

# The numéraire portfolio, asymmetric information and entropy

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September 23, 2010

## Abstract

We consider simple models of financial markets with less and better informed investors described by a smaller and a larger filtration on a general stochastic basis that describes the market dynamics, including continuous and jump components. We study the relation between different forms of non-existence of arbitrage and the characteristics of the stochastic basis under the different filtrations. This is achieved through the analysis of the properties of the numéraire portfolio. Furthermore, we focus on the problem of calculating the additional logarithmic utility of the better informed investor in terms of the Shannon entropy of his additional information. The information drift, i.e. the drift to eliminate in order to preserve the martingale property in the larger filtration turns out to be the crucial quantity needed to tackle these problems. We show that the expected logarithmic utility increment due to better information equals its Shannon entropy also in case of a pure jump basis with jumps that are quadratically hedgeable, and so extend a similar result known for bases consisting of continuous semimartingales. An example illustrates that the equality may not persist if both continuous and jump components are present in the underlying.

**2010 AMS subject classifications:** primary 60 H 30, 91 G 10; secondary 60 G 48, 94 A 17.

**Key words and phrases:** insider trading; asymmetric information; enlargement of filtrations; logarithmic utility; Shannon entropy; free lunch; arbitrage; equivalent martingale measure; numéraire portfolio; special semimartingale.

## 1 Introduction

Simple models of financial markets featuring small agents with asymmetric information have been studied for a while. In the simplest version, two small investors face a portfolio optimization problem with respect to their different information levels, described by two filtrations  $\mathcal{F}$  and  $\mathcal{G}$ , where for instance  $\mathcal{F}_t \subset \mathcal{G}_t$  for all times  $t$  in the trading period. It is understood that the smaller  $\mathcal{F}_t$  captures the knowledge of the less informed investor at time  $t$ , while  $\mathcal{G}_t$  corresponds to the information of the better informed one. In this context, the use of enlargement of filtrations' techniques and Malliavin's calculus has given access to problems such as the description of the expected utility advantage the better informed trader has with

respect to the less informed one. Mostly in the context of stochastic bases with continuous processes underlying the description of the market dynamics, the thesis by Ankirchner and subsequent papers ([Ankirchner, 2005], for instance [Ankirchner et al., 2006]) have clarified the relationship between the additional expected utility of a better informed trader and generalized forms of the entropy of his information advantage, which in case of logarithmic utility coincides with the Shannon information.

The analysis in fact has taken into account questions that led to a deeper study of the relationship between finiteness of logarithmic utility and the semimartingale property of the price dynamics of the market ([Ankirchner and Imkeller, 2005]), and consequently had to touch the fundamental problem of asset pricing related to the concept of *arbitrage* (see [Delbaen and Schachermayer, 1995]). The famous general fundamental theorem of asset pricing ([Delbaen and Schachermayer, 1994]) relates the non-existence of arbitrage to the existence of an equivalent martingale measure under which pricing and hedging of claims for general small agents can be done. It triggered the investigation of a hierarchy of different arbitrage notions taken up and further enriched in the more recent paper [Karatzas and Kardaras, 2007] on a stochastic basis as general as possible. The most common no arbitrage concept, *no free lunch with vanishing risk* (NFLVR), was introduced in [Delbaen and Schachermayer, 1995]. In this seminal paper (NFLVR) is shown to be equivalent with the existence of an *equivalent martingale measure*. Under this martingale measure the dynamics of the market assets  $S$  discounted by a risk free bond  $B$  are seen to be martingales. A number of less restrictive concepts of arbitrage have been introduced in the sequel. They are summarized in Karatzas and Kardaras [Karatzas and Kardaras, 2007], and include the notion of *no unbounded profit with bounded risk* (NUPBR), which allows for some forms of arbitrage to exist in the market, thus making the modeling more realistic. This is achieved by choosing a discounting process different from the bond  $B$ , later studied under the name *numéraire portfolio* (see also [Becherer, 2001]). From a mathematical point of view, the requirement of the existence of an *equivalent martingale measure* (EMM) under (NFLVR) is weakened in a (NUPBR) market by the existence of numéraire  $W$ . The latter has the property of transforming the discounted asset,  $\frac{S}{W}$ , into a supermartingale under the original market measure.

The aim of this paper is to understand and describe the utility advantage a better informed investor may have in terms of the underlying market structure, and to interpret it as in [Ankirchner, 2005] by entropy notions such as the Shannon entropy in the case of logarithmic utility. In contrast to previous work this is to be achieved in a setting as general as possible, in the sense of [Karatzas and Kardaras, 2007]. One of the main features to be observed in stepping from the world of a less informed investor to the one of the better informed one is that the additional information acts as an additional drift (the *information drift*) augmenting the bounded variation part in the semimartingale description of the underlying price processes. The integrability properties of this drift are intimately connected to the existence of different forms of arbitrage, to the existence of an equivalent martingale measure, or a numéraire portfolio. So after explaining the market set-up in Section 2, our first task will be to discuss (special) semimartingales playing the role of underlying price dynamics for the market in Section 3. The different notions of arbitrage ((NFLVR), (NUPBR), *no arbitrage* (NA) and *immediate arbitrage*, martingale measures and numéraire portfolios will be studied in Section 4 in terms of properties of the drift  $\alpha \cdot \langle X \rangle$  of the underlying, featuring the market price of risk  $\alpha$ . In particular, we shall discuss examples in which the explicit form of the numéraire portfolio can be given. This also allows an easier characterization of the existence of arbitrage in the market than in [Karatzas and Kardaras, 2007]. We shall see that the existence of

arbitrage does not only depend on the integrability of the market price of risk, as in the case of continuous underlying, but also on the characteristic triplet of the semimartingales, especially the structure of their jump measure. With the ensuing descriptions especially of the related numéraire portfolios, in Section 5 we are then in a position to discuss the structure of the information drift of an (initially enlarged) filtration  $\mathcal{G}$ , and therefore the expected logarithmic utility advantage of the better informed investor. We are able to identify the extra expected logarithmic utility in a purely discontinuous setting, in which the squares of the jumps are hedgeable, with the Shannon entropy of the additional information, extending this striking equality beyond the continuous case, see [Ankirchner and Imkeller, 2005], [Ankirchner, 2008]. See also [Ankirchner and Zwierz, 2008] for a related approach addressing pure jump bases. We show by an example that the equality does not hold in case the stochastic basis contains both a continuous and a jump component in general.

## 2 Market set-up

We work in a market characterized by a complete probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  satisfies the usual conditions and the time horizon  $T$  is finite. The market consists of a risk free asset  $S^0$  and  $d$  risky assets  $S^1, \dots, S^d$ . With no loss of generality we assume that  $S^1, \dots, S^d$  are strictly positive special semimartingales and  $S^0 = 1$ . Therefore we may state that for every  $i$  there exists a special semimartingale  $X^i$ , with  $X_0^i = 0$  and  $\Delta X^i > -1$  such that

$$S^i = S_0^i \mathcal{E}(X^i),$$

where  $\mathcal{E}(X)$  is the Dolean-Dade exponential. The exponential has the form

$$\mathcal{E}(X) = \exp\left(X - \frac{1}{2}\langle X^c, X^c \rangle\right) \prod_{s \leq \cdot} (1 + \Delta X_s) \exp(-\Delta X_s),$$

with  $X^c$  denoting the continuous part of  $X$ ,  $\Delta X_t^i = X_t^i - X_{t-}^i$ ,  $t \in [0, T]$  the jump part and  $\langle X^c, X^c \rangle$  the covariance of  $X^c$ .

With the  $d + 1$  assets of our market we create a portfolio, normalized by the initial value  $W_0 = 1$ , which at time  $t \in [0, T]$  is given by

$$W_t^\pi = \sum_{i=0}^d \gamma_t^i S^i, \quad t \in [0, T]$$

. For  $0 \leq i \leq d$  the process  $\gamma^i$  is predictable and describes the number of units of asset  $i$  in the portfolio. Let  $X = (X^1, \dots, X^d)^*$  and  $\pi = (\pi^1, \dots, \pi^d)^*$ , where  $\pi^i = \frac{\gamma^i S^i}{W^\pi}$  denotes the proportion of the portfolio value invested in asset  $i$ ,  $i = 1, \dots, d$ , and  $\pi^0 = 1 - \sum_{i=1}^d \pi^i$  the proportion of the portfolio invested in the risk free asset. Then the dynamics of the portfolio satisfy the equation

$$\frac{dW_t^\pi}{W_t^\pi} = \sum_{i=1}^d \pi_t^i \frac{dS_t^i}{S_t^i} = \pi_t^* dX_t,$$

hence  $W^\pi = \mathcal{E}(\pi^* X)$ . As in [Karatzas and Kardaras, 2007], we impose a “credit limit” in order to avoid “doubling strategies”. This limit is a uniform lower bound on the wealth

process  $W^\pi$ , which we set equal to zero, i.e. we impose  $W^\pi > 0$ . Again this implies no loss of generality, allowing us to define the set of admissible portfolios  $\mathcal{W}$  as

$$\mathcal{W} = \left\{ W^\pi = \mathcal{E}(\pi^* X) \mid \pi \in L(X) \text{ and } \pi^* \Delta X > -1 \right\},$$

with  $L(X)$  denoting the set of  $\mathbb{R}^d$ -valued predictable processes that are integrable with respect to  $X$ .  $\mathcal{W}$  may include strategies that take advantage of possible arbitrage opportunities in the market. Arbitrage has been formulated in a hierarchy of different forms, some of which are presented in the following definition.

