Perturbation analysis for Hermitian, skew-Hermitian, H-even and H-odd matrix polynomials†

Sk Safique Ahmad‡ Volker Mehrmann*

November 13, 2010

Abstract

We discuss the perturbation analysis for eigenvalues and eigenvectors of structured homogeneous matrix polynomials with Hermitian, skew-Hermitian, H-even and H-odd structure. We construct minimal structured perturbations (structured backward errors) such that an approximate eigenpair is an exact eigenpair of an appropriate perturbed structured matrix polynomial. We present various comparisons with unstructured backward errors and previous error bounds derived for the non-homogeneous case and show that our bounds present a significant improvement.

Keywords. Nonlinear eigenvalue problem, H-even matrix polynomial, H-odd matrix polynomial, Hermitian matrix polynomial, skew-Hermitian matrix polynomial, perturbation analysis, backward error, homogeneous polynomial.
AMS subject classification. 65F15, 15A18, 65F35, 15A12

1 Introduction

In this paper we study the perturbation analysis for eigenvalues and eigenvectors of matrix polynomials of degree m in homogeneous form

$$\mathbf{L}(c,s) := \sum_{j=0}^{m} c^{m-j} s^j A_j, \tag{1}$$

with coefficient matrices, $A_j \in \mathbb{C}^{n\times n}$, $j = 0, 1, \ldots, m$, where the eigenvalues are represented as pairs $(c,s) \in \mathbb{C}^2 \setminus \{(0,0)\}$, which for $c \neq 0$ correspond to finite eigenvalues $\lambda = \dfrac{s}{c}$, while $(0,s)$ corresponds to the eigenvalue ∞.

The eigenvalue problem for matrix polynomials arises naturally in a large number of applications, see [15, 19, 21, 26] and the references therein. Furthermore, in many applications, the problem has further structure which reflects the properties of the underlying physical model, see [8, 10, 20, 22, 23, 26]. The coefficients of the polynomial eigenvalue problem and therefore also the computed eigenvalues and eigenvectors are usually not exact, due to modeling,

*Corresponding author. Institut für Mathematik, Ma 4-5, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, Germany; email: mehrmann@math.tu-berlin.de

‡Department of Mathematics, Indian Institute of Technology Indore, Indore-452017, India; email: safique@iiti.ac.in

†Research supported by *Deutsche Forschungsgemeinschaft*, via the DFG Research Center MATHEON in Berlin.

discretization, approximation, and round-off errors. Thus it is very important to study the the sensitivity of the eigenvalue/eigenvectors under perturbation in order to judge whether the computed results are trustworthy. While the perturbation analysis for classical and generalized eigenvalue problems is well established, see [17, 24, 27], for polynomial eigenvalue problems the situation is much less satisfactory and most research is very recent, see e.g., [1, 9, 18, 25]. In this paper we are particularly interested in the behavior of the eigenvalues and eigenvectors under perturbations which preserve the structure of the matrix polynomial and we will present a systematic extension of the work in [1, 2, 3, 5, 6, 9, 14, 18, 24]. In contrast to previous work in [7], here we focus on complex matrix polynomials, where the coefficient matrices are Hermitian or skew-Hermitian, i.e., $\mathbf{L}^H(c,s) = \pm\mathbf{L}(\overline{c},\overline{s})$ or where the matrix polynomial is H-even or H-odd, i.e., $\mathbf{L}^H(c,s) = \pm\mathbf{L}(\overline{c},-\overline{s})$. Linearization techniques and applications of such problems have been studied in [20]. In many applications only finite eigenvalues and the associated eigenvectors are desired but often the eigenvectors associated with the eigenvalue infinity play an important role as well [11, 12].

In our perturbation analysis, for a given approximation to an eigenvalue/eigenvector pair of a matrix polynomial $\mathbf{L}$, we will construct an appropriately structured minimal (in Frobenius and spectral norm) perturbation polynomial $\Delta\mathbf{L}$ such that the given eigenvalue/eigenvector pair is exact for $\mathbf{L} + \Delta\mathbf{L}$. It will turn out that the so constructed minimal perturbation is unique in the case of the Frobenius norm and that there are infinitely many such minimal perturbations in the case of the spectral norm. We will compare the so constructed perturbations with those constructed for matrix pencils and matrix polynomials in [1, 2].

The paper is organized as follows. In Section 2, we review some of the techniques developed in [4, 5, 6] that will be used in the analysis of structured polynomial eigenvalue problems. In Sections 3 and 4 we study the structured backward error analysis of an approximate eigenvalues/eigenvector pair for Hermitian, skew-Hermitian, H-even, and H-odd matrix polynomials and compare these with the corresponding unstructured backward errors. We also present a systematic general procedure for the construction of an appropriate structured minimal Hermitian, skew-Hermitian, H-even, or H-odd polynomial $\Delta\mathbf{L}$ such that the given approximate (finite or infinite) eigenvalue and eigenvector are exact for $\mathbf{L} + \Delta\mathbf{L}$ and we show that our results generalize the results of [1, 2].

2 Notation and preliminaries

We denote by $\mathbb{R}^{n\times n}, \mathbb{C}^{n\times n}$ the real and complex $n\times n$ matrices, respectively. For a nonnegative vector $w = [w_i] \in \mathbb{R}^n$, $x = [x_i] \in \mathbb{R}^n$ and an integer p, $1 \le p \le \infty$, we define the weighted p-(semi)norm via

$$\|x\|_{w,p} := \|[w_1x_1, w_2x_2, \ldots, w_nx_n]^T\|_p,$$

with $\|x\|_p$ denoting the p-norm of $x \in \mathbb{R}^n$. If w is strictly positive, then this is a norm, and if w has zero components then it is a seminorm. We will consider structured and non-structured backward errors both in spectral norm and Frobenius norm on $\mathbb{C}^{n\times n}$,

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2,\ \ \|A\|_F := (\mathrm{trace} A^*A)^{1/2},$$

respectively. For a nonnegative vector $w = [w_i] \in \mathbb{R}^n$ we define the componentwise inverse via $w^{-1} := [w_1^{-1}, \ldots, w_n^{-1}]^T$, where we use the convention that $w_i^{-1} = 0$ if $w_i = 0$. By $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ we denote the largest and smallest singular value of a matrix A, respectively.

The identity matrix is denoted by I, its columns by e_j and $\overline{A}$, A^T, and A^H stand for the conjugate, transpose, and conjugate transpose of a matrix A, respectively.

The set of all matrix polynomials of degree $m \geq 0$ with coefficients in $\mathbb{C}^{n\times n}$ is denoted by $\mathbf{L}_m(\mathbb{C}^{n\times n})$. This is a vector space which we can equip with a weighted norm/seminorm defined as

$$\|\mathbf{L}\|_{w,F} \;:=\; \|(A_0,\ldots,A_m)\|_{w,F} = (w_0^2\|A_0\|_F^2 + \ldots + w_m^2\|A_m\|_F^2)^{1/2},$$

for the Frobenius norm and

$$\|\mathbf{L}\|_{w,2} \;:=\; \|(A_0,\ldots,A_m)\|_{w,2} = (w_0^2\|A_0\|_2^2 + \ldots + w_m^2\|A_m\|_2^2)^{1/2},$$

for the spectral norm. Note that here $w := [w_0, w_1, \ldots, w_m]^T \in \mathbb{R}^{m+1} \setminus \{0\}$ is again a nonnegative weight vector.

For a matrix $M \in \mathbb{C}^{\ell\times n}$ we denote by M^+ the *Moore-Penrose pseudo-inverse* of M.

A matrix polynomial in $\mathbf{L}_m(\mathbb{C}^{n\times n})$ is called *regular* if $\det(\mathbf{L}(\lambda,\mu)) \neq 0$ for some $(\lambda,\mu) \in \mathbb{C}^2\setminus\{(0,0)\}$; otherwise it is called *singular*. The *spectrum* of a homogeneous matrix polynomial is defined as

$$\Lambda(\mathbf{L}) := \{(c,s) \in \mathbb{C}^2 \setminus \{(0,0)\} :\ \operatorname{rank}(\mathbf{L}(c,s)) < n\}.$$

Throughout the paper we assume that the set of points $(c,s) \in \mathbb{C}^2\setminus\{(0,0)\}$ is normalized such that $|c|^2 + |s|^2 = 1$ and that c is real. With this normalization, it follows that the spectrum $\Lambda(\mathbf{L})$ can be identified with a subset of the Riemann sphere, see e.g. [5].

