SUPERCONVERGENCE PROPERTIES OF OPTIMAL CONTROL PROBLEMS *

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Abstract. An optimal control problem for a 2-d elliptic equation is investigated with pointwise control constraints. This paper is concerned with discretization of the control by piecewise constant functions. The state and the adjoint state are discretized by linear finite elements. Approximations of the optimal solution of the continuous optimal control problem will be constructed by a projection of the discrete adjoint state. It is proved that these approximations have convergence order h^2.

Keywords: Linear-quadratic optimal control problems, error estimates, elliptic equations, numerical approximation, control constraints, superconvergence.

AMS subject classification: 49K20, 49M25, 65N30

1. Introduction. The paper is concerned with the discretization of the 2-d elliptic optimal control problem

$$J(u) = F(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to the state equations

$$\begin{aligned} Ay + a_0 y &= u && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma \end{aligned} \tag{1.2}$$

and subject to the control constraints

$$a \le u(t,x) \le b \qquad \text{for a.a. } x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain and Γ is the boundary of Ω; A denotes a second order elliptic operator of the form

$$Ay(x) = -\sum_{i,j=1}^{2} D_i(a_{ij}(x) D_j y(x))$$

where D_i denotes the partial derivative with respect to x_i, and a and b are real numbers. Moreover, $\nu > 0$ is a fixed positive number. We denote the set of admissible controls by U_{ad}:

$$U_{ad} = \{u \in L^2(\Omega) : \ a \le u \le b \ \text{ a.e. in } \Omega\}.$$

We discuss here the full discretization of the control and the state equations by a finite element method. The asymptotic behaviour of the discretized problem is studied, and superconvergence results are established.

The approximation of the discretization for semilinear elliptic optimal control problems is discussed in Arada, Casas, and Tröltzsch [1]. The optimal control problem (1.1)–(1.3) is a linear-quadratic

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counterpart of such a semilinear problem. Our aim is to construct controls $\tilde{u}$ which have an approximation order of h^2. This higher convergence order is the difference to [1].

The discretization of optimal control problems by piecewise constant functions is well investigated, we refer to Falk [7], Geveci [8]. Piecewise constant and piecewise linear discretization in space are discussed for a parabolic problem in Malanowski [12]. Theory and numerical results for elliptic boundary control problems are contained in Casas and Tröltzsch [5] and Casas, Mateos, and Tröltsch [4].

Piecewise linear control discretizations for elliptic optimal control problems are studied by Casas and Tröltzsch, see [5]. In an abstract optimization problem, piecewise linear approximations are investigated in Rösch [14]. In all papers, the convergence order is h or $h^{3/2}$.

A quadratic convergence result is proved by Hinze [10]. In that approach only the state equation is discretized. The control is obtained by a projection of the adjoint state to the set of admissible controls.

In this paper, we combine the advantages of the different approaches. After solving a fully discretized optimal control problem, a control $\tilde{u}$ is calculated by the projection of the adjoint state p_h in a post-processing step. Although the approximation of the discretized solution is only of order h, we will show that this post-processing step improves the convergence order to h^2.

The paper is organized as follows: In section 2 the discretizations are introduced and the main results are stated. Section 3 contains auxiliary results. The proofs of the superconvergence results are placed in section 4. The paper ends with numerical experiments shown in section 5.

2. Discretization and superconvergence results. Throughout this paper, Ω denotes a convex bounded open subset in $\mathbb{R}^2$ of class $C^{1,1}$. The coefficients a_{ij} of the operator A belong to $C^{0,1}(\bar{\Omega})$ and satisfy the ellipticity condition

$$m_0|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \qquad \forall(\xi,x) \in \mathbb{R}^2 \times \bar{\Omega}, \quad m_0 > 0.$$

Moreover, we require $a_{ij}(x) = a_{ji}(x)$ and $y_d \in L^p(\Omega)$ for some $p > 2$. For the function $a_0 \in L^\infty(\Omega)$, we assume $a_0 \ge 0$. Next, we recall some results from Bonnans and Casas [2].

LEMMA 2.1. *[2] For every function $g \in L^p(\Omega)$, the solution y of*

$$Ay + a_0 y = g \qquad \text{in } \Omega, \quad y|_\Gamma = 0,$$

belongs to $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for every $p > n$. Moreover, there exists a positive constant c, independent of a_0 such that

$$\|y\|_{W^{2,p}(\Omega)} \le c\|g\|_{L^p(\Omega)}.$$

Next, we introduce the adjoint equation

$$\begin{aligned} Ap + a_0 p &= y - y_d && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma \end{aligned} \tag{2.1}$$

Due to Lemma 2.1, the state equation and the adjoint equation admit unique solutions in $H_0^1(\Omega) \cap W^{2,p}(\Omega)$, if $y_d \in L^p(\Omega)$ for $p > 2$. This space is embedded in $C^{0,1}(\bar{\Omega})$.

We call the solution y of (1.2) for a control u associated state to u and write $y(u)$. In the same way, we call the solution p of (2.1) corresponding to $y(u)$ associated adjoint state to u and write $p(u)$.

Introducing the projection

$$\Pi_{[a,b]}(f(x)) = \max(a, \min(b, f(x))),$$

we can formulate the necessary and sufficient first-order optimality condition for (1.1)–(1.3).

LEMMA 2.2. *A necessary and sufficient condition for the optimality of a control $\bar{u}$ with corresponding state $\bar{y} = y(\bar{u})$ and adjoint state $\bar{p} = p(\bar{u})$, respectively, is that the equation*

$$\bar{u}(x) = \Pi_{[a,b]}(-\frac{1}{\nu}\bar{p}) \tag{2.2}$$

holds.

Since the optimal control problem is strictly convex, we obtain the existence of a unique optimal solution. The optimality condition can be formulated as variational inequality (3.11). A standard pointwise a.e. discussion of this variational inequality leads to the above formulated projection formula.

We are now able to introduce the discretized problem. We define a finite-element based approximation of the optimal control (1.1)–(1.3). To this aim, we consider a family of triangulations $(T_h)_{h>0}$ of $\bar{\Omega}$. With each element $T \in T_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. The mesh size of the grid is defined by $h = \max_{T \in T_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(A1) There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \leq \sigma, \qquad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in T_h$ and all $h > 0$.