**Definition 2.1** [Karatzas and Kardaras, 2007] We consider the following types of arbitrage.

1. A portfolio  $W^\pi \in \mathcal{W}$  is said to generate an *arbitrage opportunity*, if it satisfies  $P[W_T^\pi \geq 1] = 1$  and  $P[W_T^\pi > 1] > 0$ . If such a portfolio does not exist, we have *no arbitrage* (NA).
2. A sequence  $(W^{\pi_n})_{n \in \mathbb{N}}$  of admissible portfolios is said to generate an *unbounded profit with bounded risk* (UPBR), if the collection of positive random variables  $(W_T^{\pi_n})_{n \in \mathbb{N}}$  is unbounded in probability, i.e. if

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} P[W_T^{\pi_n} > m] > 0.$$

If such a sequence does not exist, we say that there is *no unbounded profit with bounded risk* (NUPBR).

3. A sequence  $(W^{\pi_n})_{n \in \mathbb{N}}$  of admissible portfolios is said to be a *free lunch with vanishing risk* (FLVR), if there exist an  $\epsilon > 0$  and an increasing sequence  $(\delta_n)_{n \in \mathbb{N}}$  with  $0 \leq \delta_n \uparrow 1$ , such that  $P[W_T^{\pi_n} > \delta_n] = 1$  as well as  $P[W_T^{\pi_n} > 1 + \epsilon] \geq \epsilon$ . If such a sequence does not exist, we say that there is *no free lunch with vanishing risk* (NFLVR).
4. An admissible portfolio  $W^\pi$  is said to generate an *unbounded increasing profit* if the wealth process is increasing, i.e., if  $P[W_s^\pi \leq W_t^\pi, \forall 0 \leq s < t \leq T] = 1$ , and if  $P[W_T^\pi > 1] > 0$ . If such a portfolio does not exist, *no unbounded increasing profit* (NUIP) is said to hold.

Some of the notions of arbitrage are more restrictive than others. Considering their hierarchical ordering according to [Karatzas and Kardaras, 2007] and [Delbaen and Schachermayer, 1995], we can state that (NUIP) is a weaker notion than (NUPBR) and (NA), which in turn are weaker notions than (NFLVR). However, in general it is not possible to compare (NUPBR) and (NA).

(NFLVR) is known to be linked to the existence of an equivalent (local) martingale measure. The (NUPBR) condition, however, is closely linked to a specific portfolio in  $\mathcal{W}$ , the portfolio with the largest return. The notion of optimal return can be defined in several ways. Initially, it was defined in terms of the growth optimal portfolio (GOP). The (GOP) is the admissible portfolio with the highest expected logarithmic utility  $u := \sup_{W^\pi \in \mathcal{W}} E[\ln W^\pi]$ . This definition is rather restrictive. In case the expected logarithmic utility is not finite, the (GOP) can not be uniquely defined, see for instance example 2.20 in [Karatzas and Kardaras, 2007]. This shortcoming has led to the introduction of alternative notions of optimality, namely the *relative growth optimal portfolio* and the *numéraire portfolio*, see [Christensen and Larsen, 2007] and [Becherer, 2001] respectively. In what follows we adopt the definitions of [Karatzas and Kardaras, 2007] for the (relative) growth optimal portfolio and numéraire for a finite time horizon.

**Definition 2.2** An admissible portfolio  $W^\pi$  is called *(relative) growth optimal* (GOP), if

$$E \left[ \log \left( \frac{W_T^\rho}{W_T^\pi} \right) \right] \leq 0$$

for all  $W^\rho \in \mathcal{W}$ .

**Definition 2.3** The admissible portfolio  $W^\pi$  is called the *numéraire portfolio*, if for every  $W^\rho \in \mathcal{W}$  the process  $\frac{W^\rho}{W^\pi}$  is a supermartingale.

As is seen in the following proposition, these two versions of optimal portfolios are equivalent.

**Proposition 2.1** ([Karatzas and Kardaras, 2007]) A numéraire portfolio exists if and only if a (relative) growth optimal portfolio exists, in which case the two are identical.

Returning to the notions of arbitrage introduced above, the relationship between (NUPBR) and the numéraire (or relative growth optimal) portfolio is described in the following theorem.

**Theorem 2.1** (Kardaras and Karatzas [Karatzas and Kardaras, 2007]) For a financial market described by the asset price process  $S$  the following are equivalent:

1. The numéraire portfolio  $W^\pi$  exists and is finite.
2. The (NUPBR) condition holds.

**Remark 2.1** In this section we started by describing the assets in the market as semimartingales. This assumption can be omitted in markets generated by continuous price dynamics under the condition of finite logarithmic utility. In this setting it is proven by [Ankirchner and Imkeller, 2005] that for simple buy and hold strategies, finiteness of the logarithmic utility implies that the continuous processes in the market are semimartingales, with no assumption on the existence of arbitrage. [Larsen and Zitkovic, 2006] elaborates on this by showing that finite utility not only implies that  $S$  is a semimartingale for any admissible trading strategy, but that it also has a canonical decomposition of the form  $S = M + \alpha \langle M \rangle$ , where  $M$  is a (local) martingale and  $\alpha$  a square integrable predictable process. Furthermore it is proven that there exists a GOP that is given by  $W^\alpha$ , i.e. by investing on  $S$  according to the strategy  $\alpha$ . Hence from Theorem 2.1 we can conclude that finiteness of the logarithmic utility in this market implies (NUPBR), or even (NFLVR) if  $S$  satisfies some further technical conditions. However these nice properties do not translate to the non-continuous setting, as is illustrated in [Larsen and Zitkovic, 2006] by a counterexample. The authors show that finiteness of logarithmic utility not only does not imply a decomposition for the process as stated before, but not even that the semimartingale property of the underlying process  $S$  holds.

### 3 Semimartingale decomposition

Having introduced the setting of the market, in this section we turn our attention to the dynamics of the underlying processes, more specifically the special semimartingale  $X$ . In the analysis hereafter the notation and results are based on [Jacod and Shiryaev, 2003].

### 3.1 Market price of risk

The special semimartingale  $X$  takes the form

$$X = M + L, \tag{1}$$

where  $M = (M^1, \dots, M^d)^*$  is a  $d$ -dimensional local martingale and  $L = (L^1, \dots, L^d)^*$  is a  $d$ -dimensional predictable process with finite variation.

In the case of a market generated by a continuous semimartingale  $X$ , the existence of arbitrage is closely linked to the properties of the process  $L$ . A number of papers deals with this subject. In [Delbaen and Schachermayer, 1995] the authors prove that if (NFLVR) holds then  $X$  is a semimartingale and  $dL_t^i \ll d\langle X^i \rangle_t$  for  $1 \leq i \leq d$ . The authors also prove that the market has no immediate arbitrage iff  $dL_t^i \ll d\langle X^i \rangle_t$ .

In the more general setting of discontinuous semimartingales, we can also conclude that the finite variation part of  $X$  is related to its predictable covariance process  $\langle X \rangle$ . In order to reach this conclusion we need to introduce the notion of *immediate arbitrage*. The definition we provide is a slight modification of the one in [Karatzas and Kardaras, 2007], that fits our setting.

**Definition 3.1** A strategy  $\xi$  is called an *immediate arbitrage opportunity*, if for all  $t \in [0, T]$  it satisfies

$$\xi_t^* d\langle X^c \rangle_t = 0, \quad \xi_t^* \Delta X_t \geq 0 \text{ and } \xi_t^* dL_t \geq 0 \quad \mathbb{P} - a.s.$$

Immediate arbitrage is the weakest notion of arbitrage and its existence in the market leads to the violation of (NA) and (NUPBR), and consequently of (NFLVR).

In the case of discontinuous semimartingales, as is pointed out in [Karatzas and Kardaras, 2007], Remark 3.13, the condition  $dL_t^i \ll d\langle X^i \rangle_t$  for  $i = 1, \dots, d$ , is necessary for the absence of immediate arbitrage, and hence the absence of (UPBR) and (FLVR), but not sufficient. Therefore we introduce the following assumption.

**Assumption 1** There exists a predictable process  $\alpha$  with values in  $\mathbb{R}^d$  such that  $dL = \alpha d\langle X \rangle$ .

This assumption provides us with a process that captures the market price of risk. Moreover, it is not restrictive, since if it fails, there is already immediate arbitrage in the market and there is not much that we can say about it.

Moving on from the assumption of the existence and predictability of the market price of risk  $\alpha$ , we come to the question of its integrability and its impact on arbitrage in the market. In the continuous case it is proven, see [Ankirchner and Imkeller, 2005], that (NFLVR) is violated in case  $\alpha$  is not integrable. As the next theorem illustrates, the integrability of  $\alpha$  is only relevant, if the strategy  $\alpha$  produces a positive portfolio  $W^\alpha > 0$ .

**Theorem 3.1** Let  $\alpha$  be the market price of risk such that  $W_t^\alpha > 0$   $P$ -a.s. for all  $t \in [0, T]$ . Then, if  $P(\int_0^T \alpha_s d\langle X \rangle_s \alpha_s' = \infty) > 0$ , (NUPBR) is violated.

