det F_i'(\hat{x})}\,F_i'(\hat{x})\,e_j \quad \text{for } \hat{x}\in Z_1 \text{ and } j=1,\ldots,n-1\,. \tag{147}$$

Using formula to find for instance in [Mon03], Lemma 3.59 and formula (3.80), one shows for $j=1,\ldots,n-1$ that

$$\operatorname{div} T^{(i,j)} = [\det F']^{-1}\,\operatorname{div}(e_j) = 0 \text{ in } U_i\cap G,\quad T^{(i,j)}\cdot n_G = 0 \text{ on } U_i\cap\partial G\,,$$

proving (104). It remains to prove the representation (105). Let $\tau\in[C^{1,\alpha}(\partial G)]^n$. Fixing $i\in\{1,\ldots,m\}$, we define $\tau^{(i)}:=\zeta_i\,\tau$, and we associate with $\tau^{(i)}$ the vector field $\xi^{(i)}\in[C^{1,\alpha}(B_1(0;\mathbb{R}^{n-1})\times\{0\})]^n$ defined by

$$\xi^{(i)}(\hat{x}) := \det(F_i'(\hat{x}))\,[F_i'(\hat{x})]^{-1}\,\tau^{(i)}(F_i(\hat{x}))\,.$$

We define an extension $\xi^{(i)}\in[C^{1,\alpha}(Z_1)]^n$ by setting $\xi(\hat{x}) := \xi^{(i)}(\hat{x}_1,\ldots,\hat{x}_{n-1},0)$. It follows from (147) that

$$\begin{aligned} \tau^{(i)}(F_i(\hat{x})) &= \sum_{j=1}^{n-1}\xi_j^{(i)}(\hat{x})\,T^{(i,j)}(F_i(\hat{x}))\,,\\ \tau(x) &= \sum_{i=1}^{m}\sum_{j=1}^{n-1}\xi_j^{(i)}(F_i^{-1}(x))\,T^{(i,j)}(F_i^{-1}(x)) \quad \text{for } x\in\overline{G}\,. \end{aligned}$$

□

In the following Lemma, we prove the Fréchet-differentiability claim needed for the Theorem 7.2. The proof uses the following straightforward property.

Lemma D.3. *Let $g,\,h\in C^{0,\beta}(\overline{G})$ with $0\le\beta\le 1$. Let $f\in C^{0,1}(\mathbb{R})$. Then, $g,\,h\in C^{0,\beta}(\overline{G})$, and $f\circ g\in C^{0,\beta}(\overline{G})$.*

Lemma D.4. *Under the conditions of Theorem 7.2, the solution mapping S to the problem* (119) *is well-defined, continuous and Fréchet-differentiable from $C^{2,\beta}(\overline{G})$ into itself for all $0<\beta\le\alpha$.*

Proof. We at first prove the unique solvability of (119) in $C^{2,\beta}(\overline{G})$. For $v \in L^\infty(G)$, define

$$a(x,\, v) := \int_0^1 \phi_{x_{n+1}}(x,\, t\,v)\,dt\,. \tag{148}$$

Due to the condition (7), $a(x,\, v) \leq -\gamma_0$ uniformely in G. On the other hand, the fundamental Lemma of integration yields

$$\phi(x,\, u) = a(x,\, u)\,u + \phi(x,\, 0)\,.$$

Let $w \in C^{2,\beta}(\overline{G})$ be fixed. Let $p > n$ such that $1 - n/p > \beta$. We denote by $T :\ L^\infty(G) \rightarrow W^{1,p}(\overline{G})$ the solution operator $v \mapsto u$ to the linear problem

$$\begin{aligned} -\partial_{x_i}(a_{i,j}(x,\, \nabla w)\,\partial_{x_j}u) - a(x,\, v)\,u &= \phi(x,\, 0) + R(x,\, \nabla w) + \partial_{x_i}b_i(x) \quad &&\text{in } G\\ -n_{G,i}(x)\,a_{i,j}(x,\, \nabla w)\,\partial_{x_j}u &= \kappa(x) + n_{G,i}(x)\,b_i(x) &&\text{on } \partial G\,. \end{aligned} \tag{149}$$

Due to the estimate (20), the case (3) of Theorem C.1 induces the existence of the solution $u \in W^{1,p}(G)$, and a continuity estimate. The uniqueness of the solution follows from the uniform negativity of the coefficient a. Clearly, T is compact from $L^\infty(G)$ into itself due to the choice of p and the Sobolev embedding theorem. We now prove that T maps a closed, convex and bounded set of $L^\infty(G)$ into itself.