In the following we will compute *backward errors* for matrix polynomials. These were introduced, e.g., in [16, 25], but here we follow [4, 5]. Let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ be an approximate eigenvalue of $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ with corresponding approximate right eigenvector $x \neq 0$, i.e., $\mathbf{L}(\lambda,\mu)x = 0$. We consider the Frobenius and spectral norm backward errors

$$\begin{aligned}\eta_{w,F}(\lambda,\mu,x,\mathbf{L}) \;&:=\; \inf\{\|\Delta\mathbf{L}\|_{w,F},\ \Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n}),\\ &\qquad (\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0\},\\ \eta_{w,2}(\lambda,\mu,x,\mathbf{L}) \;&:=\; \inf\{\|\Delta\mathbf{L}\|_{w,2},\ \Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n}),\\ &\qquad (\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0\},\end{aligned}$$

respectively.

In order to compute these backward errors, we will need the partial gradient $\nabla_i\|z\|_{w,2}$ of the map $\mathbb{C}^{m+1} \to \mathbb{R}$, $z \mapsto \|z\|_{w,2}$ which is just the derivative of the map $\mathbb{C} \to \mathbb{R}$, $z_i \mapsto \|[z_0, z_1, \ldots, z_m]^T\|_{w,2}$ with the variables $z_0, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$ fixed as constants. The gradient of the map $\mathbb{C}^{m+1} \to \mathbb{R}, z \mapsto \|z\|_{w,2}$, is then defined as

$$\nabla(\|z\|_{w,2}) = [\nabla_0\|z\|_{w,2}, \nabla_1\|z\|_{w,2}, \ldots, \nabla_m\|z\|_{w,2}]^T \in \mathbb{C}^{m+1}.$$

Now consider $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $x \in \mathbb{C}^n$ such that $x^Hx = 1$, and let $k := -\mathbf{L}(\lambda,\mu)x$. Then, setting, $H_{w,2} := \|[\lambda^m, \lambda^{m-1}\mu, \ldots, \mu^m]^T\|_{w,2}$, we have

$$\eta_{w,2}(\lambda,\mu,x,\mathbf{L}) = \frac{\|\mathbf{L}(\lambda,\mu)x\|}{H_{w^{-1},2}(\lambda,\mu)}.$$

Introducing

$$z_{A_j} := \frac{\nabla_j H_{w^{-1},2}}{H_{w^{-1},2}}, \tag{2}$$

where the partial gradient is evaluated at $[\lambda^m, \lambda^{m-1}\mu, \dots, \mu]^T$, we define $\Delta A_j := \overline{z_{A_j}} k x^H$. Then, for $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$ we have $\|\!|\Delta\mathbf{L}|\!\|_2 = \dfrac{\|k\|_2}{H_{w^{-1},2}}$, see [7].

The structured backward errors for structured matrix polynomials from a set $\mathbf{S} \subset \mathbf{L}_m(\mathbb{C}^{n\times n})$ are defined analogously as

$$\begin{aligned} \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) &:= \inf\{\|\!|\Delta\mathbf{L}|\!\|_{w,F},\ \Delta\mathbf{L}\in\mathbf{S},\ (\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0\}, \\ \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) &:= \inf\{\|\!|\Delta\mathbf{L}|\!\|_{w,2},\ \Delta\mathbf{L}\in\mathbf{S},\ (\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0\}, \end{aligned}$$

respectively. If $w := [1,1,\dots,1]^T$ then we just omit the index w.

In this paper we will construct backward errors for the following classes of structured matrix polynomials in homogeneous form as they were introduced in [20] in non-homogeneous form.

Definition 2.1 *Let $(c,s)\in\mathbb{C}^2\setminus\{(0,0)\}$. A matrix polynomial $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ is called*

1. Hermitian/skew-Hermitian *if* $\mathbf{L}^H(c,s) = \pm\mathbf{L}(\overline{c},\overline{s})$,
2. H-even/H-odd *if* $\mathbf{L}^H(c,s) = \pm\mathbf{L}(\overline{c},-\overline{s})$.

The classes of matrix polynomials introduced in Definition 2.1 have the following well-known spectral properties, see e.g. [20].

Proposition 2.2
1. *Let $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ be an Hermitian/skew-Hermitian matrix polynomial of the form (1). Let $x\in\mathbb{C}^n$ and $y\in\mathbb{C}^n$ be right and left eigenvectors corresponding to an eigenvalue $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$ of $\mathbf{L}(c,s)$. Then y and x will be right and left eigenvector corresponding to an eigenvalue $(\overline{\lambda},\overline{\mu})$.*
2. *Let $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ be an H-even or H-odd matrix polynomial of the form (1). Let $x\in\mathbb{C}^n$ and $y\in\mathbb{C}^n$ be a right and left eigenvector corresponding to an eigenvalue $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$ of $\mathbf{L}(c,s)$. Then y and x will be the right and left eigenvector corresponding to an eigenvalue $(\overline{\lambda},-\overline{\mu})$.*

For the construction of the backward errors we need the following two propositions which have been proved in [7] and extend results for matrix pencils from [4, 5].

Proposition 2.3 *Let m be an integer, let $\tilde{m}_K = \dfrac{m}{2}+1$, $\hat{m}_K = m$ if m is even and $\tilde{m}_K = \frac{m+1}{2}$, $\hat{m}_K = m-1$ if m is odd, and let $\tilde{m}_N = \frac{m}{2}$, $\hat{m}_N = m$ if m is even and $\tilde{m}_N = \frac{m+1}{2}$, $\hat{m}_N = m-1$ if m is odd. Consider the maps*

- $H_{w,2} : \mathbb{C}^{m+1}\to\mathbb{R}$ *given by* $z\mapsto H_{w,2}(z) := \|\!|[z_0,z_1,\dots,z_m]^T|\!\|_{w,2}$,
- $K_{w,2} : \mathbb{C}^{\tilde{m}_K}\to\mathbb{R}$ *given by* $z\mapsto \|[w_0z_0, w_2z_2, w_4z_4,\dots,w_{\hat{m}_k}z_{\hat{m}_K}]^T\|_2$,
- $N_{w,2} : \mathbb{C}^{\tilde{m}_N}\to\mathbb{R}$ *given by* $z\mapsto \|[w_1z_1, w_3z_3, w_5z_5,\dots,w_{\hat{m}_N}z_{\hat{m}_N}]^T\|_2$.

Then $H_{w,2}, K_{w,2}$, and $N_{w,2}$ are differentiable and

$$\begin{aligned} \nabla_i H_{w,2}(z) &= \frac{w_i^2 z_i}{H_{w,2}(z)},\ i = 1,2,\dots,m, \\ \nabla_i K_{w,2}(z) &= \frac{w_i^2 z_i}{K_{w,2}(z)},\ i = 0,2,4,\dots,\hat{m}_K, \\ \nabla_i N_{w,2}(z) &= \frac{w_i^2 z_i}{N_{w,2}(z)},\ i = 1,3,5,\dots,\hat{m}_N. \end{aligned}$$

Proposition 2.4 *Consider the functions*

$$\begin{aligned}
H_{w,2}(c^m, c^{m-1}s, \ldots, s^m) &= \|[c^m, c^{m-1}s, \ldots, s^m]^T\|_{w,2}, \\
K_{w,2}(c^m, c^{m-2}s^2, \ldots, s^m) &= \|[w_0c^m, w_2c^{m-2}s^2, \ldots, w_ms^m]^T\|_2 \text{ if } m \text{ is even}, \\
K_{w,2}(c^m, c^{m-2}s^2, \ldots, s^{m-1}) &= \|[w_0c^m, w_2c^{m-2}s^2, \ldots, w_{m-1}cs^{m-1}]^T\|_2 \text{ if } m \text{ is odd}, \\
N_{w,2}(c^{m-1}s, c^{m-3}s^3, \ldots, cs^{m-1}) &= \|[w_1c^{m-1}, w_3c^{m-3}s^3, \ldots, w_{m-1}cs^{m-1}]^T\|_2 \text{ if } m \text{ is even}, \\
N_{w,2}(c^{m-1}s, c^{m-3}s^3, \ldots, s^m) &= \|[w_1c^{m-1}, w_3c^{m-3}s^3, \ldots, w_ms^m]^T\|_2 \text{ if } m \text{ is odd}.
\end{aligned}$$