(A2) Let us define $\bar{\Omega}_h = \bigcup_{T \in T_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\bar{\Omega}_h$ is convex and that the vertices of T_h placed on the boundary of Γ_h are points of Γ. From [13], estimate (5.2.19), it is known that

$$|\Omega \setminus \Omega_h| \leq Ch^2,$$

where $|.|$ denotes the measure of the set. Next, to every boundary triangle T of T_h we associate another triangle $\hat{T}$ with curved boundary as follows: The edge between the two boundary nodes of T is substituted by the corresponding curved part of Γ. We denote by $\hat{T}_h$ the union of these curved boundary triangles with the interior triangles to Ω of T_h, such that $\bar{\Omega} = \bigcup_{\hat{T} \in \hat{T}_h} \hat{T}$. Moreover, we set

$$\begin{aligned} U_h &= \{u \in L^\infty(\Omega) : \ u|_{\hat{T}} \text{ is constant on all } \hat{T} \in \hat{T}_h\}, \quad U_h^{ad} = U_h \cap U_{ad}, \\ V_h &= \{y_h \in C(\bar{\Omega}) : \ y_h \in \mathcal{P}_1 \text{ for all } T \in T_h, \text{ and } y_h = 0 \text{ on } \bar{\Omega} \setminus \Omega_h\}, \end{aligned}$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. For each $u_h \in U_h$, we denote by $y_h(u_h)$ the unique element of V_h that satisfies

$$a(y_h(u_h), v_h) = \int_\Omega u_h v_h \, dx \qquad \forall v_h \in V_h, \tag{2.3}$$

where $a: V_h \times V_h \to \mathbb{R}$ is the bilinear form defined by

$$a(y_h, v_h) = \int_\Omega \left(a_0(x) y_h(x) v_h(x) + \sum_{i,j=1}^{2} a_{ij}(x) D_i y_h(x) D_j v_h(x) \right) dx.$$

In other words, $y_h(u_h)$ is the approximated state associated with u_h. Because of $y_h = v_h = 0$ on $\bar{\Omega} \setminus \Omega_h$ the integrals over Ω can be replaced by integrals over Ω_h. The finite dimensional approximation of the optimal control problem is defined by

$$\inf J(u_h) = \frac{1}{2}\|y_h(u_h) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Omega)} \qquad u_h \in U_h^{ad}. \tag{2.4}$$

The adjoint equation is discretized in the same way

$$a(p_h(u_h), v_h) = \int_\Omega (y_h(u_h) - y_d) v_h \, dx \qquad \forall v_h \in V_h. \tag{2.5}$$

The approximation order of the solutions of (2.4) in L^2-sense is investigated in [1].

We will go a different way. For our superconvergence result we need an additional assumption for $\bar{u}$. We know already that the associated adjoint state $\bar{p}$ belongs to a space $W^{2,p}(\Omega)$ for a certain $p > 2$. The optimal control $\bar{u}$ is obtained by the projection formula (2.2). Therefore, we can classify the triangles T_i in two sets K_1 and K_2:

$$K_1 = \{T_i : \ \bar{u} \text{ is only Lipschitz continuous on } T_i\}, \qquad K_2 = \{T_i : \ \bar{u} \in W^{2,p}(T_i)\}$$

This classification is correct: $W^{2,p}(\Omega)$ is embedded in $C^{0,1}(\bar{\Omega})$. Moreover, the projection operator is continuous from $C^{0,1}(\bar{\Omega})$ to $C^{0,1}(\bar{\Omega})$. Clearly, the number of triangles in K_1 grows for decreasing h. Nevertheless, the following additional assumption is fulfilled in many practical cases:

(A3) $|K_1| \le c \cdot h$.

Let $\bar{u}$ be the optimal solution of (1.1)–(1.3). Next, we denote by S_i the centroid of the triangle T_i. We define a piecewise constant function by the values of $\bar{u}(S_i)$

$$w_h(x) = \bar{u}(S_i) \quad \text{ if } x \in T_i. \tag{2.6}$$

It is easy to verify that $w_h \in U_h^{ad}$.

Now we are able to formulate our first superconvergence result.

THEOREM 2.3. *Assume that the assumptions (A1)–(A3) hold. Let u_h be the solutions of (2.4). Then the estimate*

$$\|u_h - w_h\|_{L^2(\Omega)} \le ch^2 \tag{2.7}$$

holds true. The proof of Theorem 2.3 is contained in section 4.

Moreover, we can construct controls in a post-processing step. We start by the solution u_h of (2.4). The control $\tilde{u}$ is calculated by a projection of the discrete adjoint state $p_h(u_h)$ to the admissible set

$$\tilde{u}(x) = \Pi_{[a,b]}(-\frac{1}{\nu}(p_h(u_h))(x)).$$

THEOREM 2.4. *Assume that the assumptions (A1)–(A3) hold. Let $\tilde{u}$ be the control constructed above. Then the estimate*

$$\|\bar{u} - \tilde{u}\|_{L^2(\Omega)} \le ch^2 \tag{2.8}$$

holds true. The proof of Theorem 2.4 is also derived in section 4.

3. Auxiliary results. First, we recall some well known results for FEM-approximations [6]. We start with the so-called Aubin-Nitsche Lemma.

LEMMA 3.1. *Let (A1) and (A2) be fulfilled and $u \in L^2(\Omega)$. Then we have*

$$\|y(u) - y_h(u)\|_{L^2(\Omega)} \leq ch^2 \|u\|_{L^2(\Omega)} \tag{3.1}$$

$$\|p(u) - p_h(u)\|_{L^2(\Omega)} \leq ch^2 (\|u\|_{L^2(\Omega)} + \|y_d\|_{L^2(\Omega)}). \tag{3.2}$$

Next, we prove an estimate for the numerical integration.

LEMMA 3.2. *Let f be a function belonging to $H^2(T_i)$ for all i in a certain index set I. Then the estimates*

$$\left| \int_{T_i} (f(x) - f(S_i)) \ dx \right| \leq ch^2 \sqrt{|T_i|} |f|_{H^2(T_i)}$$

and

$$\sum_{i \in I} \left| \int_{T_i} (f(x) - f(S_i)) \ dx \right| \leq ch^2 \left(\sum_{i \in I} |f|^2_{H^2(T_i)} \right)^{1/2}$$

are valid.