For $k \in \mathbb{N}$, multiply (149) with the testfunction $\xi_k := (u - T_k(u))$. Integration by parts yields

$$\begin{aligned} &\int_G a_{i,j}(x,\, \nabla w)\,\partial_{x_j}(u - T_k(u))\,\partial_{x_i}(u - T_k(u)) - \int_G a(x,\, v)\,u\,(u - T_k(u))\\ &\quad = -\int_{\partial G} \kappa\,\xi_k - \int_G b_i(x)\,\partial_{x_i}\xi_k + \int_G (\phi(x,\, 0) + R)\,\xi_k\,. \end{aligned}$$

It follows that

$$\begin{aligned} &\inf\{\gamma_0,\, \alpha_0(w)\}\,\|u - T_k(u)\|^2_{W^{1,2}(G)}\\ &\quad \leq \int_{\partial G} |\kappa|\,|u - T_k(u)| + \int_G |b_i|\,|\partial_{x_i}(u - T_k(u))| + \int_G |\phi(\cdot,\, 0) + R|\,|u - T_k(u)|\,. \end{aligned}$$

For $r_0 > n-1$ and $q_0 > n$ arbitrary, it follows from Lemma C.2 that

$$\|u\|_{L^\infty(G)} \leq c\,(\|\kappa\|_{L^{r_0}(\partial G)} + \|b_i\|_{L^{q_0}(G)} + \|\phi(\cdot,\, 0) + R\|_{L^{q_0/2}(G)})\,, \tag{150}$$

with a constant c independent of v, that is, due also to (20), $\|Tv\|_{L^\infty(G)} \leq K$, with K independent of v. Thus, invoking the Schauder fixed-point theorem, there is $u \in L^\infty(G) \cap W^{1,p}(G)$ such that $Tu = u$, that is u solves the problem (119). Due to the choice of p, $u \in C^{0,\beta}(\overline{G})$. Since ϕ is Lipschitz continuous, it follows that $\phi(x,\, u) \in C^{0,\beta}(\overline{G})$, and the case (4) of Theorem C.1 implies even that $u \in C^{2,\beta}(\overline{G})$, with a corresponding continuous estimate.

We now prove the Fréchet differentiability of the solution mapping $Sw := u$. Let $h \in C^{2,\beta}(\overline{G})$. Denote $u_h = S(w+h) \in C^{2,\beta}(\overline{G})$. Then, defining

$$\begin{aligned} g_i &:= (a_{i,j}(x,\, \nabla(w+h)) - a_{i,j}(x,\, \nabla w))\,\partial_{x_j}u_h \quad \text{for } i = 1, \ldots, n\,,\\ g_0 &:= R(x,\, \nabla(w+h)) - R(x,\, \nabla w)\,, \end{aligned}$$

one proves that

$$
\begin{aligned}
-\partial_{x_i}(a_{i,j}(x,\,\nabla w)\,\partial_{x_j}(u_h-u)) &= \phi(x,\,u_h)-\phi(x,\,u)+\partial_{x_i}g_i+g_0 && \text{in } G \\
-n_{G,i}(x)\,a_{i,j}(x,\,\nabla w)\,\partial_{x_j}(u_h-u) &= n_{G,i}(x)\,g_i && \text{on } \partial G\,,
\end{aligned}
\tag{151}
$$

With the notation

$$
\begin{aligned}
\phi(x,\,u_h)-\phi(x,\,u) &= \int_0^1 \phi_{n+1}(x,\,t\,u_h+(1-t)\,u)\,dt\,(u_h-u) \\
&=: a(x,\,u,\,u_h)\,(u_h-u)\,,
\end{aligned}
$$

one equivalently obtains for all $\xi\in W^{1,2}(G)$ that

$$
\int_G (a_{i,j}(x,\,\nabla w)\,\partial_{x_j}(u_h-u)\,\partial_{x_i}\xi - a(x,\,u,\,u_h)\,(u_h-u)\,\xi) = \int_G (g_0\,\xi - g_i\,\partial_{x_i}\xi)\,. \tag{152}
$$