For even m then the following formulas hold:

$$\begin{aligned}
&\sum_{j=0, j \text{ even}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{K_{w,2}^2}{H_{w,2}^2} \text{ and } \sum_{j=0}^{m} w_j^{-2}|\nabla_j K_{w,2}|^2 = 1, \\
&\sum_{j=1, j \text{ odd}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{N_{w,2}^2}{H_{w,2}^2} \text{ and } \sum_{j=1}^{m-1} w_j^{-2}|\nabla_j N_{w,2}|^2 = 1, \\
&\sum_{j=0, j \text{ even}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j K_{w,2}}}{K_{w,2}} = 1, \sum_{j=1, j \text{ odd}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j N_{w,2}}}{N_{w,2}} = 1, \textit{ and} \\
&\sum_{j=0, j \text{ even}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} + \sum_{j=1,\, j \textit{ odd}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1.
\end{aligned}$$

For odd m then the following formulas hold:

$$\begin{aligned}
&\sum_{j=0, j \text{ even}} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{K_{w,2}^2}{H_{w,2}^2}, \sum_{j=1, j \text{ odd}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{N_{w,2}^2}{H_{w,2}^2}, \\
&\sum_{j=0, j \text{ even}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j K_{w,2}}}{K_{w,2}} = 1, \sum_{j=1 j \text{ odd}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j N_{w,2}}}{N_{w,2}} = 1, \textit{ and} \\
&\sum_{j=0, j \text{ even}}^{m-1} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} + \sum_{j=1, j \text{ odd}}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1.
\end{aligned}$$

For all m the following formulas hold:

$$\sum_{j=0}^{m} c^{m-j}s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1, \sum_{j=0}^{m} w_j^{-2}|\nabla_j H_{w,2}|^2 = 1.$$

We will also frequently use the following completion theorem from [13].

Theorem 2.5 *Consider a complex block matrix* $T := \begin{bmatrix} A & C \\ B & X \end{bmatrix}$. *Then for any positive number*

$$\chi \geq \max\left\{ \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2, \left\| \begin{bmatrix} A & C \end{bmatrix} \right\|_2 \right\},$$

the block X can be chosen such that

$$\left\| \begin{bmatrix} A & C \\ B & X \end{bmatrix} \right\|_2 \leq \chi.$$

and X is of the form $X = -KA^HL + \chi(I - KK^H)^{1/2}Z(I - L^HL)^{1/2}$, where $K := ((\chi^2 I - A^HA)^{-1/2}B^H)^H$, $L := (\chi^2 I - A^HA)^{-1/2}C$ and Z is an arbitrary matrix such that $\|Z\|_2 \le 1$.

As a Corollary one has the following result for Hermitian/skew-Hermitian matrices.

Corollary 2.6 *For $A = \pm A^H, B = \pm B^H$. Set $\chi := \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2$. Then there exists an Hermitian/skew-Hermitian matrix D such that $\left\| \begin{bmatrix} A & \pm B^H \\ B & D \end{bmatrix} \right\|_2 = \chi$ and D is of the form $D := -KAK^H + \chi(I - KK^H)^{1/2}Z(I - KK^H)^{1/2}$, where $K := B(\chi^2 I - A^2)^{-1/2}$ and $Z = \pm Z^H$ is an arbitrary matrix such that $\|Z\|_2 \le 1$.*

After recalling these preliminary results, in the following section we derive structured backward errors for complex structured matrix polynomials.

3 Backward errors for Hermitian and skew-Hermitian matrix polynomials

In this section we derive backward error formulas for homogeneous Hermitian and skew-Hermitian matrix polynomials. Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a matrix polynomial of the form (1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$. Introduce $k := -\mathbf{L}(\lambda, \mu)x$, $P_x := I - xx^H$, and furthermore, for

$$M := \begin{bmatrix} \frac{\Re(\lambda^m)}{w_0} & \frac{\Re(\lambda^{m-1}\mu^m)}{w_1} & \cdots & \frac{\Re(\mu^m)}{w_m} \\ \frac{\Im(\lambda^m)}{w_0} & \frac{\Im(\lambda^{m-1}\mu)}{w_1} & \cdots & \frac{\Im(\mu^m)}{w_m} \end{bmatrix},$$

set

$$t := [t_0, t_1, \ldots, t_m]^T := M^+ \begin{bmatrix} \Re(x^Hk) \\ \Im(x^Hk) \end{bmatrix}. \tag{3}$$

Theorem 3.1 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a matrix polynomial of the form (1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$. For $j = 0, 1, 2, \ldots, m$ define the following coefficients of the perturbation matrix polynomial.*

i) In the case that $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ choose

$$\Delta A_j := \begin{cases} -xx^HA_jxx^H + \overline{z_{A_j}}P_xkx^H + z_{A_j}xk^HP_x, & \text{if } A_j = A_j^H, \\ -xx^HA_jxx^H + \overline{z_{A_j}}P_xkx^H - z_{A_j}xk^HP_x, & \text{if } A_j = -A_j^H. \end{cases},$$

ii) In the case that $\lambda^{m-j}\mu^j \in \mathbb{R}$ choose

$$\Delta A_j := \begin{cases} z_{A_j}\left[(x^Hk)(xx^H) + P_xkx^H + xk^HP_x\right], & \text{if } A_j = A_j^H, \\ z_{A_j}\left[i(x^Hk)(xx^H) + P_xkx^H - xk^HP_x\right], & \text{if } A_j = -A_j^H. \end{cases}$$

Then

$$\Delta\mathbf{L}(c, s) = \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j \in \mathbf{L}_m(\mathbb{C}^{n\times n})$$

is an Hermitian or skew-Hermitian matrix polynomial, respectively, such that $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$.

Proof. If $\mathbf{L}$ is Hermitian, then in all suggested cases $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$ is also Hermitian. By Proposition 2.4 we have that $\sum_{j=0}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}} = 1$, and thus we have

$$\begin{aligned}
(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j + \Delta A_j)x \\
&= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j - xx^H A_j xx^H + [z_{A_j} x k^H (I - xx^H) + \overline{z_{A_j}}(I - xx^H)kx^H])x \\
&= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j x - xx^H A_j x + \overline{z_{A_j}}(I - xx^H)k) \\
&= (I - xx^H)(\sum_{j=0}^{m} \lambda^{m-j}\mu^j A_j)x + \sum_{j=0}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}}(I - xx^H)k \\
&= -(I - xx^H)k + (I - xx^H)k = 0.
\end{aligned}$$

If $\mathbf{L}$ is skew-Hermitian, then by construction also $\Delta\mathbf{L}$ is skew-Hermitian and we have

$$\begin{aligned}
(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j + \Delta A_j)x \\
&= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j - xx^H A_j xx^H - [z_{A_j} x k^H (I - xx^H) - \overline{z_{A_j}}(I - xx^H)kx^H])x \\
&= \sum_{j=0}^{m} \lambda^{m-j}\mu^j (A_j x - xx^H A_j x + \overline{z_{A_j}}(I - xx^H)k) \\
&= (I - xx^H)(\sum_{j=0}^{m} \lambda^{m-j}\mu^j A_j)x + \sum_{j=0}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}}(I - xx^H)k \\
&= -(I - xx^H)k + (I - xx^H)k = 0.
\end{aligned}$$

The proof for the real case follows analogously. □

Using Theorem 3.1 we then obtain the following backward errors.

Theorem 3.2 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be an Hermitian matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and let $x \in \mathbb{C}^n$ be such that $x^H x = 1$. With $k = -\mathbf{L}(\lambda,\mu)x$ and $t = [t_0, t_1, \ldots, t_m]^T$ as in (3) we have the following backward errors.*

i) The structured backward error in Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) := \begin{cases} \dfrac{\sqrt{2\|k\|_2^2 - |x^H k|^2}}{H_{w^{-1},2}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{R}, \\ \sqrt{\displaystyle\sum_{j=0}^{m} \left|\frac{t_j}{w_j}\right|^2 + \frac{2\|k\|_2^2 - |x^H k|^2}{H^2_{w^{-1},2}}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

ii) The structured backward error in spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) := \begin{cases} \dfrac{\|k\|_2}{H_{w^{-1},2}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{R}, \\ \sqrt{\displaystyle\sum_{j=0}^{m}\left|\frac{t_j}{w_j}\right|^2 + \frac{\|k\|_2^2 - |x^H k|^2}{H^2_{w^{-1},2}}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{C}\setminus\mathbb{R}. \end{cases}$$

If we introduce the coefficient matrices of the perturbation polynomial as

$$\Delta A_j := \begin{cases} z_{A_j}\left[xx^H kx^H + kx^H P_x + P_x x k^H\right], & \text{if } \lambda^{m-j}\mu^j \in \mathbb{R}, \\ w_j^{-1} t_j x x^H + \overline{z_{A_j}} P_x k x^H + z_{A_j} P_x x k^H, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{C}\setminus\mathbb{R}, \end{cases} \tag{4}$$

then $\Delta\mathbf{L} = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$ *is the unique minimal (in Frobenius norm) Hermitian perturbation polynomial such that* $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ *and* $|||\Delta\mathbf{L}|||_{w,F} = \eta_{w,F}(\lambda,\mu,x,\mathbf{L})$.