**Proof**

Since there is a positive probability that  $\int_0^T \alpha_s d\langle X \rangle_s \alpha_s' = \infty$ , we have  $\alpha \notin L(X)$ . From Proposition 4.16 in [Karatzas and Kardaras, 2007] the non-integrability of  $\alpha$  implies  $P(W_T^\alpha = \infty) > 0$ . This in turn implies that (NUPBR) is violated. •

**Remark 3.1** Under Assumption 1 we have  $X = M + \alpha \langle X \rangle$ . Hence,  $\alpha \in L(X)$  iff  $\alpha \in L^2(X)$ .

### 3.2 Characteristics of the market

Since  $X$  is a special semimartingale, it possesses a canonical representation. However, this representation restricts the spectrum of models that we are able to study, specifically it imposes a specific structure on the jump size. So in order to include a wide range of jump models the following assumption is introduced.

**Assumption 2** The  $d$ -dimensional local martingale  $M = (M^1, \dots, M^d)$  from 1 supports the probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ . Furthermore, the characteristic triplet of  $M$  is given by  $(0, C, \eta)$ .

From classical semimartingale theory, see for instance [Jacod and Shiryaev, 2003], it is known that  $B$  is a  $d$ -dimensional predictable process with finite variation,  $C = \langle M^c, M^c \rangle$  is a process in  $\mathbb{R}^{d \times d}$ , denoting the quadratic variation of the continuous local martingale part of  $M$  by  $M^c$ ,  $\eta$  the compensator of the  $d$ -dimensional random measure  $\mu$  associated with the jumps of  $M$ . Also, according to [Jacod and Shiryaev, 2003] Proposition II.2.9, the triplet  $(B, C, \eta)$  can take the form

$$B = b \cdot A \tag{2}$$

$$C = c \cdot A \tag{3}$$

$$\eta(dt, dz) = \nu_t(dz) dA_t, \tag{4}$$

where  $c$  and  $b$  are predictable processes in  $\mathbb{R}^{d \times d}$  and  $\mathbb{R}^d$  respectively, with  $c$  positive definite.  $A$  is a  $d$ -dimensional continuous predictable process in  $\mathbb{R}^d$ , with  $A_0^i = 0$  for  $i = 1, \dots, d$  and non decreasing paths.

From the representation property, see [Jacod and Shiryaev, 2003] Section III.4.4.c., the local martingale part of  $X$  can be represented by elements of  $Y$ ; i.e.

$$X = M^c + H * (\mu - \eta) + \alpha \cdot \langle X \rangle,$$

where  $H$  is a  $d$ -dimensional process that is in  $G_{loc}(\mu)^1$ . We use the notation  $H * \mu = \int_0 \int_{\mathbb{R}_0} H(s, z) \mu(ds, dz)$ .

Recapitulating, the martingale part of  $X$  is decomposed with respect to the characteristics of  $M$ , which is the process generating the probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ . The covariance of  $X$ , from the preceding analysis, takes the form

$$d\langle X \rangle_t = \left( c_t + \int_{\mathbb{R}_0} H^2(t, z) \nu_t(dz) \right) dA_t. \tag{5}$$

Hence

$$X_t = \int_0^t dM_s^c + \int_0^t \int_{\mathbb{R}_0} H(s, z) (\mu(dz, ds) - \nu_s(dz) dA_s) + \int_0^t \alpha_s \left( I_s^2 c_s + \int_{\mathbb{R}_0} H^2(s, z) \nu_s(dz) \right) dA_s$$

“Abusing” the notation in what follows, we denote by  $(\alpha(c + H^2 \cdot \nu), c, H \cdot \nu)$  the “characteristics” of  $X$ .

**Remark 3.2** The characteristic triplet  $(B, C, \eta)$  of any special semimartingale can be represented as in the system of equations (2), (3) and (4). The condition of  $Y$  being quasi-left continuous in Assumption 2 is necessary for  $A$  to be a continuous process. This condition is introduced in order to ease the presentation and the analysis in the forthcoming sections, since the choice of a continuous  $A$  provides a tractable version of  $\langle X \rangle$  as given in (5).

<sup>1</sup>For a definition see Definition II.1.27 p.72 in [Jacod and Shiryaev, 2003].

## 4 Numéraire portfolio.

So far we have introduced a market, the assets of which are driven by special semimartingales. We have also presented different notions of arbitrage and defined the numéraire portfolio. In this section we study the relationship between the characteristics of  $X$  and the existence of arbitrage in its various forms. For this reason we need to consider a larger class of portfolios than the admissible ones denoted by  $\mathcal{W}$ . Namely, we study the strategies that provide a positive portfolio for (almost) all trajectories, irrelevant of their integrability. In this case a strategy is considered to be optimal which at any point  $(t, \omega) \in [0, T] \times \Omega$  creates a positive portfolio with the greatest return in the market. Thus the set of optimal strategies contains not only all admissible strategies but also all strategies that possibly lead to arbitrage. From this point onwards, to simplify notation and computations, we consider a market consisting of only one risky asset.

In the following Lemma, which is based on the results of [Christensen and Larsen, 2007], we present the interval in which the optimal strategy lives.

**Lemma 4.1** Let  $t \in [0, T]$ . Define the *max* fraction  $\bar{\pi}_t$  and the *min* fraction  $\underline{\pi}_t$  as

$$\begin{aligned}\underline{\pi}_t &= \inf\{\pi | 1 + \pi H(t, z) > 0, \nu - \text{almost everywhere}\} \\ \bar{\pi}_t &= \sup\{\pi | 1 + \pi H(t, z) > 0, \nu - \text{almost everywhere}\}.\end{aligned}$$

Then all candidates for optimal strategies  $\pi$  take their values in the interval  $[\underline{\pi}, \bar{\pi}] = \left\{ [\underline{\pi}_t, \bar{\pi}_t], t \in [0, T] \right\}$ .

### Proof

Follow the reasoning of [Christensen and Larsen, 2007]. •

The explicit form of any candidate for an optimal portfolio is given in the following lemma.

**Lemma 4.2** Let  $X$  be a semimartingale with characteristics  $(\alpha(c + H^2 \cdot \nu), c, H \cdot \nu)$ , such that  $\Delta X > -1$ . Then for  $\pi \in [\underline{\pi}, \bar{\pi}]$  we have

$$\begin{aligned}W^\pi &= \exp(\pi X^c + [\ln(1 + \pi X)] * \mu) \\ &\times \exp\left\{\left(-\frac{1}{2}\pi(\pi - 2\alpha)c + [\alpha\pi H^2 - \pi H] * \nu\right) \cdot A\right\},\end{aligned}$$

and for any  $\pi_t, \rho_t \in [\underline{\pi}_t, \bar{\pi}_t]$

$$\begin{aligned}d\frac{W_t^\rho}{W_t^\pi} &= \frac{W_{t-}^\rho}{W_{t-}^\pi} \left\{ (\rho_t - \pi_t) dX_t^c + (\rho_t - \pi_t) \int_{\mathbb{R}_0} \frac{\pi_t H(t, z)}{1 + \pi_t H(t, z)} \tilde{\mu}(dz, dt) \right. \\ &\left. + (\pi_t - \rho_t) \left( (\pi_t - \alpha_t)c_t + \int_{\mathbb{R}_0} \left( \frac{\pi_t H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz) \right) dA_t \right\}, t \in [0, T].\end{aligned}$$

### Proof



For  $\pi \in [\underline{\pi}, \bar{\pi}]$  we have

$$\begin{aligned}
W^\pi &= \mathcal{E}(\pi X) \\
&= \exp\left(\pi X - \frac{1}{2}\pi^2 \langle X^c, X^c \rangle\right) \Pi_{s \leq \cdot} (1 + \pi \Delta X_s) \exp(-\pi \Delta X_s) \\
&= \exp\left(\pi X^c + [\pi H] * (\mu - \nu) + \alpha \pi (c + [H^2] * \nu) \cdot A - \frac{1}{2}\pi^2 c \cdot A\right) \\
&\quad \times \exp([\ln(1 + \pi H)] * \mu - [\pi H] * \mu) \\
&= \exp(\pi X^c + [\ln(1 + \pi H)] * \mu) \\
&\quad \times \exp\left\{\left(-\frac{1}{2}\pi(\pi - 2\alpha)c + [\alpha \pi H^2 - \pi H] * \nu\right) \cdot A\right\}
\end{aligned}$$

For  $\rho \in [\underline{\pi}, \bar{\pi}]$  we therefore have

$$\begin{aligned}
\frac{W^\rho}{W^\pi} &= \exp\left((\rho - \pi)X^c + \left[\ln \frac{1 + \rho H}{1 + \pi H}\right] * \mu\right) \\
&\quad \times \exp\left\{\left(-\frac{1}{2}(\rho - \pi)(\rho + \pi - 2\alpha)c + [(\rho - \pi)(\alpha H^2 - H)] * \nu\right) \cdot A\right\}.
\end{aligned}$$