Proof. The proof is almost standard. First, we remark that $|.|_{H^2(T_i)}$ denotes the H^2-seminorm. Next, we transform the integral to an integral over the reference element by $E\hat{x} = x$ and apply the Bramble-Hilbert-Lemma:

$$\begin{aligned}
\left| \int_{T_i} (f(x) - f(S_i)) \ dx \right| &= \frac{|T_i|}{|\hat{T}|} \left| \int_{\hat{T}} (f(E\hat{x}) - f(S_i)) \ d\hat{x} \right| \\
&\leq c|T_i| \left(\int_{\hat{T}} \sum_{|\alpha|=2} |D^\alpha_{\hat{x}} f(E\hat{x})|^2 \ d\hat{x} \right)^{1/2} \\
&\leq ch^2 |T_i| \left(\frac{|\hat{T}|}{|T_i|} \int_{T_i} \sum_{|\alpha|=2} |D^\alpha_x f(x)|^2 \ dx \right)^{1/2} \\
&\leq ch^2 \sqrt{|T_i|} |f|_{H^2(T_i)}
\end{aligned}$$

This implies

$$\sum_{i \in I} \left| \int_{T_i} (f(x) - f(S_i)) \ dx \right| \leq ch^2 \left(\sum_{i \in I} |f|^2_{H^2(T_i)} \right)^{1/2}$$

by the Cauchy-Schwarz inequality. ∎

LEMMA 3.3. *Let w_h be the functions defined by (2.6). In addition, we assume that the assumptions (A1)–(A3) are fulfilled. Then the estimate*

$$\|y_h(\bar{u}) - y_h(w_h)\|_{L^2(\Omega)} \leq ch^2 \|\bar{p}\|_{W^{2,p}(\Omega)} \tag{3.3}$$

holds true.

Proof. We start with the transformation

$$\begin{aligned}\|y_h(\bar{u}) - y_h(w_h)\|^2_{L^2(\Omega)} &= (y_h(\bar{u}) - y_h(w_h), y_h(\bar{u}) - y_h(w_h))_{L^2(\Omega)} \\ &= (p_h(\bar{u}) - p_h(w_h), \bar{u} - w_h)_{L^2(\Omega)} \\ &= \int_{K_1} (p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h)\, dx \\ &+ \int_{K_2} (p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h)\, dx \end{aligned} \tag{3.4}$$

It remains to estimate these two integrals. The K_1-part can be estimated by the following arguments: The function $\bar{p}$ belongs to $W^{2,p}(\Omega)$ with $p > 2$. Hence, we have

$$\|\bar{u}\|_{C^{0,1}(\bar{\Omega})} \leq \frac{1}{\nu}\|\bar{p}\|_{C^{0,1}(\bar{\Omega})} \leq c\|\bar{p}\|_{W^{2,p}(\Omega)}.$$

Because of $\bar{u}(S_i) = w_h(S_i)$ and the fact that w_h is constant on T_i, this implies $|\bar{u}(x) - w_h(x)| \leq c\|\bar{p}\|_{W^{2,p}(\Omega)} \cdot |x - S_i| \leq ch\|\bar{p}\|_{W^{2,p}(\Omega)}$. Consequently, we obtain

$$\begin{aligned}|\int_{K_1} (p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h)\, dx| &\leq \sum_{T_i \in K_1} \int_{T_i} |(p_h(\bar{u}) - p_h(w_h))(\bar{u} - w_h)|\, dx \\ &\leq \sum_{T_i \in K_1} ch\|\bar{p}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{C(\bar{\Omega})} \int_{T_i} dx \\ &\leq ch\|\bar{p}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{C(\bar{\Omega})} \int_{K_1} dx \\ &\leq ch^2\|\bar{p}\|_{W^{2,p}(\Omega)} \|p_h(\bar{u}) - p_h(w_h)\|_{C(\bar{\Omega})} \end{aligned} \tag{3.5}$$

by means of assumption (A3). For a triangle T_i of the K_2-part we have for an arbitrary function $v_h \in V_h$

$$\int_{T_i} w_h v_h\, dx = \int_{T_i} \bar{u}(S_i) v_h\, dx = \int_{T_i} \bar{u}(S_i) v_h(S_i)\, dx.$$

This is a formula for the numerical integration of $\bar{u}v_h$. Consequently, we obtain by Lemma 3.2

$$\begin{aligned}|\int_{K_2} (\bar{u} - w_h) v_h\, dx| &\leq \sum_{T_i \in K_2} |\int_{T_i} (\bar{u} - \bar{u}(S_i)) v_h\, dx| \\ &\leq ch^2 \left(\sum_{T_i \in K_2} |\bar{u}v_h|^2_{H^2(T_i)} \right)^{1/2}. \end{aligned} \tag{3.6}$$

Next, we divide each triangle T_i of K_2 in an "active" (A_i) and an "inactive" part (I_i) with $A_i \cup I_i = T_i$. The optimal control $\bar{u}$ is constant on the active component A_i ($\bar{u} = a$ or $\bar{u} = b$). Therefore, the seminorm is 0 on these parts. On the inactive parts I_i, we have

$$\bar{u} = -\frac{1}{\nu}\bar{p}.$$

Therefore, we can estimate

$$|\bar{u}v_h|_{H^2(T_i)} = |\bar{u}v_h|_{H^2(I_i)} = \frac{1}{\nu}|\bar{p}v_h|_{H^2(I_i)} \leq c|\bar{p}v_h|_{H^2(T_i)}.$$