If for ΔA_j *as in (4), we introduce the coefficient matrices of the perturbation polynomial as*

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{R}, \\ \Delta A_j - \dfrac{t_j P_x k k^H P_x}{w_j \gamma^2}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{C}\setminus\mathbb{R}, \end{cases}$$

then $\Delta\mathbf{L}(c,s) = \displaystyle\sum_{j=0}^{m} c^{m-j}s^j \Delta E_j$ *is a minimal (in spectral norm) Hermitian perturbation polynomial such that* $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ *and* $|||\Delta\mathbf{L}|||_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$.

Proof. For the given perturbation coefficients, by Theorem 3.1 we have that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$. Choose a unitary matrix $U = [x, U_1], U_1 \in \mathbb{C}^{n\times n-1}$ and let

$$\widetilde{\Delta A_j} := U^H \Delta A_j U = \begin{bmatrix} d_{j,j} & d_j^H \\ d_j & D_{j,j} \end{bmatrix},$$

where $D_{j,j} = D_{j,j}^H \in \mathbb{C}^{n-1\times n-1}$. Then

$$U\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H = UU^H(\Delta\mathbf{L}(\lambda,\mu))U^H U = \Delta\mathbf{L}(\lambda,\mu).$$

This implies that $U\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H x = \Delta\mathbf{L}(\lambda,\mu)x = k$, and hence

$$\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H x = U^H k = \begin{bmatrix} x^H k \\ U_1^H k \end{bmatrix}.$$

Since $U^H x = e_1$, we get that

$$\begin{bmatrix} \displaystyle\sum_{j=0}^{m}\lambda^{m-j}\mu^j d_{j,j} \\ \displaystyle\sum_{j=0}^{m}\lambda^{m-j}\mu^j d_j \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{j=0}^{m} w_j d_{j,j}\frac{\lambda^{m-j}\mu^j}{w_j} \\ \displaystyle\sum_{j=0}^{m} w_j \lambda^{m-j}\mu^j \frac{d_j}{w_j} \end{bmatrix} = \begin{bmatrix} x^H k \\ U_1^H k \end{bmatrix}.$$

To minimize the perturbation, we solve this system for the parameters $d_{j,j}, d_j$ in a least square sense and, applying Proposition 2.3, we obtain

$$\begin{bmatrix} d_{0,0} \\ d_{1,1} \\ \vdots \\ d_{m,m} \end{bmatrix} = \begin{bmatrix} \overline{\frac{\nabla_0 H_{w^{-1},2}}{H_{w^{-1},2}}} \\ \overline{\frac{\nabla_1 H_{w^{-1},2}}{H_{w^{-1},2}}} \\ \vdots \\ \overline{\frac{\nabla_m H_{w^{-1},2}}{H_{w^{-1},2}}} \end{bmatrix} x^H k, \quad \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} \overline{\frac{\nabla_0 H_{w^{-1},2}}{H_{w^{-1},2}}} \\ \overline{\frac{\nabla_1 H_{w^{-1},2}}{H_{w^{-1},2}}} \\ \vdots \\ \overline{\frac{\nabla_m H_{w^{-1},2}}{H_{w^{-1},2}}} \end{bmatrix} U_1^H k,$$

and from this

$$\begin{bmatrix} d_{0,0} \\ d_{1,1} \\ \vdots \\ d_{m,m} \end{bmatrix} = \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} x^H k, \quad \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_m \end{bmatrix} = \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} U_1^H k.$$

We now have to consider cases. If $\lambda^{m-j}\mu^j \in \mathbb{R}$, then $d_{j,j} = z_{A_j} x^H k \in \mathbb{R}$. Then the Frobenius norm can be minimized by taking $D_{j,j} = 0$ and hence,

$$\|\Delta A_j\|_F^2 = |d_{j,j}|^2 + 2\|d_j\|_2^2 = |z_{A_j}|^2(2\|k\|_2^2 - |x^H k|^2) = |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}^2}$$

as $\|U^H k\|^2 = |x^H k|^2 + \|U_1^H k\|_2^2$, and by Proposition 2.4, we have that $\sum_{j=0}^{m} w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$.

Then

$$\|\Delta \mathbf{L}\|_{w,F} = \sqrt{\sum_{j=0}^{m} w_j^2 \|\Delta A_j\|_F^2} = \sqrt{\frac{2\|k\|_2^2 - |x^H k|^2}{H_{w^{-1},2}^2}},$$

with the Hermitian perturbation

$$\begin{aligned} \Delta A_j &= U\widetilde{\Delta A}U^H = \begin{bmatrix} x & U_1 \end{bmatrix} \begin{bmatrix} d_{j,j} & d_j^H \\ d_j & D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\ &= (x d_{j,j} + U_1 d_j) x^H + (x d_j^H + U_1 D_{j,j}) U_1^H \\ &= z_{A_j}[(x x^H k x^H) + U_1 U_1^H k x^H + x k^H U_1 U_1^H)] + U_1 D_{j,j} U_1^H \\ &= z_{A_j}[(x x^H k x^H) + P_x k x^H + x k^H P_x] + U_1 D_{j,j} U_1^H && (5) \\ &= z_{A_j}\left[k x^H + x k^H - (k^H x) x x^H\right] + U_1 D_{j,j} U_1^H. && (6) \end{aligned}$$

With the minimal Frobenius norm choice $D_{j,j} = 0$, we then have

$$\Delta A_j = z_{A_j}\left[k x^H + x k^H - x k^H x x^H\right]. \tag{7}$$

and, since $\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}} = 1$, we have

$$\begin{aligned}(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m}\lambda^{m-j}\mu^j(A_j+\Delta A_j)x \\ &= -k+\sum_{j=0}^{m}\lambda^{m-j}\mu^j z_{A_j}[kx^H+xk^H-xk^Hxx^H]x \\ &= -k+k+xk^Hx-xk^Hx=0.\end{aligned}$$

This implies that x is an eigenvector corresponding to an eigenvalue (λ,μ).

To find the smallest spectral norm perturbation, by Corollary 2.6, we get

$$D_{j,j} = -z_{A_j}\frac{x^HkU_1^Hkk^HU_1}{\gamma^2}$$

with $\gamma^2 = \|k\|_2^2 - |x^Tk|^2$. Hence for the spectral norm we get

$$\Delta E_j = \Delta A_j - \frac{z_{A_j}x^HkP_xkk^HP_x}{\gamma^2},$$

where ΔA_j is given by (7).