Since $|g_0| \le \|g\|_{C^1(\overline{G}\times\mathbb{R}^n)}\,|\nabla h|$ and $|g_i| \le \|\nabla u_h\|_{L^\infty(G)}\,\|\nabla_p a_{i,j}\|_{L^\infty(G)}\,\nabla h$ (cp. (19)), it follows for $1\le q\le\infty$ that

$$
\|g_i\|_{L^q(G)}+\|g_0\|_{L^q(G)} \le (\|\nabla u_h\|_{L^\infty(G)}\,\|\nabla_p a_{i,j}\|_{L^\infty(G)}+\|g\|_{C^1(\overline{G}\times\mathbb{R}^n)})\,\|\nabla h\|_{L^q(G)}\,.
$$

With arguments similar to the ones used for the problem (149), we can prove that if u_h-u satisfies (152), and if $\beta\le 1-n/q_0$, there is a continuous function $c(\cdot,\cdot)$ on $[C^{2,\beta}(\overline{G})]^2$ such that

$$
\|u_h-u\|_{C^{0,\beta}(\overline{G})} \le c(w,\,w+h)\,\|\nabla h\|_{L^{q_0}(G)}\,, \tag{153}
$$

We now prove that there even is $c(w,\,w+h)$ such that

$$
\|u_h-u\|_{C^{2,\beta}(\overline{G})} \le c(w,\,w+h)\,\|h\|_{C^{2,\beta}(\overline{G})}\,. \tag{154}
$$

For $k=1,\ldots,n$, observe that

$$
\begin{aligned}
\frac{d}{dx_k}(a_{i,j}(x,\,\nabla(w+h))-a_{i,j}(x,\,\nabla w)) &= \partial_{x_k}a_{i,j}(x,\,\nabla(w+h))-\partial_{x_k}a_{i,j}(x,\,\nabla w) \\
&\quad +\partial_{p_l}a_{i,j}(x,\,\nabla(w+h))\,\frac{\partial^2(w+h)}{\partial x_l\partial x_k}-\partial_{p_l}a_{i,j}(x,\,\nabla w)\,\frac{\partial^2 w}{\partial x_l\partial x_k}\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\partial_{x_k}g_i =&\{\partial_{x_k}a_{i,j}(x,\,\nabla(w+h))-\partial_{x_k}a_{i,j}(x,\,\nabla w)+\partial_{p_l}a_{i,j}(x,\,\nabla(w+h))\,\frac{\partial^2 h}{\partial x_l\partial x_k} \\
&+(\partial_{p_l}a_{i,j}(x,\,\nabla(w+h))-\partial_{p_l}a_{i,j}(x,\,\nabla w))\,\frac{\partial^2 w}{\partial x_l\partial x_k}\}\,\partial_{x_j}u_h \\
&+\frac{\partial^2 u_h}{\partial x_j\partial x_k}\,(a_{i,j}(x,\,\nabla(w+h))-a_{i,j}(x,\,\nabla w))\,.
\end{aligned}
$$

The latest together with the property of Lemma D.3 and (19) yields

$$\|\nabla g_i\|_{C^{0,\beta}(\overline{G})} \leq c(w,\, w+h)\, \|h\|_{C^{2,\beta}(\overline{G})}\,.$$

Similarily, using the regularity (19)

$$\|g_0\|_{C^{0,\beta}(\overline{G})} \leq c(w,\, w+h)\, \|h\|_{C^{1,\beta}(\overline{G})}\,.$$

Therefore, the case (4) of Theorem C.1 on the regularity of the problem (151) yields

$$\begin{aligned}
&\|S(w+h) - S(w)\|_{C^{2,\beta}(\overline{G})} \\
&\quad \leq c(w, w+h)\, (\|g_0\|_{C^{0,\beta}(\overline{G}} + \|g_i\|_{C^{1,\beta}(\overline{G}} + \|\phi(x,\, u_h) - \phi(x,\, u)\|_{C^{0,\beta}(\overline{G})}) \\
&\quad \leq c(w, w+h)\, (\|h\|_{C^{2,\beta}(\overline{G})} + \|\phi\|_{C^{0,1}}\, \|u_h - u\|_{C^{0,\beta}(\overline{G})})\,.
\end{aligned}$$

The claim follows using (153). □

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