Applying Itô's formula the dynamics of the portfolio are given by

$$\begin{aligned}
d \frac{W_t^\rho}{W_t^\pi} &= \frac{W_{t^-}^\rho}{W_{t^-}^\pi} \left\{ (\rho_t - \pi_t) dX_t^c + \int_{\mathbb{R}_0} \left( \frac{1 + \rho_t H(t, z)}{1 + \pi_t H(t, z)} - 1 \right) \mu(dz, dt) + \frac{1}{2}(\rho_t - \pi_t)^2 c_t dA_t \right. \\
&\quad \left. - (\rho_t - \pi_t) \left( \frac{1}{2}(\rho_t + \pi_t - 2\alpha_t) c_t c_t - \int_{\mathbb{R}_0} (\alpha_t H^2(t, z) - H(t, z)) \nu_t(dz) \right) dA_t \right\} \\
&= \frac{W_{t^-}^\rho}{W_{t^-}^\pi} \left\{ (\rho_t - \pi_t) dX_t^c + \int_{\mathbb{R}_0} \frac{(\rho_t - \pi_t) H(t, z)}{1 + \pi_t H(t, z)} \mu(dz, dt) \right. \\
&\quad \left. + (\rho_t - \pi_t) \left( (\alpha_t - \pi_t) c_t + \int_{\mathbb{R}_0} (\alpha_t H^2(t, z) - H(t, z)) \nu_t(dz) \right) dA_t \right\} \\
&= \frac{W_{t^-}^\rho}{W_{t^-}^\pi} \left\{ (\rho_t - \pi_t) dX_t^c + \int_{\mathbb{R}_0} \frac{(\rho_t - \pi_t) H(t, z)}{1 + \pi_t H(t, z)} \tilde{\mu}(dz, dt) + (\rho_t - \pi_t) (\alpha_t - \pi_t) c_t dA_t \right. \\
&\quad \left. + (\rho_t - \pi_t) \int_{\mathbb{R}_0} \left( \alpha_t H^2(t, z) - H(t, z) + \frac{H(t, z)}{1 + \pi_t H(t, z)} \right) \nu_t(dz) dA_t \right\} \\
&= \frac{W_{t^-}^\rho}{W_{t^-}^\pi} \left\{ (\rho_t - \pi_t) dX_t^c + (\rho_t - \pi_t) \int_{\mathbb{R}_0} \frac{H(t, z)}{1 + \pi_t H(t, z)} \tilde{\mu}(dz, dt) \right. \\
&\quad \left. + (\pi_t - \rho_t) \left( (\pi_t - \alpha_t) c_t + \int_{\mathbb{R}_0} \left( \frac{\pi_t H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz) \right) dA_t \right\},
\end{aligned}$$

$t \in [0, T]$ . •

If an optimal portfolio  $W^\pi$  exists, the drift of  $\frac{W_t^\rho}{W_t^\pi}$  at each time point  $t$  must be non-positive. Hence we have to study the properties of the process

$$D_t(\rho_t) = (\pi_t - \rho_t) \left( (\pi_t - \alpha_t) c_t + \int_{\mathbb{R}_0} \left( \frac{\pi_t H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz) \right), \quad t \in [0, T],$$

and more specifically find conditions under which  $D \leq 0$ .

In case the jump measure is trivial, i.e.  $\nu_t = 0$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , the drift has the

form

$$D_t(\rho_t) = (\pi_t - \rho_t)(\pi_t - \alpha_t)c_t, \quad t \in [0, T].$$

Clearly, the optimal portfolio is the one that follows the strategy  $\alpha$ . If  $\alpha \in L(X)$ , then  $W^\alpha$  is not only the optimal portfolio but also the numéraire. Furthermore, from Remark 3.1 it follows that  $\frac{1}{W^\alpha}$  is a martingale and the density of an equivalent martingale measure, implying (NFLVR) in the market. Otherwise, from Theorem 3.1 portfolio  $W^\alpha$  takes advantage of arbitrage opportunities in the market, leading to the violation of (NUPBR).

If the jump measure is not trivial, we need to study the functions

$$E_t(\pi_t) = \int_E \left( \frac{\pi_t H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz),$$

and

$$F_t(\pi_t) = (\pi_t - \alpha_t)c_t + E_t(\pi_t), \quad t \in [0, T].$$

Both  $x \mapsto E_t(x)$  and  $x \mapsto F_t(x)$  are increasing functions, a property that is critical for the analysis in the sequel.

Fix  $(\omega, t) \in \Omega \times [0, T]$ . Then:

1. If  $0 < \lim_{\pi \rightarrow \underline{\pi}_t} E_t(\pi)$ , for any  $\pi_t \in [\underline{\pi}_t, \bar{\pi}_t]$ ,  $E_t(\pi_t) > 0$  holds. Hence the sign of  $F_t(\cdot)$  depends on the market price of risk  $\alpha_t$ :
  - (a) If  $\alpha_t < \underline{\pi}_t$ , then  $F_t(\pi_t) > 0$  for any  $\pi_t \in [\underline{\pi}_t, \bar{\pi}_t]$ . For the optimal portfolio to exist we need to have  $D_t(\pi_t) < 0$ , which makes  $\pi_t = \underline{\pi}_t$  the optimal strategy. The analysis hereafter follows the same logic.
  - (b) If  $\underline{\pi}_t \leq \alpha_t$ , since the function  $F_t(\cdot)$  is increasing, the following cases are possible:
    - i. If  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi) > 0$ , the optimal strategy is  $\pi_t = \underline{\pi}_t$ .
    - ii. If  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi) < 0$ , the optimal strategy is  $\pi_t = \bar{\pi}_t$ .
    - iii. Otherwise,  $F$  takes both positive and negative values in  $\pi_t \in [\underline{\pi}_t, \bar{\pi}_t]$ , hence the optimal strategy is the unique solution of the equation  $F_t(\pi_t) = 0$ .
2. If  $\lim_{\pi \rightarrow \underline{\pi}_t} E_t(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} E_t(\pi)$ , the drift behaves as follows:
  - (a) If  $\alpha_t < \underline{\pi}_t$ , then  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi) \geq 0$ . The sign of  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi)$  is crucial for the possible scenarios. Since  $F_t(\cdot)$  is an increasing function, there exist two cases
    - i. If  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi)$  the equation  $F_t(\pi_t) = 0$  has a solution in  $[\underline{\pi}_t, \bar{\pi}_t]$ , which is also the optimal strategy.
    - ii. If  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi) > 0$  the optimal strategy is  $\underline{\pi}_t$ .
  - (b) If  $\underline{\pi}_t \leq \alpha_t \leq \bar{\pi}_t$  the conclusion is the same as in (a).i).
  - (c) If  $\bar{\pi}_t < \alpha_t$ , then  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi) \leq 0$ . Again the sign of  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi)$  is crucial. Since  $F_t(\cdot)$  is an increasing function, there exist two cases
    - i. If  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi)$  the equation  $F_t(\pi_t) = 0$  has a solution in  $[\underline{\pi}_t, \bar{\pi}_t]$ , which is also the optimal strategy.
    - ii. If  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi) < 0$  the optimal strategy is  $\bar{\pi}_t$ .
3.  $\lim_{\pi \rightarrow \bar{\pi}_t} E_t(\bar{\pi}_t) < 0$ .  
In this case  $E_t(\pi_t) < 0$  for all  $\pi \in [\underline{\pi}_t, \bar{\pi}_t]$ . Then we have the following cases:

- (a) Let  $\alpha_t > \bar{\pi}_t$ , then the optimal strategy is given by  $\pi_t = \bar{\pi}_t$ .
- (b)  $\bar{\pi}_t \geq \alpha_t$ .

Since the function  $F_t(\cdot)$  is increasing, we face the following cases.

- i. Let  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi) > 0$ . Then the optimal strategy is given by  $\pi_t = \underline{\pi}_t$ .
- ii. Let  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi) < 0$ . Then the optimal strategy is  $\pi_t = \bar{\pi}_t$ .
- iii. Otherwise, there exist a solution of the equation  $F_t(\pi_t) = 0$ .

**Remark 4.1** As is obvious from the previous analysis, there exists an optimal portfolio for any  $(\omega, t) \in \Omega \times [0, T]$ . However, this does not imply the existence of a numéraire in the market. The latter depends on the integrability of the optimal strategy.

**Remark 4.2** In the special case in which  $\underline{\pi}, \bar{\pi} \in L(X)$ , the optimal strategy belongs to the set of admissible ones, making the optimal portfolio also the numéraire.

The results of this analysis are summarized in the following theorems, after additional notation is introduced.

We define the following predictable subsets of  $\Omega \times [0, T]$ :

$$\mathcal{E} = \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} E_t(\pi_t) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} E_t(\pi_t) \right\}$$

$$\underline{\mathcal{E}} = \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} E_t(\pi_t) > 0 \right\}$$

$$\bar{\mathcal{E}} = \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \bar{\pi}_t} E_t(\pi_t) < 0 \right\}$$

$$\mathcal{F} = \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi_t) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi_t) \right\}$$

$$\underline{\mathcal{F}} = \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} F_t(\pi_t) = 0 \right\}$$

$$\bar{\mathcal{F}} = \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \bar{\pi}_t} F_t(\pi_t) = 0 \right\}$$

The following theorem is the first main result of this paper.