In the general complex case that $\lambda^{m-j}\mu^j \in \mathbb{C}\setminus\mathbb{R}$, we can rewrite $\sum_{j=0}^{m}\frac{\lambda^{m-j}\mu^j}{w_j}w_jd_{j,j} = x^Hk$ as

$$\begin{bmatrix}\frac{\Re(\lambda^m)}{w_0} & \frac{\Re(\lambda^{m-1}\mu)}{w_1} & \cdots & \frac{\Re(\mu^m)}{w_m} \\ \frac{\Im(\lambda^m)}{w_0} & \frac{\Im(\lambda^{m-1}\mu)}{w_1} & \cdots & \frac{\Im(\mu^m)}{w_m}\end{bmatrix}\begin{bmatrix}w_0d_{0,0}\\ w_1d_{1,1}\\ \vdots \\ w_md_{m,m}\end{bmatrix} = \begin{bmatrix}\Re(x^Hk)\\ \Im(x^Hk)\end{bmatrix}.$$

The least squares solution of this system is given by

$$\begin{bmatrix}w_0d_{0,0}\\ w_1d_{1,1}\\ \vdots \\ w_md_{m,m}\end{bmatrix} = M^+\begin{bmatrix}\Re(x^Hk)\\ \Im(x^Hk)\end{bmatrix} = t,$$

and thus $d_{j,j} = w_j^{-1}t_j$, and

$$\begin{aligned}\Delta A_j &= U\widetilde{\Delta A}U^H = \begin{bmatrix}x & U_1\end{bmatrix}\begin{bmatrix}w_j^{-1}t_j & \left(\overline{z_{A_j}}U_1^Hk\right)^H \\ \overline{z_{A_j}}U_1^Hk & D_{j,j}\end{bmatrix}\begin{bmatrix}x & U_1\end{bmatrix}^H \\ &= \left[w_j^{-1}t_jx+U_1\overline{z_{A_j}}U_1^Hk\right]x^H + \left[xz_{A_j}(U_1^Hk)^H+U_1D_{j,j}\right]U_1^H \\ &= w_j^{-1}t_jxx^H+\overline{z_{A_j}}U_1U_1^Hkx^H+z_{A_j}xk^HU_1U_1^H+U_1D_{j,j}U_1^H \\ &= w_j^{-1}t_jxx^H+\overline{z_{A_j}}P_xkx^H+z_{A_j}xk^HP_x+U_1D_{j,j}U_1^H.\end{aligned}$$

For the Frobenius norm we choose $D_{j,j} = 0$, and then the perturbation matrix and the backward error are given by

$$\Delta A_j = w_j^{-1} t_j x x^H + \overline{z_{A_j}} P_x k x^H + z_{A_j} x k^H P_x, \tag{8}$$

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \sqrt{\sum_{j=0}^{m} \left| \frac{t_j}{w_j} \right|^2 + 2 \frac{\|k\|_2^2 - |k^H x|^2}{H^2_{w^{-1},2}}},$$

respectively.

For the spectral norm, by Corollary 2.6, we have

$$U_1 D_{j,j} U_1^H = -\frac{w_j^{-1} t_j P_x k k^H P_x}{\|k\|_2^2 - |x^H k|^2}$$

and thus, the perturbation matrix and the backward error are given by

$$\Delta E_j = \Delta A_j - \frac{w_j^{-1} t_j P_x k k^H P_x}{\|k\|_2^2 - |x^H k|^2},$$

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \sqrt{\sum_{j=0}^{m} \left| \frac{t_j}{w_j} \right|^2 + \frac{\|k\|_2^2 - |x^H k|^2}{H^2_{w^{-1},2}}},$$

respectively, where ΔA_j is as in (8). With this perturbation matrix we obtain

$$\begin{aligned}
(\sum_{j=0}^{m} \lambda^{m-j} \mu^j (A_j + \Delta E_j))x &= -k + \sum_{j=0}^{m} \lambda^{m-j} \mu^j \left[w_j^{-1} t_j x + \overline{z_{A_j}} P_x k \right] \\
&= -k + \sum_{j=0}^{m} \lambda^{m-j} \mu^j w_j^{-1} t_j x + P_x k \\
&= \sum_{j=0}^{m} \lambda^{m-j} \mu^j w_j^{-1} t_j x - x x^H k \\
&= \sum_{j=0}^{m} \lambda^{m-j} \mu^j w_j^{-1} t_j x - x \sum_{j=0}^{m} \lambda^{m-j} \mu^j x^H \Delta A_j x \\
&= \sum_{j=0}^{m} \lambda^{m-j} \mu^j w_j^{-1} t_j x - \sum_{j=0}^{m} \lambda^{m-j} \mu^j w_j^{-1} t_j = 0,
\end{aligned}$$

and hence the proof is finished. ∎

Note that if $w_j = 0$ then $z_{A_j} = 0$, and by Theorem 3.2 we have $\Delta A_j = 0$. Thus if $w_j = 0$ then A_j remains unperturbed.

We do not present here the results for the skew-Hermitian case, which can be obtained directly from the Hermitian case, either by scaling the matrix polynomial with i or by going through the same arguments as in the Hermitian case, using the skew-Hermitian formulas in Theorem 3.1.

As a corollary we obtain relations between structured and unstructured backward errors.

Corollary 3.3 *et* $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ *be an Hermitian/skew-Hermitian matrix polynomial of the form (1), let* $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$*, and let* $x \in \mathbb{C}^n$ *be such that* $x^H x = 1$*. Then we have the following relations between the structured and unstructured backward errors.*

i) $\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L}) \le \sqrt{2}\ \eta_{w,2}(\lambda, \mu, x, \mathbf{L})$,

ii) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \lambda^{m-j}\mu^j \in \mathbb{R}$,

iii) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) \le \eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$.

As immediate corollary one obtains the results of [4, 6] for matrix pencils $\mathbf{L}(c, s) = cA_0 + sA_1$ and $w = [1, 1]^T$.

The results for the case of non-homogeneous matrix polynomials without infinite eigenvalues [1] follow by setting $c = 1, \mathbf{L}(s) := \mathbf{L}(1, s), \Lambda := [1, \mu, \dots, \mu^m]^T$, and by choosing the weight vector $w := [1, 1, \dots, 1]^T$.

4 Backward errors for H-even and H-odd matrix polynomials

In this section we derive backward error formulas for homogeneous H-odd/H-even matrix polynomials. To simplify the presentation, we assume without loss of generality that the coefficient matrix A_0 is always in the even-position which is Hermitian for a H-even matrix polynomial and skew-Hermitian for H-odd matrix polynomial (the other case can be obtained by scaling the matrix polynomial by i.)

In the following, similar to the Hermitian/skew Hermitian case, for $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $x \in \mathbb{C}^n$ with $x^H x = 1$, we introduce $k = -\mathbf{L}(\lambda, \mu)x$, $P_x := I - xx^H$, and slightly different than in the Hermitian/skew Hermitian case,

$$\hat{M} = \begin{bmatrix} \Re(\lambda^m) & -\Im(\lambda^{m-1}\mu) & \Re(\lambda^{(m-2)}\mu^2) & \dots & -\Im(\lambda\mu^{m-1}) & \Re(\mu^m) \\ \Im(\lambda^m) & \Re(\lambda^{m-1}\mu) & \Im(\lambda^{(m-2)}\mu^2) & \dots & \Re(\lambda\mu^{m-1}) & \Im(\mu^m) \end{bmatrix},$$

and

$$\hat{t} = [\hat{t}_0, \hat{t}_1, \dots, \hat{t}_n]^T = \hat{M}^+ \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix}.$$

Theorem 4.1 *Let* $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ *be H-even and of the form (1). Let* $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *and* $x \in \mathbb{C}^n$ *with* $x^H x = 1$*. We then have the following backward errors.*

i) The structured backward error for the Frobenius norm is

$$\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L}) := \begin{cases} \dfrac{\sqrt{2\|k\|_2^2 - |x^H k|^2}}{H_{w^{-1},2}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{R}, \\ \sqrt{\displaystyle\sum_{j=0}^m \left|\frac{\hat{t}_j}{w_j}\right|^2 + 2\frac{\|k\|_2^2 - |x^H k|^2}{H^2_{w^{-1},2}}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}. \end{cases} \tag{9}$$

ii) The structured backward error for the spectral norm is

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) := \begin{cases} \dfrac{\|k\|_2}{H_{w^{-1},2}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{R}, \\ \sqrt{\displaystyle\sum_{j=0}^m \left|\frac{\hat{t}_j}{w_j}\right|^2 + \frac{\|k\|_2^2 - |x^H k|^2}{H^2_{w^{-1},2}}}, & \text{if } \lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}. \end{cases} \tag{10}$$

For the Frobenius norm we introduce the following perturbation matrices.