Hence, we can continue by

$$
\begin{aligned}
|\int_{K_2}(\bar{u}-w_h)v_h\,dx| &\le ch^2\left(\sum_{T_i\in K_2}|\bar{u}v_h|^2_{H^2(T_i)}\right)^{1/2}\\
&\le ch^2\left(\sum_{T_i\in K_2}|\bar{p}v_h|^2_{H^2(T_i)}\right)^{1/2}\\
&\le ch^2\left(\sum_{T_i\in K_2}\sum_{|\alpha|,|\beta|=1}\|D^{\alpha+\beta}\bar{p}v_h\|^2_{L^2(T_i)}+\|D^{\alpha}\bar{p}D^{\beta}v_h\|^2_{L^2(T_i)}\right)^{1/2}\\
&\le ch^2\|\bar{p}\|_{W^{2,p}(\Omega)}\|v_h\|_{H^1_0(\Omega)}, \qquad (3.7)
\end{aligned}
$$

by means of Hölder's inequality in the last step. Next, we set $v_h = p_h(\bar{u}) - p_h(w_h)$ and obtain

$$
|\int_{K_2}(\bar{u}-w_h)(p_h(\bar{u})-p_h(w_h))\,dx| \le ch^2\|\bar{p}\|_{W^{2,p}(\Omega)}\|p_h(\bar{u})-p_h(w_h)\|_{H^1_0(\Omega)}. \qquad (3.8)
$$

Inserting (3.5) and (3.8) in (3.4), we get

$$
\|y_h(\bar{u})-y_h(w_h)\|^2_{L^2(\Omega)} \le ch^2\|\bar{p}\|_{W^{2,p}(\Omega)}(\|p_h(\bar{u})-p_h(w_h)\|_{C(\bar{\Omega})}+\|p_h(\bar{u})-p_h(w_h)\|_{H^1_0(\Omega)}).
$$

We benefit now from the fact that $p_h(\bar{u})$ and $p_h(w_h)$ are the solutions of the discretized adjoint equation (2.5) . Hence, we have

$$
\|p_h(\bar{u})-p_h(w_h)\|_{C(\bar{\Omega})}+\|p_h(\bar{u})-p_h(w_h)\|_{H^1_0(\Omega)} \le c\|y_h(\bar{u})-y_h(w_h)\|_{L^2(\Omega)}
$$

with a positive constant c independent of h. The C-estimate can be obtained as follows: Take the adjoint equation (2.1) and the discretized adjoint equation (2.5), but with a right hand side $f_h \in V_h$ instead of $y - y_d$. Then we find for the corresponding solutions z and z_h of the continuous and discretized adjoint equation

$$
\begin{aligned}
\|z_h\|_{C(\bar{\Omega})} &\le \|z_h - z\|_{C(\bar{\Omega})} + \|z\|_{C(\bar{\Omega})}\\
&\le ch\|f_h\|_{L^2(\Omega)} + c\|z\|_{H^2(\Omega)}\\
&\le ch\|f_h\|_{L^2(\Omega)} + c\|f_h\|_{L^2(\Omega)}.
\end{aligned}
$$

Substituting $f_h = y_h(\bar{u}) - y_h(w_h)$ and $z_h = p_h(\bar{u}) - p_h(w_h)$ delivers the desired estimate. For the estimate of the first expression, we refer to Braess [3].

Finally, we get

$$
\|y_h(\bar{u})-y_h(w_h)\|_{L^2(\Omega)} \le ch^2\|\bar{p}\|_{W^{2,p}(\Omega)}
$$

which is exactly inequality (3.3). □

COROLLARY 3.4. *Assume that the assumptions of Lemma 3.3 are fulfilled. Then, we have*

$$
\|p_h(\bar{u})-p_h(w_h)\|_{L^2(\Omega)} \le ch^2\|\bar{p}\|_{W^{2,p}(\Omega)}. \qquad (3.9)
$$

By means of Lemma 3.1, we obtain

$$
\|\bar{p}-p_h(w_h)\|_{L^2(\Omega)} \le ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)}+\|y_d\|_{L^2(\Omega)}). \qquad (3.10)
$$

LEMMA 3.5. *The following variational inequalities are necessary and sufficient for the optimality of the unique solutions of (1.1)–(1.3) and (2.4), respectively.*

$$(\bar{p} + \nu\bar{u}, u - \bar{u})_{L^2(\Omega)} \geq 0 \qquad \textit{for all } u \in U_{ad}, \tag{3.11}$$

$$(p_h(u_h) + \nu u_h, \zeta_h - u_h)_{L^2(\Omega)} \geq 0 \qquad \textit{for all } \zeta_h \in U_h^{ad}. \tag{3.12}$$

The variational inequality (3.11) is an equivalent formulation for the projection formula (2.2).

Next, we derive a variational inequality for the function w_h. First, formula (3.11) is true for all $u \in U_{ad}$. Therefore, we have pointwise a.e.

$$(\bar{p}(x) + \nu\bar{u}(x)) \cdot (u - \bar{u}(x)) \geq 0 \qquad \forall u \in [a, b].$$

We apply this formula for $x = S_i$ and $u = u_h(S_i)$. This is correct because of the continuity of $\bar{u}$, $\bar{p}$, and u_h in these points. We arrive at

$$(\bar{p}(S_i) + \nu\bar{u}(S_i)) \cdot (u_h(S_i) - \bar{u}(S_i)) \geq 0 \qquad \text{for all } S_i.$$

Due to (2.6), this is equivalent to

$$(\bar{p}(S_i) + \nu w_h(S_i)) \cdot (u_h(S_i) - w_h(S_i)) \geq 0 \qquad \text{for all } S_i.$$

We integrate this formula over T_i and add up over all i

$$(\hat{p} + \nu w_h, u_h - w_h)_{L^2(\Omega)} \geq 0 \tag{3.13}$$

with

$$\hat{p}(x) = \bar{p}(S_i) \qquad \text{if } x \in T_i.$$

Moreover, we can test inequality (3.12) with the function w_h and get

$$(p_h(u_h) + \nu u_h, w_h - u_h)_{L^2(\Omega)} \geq 0. \tag{3.14}$$

We add these two inequalities and obtain

$$(\hat{p} - p_h(u_h) + \nu(w_h - u_h), u_h - w_h)_{L^2(\Omega)} \geq 0.$$

This is equivalent to

$$\nu\|w_h - u_h\|^2_{L^2(\Omega)} \leq (\hat{p} - p_h(u_h), u_h - w_h)_{L^2(\Omega)}. \tag{3.15}$$

4. Superconvergence properties. Inequality (3.15) is the starting point for the proofs of the superconvergence results. Now, we are ready to prove Theorem 2.3.