**Theorem 4.1** Let  $X$  be a special semimartingale with characteristic triplet  $(\alpha (c + H^2 \cdot \nu), c, H \cdot \nu)$  and  $\underline{\pi}, \bar{\pi} \in L(X)$ . Then there exist a numéraire portfolio  $W_T^\pi < \infty$ , hence (NUPBR) is satisfied. Moreover,

- (i). If  $\mathcal{F}^c$  has measure  $T$ , then the fraction  $\pi_t$  invested in the numéraire at time  $t$  takes values in  $\{\underline{\pi}_t, \bar{\pi}_t\}$  for all  $t \in [0, T]$ . Furthermore,  $\frac{1}{W^\pi}$  is a strict supermartingale.
- (ii). If  $\mathcal{F}$  has measure  $T$ , then the fraction  $\pi_t$  invested in the numéraire at time  $t$  is the solution of  $F_t(\pi) = 0$  for all  $t \in [0, T]$ . Furthermore,  $\frac{1}{W^\pi}$  is the density of an equivalent local martingale measure implying that (NFLVR) is also satisfied.
- (iii). Let  $\alpha_t \in [\underline{\pi}_t, \bar{\pi}_t]$  for all  $t \in [0, T]$ . Then  $W^\alpha$  is the numéraire portfolio and
  - (a) if  $X$  is a continuous semimartingale, (NFLVR) is satisfied and  $\frac{1}{W^\alpha}$  is the density of the equivalent martingale measure;
  - (b) if  $E(\alpha_t) = 0$ ,  $P \times dt$ -a.s. , (NFLVR) is satisfied and there exists an equivalent minimal martingale measure  $\mathbb{Q}$ , such that  $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{1}{W^\alpha}$ .

**Proof**

The fact that the numéraire exists and (NUPBR) is satisfied follows from Remarks 4.1 and 4.2.

Part (i) follows from cases 1(a), 1(b,i), 1(b,ii), 2(a,ii), 2(c,ii), 3(a), 3(b,i), 3(b,ii).

Part (ii) follows from the combination of cases 1(b,iii), 2(a,i), 2(b), 2(c,i), 3(b,iii).

Part (iii), (a) follows from the pre-existing analysis. Part (iii), (b) is a combination of part 2, (b), the Remark 3.1 and the definition of the Föllmer-Schweizer minimal martingale measure.

•

The following theorem covers the case in which  $\underline{\pi}$  and  $\bar{\pi}$  are not integrable.

**Theorem 4.2** (i). If  $\bar{\pi}, \underline{\pi}$  are not in  $L(X)$  and  $\mathcal{F}^c$  or  $\underline{\mathcal{F}} \cup \bar{\mathcal{F}}$  has positive measure, then (NUPBR) is violated.

(ii). If  $\bar{\pi}$  (resp.  $\underline{\pi}$ ) is not in  $L(X)$  and  $\underline{\mathcal{F}}$  (resp.  $\bar{\mathcal{F}}$ ) has a positive measure, then (NUPBR) is violated.

**Proof**

This follows from Theorem 3.1 and the cases of the analysis of the drift, where  $\bar{\pi}$  or  $\underline{\pi}$  is selected as an optimal strategy. •

**4.1 Examples**

In the following examples we examine the properties of the characteristics of  $X$  and their relationship to arbitrage properties.

**Example 4.1 From Karatzas and Kardaras [Karatzas and Kardaras, 2007]** Let us assume that  $S_t = \mathcal{E}(N_t)$ , where  $N$  is a Poisson process with intensity  $\lambda = 1$ . The market is characterized by the triplet  $(1, 0, 1)$  and the range of optimal strategies is  $[-1, +\infty]$  for all  $t \in [0, T]$ . Furthermore, the market price of risk is  $\alpha_t = 1$  for all  $t \in [0, T]$ , and we have  $E_t(\pi_t) = \frac{-1}{1+\pi_t}$ , where  $\pi_t \in [-1, +\infty]$ . Clearly  $E_t(\cdot)$  is strictly negative with  $\lim_{\pi_t \rightarrow +\infty} E_t(\pi_t) = 0$ . Thus we are in case (ii) of Theorem 4.2, and we conclude that there exists no numéraire portfolio. (NUPBR) is violated.

**Example 4.2 From Becherer [Becherer, 2001]** This example is a continuous time version of ex. 6 in [Becherer, 2001]. Let  $S_t = \Pi_{s \leq t} Y_s$ , where  $t \in [0, T]$  and  $Y$  is lognormally distributed,  $\log Y \sim \mathcal{N}(\mu, \sigma^2)$ . The semimartingale that generates the market is given by

$$X_t = \int_0^t \int_{\mathbb{R}_0} (e^z - 1) \tilde{\mu}(dz, ds) + \left( e^{\mu + \frac{\sigma^2}{2}} - 1 \right) t,$$

with the characteristic triplet  $\left( e^{\mu + \frac{\sigma^2}{2}} - 1, 0, \int_{\mathbb{R}_0} (e^z - 1) \nu(dz) \right)$ , where  $\nu$  is the density of the standard normal distribution. It follows that the market price of risk is given by  $\alpha_t = \frac{e^{\mu + \frac{\sigma^2}{2}} - 1}{(e^{\sigma^2} - 1)e^{\mu + \sigma^2} + (e^{\mu + \frac{\sigma^2}{2}} - 1)}$ . There is no short sale in the market, hence the range of

the optimal strategies is  $[0, 1]$ . Under these assumptions the conditions of Theorem 4.1 are satisfied. This implies that (NUPBR) is satisfied and a numéraire portfolio exists. Since this is a pure jump market, we study the properties of  $E_t(\cdot)$ . We have  $E_t(0) = 1 - e^{\mu + \frac{\sigma^2}{2}}$  and

$$E_t(1) = e^{-\mu + \frac{\sigma^2}{2}} - 1, \quad t \in [0, T].$$

If  $\mu \leq -\frac{\sigma^2}{2}$ ,  $E_t(0) > 0$  and  $\alpha_t < 0 = \underline{\pi}_t$  for all  $t \in [0, T]$ . Hence we are in case 1,(i), or case (i) of Theorem 4.1, which implies that the optimal strategy, which also describes the numéraire, is given by  $\underline{\pi} = 0$ , and the numéraire is a strict supermartingale.

For  $-\frac{\sigma^2}{2} \leq \mu \leq \frac{\sigma^2}{2}$ , since  $E_t(0) < 0 < E_t(1)$  for all  $t \in [0, T]$  we are in case (ii) of Theorem 4.1, the numéraire portfolio exists and  $\frac{1}{W^\pi}$  is a martingale.

For  $\mu \leq \frac{\sigma^2}{2}$ , we are in case 3,(b),ii), since  $E_t(1) < 0$  and  $\bar{\pi}_t > \alpha_t$  for all  $t \in [0, T]$ . This implies that the optimal strategy is  $\bar{\pi} = 1$  and the numéraire is a strict supermartingale.

### Example 4.3 Christensen-Platen[Christensen and Platen, 2005]

Here we consider a one dimensional version of the setting in [Christensen and Platen, 2005]. The market asset satisfies the sde

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= \left( \theta_t^2 + \int_E \frac{\psi^2(t, z)}{1 - \psi(t, z)} \nu(dz) \right) dt + \theta_t dW_t \\ &+ \int_E \frac{\psi(t, z)}{1 - \psi(t, z)} \tilde{\mu}(dz, dt), \end{aligned}$$

where  $\theta$  is a predictable and square integrable process,  $\psi(\cdot, \cdot)$  is predictable and  $\psi(t, z) < 1$  a.e.. Furthermore, the Lévy measure  $\nu$  is finite.

In this case  $\alpha_t = \frac{\theta_t^2 + \int_E \frac{\psi^2(t, z)}{1 - \psi(t, z)} \nu(dz)}{\theta_t^2 + \int_E \frac{\psi^2(t, z)}{(1 - \psi(t, z))^2} \nu(dz)}$ , the characteristics are given by

$\left( \left( \theta_t^2 + \int_E \frac{\psi^2(t, z)}{1 - \psi(t, z)} \nu(dz) \right), \theta_t^2, \int_E \frac{\psi(t, z)}{1 - \psi(t, z)} \nu(dz) \right)$ , and the range of optimal strategies is  $[0, 1]$ . Then

$$\begin{aligned} F_t(\pi_t) &= (\pi_t - 1)\theta_t^2 + \int_E \left( \frac{\pi_t \left( \frac{\psi(t, z)}{1 - \psi(t, z)} \right)^2}{1 + \pi_t \frac{\psi(t, z)}{1 - \psi(t, z)}} - \frac{\psi^2(t, z)}{1 - \psi(t, z)} \right) \nu(dz) \\ &= (\pi_t - 1) \left\{ \theta_t^2 + \int_E \frac{\psi^2(t, z)}{1 + (1 - \pi_t)\psi(t, z)} \nu(dz) \right\}. \end{aligned}$$

Hence it is easy to see that we are in case (ii) of Theorem 4.1, and the numéraire portfolio is given by  $\pi_t = 1$ . In this case  $\frac{1}{S^\pi}$  and  $\frac{S^\rho}{S^\pi}$  are local martingales for all  $\rho \in \mathcal{W}$ .

## 5 Enlarged filtration

In this section we are interested in identifying the difference in return due to asymmetric information. The classical approach to this problem compares the logarithmic utilities under different information structures. To this end, under the assumption of finite logarithmic utilities, we calculate the additional logarithmic utility of a trader with larger information flow  $\mathcal{G}$  than the rest of the market, possessing information described by a smaller filtration  $\mathcal{F} \subset \mathcal{G}$ . Optimal logarithmic utility is linked to the existence of a GOP and in essence to the existence of a numéraire, see Proposition 2.1. For this reason subsection 5.1 summarizes results on the link between the the optimal logarithmic utility of the portfolio and the numéraire. In subsection 5.2 the characteristics of the underlying semimartingale  $X$  under  $\mathcal{G}$  are derived, the available results on the relationship between the characteristics of  $X$  and the

existence of the numéraire portfolio are extended to the setting in the large filtration  $\mathcal{G}$ . In a final step we aim at comparing the additional logarithmic utility with the relative entropy of the filtrations. From [Ankirchner et al., 2006] we know that in a continuous semimartingale framework the extra logarithmic utility of an insider is equal to the Shannon entropy of his additional information. This property also holds true in markets with purely discontinuous semimartingale basis under further assumptions. However, if the semimartingale basis contains both continuous and jump components, this equality may not persist.