1. *If* $\lambda^{m-j}\mu^j \in \mathbb{R}$, *then choose*

$$\Delta A_j := \begin{cases} z_{A_j}\left[(x^H k)(xx^H) + P_x k x^H + x k^H P_x\right], & \text{for } j \text{ even,} \\ z_{A_j}\left[i(x^H k)(xx^H) + P_x k x^H - x k^H P_x\right], & \text{for } j \text{ odd.} \end{cases} \tag{11}$$

2. *For* $(\lambda, \mu) \in (\mathbb{R}, i\mathbb{R})$ *purely imaginary, we choose*

$$\Delta A_j = \overline{z_{A_j}}\left[(x^H k)(xx^H) + P_x k x^H + x k^H P_x\right]. \tag{12}$$

3. *If* $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ *is not purely imaginary, then choose*

$$\Delta A_j := \begin{cases} \left[\dfrac{\hat{t}_j}{w_j}(xx^H) + \overline{z_{A_j}} P_x k x^H + z_{A_j} x k^H P_x\right], & \text{for } j \text{ even,} \\ \left[\dfrac{i\hat{t}_j}{w_j}(xx^H) + \overline{z_{A_j}} P_x k x^H - z_{A_j} x k^H P_x\right], & \text{for } j \text{ odd.} \end{cases} \tag{13}$$

This gives the unique (in Frobenius norm) H*-even matrix polynomial* $\Delta\mathbf{L}(c, s) = \sum_{j=0}^{m} c^{m-j} s^j \Delta A_j$ *such that* $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$, *and* $\|\Delta\mathbf{L}\|_{w,F} = \eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L})$.

For the spectral norm, let $\gamma^2 := \|k\|_2^2 - |x^H k|^2$ *and introduce the following perturbation matrices.*

1. *For* $\lambda^{m-j}\mu^j \in \mathbb{R}$ *choose*

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{for } j \text{ even,} \\ \Delta A_j + i\dfrac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{for } j \text{ odd,} \end{cases}$$

   where ΔA_j *is given by (11).*

2. *If* $(\lambda, \mu) \in (\mathbb{R}, i\mathbb{R})$ *is purely imaginary, then choose*

$$\Delta A_j = \Delta A_j + \frac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, \tag{14}$$

   with ΔA_j *given by (12).*

3. *If* $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ *is not purely imaginary, then choose*

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\hat{t}_j P_x k k^H P_x}{w_j \gamma^2}, & \text{for } j \text{ even,} \\ \Delta A_j + \dfrac{i\hat{t}_j P_x k k^H P_x}{w_j \gamma^2}, & \text{for } j \text{ odd,} \end{cases}$$

   where ΔA_j *is given by (13).*

This gives a minimal (in spectral norm) H-even matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j\Delta E_j$ *such that* $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$, *and* $|||\Delta\mathbf{L}|||_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$.

Proof. Consider an H-even matrix polynomial $\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^jA_j$. Then we have $A_j = A_j^H$ for j even and $A_j = -A_j^H$ for j odd. By Theorem 3.1 there exists an H-even matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$ such that $\Delta\mathbf{L}(\lambda,\mu)x = -\mathbf{L}(\lambda,\mu)x = k$. Choosing a unitary matrix $U = \begin{bmatrix} x & U_1\end{bmatrix}$ and then $\Delta A_j := U\begin{bmatrix} d_{j,j} & d_j^H \\ d_j & D_{j,j}\end{bmatrix}U^H$, $B_j := U\begin{bmatrix} id_{j,j} & d_j^H \\ -d_j & B_{j,j}\end{bmatrix}U^H$, where $D_{j,j} = D_{j,j}^H$, $B_{j,j} = -B_{j,j}^H \in \mathbb{C}^{n-1\times n-1}$. Then $\Delta\mathbf{L}(\lambda,\mu)x = k$ implies that

$$\sum_{j=0,\, j\,\text{even}}^{m} \lambda^{m-j}\mu^j d_{j,j} + i\sum_{j=1,\, j\,\text{odd}}^{m-1} \lambda^{m-j}\mu^j d_{j,j} = x^Hk, \tag{15}$$

$$\sum_{j=0,\, j\,\text{even}}^{m} \lambda^{m-j}\mu^j d_j - \sum_{j=1,\, j\,\text{odd}}^{m-1} \lambda^{m-j}\mu^j d_j = U_1^Hk. \tag{16}$$

To minimize the perturbation we first solve (16) for the vector d_j. The minimum norm solution is given by $d_j = \overline{z_{A_j}}U_1^Hk$.

For the computation of the parameters $d_{j,j}, D_{j,j}$ we consider three different cases.

1) If $\lambda^{m-j}\mu^j \in \mathbb{R}$, then we solve (15) in a least squares sense and then obtain $d_{j,j} = z_{A_j}x^Hk$ for j even, and $d_{j,j} = -iz_{A_j}x^Hk$ for j odd. Then for even j we have

$$\begin{aligned}\Delta A_j &= \begin{bmatrix} x & U_1\end{bmatrix}\begin{bmatrix} z_{A_j}x^Hk & \left(z_{A_j}U_1^Hk\right)^H \\ z_{A_j}U_1^Hk & D_{j,j}\end{bmatrix}\begin{bmatrix} x^H \\ U_1^H\end{bmatrix} \\ &= z_{A_j}\left[(x^Tk)(xx^H) + U_1(U_1^H)kx^H + xk^HU_1U_1^H\right] + U_1D_{j,j}U_1^H.\end{aligned}$$

Similarly, when j is odd we have

$$\Delta A_j = z_{A_j}\left[(x^Hk)(xx^H) + P_xkx^H - xk^HP_x\right] + U_1B_{j,j}U_1^H.$$

For the Frobenius norm we obtain the minimum by setting $D_{j,j} = 0$ and $B_{j,j} = 0$ and thus,

$$\|\Delta A_j\|_F = \sqrt{\|d_{j,j}\|_2^2 + 2\|d_j\|^2}.$$

which implies that

$$\begin{aligned}\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,\mathbf{L}) &= \sum_{j=0}^{m}\sqrt{\|z_{A_j}\|(2\|k\|_2^2 - \|x^Hk\|_2^2)} \\ &= \sqrt{\frac{(2\|k\|_2^2 - \|x^Hk\|_2^2)}{H^2_{w^{-1},2}}}.\end{aligned}$$

To show that $(\mathbf{L}+\Delta\mathbf{L})x=0$ consider first the coefficients with even index.

$$\begin{aligned}
(\mathbf{L}+\Delta\mathbf{L})x &= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j (A_j+\Delta A_j)x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j (A_j+\Delta A_j)x \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j A_j x + \sum_{j=1\, j \text{ odd}} \lambda^{m-j}\mu^j A_j x \\
&\quad + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \Delta A_j x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \Delta A_j x \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \Delta A_j x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \Delta A_j x \\
&= -k + \sum_{j=0}^{m} \lambda^{m-j}\mu^j z_{A_j} \left[(x^H k)(xx^H) + P_x k x^H + x k^H P_x\right] x \\
&= -k + \left[(x^H k)(xx^H) + P_x k x^H + x k^H P_x\right] x \\
&= -k + (x^H k)x + P_x k = -k + (x^H k)x + k - xx^H k = 0.
\end{aligned}$$

The proof for the coefficients with odd index is analogous.

For the spectral norm set $\gamma^2 = \|k\|_2^2 - |x^H k|^2$ and apply Corollary 2.6. For Hermitian coefficients we obtain that

$$D_{j,j} = -\overline{z_{A_j}} \frac{x^H k U_1^H k k^H U_1}{\gamma^2},$$

while for skew-Hermitian coefficients we get

$$B_{j,j} = \overline{z_{A_j}} \frac{i x^H k U_1^H k k^H U_1}{\gamma^2},$$

Hence,

$$\Delta E_j := \begin{cases} \Delta A_j - \frac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{for } j \text{ even}, \\ \Delta A_j + i\frac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{for } j \text{ odd}. \end{cases}$$

It remains to show that $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0$. We have

$$\begin{aligned}
&(\mathbf{L}+\Delta\mathbf{L})x \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j (A_j+\Delta A_j)x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j (A_j+\Delta A_j)x \\
&= \sum_{j=0, j \text{ even}} \lambda^{m-j}\mu^j A_j x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j A_j x \\
&\quad + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \Delta A_j x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \Delta A_j x \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} \left[x^H k (xx^H) + P_x k x^H + x k^H P_x\right] x + \\
&\quad \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j z_{A_j} \left[i x^H k (xx^H) + P_x k x^H - x k^H P_x\right] x
\end{aligned}$$

$$
\begin{aligned}
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} x^H kx + \sum_{j=1,\, j \text{ odd}} i\lambda^{m-j}\mu^j z_{A_j} x^H kx + \\
&\quad [\sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j z_{A_j}] P_x k \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} x^H kx + \sum_{j=1,\, j \text{ odd}} i\lambda^{m-j}\mu^j z_{A_j} x^H kx + P_x k \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} x^H kx + \sum_{j=1,\, j \text{ odd}} i\lambda^{m-j}\mu^j z_{A_j} x^H kx \\
&\qquad +(I - xx^H) \sum_{j=0}^{m} \lambda^{m-j}\mu^j \Delta A_j x \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} x^H kx + \sum_{j=1,\, j \text{ odd}} i\lambda^{m-j}\mu^j z_{A_j} x^H kx \\
&\quad - \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j z_{A_j} x^H kx - \sum_{j=1,\, j \text{ odd}} i\lambda^{m-j}\mu^j z_{A_j} x^H kx = 0.
\end{aligned}
$$

2) If $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ is purely imaginary, then the construction of the perturbation coefficients and the proof follows exactly as in the real case.