Proof. For the right-hand side of (3.15), we find

$$\begin{aligned}(\hat{p} - p_h(u_h), u_h - w_h)_{L^2(\Omega)} &= (p_h(w_h) - p_h(u_h), u_h - w_h)_{L^2(\Omega)} \\ &\quad + (\bar{p} - p_h(w_h), u_h - w_h)_{L^2(\Omega)} \\ &\quad + (\hat{p} - \bar{p}, u_h - w_h)_{L^2(\Omega)}. \end{aligned} \tag{4.1}$$

Next we estimate these three terms. We start with

$$(p_h(w_h) - p_h(u_h), u_h - w_h)_{L^2(\Omega)} = (y_h(w_h) - y_h(u_h), y_h(u_h) - y_h(w_h))_{L^2(\Omega)} \leq 0. \tag{4.2}$$

The second term can be estimated by formula (3.10)

$$(\bar{p} - p_h(w_h), u_h - w_h)_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}) \cdot \|w_h - u_h\|_{L^2(\Omega)}. \tag{4.3}$$

The third term represents again a formula for the numerical integration. Using that u_h and w_h are constant on each triangle T_i, we obtain by Lemma 3.2

$$\begin{aligned}
(\hat{p} - \bar{p}, u_h - w_h)_{L^2(\Omega)} &= \sum_i \int_{T_i} (\hat{p}(x) - \bar{p}(x))(u_h(x) - w_h(x))\, dx \\
&= \sum_i (u_h(S_i) - w_h(S_i)) \int_{T_i} (\bar{p}(S_i) - \bar{p}(x))\ dx \\
&\leq \sum_i ch^2 |u_h(S_i) - w_h(S_i)| \sqrt{|T_i|} \cdot |\bar{p}|_{H^2(T_i)} \\
&\leq ch^2 \cdot \|w_h - u_h\|_{L^2(\Omega)} \cdot \|\bar{p}\|_{W^{2,p}(\Omega)}.
\end{aligned} \tag{4.4}$$

Inserting (4.2)–(4.4) in (4.1), we get

$$(\hat{p} - p_h(u_h), u_h - w_h)_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}) \cdot \|w_h - u_h\|_{L^2(\Omega)}.$$

Next, we combine this inequality with (3.15)

$$\nu \|w_h - u_h\|^2_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}) \cdot \|w_h - u_h\|_{L^2(\Omega)}.$$

This formula is equivalent to

$$\|w_h - u_h\|_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}),$$

which was the assertion of Theorem 2.3. □

Theorem 2.3 means that the values of the numerical solution u_h in the centroids have already quadratic convergence rate. By the projection of the associated adjoint state $p_h(u_h)$, we obtain an admissible control $\tilde{u}$ that has a quadratic convergence order with respect to the L^2-norm. This was the assertion of Theorem 2.4.

Proof. We start with the result of Theorem 2.3

$$\|w_h - u_h\|_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

This inequality implies

$$\|p_h(w_h) - p_h(u_h)\|_{L^2(\Omega)} \leq c\|w_h - u_h\|_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

From Corollary 3.4, we know formula (3.10)

$$\|\bar{p} - p_h(w_h)\|_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

Therefore, we obtain by the triangle inequality

$$\|\bar{p} - p_h(u_h)\|_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

The projection operator $\Pi_{[a,b]}$ is Lipschitz continuous with constant 1 from $L^2(\Omega)$ to $L^2(\Omega)$. Finally, we get

$$\|\bar{u} - \tilde{u}\|_{L^2(\Omega)} \leq ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

The superconvergence result is proved. □

COROLLARY 4.1. *By the arguments of the proof of Theorem 2.4, we get another result. We find for the L^∞-error*

$$\|\bar{u} - \tilde{u}\|_{L^\infty(\Omega)} \le ch(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

Proof. From formula (3.3)

$$\|y_h(w_h) - y_h(\bar{u})\|_{L^2(\Omega)} \le ch^2\|\bar{p}\|_{W^{2,p}(\Omega)}$$

and the inequality

$$\|y_h(w_h) - y_h(u_h)\|_{L^2(\Omega)} \le c\|w_h - u_h\|_{L^2(\Omega)} \le ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}),$$

we get by the triangle inequality

$$\|y_h(\bar{u}) - y_h(u_h)\|_{L^2(\Omega)} \le ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

This inequality implies

$$\|p_h(\bar{u}) - p_h(u_h)\|_{L^\infty(\Omega)} \le c\|y_h(\bar{u}) - y_h(u_h)\|_{L^2(\Omega)} \le ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

Denoting the solution of (2.5) with $\bar{y}$ instead of $y_h(u_h)$ by $\bar{p}_h$, we continue with

$$\|\bar{p} - p_h(u_h)\|_{L^\infty(\Omega)} \le \|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} + \|\bar{p}_h - p_h(\bar{u})\|_{L^\infty(\Omega)} + \|p_h(\bar{u}) - p_h(u_h)\|_{L^\infty(\Omega)}. \tag{4.5}$$

The first term can be estimated by $ch\|\bar{p}\|_{H^2(\Omega)}$ see [3]. For the second term, we use the argumentation of Lemma 3.1 with $z_h = p_h(\bar{u}) - p_h(u_h)$ and $f_h = y_h(\bar{u}) - \bar{y}$

$$\begin{aligned}\|\bar{p} - p_h(u_h)\|_{L^\infty(\Omega)} &\le ch\|\bar{p}\|_{H^2(\Omega)} + c\|y_h(\bar{u}) - \bar{y}\|_{L^2(\Omega)} + ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}) \\ &\le ch\|\bar{p}\|_{H^2(\Omega)} + ch^2(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}) \\ &\le ch(\|\bar{p}\|_{W^{2,p}(\Omega)} + \|y_d\|_{L^2(\Omega)}).\end{aligned}$$

The properties of the projection operator implies the assertion. □

COROLLARY 4.2. *The first estimate can be improved, if all data are sufficiently smooth*

$$\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \le ch^2|\ln h|^{3/2}\|\bar{p}\|_{W^{2,\infty}(\Omega)},$$

see Braess [3]. In this case, formula (4.5) implies

$$\|\bar{u} - \tilde{u}\|_{L^\infty(\Omega)} \le ch^2|\ln h|^{3/2}(\|\bar{p}\|_{W^{2,\infty}(\Omega)} + \|y_d\|_{L^2(\Omega)}).$$

5. Numerical tests. We have tested the convergence theory by two examples. In both cases, the Laplace operator $-\Delta$ was chosen for the elliptic operator A. The first example fits exactly to the presented theory.