## 5.1 Log-utility

The description of the logarithmic utility under (NFLVR) involves the set of (local) equivalent martingale measure, and in the extended framework of (NUPBR) the set of supermartingale densities. The definition of these sets is taken from [Becherer, 2001].

**Definition 5.1** 1. With  $\mathcal{M}$  we denote the set of all probability measures  $Q$ , such that  $Q \sim P$  and  $W^\rho$  is a  $Q$ -local martingale for any  $W^\rho \in \mathcal{W}$ .

2. The set of all  $P$ -supermartingale densities is denoted by

$$\mathcal{SM} := \{Z \mid Z \geq 0, Z_0 = 1, ZW^\rho \text{ is a } P\text{-supermartingale for all } W^\rho \in \mathcal{W}\}.$$

Then the following basic result holds.

**Proposition 5.1** Let (NUPBR) be satisfied and  $u < \infty$ . Then there exists a numéraire portfolio  $W^\pi \in \mathcal{W}$  (i.e. a (GOP)), that satisfies

$$\begin{aligned} E[\log W^\pi] &= \sup_{W^\rho \in \mathcal{W}} E[\log W^\rho] \\ &= \inf_{Z \in \mathcal{SM}} E\left[\log \frac{1}{Z_T}\right] \end{aligned}$$

Furthermore, if (NFLVR) holds, we have

$$E[\log W^\pi] = \inf_{Q \in \mathcal{M}} H(\mathbb{P}|\mathbb{Q}).$$

From the results of the previous section we also obtain the following Lemma.

**Lemma 5.1** Let (NUPBR) hold and  $u < \infty$ . Then the return of the (GOP) for a market with characteristics  $(\alpha(c + H^2 \cdot \nu), c, H \cdot \nu)$  is given by

$$\begin{aligned} E[\log W_T^\pi] &= E\left[\int_0^T -\frac{1}{2}(\pi_t^2 - 2\alpha_t) c_t dA_t\right] \\ &+ E\left[\int_E (\ln(1 + \pi_t H(t, z)) + \pi_t H(t, z)(\alpha_t H(t, z) - 1)) \nu_t(dz) dA_t\right]. \end{aligned}$$

## 5.2 Asymmetric filtration

To describe the additional logarithmic utility, in this subsection start in the following enlargement of filtrations setting. Let  $\mathcal{G}$  be a filtration such that  $\mathcal{F} \subset \mathcal{G}$ . We work under the following assumption concerning the decomposition of the underlying  $X$  in the larger filtration.

**Assumption 3**  $X$  is a quasi-left-continuous semimartingale under  $\mathcal{G}$  and has the representation,

$$X = N + \beta \cdot \langle X, X \rangle,$$

where  $N$  is a local martingale with respect to the filtration  $\mathcal{G}$  and  $\beta$  is a predictable process with respect to  $\mathcal{G}$ .

In the previous sections, under Assumptions 1 and 2 we have deduced the characteristics of  $X$  with respect to  $\mathcal{F}$ , studied their relationship with arbitrage properties, and evaluated the optimal logarithmic utility in Lemma 5.1. To extend this to the enlarged filtration framework we determine the characteristics of  $X$  under  $\mathcal{G}$  in the following theorem.

**Theorem 5.1** Let  $X$  be a semimartingale with characteristics  $(\alpha(c + H^2 \cdot \nu), \cdot c, H \cdot \nu)$  with respect to the filtration  $\mathcal{F}$ . Let  $\mathcal{G}$  be a filtration such that  $\mathcal{F} \subseteq \mathcal{G}$ . Then the characteristic triplet of  $X$  under  $\mathcal{G}$  is given by  $(\beta(c + H^2 \cdot \nu), c, H[1 - (\alpha - \beta)H] \cdot \nu)$ .

### Proof

From the representations of  $X$  under the different filtrations we have

$$\begin{aligned} N &= M + (\alpha - \beta) \cdot \langle X, X \rangle \\ &= M^c + [H] * (\mu - \nu \cdot A) + (\alpha - \beta) (c \cdot A + [H^2] * \nu \cdot A) \\ &= M^c + (\alpha - \beta)c \cdot A + [H] * \mu - [H(1 - (\alpha - \beta)H)] * \nu \cdot A. \end{aligned}$$

Using orthogonality arguments the result follows. •

From the previous theorem, we conclude that the structure of the jump size with respect to the original filtration is preserved in the enlarged filtration. Hence the following lemma is immediate.

**Lemma 5.2** The interval of optimal strategies with respect to the filtration  $\mathcal{G}$  coincides with the interval  $[\underline{\pi}, \bar{\pi}]$  under  $\mathcal{F}$ .

By  $\mathcal{W}^{\mathcal{F}}, \mathcal{W}^{\mathcal{G}}$  we denote the sets of admissible portfolios under the filtrations  $\mathcal{F}$  and  $\mathcal{G}$  respectively.

**Proposition 5.2** Let  $X$  be as in Theorem 5.1, such that  $\Delta X > -1$ . Then for  $\pi \in [\underline{\pi}, \bar{\pi}]$  we have

$$\begin{aligned} W^\pi &= \exp(\pi N^c + [\ln(1 + \pi H)] * \mu) \\ &\times \exp \left\{ \left( -\frac{1}{2} \pi (\pi - 2\beta) c + [\pi H (\alpha H - 1)] * \nu \right) \cdot A \right\}, \end{aligned}$$

and for any  $\rho_t \in [\underline{\pi}_t, \bar{\pi}_t]$

$$\begin{aligned} \frac{d\frac{W^\rho}{W^\pi}}{\frac{W^\rho}{W^\pi}} &= (\pi - \rho) \left\{ (\pi - \beta)c + \left[ \frac{(\pi + \alpha - \beta)H^2}{1 + \pi H} - \alpha H^2 \right] * \nu \right\} dA \\ &+ (\rho - \pi)dN^c + \left[ \frac{(\rho - \pi)H}{1 + \pi H} \right] * \tilde{\mu}^{\mathcal{G}}, \quad t \in [0, T]. \end{aligned}$$

**Proof**

For  $\pi \in [\underline{\pi}, \bar{\pi}]$  we have

$$\begin{aligned} W^\pi &= \mathcal{E}(\pi X) \\ &= \exp \left( \pi N^c + [\pi H] * \mu - [\pi H(1 - (\alpha - \beta)H)] * \nu + \beta\pi(c + [H^2] * \nu) \cdot A - \frac{1}{2}\pi^2 c \cdot A \right) \\ &\times \exp([\ln(1 + \pi H)] * \mu - [\pi H] * \mu) \\ &= \exp(\pi N^c + [\ln(1 + \pi X)] * \mu) \\ &\times \exp \left\{ \left( -\frac{1}{2}\pi(\pi - 2\beta)c + [\pi H(\alpha H - 1)] * \nu \right) \cdot A \right\}. \end{aligned}$$

Let  $\rho \in [\underline{\pi}, \bar{\pi}]$ . Then

$$\begin{aligned} \frac{W^\rho}{W^\pi} &= \exp \left( (\rho - \pi)N^c + \left[ \ln \frac{1 + \rho H}{1 + \pi H} \right] * \mu \right) \\ &\times \exp \left\{ \left( -\frac{1}{2}(\rho - \pi)(\rho + \pi - 2\beta)c + [(\rho - \pi)H(\alpha H - 1)] * \nu \right) A \right\}, \end{aligned}$$

and by applying Itô's rule we have

$$\begin{aligned} d\frac{W_t^\rho}{W_t^\pi} &= \frac{W_t^\rho}{W_t^\pi} \left\{ (\rho_t - \pi_t)dX_t^c + \int_{\mathbb{R}_0} \left( \frac{1 + \rho_t H(t, z)}{1 + \pi_t H(t, z)} - 1 \right) \mu(dz, dt) + \frac{1}{2}(\rho_t - \pi_t)^2 c_t dA_t \right. \\ &- \left. (\rho_t - \pi_t) \left( \frac{1}{2}(\rho_t + \pi_t - 2\beta_t)c_t - \int_{\mathbb{R}_0} H(t, z) (\alpha_t H(t, z) - 1) \nu_t(dz) \right) dA_t \right\} \\ &= \frac{W_t^\rho}{W_t^\pi} \left\{ (\rho_t - \pi_t)dX_t^c + \int_{\mathbb{R}_0} \frac{(\rho_t - \pi_t)H(t, z)}{1 + \pi_t H(t, z)} \mu(dz, dt) \right. \\ &+ \left. (\rho_t - \pi_t) \left( (\beta_t - \pi_t)c_t + \int_{\mathbb{R}_0} H(t, z) (\alpha_t H(t, z) - 1) \nu_t(dz) \right) dA_t \right\} \\ &= \frac{W_t^\rho}{W_t^\pi} \left\{ (\rho_t - \pi_t)dX_t^c + \int_{\mathbb{R}_0} \frac{(\rho_t - \pi_t)H(t, z)}{1 + \pi_t H(t, z)} \tilde{\mu}^{\mathcal{G}}(dz, dt) + (\rho_t - \pi_t)(-\pi_t + \beta_t)c_t dA_t \right. \\ &+ \left. (\rho_t - \pi_t) \int_{\mathbb{R}_0} \left( \alpha_t H^2(t, z) - H(t, z) + \frac{H(t, z)(1 - (\alpha_t - \beta_t)H(t, z))}{1 + \pi_t H(t, z)} \right) \nu_t(dz) dA_t \right\} \\ &= \frac{W_t^\rho}{W_t^\pi} \left\{ (\rho_t - \pi_t)dX_t^c + (\rho_t - \pi_t) \int_{\mathbb{R}_0} \frac{\pi_t H(t, z)}{1 + \pi_t H(t, z)} \tilde{\mu}^{\mathcal{G}}(dz, dt) \right. \\ &+ \left. (\pi_t - \rho_t) \left( (\pi_t - \beta_t)c_t + \int_{\mathbb{R}_0} \left( \frac{(\pi_t + \alpha_t - \beta_t)H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz) \right) dA_t \right\}. \end{aligned}$$