3) If $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ is not purely imaginary, then we have to solve

$$
\begin{bmatrix}
\sum\limits_{j=0,\, j \text{ even}}^{m} \Re(\lambda^{m-j}\mu^j) d_{j,j} - \sum\limits_{j=1,\, j \text{ odd}}^{m-1} \Im(\lambda^{m-j}\mu^j) d_{j,j} \\
\sum\limits_{j=0,\, j \text{ even}}^{m} \Im(\lambda^{m-j}\mu^j) d_{j,j} + \sum\limits_{j=1,\, j \text{ odd}}^{m} \Re(\lambda^{m-j}\mu^j) d_{j,j}
\end{bmatrix}
= \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix},
$$

which is just

$$
\hat{M} \begin{bmatrix} w_0 a_{0,0} \\ w_1 a_{1,1} \\ \vdots \\ w_m a_{m,m} \end{bmatrix} = \begin{bmatrix} \Re(x^H k) \\ \Im(x^H k) \end{bmatrix},
$$

and hence $d_{jj} = w_j^{-1} \hat{t}_j$. This gives the perturbation coefficients

$$
\Delta A_j := \begin{cases}
U \begin{bmatrix} {w_j}^{-1} \hat{t}_j & \left(\overline{z_{A_j}} U_1^H k \right)^H \\ \overline{z_{A_j}} U_1^H k & D_{j,j} \end{bmatrix} U^H, & \text{for even } j \\
U \begin{bmatrix} i {w_j}^{-1} \hat{t}_j & -\left(\overline{z_{A_j}} U_1^H k \right)^H \\ \overline{z_{A_j}} U_1^H k & B_{j,j} \end{bmatrix} U^H, & \text{for odd } j.
\end{cases}
$$

For even j we have

$$
\begin{aligned}
\Delta A_j &= \begin{bmatrix} x & U_1 \end{bmatrix} \begin{bmatrix} {w_j}^{-1} \hat{t}_j & \left(\overline{z_{A_j}} U_1^H k \right)^H \\ \overline{z_{A_j}} U_1^H k & D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\
&= {w_j}^{-1} \hat{t}_j (xx^H) + \overline{z_{A_j}} U_1 (U_1^H) k x^H + z_{A_j} x k^H U_1 U_1^H + U_1 D_{j,j} U_1^H.
\end{aligned}
$$

By construction we have that $U_1U_1^H = I - xx^H$ and thus,

$$\Delta A_j = {w_j}^{-1}\hat{t}_j(xx^H) + \overline{z_{A_j}}P_xkx^H + z_{A_j}xk^HP_x + U_1D_{j,j}U_1^H.$$

Similarly, when j is odd we have

$$\Delta A_j = \left[{w_j}^{-1}\hat{t}_j(xx^H) + \overline{z_{A_j}}P_xkx^H - z_{A_j}xk^HP_x\right] + U_1B_{j,j}U_1^H.$$

Hence for $\lambda^{m-j}\mu^j \in \mathbb{C}$ and for the Frobenius norm, taking $D_{j,j} = 0$ and $B_{j,j} = 0$, we obtain the perturbation matrices

$$\Delta A_j := \begin{cases} {w_j}^{-1}\hat{t}_j(xx^H) + \overline{z_{A_j}}P_xkx^H + z_{A_j}xk^HP_x, & \text{for } j \text{ even}, \\ {w_j}^{-1}\hat{t}_j(xx^H) + \overline{z_{A_j}}P_xkx^H - z_{A_j}xk^HP_x, & \text{for } j \text{ odd}. \end{cases}$$

For the spectral norm we introduce the perturbation matrices

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\hat{t}_jP_xkk^HP_x}{w_j\gamma^2}, & \text{for } j \text{ even}, \\ \Delta A_j + \dfrac{i\hat{t}_jP_xkk^HP_x}{w_j\gamma^2}, & \text{for } j \text{ odd}. \end{cases}$$

and we have the backward error

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,\mathbf{L}) = \sqrt{\sum_{j=0}^{m}\left|\frac{\hat{t}_j}{w_j}\right|^2 + |z_{A_j}|^2(\|k\|_2^2 - |x^Hk|^2)} = \sqrt{\sum_{j=0}^{m}\left|\frac{\hat{t}_j}{w_j}\right|^2 + \frac{\|k\|_2^2 - |x^Hk|^2}{H^2_{w^{-1},2}}}.$$

It remains to show that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

$$\begin{aligned}
&(\mathbf{L} + \Delta\mathbf{L})x \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j(A_j + \Delta A_j)x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j(A_j + \Delta A_j)x \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^jA_jx + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^jA_jx \\
&\quad + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j\Delta A_jx + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j\Delta A_jx \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j\left[\frac{\hat{t}_j}{w_j}(xx^H) + \overline{z_{A_j}}P_xkx^H + z_{A_j}xk^HP_x\right]x + \\
&\sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j\left[\frac{i\hat{t}_j}{w_j}(xx^H) + \overline{z_{A_j}}P_xkx^H - z_{A_j}xk^HP_x\right]x \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j\left[\frac{\hat{t}_j}{w_j}x + \overline{z_{A_j}}P_xk\right] + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j\left[\frac{i\hat{t}_j}{w_j}x + \overline{z_{A_j}}P_xk\right] \\
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j\frac{\hat{t}_j}{w_j}x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j\frac{i\hat{t}_j}{w_j}x + P_xk
\end{aligned}$$

$$
\begin{aligned}
&= -k + \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \frac{\hat{t}_j}{w_j} x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \frac{i\hat{t}_j}{w_j} x + (I - xx^H)k \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \frac{\hat{t}_j}{w_j} x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \frac{i\hat{t}_j}{w_j} x - x \sum_{j=0}^{m} \lambda^{m-j}\mu^j x^H \Delta A_j x \\
&= \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \frac{\hat{t}_j}{w_j} x + \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \frac{i\hat{t}_j}{w_j} x - \\
&\quad \sum_{j=0,\, j \text{ even}} \lambda^{m-j}\mu^j \frac{\hat{t}_j}{w_j} x - \sum_{j=1,\, j \text{ odd}} \lambda^{m-j}\mu^j \frac{i\hat{t}_j}{w_j} x = 0. \quad \square
\end{aligned}
$$

The result for homogeneous matrix pencils ($m = 1$) of the form $\mathbf{L}(c, s) = cA_0 + sA_1 \in \mathbf{L}_m(\mathbb{C}^{n\times n})$, see [4, 5] as well as the corresponding results for the case of non-homogenous matrix polynomials ($c = 1$) that have only finite eigenvalues of [1] follow as special cases by setting and $w = [1, 1, \ldots, 1]^T$.

For completeness we also state the expressions for the case of H-odd homogeneous matrix polynomials. The proof is completely analogous to the H-even case, just exchanging the role of off and even indices in appropriate places.

Theorem 4.2 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be H-odd and of the form (1). Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $x \in \mathbb{C}^n$ with $x^H x = 1$. We then have the same backward error formulas (9)–(10) as in the H-even case.*

For the Frobenius norm we introduce the following perturbation matrices.