Both optimization problems were solved numerically by a primal-dual active set strategy, see for instance [11]. The discretization was already described in section 2: The control function u is discretized by piecewise constant functions, whereas the state y and the adjoint state p were approximated by piecewise linear functions. We used uniform meshes. The additional numerical effort for the calculation of $\tilde{u}$ is very small. We have only to evaluate the pointwise projection of the funtion $-\frac{1}{\nu}p_h$ to the interval $[a, b]$.

In contrast to this, the numerical evaluation of the L^2-norm $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$ and the graphical representation are not so simple. Therefore, we shortly sketch these aspects. We want to point out that this additional effort is only needed to confirm the theoretical results. This effort is not necessary for the computation of the approximated optimal control.

For the computation of the L^2-norms $\|\bar{u} - u_h\|_{L^2(\Omega)}$ and $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$, respectively, we introduce sets $\tilde{K}_1$, $\tilde{K}_2$ analogue to the sets K_1 and K_2:

$$\tilde{K}_1 = \{T_i : \tilde{u} \text{ is only Lipschitz continuous on } T_i\}, \quad \tilde{K}_2 = \{T_i : \tilde{u} \in C^\infty(T_i)\}.$$

Moreover, we set $M_1 = K_1 \cup \tilde{K}_1$, $M_2 = K_2 \cap \tilde{K}_2$. The numerical evaluation of $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$ differs on the sets M_1 and M_2. Therefore, we split the L^2-norm up into

$$\|\bar{u} - \tilde{u}\|^2_{L^2(\Omega)} = \|\bar{u} - \tilde{u}\|^2_{L^2(M_1)} + \|\bar{u} - \tilde{u}\|^2_{L^2(M_2)}.$$

In our examples, the part $\bar{u}|_{K_2} \in C^\infty(K_2)$ is smooth. Thus, $\|\bar{u} - \tilde{u}\|_{L^2(M_2)}$ can be evaluated with sufficient accuracy applying an appropriate quadrature formula. In contrast to this, the difference $\bar{u} - \tilde{u}$ only belongs to $C^{0,1}(T_i)$ for all triangles $T_i \in M_1$. Hence, an arbitrary accurate quadrature formula would only admit an error of order h. Therefore, we introduce a subgrid of significant smaller mesh size in each triangle $T_i \in M_1$ and evalute the norm $\|\bar{u} - \tilde{u}\|_{L^2(M_1)}$ on this subgrid to ensure sufficient accuracy. We want to point out that this subgrid is only used for the evaluation of the norm $\|\bar{u} - \tilde{u}\|_{L^2(M_1)}$ with a sufficient high accuracy.

Example 1. In this example, we investigate the Laplace equation with homogenious Dirichlet boundary conditions. Therfore, we choose $a_0 \equiv 0$ in (1.2). Thus, the state equation is given by

$$\begin{aligned} -\Delta y &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{5.1}$$

Now, we define the optimal state by

$$\bar{y} = y_a - y_g$$

with an analytical part $y_a = \sin(\pi\, x_1)\, \sin(\pi\, x_2)$ and a less smooth function y_g. The function y_g is defined as the solution of

$$\begin{aligned} -\Delta y_g &= g \quad \text{in } \Omega \\ y_g &= 0 \quad \text{on } \Gamma. \end{aligned}$$

The function g is given by

$$g(x_1, x_2) = \begin{cases} u_f(x_1, x_2) - a & , \text{if } u_f(x_1, x_2) < a \\ 0 & , \text{if } u_f(x_1, x_2) \in [a, b] \\ u_f(x_1, x_2) - b & , \text{if } u_f(x_1, x_2) > b \end{cases}$$

with $u_f(x_1, x_2) = 2\,\pi^2\, \sin(\pi\, x_1)\, \sin(\pi\, x_2)$. Due to the state equation (5.1), we obtain for the exact optimal control $\bar{u}$

$$\bar{u}(x_1, x_2) = \begin{cases} a & , \text{if } u_f(x_1, x_2) < a \\ u_f(x_1, x_2) & , \text{if } u_f(x_1, x_2) \in [a, b] \\ b & , \text{if } u_f(x_1, x_2) > b. \end{cases}$$

For the optimal adjoint state $\bar{p}$, we find

$$\bar{p}(x_1, x_2) = -2\,\pi^2\, \nu\, \sin(\pi\, x_1)\, \sin(\pi\, x_2).$$

The desired state is given by

$$y_d(x_1, x_2) = \bar{y} + \Delta\bar{p} = y_a - y_g + 4\,\pi^4\,\nu\,\sin(\pi\,x_1)\,\sin(\pi\,x_2).$$

It is easy to see that these functions fulfill the necessary and sufficient first-order optimality conditions. Moreover, the sets with

$$-\frac{1}{\nu}\bar{p} = a \quad \text{or} \quad -\frac{1}{\nu}\bar{p} = b$$

are a finite number (here two) of curves γ_i. Hence, the measure of the set K_1 is bounded by the total length of these curves

$$|K_1| \leq 2h \sum |\gamma_i|$$

and Assumption (A3) is fulfilled. We chose $\nu = 1$ for the numerical calculations.

Figures 5.1 and 5.2 show the numerical solutions $\tilde{u}$ for $h = 0.04$ and $h = 0.02$. As described above, the function $\tilde{u}$ is obtained by the pointwise projection $\Pi_{[a,b]}(-\frac{1}{\nu}p_h)$ in a post-processing step. Therefore, $\tilde{u}$ contains sharp breaks on the subset $\tilde{K}_1$. To visualize these breaks, we introduce new mesh points in all triangles $T_i \in \tilde{K}_1$. Notice that these new grid points are only used for the graphical presentation of the projection.