From Theorem 5.1 we know the characteristics of  $X$  under  $\mathcal{G}$ . Following the same steps as in Proposition 4 we obtain the desired results. •



Hence the drift under  $\mathcal{G}$  is given by

$$D_t^*(\rho_t) = (\pi_t - \rho_t) \left\{ (\pi_t - \beta_t)c + \left[ \frac{(\pi_t + \alpha_t - \beta_t)H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right] * \nu \right\}, \quad t \in [0, T].$$

As in section 4 we introduce the functions

$$E_t^*(\pi_t) = \int_E \left( \frac{(\pi_t + \alpha_t - \beta_t)H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz),$$

and

$$F_t^*(\pi_t) = (\pi_t - \beta_t)c_t + E_t^*(\pi_t), \quad t \in [0, T].$$

To proceed with the analysis, we introduce the following assumption.

**Assumption 4** The information drifts  $\alpha, \beta$  satisfy  $1 + (\beta_t - \alpha_t)H(t, z) > 0$   $P$ -a.s. for all  $t \in [0, T]$ .

Under this assumption, the functions  $x \mapsto E_t^*(x)$  and  $x \mapsto F_t^*(x)$  are increasing.<sup>2</sup>

Using the characteristic triplet under  $\mathcal{G}$  and the properties of the functions  $E_t^*(\cdot), F_t^*(\cdot)$ , the analysis of the drift is identical with the one under  $\mathcal{F}$ , and the results transfer accordingly. In case the jump measure is trivial, i.e.  $\nu_t = 0$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ , the optimal portfolio is the one that follows strategy  $\beta$ . If  $\beta \in L(X)$ , then  $W^\beta$  is the numéraire,  $\frac{1}{W^\beta}$  is a martingale and the density of an equivalent martingale measure, implying (NFLVR) in the market. Otherwise, the portfolio  $W^\beta$  takes advantage of arbitrage opportunities in the market, leading to the violation of (NUPBR).

If the jump measure is not trivial, in order to complete the exposition we describe the optimal strategies in summary. For fixed  $(\omega, t) \in \Omega \times [0, T]$  we have

1. if  $0 < \lim_{\pi \rightarrow \underline{\pi}_t} E_t^*(\pi)$  and
  - (a)  $\beta_t < \underline{\pi}_t$ , the optimal strategy is given by  $\pi_t = \underline{\pi}_t$ .
  - (b)  $\underline{\pi}_t \leq \beta_t$ , then
    - i. for  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) > 0$  the optimal strategy is described by  $\pi_t = \underline{\pi}_t$ ,
    - ii. for  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) < 0$ , the optimal strategy is  $\pi_t = \bar{\pi}_t$ ,
    - iii. otherwise, the optimal strategy is the unique solution of the equation  $F_t^*(\pi_t) = 0$ .
2. If  $\lim_{\pi \rightarrow \underline{\pi}_t} E_t^*(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} E_t^*(\pi)$  and
  - (a)  $\beta_t < \underline{\pi}_t$ , then
    - i. for  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} F_t^*(\pi)$  the optimal strategy is the unique solution of the equation  $F_t^*(\pi_t) = 0$ ,
    - ii. if  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) > 0$  the optimal strategy is  $\underline{\pi}_t$ .
  - (b)  $\underline{\pi}_t \leq \beta_t \leq \bar{\pi}_t$ , the conclusion is the same as in (a),i).
  - (c)  $\bar{\pi}_t < \beta_t$ , then
    - i. if  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} F_t^*(\pi)$  the optimal strategy is the unique solution of the equation  $F_t^*(\pi_t) = 0$ ,

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<sup>2</sup>As will become evident in the next section, Assumption 4 is also necessary for the definition of the entropy and hence not restrictive.

- ii. if  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t^*(\pi) < 0$  the optimal strategy is  $\bar{\pi}_t$ .
3. If  $\lim_{\pi \rightarrow \bar{\pi}_t} E_t^*(\bar{\pi}_t) < 0$  and
- (a)  $\beta_t > \bar{\pi}_t$ , the optimal strategy is  $\pi_t = \bar{\pi}_t$ ,
  - (b)  $\bar{\pi}_t \geq \beta_t$ ,
    - i. for  $\lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) > 0$ , the optimal strategy is  $\pi_t = \underline{\pi}_t$ ,
    - ii. for  $\lim_{\pi \rightarrow \bar{\pi}_t} F_t^*(\pi) < 0$ ,
    - iii. otherwise, the optimal strategy is the unique solution of the equation  $F_t^*(\pi_t) = 0$ .

In analogy to section 4 we define the following predictable subsets of  $\Omega \times [0, T]$ :

$$\begin{aligned} \mathcal{E}^* &= \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} E_t^*(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} E_t^*(\pi) \right\}, \\ \underline{\mathcal{E}}^* &= \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} E_t^*(\pi) > 0 \right\}, \\ \overline{\mathcal{E}}^* &= \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \bar{\pi}_t} E_t^*(\pi) < 0 \right\}, \\ \mathcal{F}^* &= \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) \leq 0 \leq \lim_{\pi \rightarrow \bar{\pi}_t} F_t^*(\pi) \right\}, \\ \underline{\mathcal{F}}^* &= \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \underline{\pi}_t} F_t^*(\pi) = 0 \right\}, \\ \overline{\mathcal{F}}^* &= \left\{ (t, \omega) \mid \lim_{\pi \rightarrow \bar{\pi}_t} F_t^*(\pi) = 0 \right\}. \end{aligned}$$

We have the following result about the existence of numéraire portfolios.

**Theorem 5.2** Let  $X$  be a special semimartingale as in Theorem 5.1.

1. If  $\underline{\pi}, \bar{\pi} \in L(X)$ , there exist a numéraire portfolio  $W_T^\pi < \infty$ , hence (NUPBR) is satisfied. Moreover,
  - (i) If  $(\mathcal{F}^*)^c$  has measure  $T$ , then the fraction  $\pi_t$  invested in the numéraire at time  $t$  takes values in  $\{\underline{\pi}_t, \bar{\pi}_t\}$  for all  $t \in [0, T]$ . Furthermore,  $\frac{1}{W^\pi}$  is a strict supermartingale.
  - (ii) If  $\mathcal{F}$  has measure  $T$ , then the fraction  $\pi_t$  invested in the numéraire at time  $t$  is the solution of  $F_t^*(\pi) = 0$  for all  $t \in [0, T]$ . Furthermore,  $\frac{1}{W^\pi}$  is a martingale implying that (NFLVR) is also satisfied.
  - (iii) Let  $\beta_t \in [\underline{\pi}_t, \bar{\pi}_t]$  for all  $t \in [0, T]$ . Then  $W^\beta$  is the numéraire portfolio and
    - i. if  $X$  is a continuous semimartingale, (NFLVR) is satisfied and  $\frac{1}{W^\beta}$  is the density of the equivalent martingale measure.
    - ii. If  $E(\beta_t) = 0$ ,  $P \times dt$ -a.s., (NFLVR) is satisfied and there exists an equivalent minimal martingale measure  $\mathbb{Q}$ , such that  $\frac{d\mathbb{Q}}{dP} = \frac{1}{W^\beta}$ .
2. If  $\bar{\pi}, \underline{\pi}$  are not in  $L(X)$  and  $(\mathcal{F}^*)^c$  or  $\underline{\mathcal{F}}^* \cup \overline{\mathcal{F}}^*$  has positive measure, then (NUPBR) is violated.

3. If  $\bar{\pi}$  (resp.  $\underline{\pi}$ ) is not in  $L(X)$  and  $\underline{\mathcal{F}}$  (resp.  $\overline{\mathcal{F}^*}$ ) has a positive measure, then (NUPBR) is violated.
4. If the market price of risk  $\beta$  satisfies  $W_t^\beta > 0$ ,  $P - a.s.$  for all  $t \in [0, T]$ , (NUPBR) is violated in case  $P(\int_0^T \beta_s^2 d\langle X \rangle_s) > 0$ .