1. *If $\lambda^{m-j}\mu^j \in \mathbb{R}$, then choose*

$$
\Delta A_j := \begin{cases} z_{A_j}\left[(x^H k)(xx^H) + P_x k x^H + x k^H P_x\right], & \text{for } j \text{ odd}, \\ z_{A_j}\left[i(x^H k)(xx^H) + P_x k x^H - x k^H P_x\right], & \text{for } j \text{ even}. \end{cases} \tag{17}
$$

2. *For $(\lambda, \mu) \in (\mathbb{R}, i\mathbb{R})$ purely imaginary, we choose*

$$
\Delta A_j = \overline{z_{A_j}}\left[(x^H k)(xx^H) + P_x k x^H + x k^H P_x\right]. \tag{18}
$$

3. *If $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ is not purely imaginary, then choose*

$$
\Delta A_j := \begin{cases} \left[\frac{\hat{t}_j}{w_j}(xx^H) + \overline{z_{A_j}} P_x k x^H + z_{A_j} x k^H P_x\right], & \text{for } j \text{ odd}, \\ \left[\frac{i\hat{t}_j}{w_j}(xx^H) + \overline{z_{A_j}} P_x k x^H - z_{A_j} x k^H P_x\right], & \text{for } j \text{ even}. \end{cases} \tag{19}
$$

This gives the unique (in Frobenius norm) H-odd matrix polynomial $\Delta\mathbf{L}(c, s) = \sum_{j=0}^{m} c^{m-j} s^j \Delta A_j$ such that $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$, and $\|\Delta\mathbf{L}\|_{w,F} = \eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L})$.

For the spectral norm, let $\gamma^2 := \|k\|_2^2 - |x^H k|^2$ and introduce the following perturbation matrices.

1. *For $\lambda^{m-j}\mu^j \in \mathbb{R}$ choose*

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{for } j \text{ odd}, \\ \Delta A_j + i\dfrac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, & \text{for } j \text{ even}, \end{cases}$$

where ΔA_j is given by (17).

2. *If $(\lambda, \mu) \in (\mathbb{R}, i\mathbb{R})$ is purely imaginary, then choose*

$$\Delta A_j = \Delta A_j + \frac{z_{A_j} x^H k P_x k k^H P_x}{\gamma^2}, \tag{20}$$

with ΔA_j given by (18).

3. *If $\lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$ is not purely imaginary, then choose*

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\hat{t}_j P_x k k^H P_x}{w_j \gamma^2}, & \text{for } j \text{ odd}, \\ \Delta A_j + \dfrac{i\hat{t}_j P_x k k^H P_x}{w_j \gamma^2}, & \text{for } j \text{ even}, \end{cases}$$

where ΔA_j is given by (19).

This gives a minimal (in spectral norm) H-odd matrix polynomial $\Delta \mathbf{L}(c, s) = \sum_{j=0}^{m} c^{m-j} s^j \Delta E_j$ such that $(\mathbf{L}(\lambda, \mu) + \Delta \mathbf{L}(\lambda, \mu))x = 0$, and $|||\Delta \mathbf{L}|||_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L})$.

Further immediate corollaries are the corresponding results for matrix pencils $\mathbf{L}(s) = A_0 + sA_1 \in \mathbf{L}(\mathbb{C}^{n\times n})$ given in [4, 5], which follows by taking $c = 1$, $m = 1$ and the results for non-homogeneous H-even and H odd matrix polynomials having no infinite eigenvalues of [1, 2].

Corollary 4.3 *Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, and let $x \in \mathbb{C}^n$ be such that $x^H x = 1$. Then we have the following relations between structured H-even/H-odd and unstructured backward errors.*

i) $\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L}) \le \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L})$,

ii) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \lambda^{m-j}\mu^j \in \mathbb{R}$,

iii) $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) \le \eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \lambda^{m-j}\mu^j \in \mathbb{C} \setminus \mathbb{R}$.

5 Conclusion

In [1, 2] the structured backward error analysis i.e., the construction of minimal structured matrix polynomials (in Frobenius and spectral norm) such that an approximate eigenpair of $\mathbf{L}$ becomes an exact eigenpair for $\mathbf{L} + \Delta\mathbf{L}$ has been studied. However, these results were derived for the case that the polynomial eigenvalue problem has no eigenvalue at ∞. In this paper we have extended the results of [1, 2] in the homogeneous setup of matrix polynomials including the case of infinite eigenvalues. We have presented a systematic general procedure for the construction of an appropriately structured minimal polynomial $\Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ such that approximate eigenvector and eigenvalue becomes exact eigenvalue and eigenvector of $\mathbf{L} + \Delta\mathbf{L}$. We have shown that the minimal perturbation is unique in the case of Frobenius norm and there are infinitely many minimal perturbations for the case of 2 norm.

References

[1] B. Adhikari. *Backward perturbation and sensitivity analysis of structured polynomial eigenvalue problem*. PhD thesis, IIT Guwahati, Dept. of Mathematics, 2008.

[2] B. Adhikari and R. Alam. Structured mapping problems for a class of linearly structured matrices. Preprint, IIT Guwahati, Dept. of Mathematics, 2009.

[3] B. Adhikari and D. Kressner. Backward errors and linearizations for palindromic matrix polynomials. Preprint, IIT Guwahati, Dept. of Mathematics, 2009.

[4] S. S. Ahmad. *Pseudospectra of matrix pencils and their applications in the perturbation analysis of eigenvalues and eigendecompositions*. PhD thesis, IIT Guwahati, Dept. of Mathematics, 2007.

[5] S. S. Ahmad and R. Alam. Pseudospectra, critical points and multiple eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 430:1171–1195, 2009.

[6] S. S. Ahmad, R. Alam, and R. Byers. On pseudospectra, critical points and multiple eigenvalues of matrix pencils. *SIAM J. Matrix Anal. Appl.*, 31:1915–1933, 2010.

[7] S. S. Ahmad and V. Mehrmann. Perturbation analysis for complex symmetric, skew-symmetric, even and odd matrix polynomials. Preprint 689, DFG Research Center Matheon, *Mathematics for key technologies* in Berlin, 2010.

[8] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, and F. Tisseur. NLEVP: A collection of nonlinear eigenvalue problems. MIMS Eprint, The University of Manchester, UK, 2008.

[9] S. Bora. Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 31:900–917, 2009.

[10] S. Bora and V. Mehrmann. Perturbation theory for structured matrix pencils arising in control theory. *SIAM J. Matrix Anal. Appl.*, 28:148–169, 2006.

[11] T. Brüll and V. Mehrmann. STCSSP: A FORTRAN 77 routine to compute a structured staircase form for a (skew-)symmetric/(skew-)symmetric pencil. Preprint 2007-31, Institut für Mathematik, TU Berlin, D-10623 Berlin, FRG, 2007. `url:` `http://www.math.tu-berlin.de/preprints/`.

[12] R. Byers, V. Mehrmann, and H. Xu. A structured staircase algorithm for skew-symmetric/symmetric pencils. *Electr. Trans. Num. Anal.*, 26:1–33, 2007.

[13] C. Davis, W. Kahan, and H. Weinberger. Norm-preserving dialations and their applications to optimal error bounds. *SIAM J. Math. Anal.*, 19:445–469, 1982.

[14] J. P. Dedieu and F. Tisseur. Perturbation theory for homogeneous polynomial eigenvalue problems. *Linear Algebra Appl.*, 358:71–94, 2003.

[15] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.

[16] D. J. Higham and N. J. Higham. Structured backward error and condition of generalized eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 20:493–512, 1998.

[17] N. J. Higham. *The Accuracy and Stability of Numerical Algorithims*. SIAM, Philadelphia, USA, 1996.

[18] N. J. Higham and F. Tisseur. More on pseudospectra for polynomial eigenvalue problems and applications in control theory. *Linear Algebra Appl.*, 351–352:435–453, 2002.

[19] P. Lancaster. *Lambda-matrices and Vibrating Systems*. Pergamon Press, Oxford, 1966.

[20] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28:1029–1051, 2006.

[21] V. Mehrmann and H. Voss. Nonlinear eigenvalue problems: a chalenge for modern eigenvalue methods. 27:121–152, 2004.

[22] J. Qian and W. W. Lin. A numerical method for quadratic eigenvalue problems of gyroscopic system. *J. Sound Vibration*, 306:284–296, 2007.

[23] C. Schröder. *Palindromic and even eigenvalue problems-analysis and numerical methods*. PhD thesis, Technical University Berlin, Germany, 2008.

[24] G.W. Stewart and J.-G. Sun. *Matrix Perturbation Theory*. Academic Press, New York, 1990.

[25] F. Tisseur. Backward error and condition of polynomial eigenvalue problem. *Linear Algebra Appl.*, 309:339–361, 2000.

[26] F. Tisseur and K. Meerbergen. A survey of the quadratic eigenvalue problem. *SIAM Rev.*, 43:234–286, 2001.

[27] J.H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, Oxford, 1965.