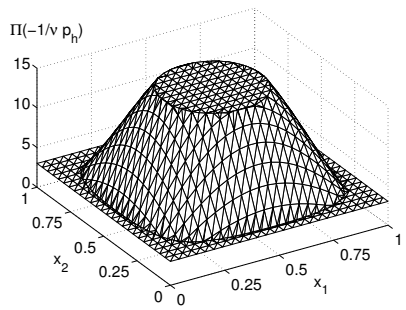

FIG. 5.1. $\tilde{u}$ *at* $h = 0.04$

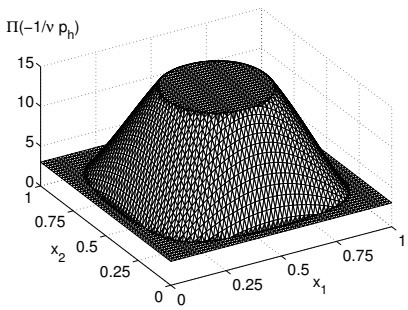

FIG. 5.2. $\tilde{u}$ *at* $h = 0.02$

The following figures show the convergence behaviour of $\|\bar{u} - u_h\|_{L^2(\Omega)}$ and $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$, respectively, for $h = 0.04$, 0.02, 0.01 and 0.005. In the figures, $\bar{u}$ is denoted by u_{opt}.

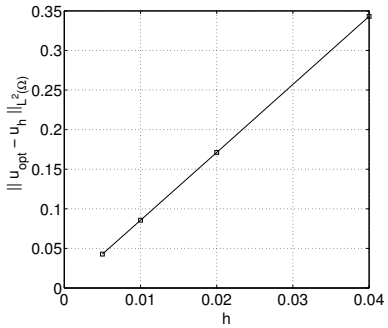

FIG. 5.3. $\|\bar{u} - u_h\|_{L^2(\Omega)}$

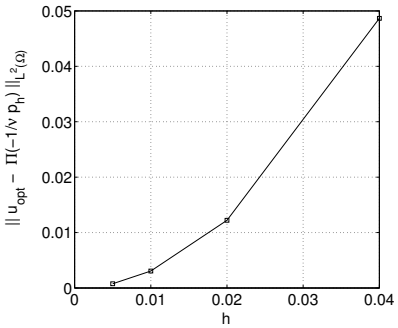

FIG. 5.4. $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$

As one can see, the theoretical predictions are fulfilled and one obtains quadratic convergence for $\|\bar{u}-\tilde{u}\|_{L^2(\Omega)}$. Furthermore, the absolute value of the error is significantly reduced by the projection, as the following table shows.

$h/\sqrt{2}$	$\|\bar{u}-u_h\|_{L^2(\Omega)}$	$\|\bar{u}-\tilde{u}\|_{L^2(\Omega)}$	$\|u_h-w_h\|_{L^2(\Omega)}$	$\|\bar{u}-u_h\|_{L^\infty(\Omega)}$	$\|\bar{u}-\tilde{u}\|_{L^\infty(\Omega)}$
0.04	0.34312	0.04856	0.05335	1.61552	0.13760
0.02	0.17155	0.01221	0.01342	0.81633	0.03513
0.01	0.08556	0.00306	0.00335	0.40975	0.00884
0.005	0.04281	0.00077	0.00084	0.20485	0.00221

Theoretical results with respect to the $L^\infty(\Omega)$-norm were addressed in the Corollaries 4.1 and 4.2. Again, we used finer subgrids for the numerical evulation of the norms.

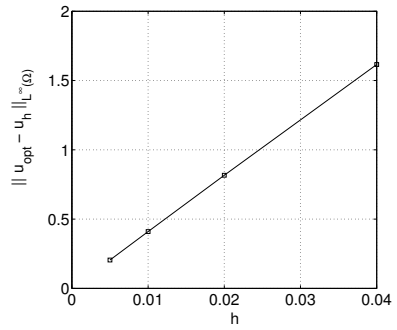

FIG. 5.5. $\|\bar{u}-u_h\|_{L^\infty(\Omega)}$

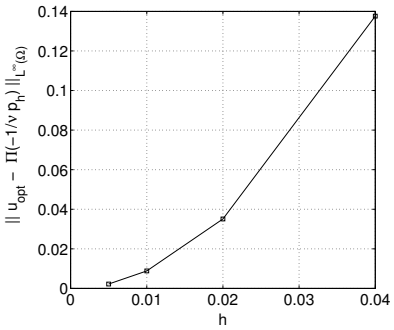

FIG. 5.6. $\|\bar{u}-\tilde{u}\|_{L^\infty(\Omega)}$

Example 2. A Neumann boundary problem is studied in this example. In this case, the theoretical results does not exactly fit to the problem. For this special structure (Laplace-operator, homogeneous Neumann data, $\Omega=\Omega_h$), we get the $W^{2,p}$-regularity of the solution corresponding to Lemma 2.1 by a result of Grisvard [9] Th. 4.4.1.2. Moreover, the proofs of our main results can be adapted. A general discussion of Neumann boundary conditions requires more detailed investigations. We discuss here the problem

$$\begin{aligned} -\Delta y + c\,y &= u \quad \text{in } \Omega \\ \partial_n y &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{5.2}$$

where ∂_n denotes the normal derivative with respect to the outward normal vector. Again, we construct the optimal state $\bar{y}$ by $\bar{y}=y_a-y_g$, with an analytical part $y_a(x_1,x_2)=\cos(\pi\,x_1)\,\cos(\pi\,x_2)$. The function y_g is now determined by the following equation

$$\begin{aligned} -\Delta y_g + c\,y_g &= g \quad \text{in } \Omega \\ \partial_n y_g &= 0 \quad \text{on } \Gamma, \end{aligned}$$

with the inhomogenity

$$g(x_1,x_2)=\begin{cases} u_f(x_1,x_2)-a & \text{, if } u_f(x_1,x_2)<a \\ 0 & \text{, if } u_f(x_1,x_2)\in[a,b] \\ u_f(x_1,x_2)-b & \text{, if } u_f(x_1,x_2)>b \end{cases}$$

and $u_f(x_1, x_2) = (2\,\pi^2 + c)\,\cos(\pi\,x_1)\,\cos(\pi\,x_2)$. The optimal control $\bar{u}$ is given by (5.2)

$$\bar{u}(x_1, x_2) = \begin{cases} a & , \text{if } u_f(x_1, x_2) < a \\ u_f(x_1, x_2) & , \text{if } u_f(x_1, x_2) \in [a, b] \\ b & , \text{if } u_f(x_1, x_2) > b. \end{cases}$$

The optimal adjoint state is defined by

$$\bar{p}(x_1, x_2) = -(2\,\pi^2 + c)\,\nu\,\sin(\pi\,x_1)\,\sin(\pi\,x_2).$$

Moreover the desired state y_d is chosen as

$$\begin{aligned} y_d(x_1, x_2) &= \bar{y} + \Delta\bar{p} - c\,\bar{p} \\ &= y_a - y_g + (4\,\pi^4\,\nu + 4\,\pi^2\,\nu\,c + \nu\,c^2)\,\sin(\pi\,x_1)\,\sin(\pi\,x_2). \end{aligned}$$

Again, it is easy to see that these functions fulfill the necessary and sufficient first-order optimality conditions. Assumption (A3) can be verified with the same arguments as in Example 1. In the numerical test, we chose $\nu = c = 1$. The projected function $\tilde{u}$ for $h = 0.04$ and $h = 0.02$ is shown in figures 5.7 and 5.8. For the visualization of this projection we introduced again new grid points.

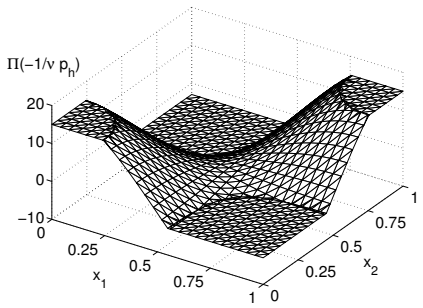

FIG. 5.7. $\tilde{u}$ *at* $h = 0.04$

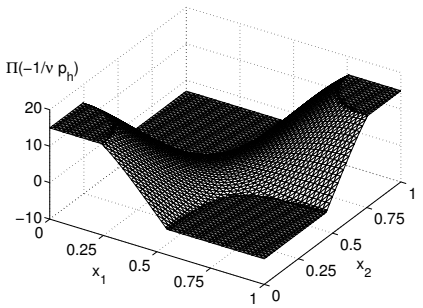

FIG. 5.8. $\tilde{u}$ *at* $h = 0.02$

The following figures illustrate that we obtain the same convergence results for this Neumann boundary example.

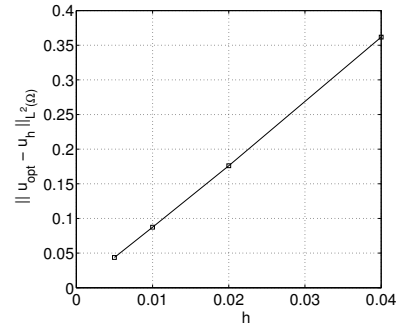

FIG. 5.9. $\|\bar{u} - u_h\|_{L^2(\Omega)}$

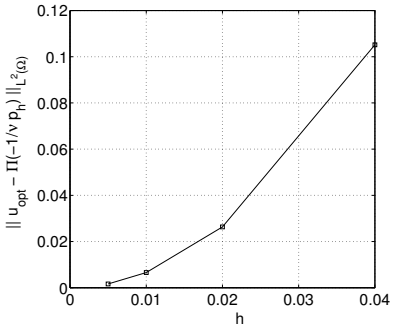

FIG. 5.10. $\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}$

Comparable with the first example, the absolute error is considerable reduced by the projection, as the following table shows.

$h/\sqrt{2}$	$\|\bar{u}-u_h\|_{L^2(\Omega)}$	$\|\bar{u}-\tilde{u}\|_{L^2(\Omega)}$	$\|u_h-w_h\|_{L^2(\Omega)}$	$\|\bar{u}-u_h\|_{L^\infty(\Omega)}$	$\|\bar{u}-\tilde{u}\|_{L^\infty(\Omega)}$
0.04	0.36168	0.10517	0.10659	1.84963	0.21285
0.02	0.17610	0.02632	0.02657	0.89765	0.05352
0.01	0.08744	0.00656	0.00663	0.44174	0.01340
0.005	0.04366	0.00164	0.00166	0.21905	0.00336

The convergence behaviour of the $L^\infty(\Omega)$-errors is illustrated in the next two figures.

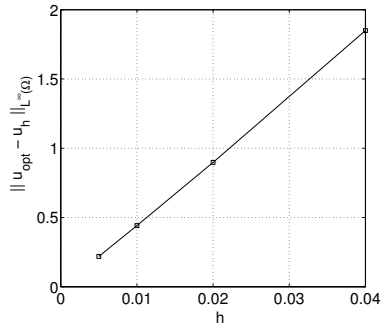

FIG. 5.11. $\|\bar{u}-u_h\|_{L^\infty(\Omega)}$

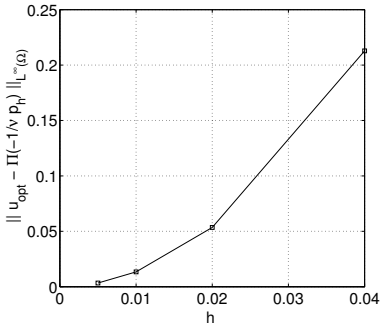

FIG. 5.12. $\|\bar{u}-\tilde{u}\|_{L^\infty(\Omega)}$

The last two figures show the convergence behaviour of $\|u_h-w_h\|_{L^2(\Omega)}$ (analyzed in Theorem 2.3) for the two examples.

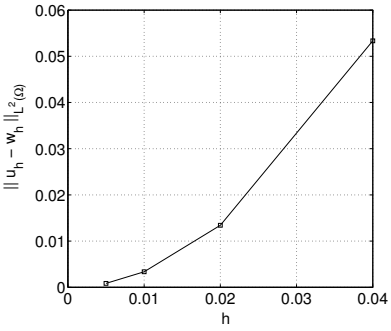

FIG. 5.13. $\|u_h-w_h\|_{L^2(\Omega)}$ *for Ex. 1*

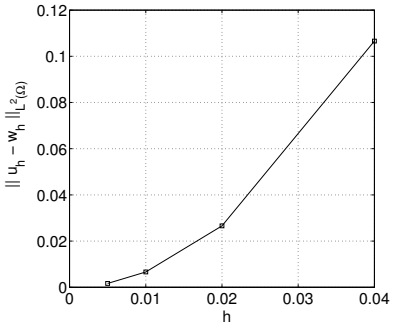

FIG. 5.14. $\|u_h-w_h\|_{L^2(\Omega)}$ *for Ex. 2*

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