**Proof**

The arguments follow the same steps as the proof of Theorem 4.1. •

**Proposition 5.3** Let  $X$  be a semimartingale with characteristic triplet  $(\alpha\langle X \rangle, C, H\eta)$  with respect to a filtration  $\mathcal{F}$  where (NUPBR) holds, and  $\mathcal{G}$  a filtration such that (NUPBR) holds and  $\mathcal{F}_t \subseteq \mathcal{G}_t$  for all  $t \in [0, T]$ . Furthermore, if  $W^\pi$  and  $W^\rho$  are the numéraire portfolios under  $\mathcal{F}$  and  $\mathcal{G}$  respectively, the difference in return is given by

$$\begin{aligned} u^\mathcal{G} - u^\mathcal{F} &= E \left[ \int_0^T \left( -\frac{1}{2}(\pi_t(\pi_t - 2\beta_t) - \rho_t(\rho_t - 2\alpha_t)) \right) c_t dA_t \right] \\ &+ E \left[ \int_0^T \left( (\pi_t - \rho_t)H(t, z)(\alpha_t H(t, z) - 1) + \ln \frac{1 + \pi_t H(t, z)}{1 + \rho_t H(t, z)} \right. \right. \\ &+ \left. \left. (\beta_t - \alpha_t)H(t, z) \ln(1 + \pi_t H(t, z)) \right) \nu_t(dz) dA_t \right]. \end{aligned}$$

**Proof**

We have

$$\begin{aligned} E[\log W_T^\pi] &= E \left[ \int_0^T \pi_t N_t^c dt + \int_0^T \int_{\mathbb{R}_0} \log(1 + \pi_t H(t, z)) \mu(dz, dt) \right] \\ &+ E \left[ -\int_0^T \frac{1}{2} \pi_t (\pi_t - 2\beta_t) c_t dA_t + \int_0^T \int_{\mathbb{R}_0} \pi_t H(t, z) (\alpha_t H(t, z) - 1) \nu_t(dz) dA_t \right] \\ &= E \left[ \int_0^T \pi_t N_t^c dt + \int_0^T \int_{\mathbb{R}_0} \log(1 + \pi_t H(t, z)) \tilde{\mu}(dz, dt) \right] \\ &+ E \left[ -\int_0^T \frac{1}{2} \pi_t (\pi_t - 2\beta_t) c_t dA_t + \int_0^T \int_{\mathbb{R}_0} [\pi_t H(t, z) (\alpha_t H(t, z) - 1) \nu_t(dz) dA_t \right] \\ &+ E \left[ \int_0^T \int_{\mathbb{R}_0} (1 - (\alpha_t - \beta_t)H(t, z)) \log(1 + \pi_t H(t, z)) \nu_t(dz) dA_t \right] \\ &= E \left[ -\int_0^T \frac{1}{2} \pi_t (\pi_t - 2\beta_t) c_t dA_t \right] + E \left[ \int_0^T \int_{\mathbb{R}_0} \beta_t H(t, z) \log(1 + \pi_t H(t, z)) \nu_t(dz) dA_t \right] \\ &+ E \left[ \int_0^T \int_{\mathbb{R}_0} [(\alpha_t H(t, z) - 1) \{ \pi_t H(t, z) - \log(1 + \pi_t H(t, z)) \}] \nu_t(dz) dA_t \right]. \end{aligned}$$

Combining the previous formula with Lemma 5.1, the result follows.

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### 5.3 Entropy

In this section we describe the entropy of the additional information that a larger filtration provides with respect to a smaller one. To simplify our presentation we assume that  $\mathcal{G}$  is obtained by an initial enlargement of  $\mathcal{F}$ .

Under the Assumption 2 the local semimartingale  $M$  generates the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$ . Let  $(\mathcal{F}_t^0)_{t \in [0, T]}$  be a filtration the  $\sigma$ -algebras of which are countably generated, and under which  $M$  is a local martingale. Assume that  $(\mathcal{F}_t)_{t \in [0, T]}$  is the smallest filtration containing  $(\mathcal{F}_t^0)_{t \in [0, T]}$  and satisfying the usual conditions. Also, let  $(\mathcal{G}_t^0)_{t \in [0, T]}$  be a filtration with countably generated  $\sigma$ -algebras, and  $(\mathcal{G}_t)_{t \in [0, T]}$  the smallest filtration satisfying the usual conditions and containing  $\mathcal{F}_t$ , i.e.  $\mathcal{G}_t \supset \mathcal{F}_t$  for all  $t \geq 0$ .

The introduction of the smaller filtrations  $\mathcal{F}^0, \mathcal{G}^0$  is a necessary condition for the regularity of the conditional probability  $P_t(\cdot, A)$  given  $\mathcal{F}_t^0$ , where  $A \in \mathcal{G}_T^0, t \in [0, T]$ . From [Ankirchner et al., 2006] and [Ankirchner, 2008] it follows that  $P(\cdot, A)$  is a  $(\mathcal{F}_t^0)$ -martingale. And by the martingale representation property, for  $t \in [0, T]$ ,  $P_t(\cdot, A)$  has the form

$$P_t(\cdot, A) = P_0(A) + \int_0^t \gamma_s(\cdot, A) dM_s^c + \int_0^t \int_{\mathbb{R}_0} \delta_u(z, \cdot, A) \tilde{\mu}(dz, dt), \quad (6)$$

where  $\gamma, \delta$  are predictable processes belonging to  $L^2(P)$  and  $L^2(P \otimes \eta)$  respectively. To continue our analysis, we introduce the following assumption.

**Assumption 5** For  $0 \leq t \leq T$  let  $\gamma_t(\omega, \cdot)|_{\mathcal{G}_t^0}$  and  $\delta_t(z, \omega, \cdot)|_{\mathcal{G}_t^0}$  be signed measures on  $\mathcal{G}_t^0$ , such that

$$\gamma_t(\omega, \cdot)|_{\mathcal{G}_t^0} \ll P_t(\omega, \cdot)|_{\mathcal{G}_t^0}, \quad P - a.s.,$$

and

$$\delta_t(z, \omega, \cdot)|_{\mathcal{G}_t^0} \ll P_t(\omega, \cdot)|_{\mathcal{G}_t^0} \quad P \otimes \eta - a.s.$$

**Theorem 5.3** Under Assumption 5 there exist  $\mathcal{F}_t \otimes \mathcal{G}_t$  predictable processes

$$c_t(\omega, \omega') = \frac{\gamma_t(\omega, d\omega')}{P_t(\omega, d\omega')}\Big|_{\mathcal{G}_t^0} \quad P - a.s.,$$

and

$$d_t(z, \omega, \omega') = \frac{\delta_t(z, \omega, d\omega')}{P_t(\omega, d\omega')}\Big|_{\mathcal{G}_t^0} \quad P \otimes \pi - a.e.$$

Furthermore  $c_t(\omega, \omega) = \beta_t(\omega) - \alpha_t(\omega)$  and  $d_t(z, \omega, \omega) = (\beta_t(\omega) - \alpha_t(\omega))H(t, z, \omega)$   $P$ -a.s.

**Proof**

From the beginning of this subsection we know that the information drift for the continuous part of the semimartingale is given by  $\beta - \alpha$  and that for  $t \in [0, T]$   $\frac{\eta_t^{\mathcal{G}}(dz, \omega)}{\eta_t(dz, \omega)} = 1 + (\beta_t(\omega) - \alpha_t(\omega))H(t, z, \omega)$ . Then using the orthogonality of  $M^c$  and  $\mu$  the result follows easily from Lemma 2.3 and Theorem 2.6 in [Ankirchner et al., 2006], as well as Lemma 2.5 and Theorem 2.6 from [Ankirchner, 2008]. •

The preceding theorem is instrumental in the computation of *additional information*, that is introduced in the following.

**Definition 5.2** Let  $\mathcal{A}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$  and  $P, Q$  two probability measures on  $\mathcal{F}$ . Then we define the *relative entropy* of  $P$  with respect to  $Q$  on the sigma field  $\mathcal{A}$  by

$$\mathcal{H}_{\mathcal{A}}(P||Q) = \begin{cases} \int \log \frac{dP}{dQ} dP, & \text{if } P \ll Q, \\ \infty & \text{else.} \end{cases}$$

Moreover, the *additional information* of  $\mathcal{A}$  relative to the filtration  $(\mathcal{F}_u)$  on  $[s, t]$ , where  $0 \leq s < t \leq T$ , is defined by

$$H_{\mathcal{A}}(s, t) = \int \mathcal{H}_{\mathcal{A}}(P_t(\omega, \cdot) || P_s(\omega, \cdot)) dP(\omega).$$

The explicit form of  $H_{\mathcal{G}}$  is provided by the following Lemma.

**Lemma 5.3** The additional information of  $\mathcal{G}_t^0$  relative to the filtration  $(\mathcal{F}_u)$  on  $[s, t]$  is given by

$$\begin{aligned} H_{\mathcal{G}_t^0}(s, t) &= E \left[ \int_s^t \frac{(\beta_t - \alpha_t)^2}{2} d\langle M^c \rangle_t \right. \\ &+ \int_s^t \int_{\mathbb{R}_0} (\beta_t - \alpha_t) H(t, z) \nu(dz) dA_t \\ &+ \left. (1 + (\beta_t - \alpha_t) H(t, z)) \ln(1 + (\beta_t - \alpha_t) H(t, z)) \nu(dz) dA_t \right]. \end{aligned}$$

**Proof**

Using Itô's rule for semimartingales we get

$$\begin{aligned} d \ln P_t(\cdot, A) &= \frac{\gamma_t}{P_t(\cdot, A)} dM_t^c - \frac{\gamma_t^2}{2P_t(\cdot, A)^2} d\langle M^c \rangle_t \\ &+ \int_{\mathbb{R}_0} [\ln(P_t(\cdot, x) + \delta_t(z)) - \ln P_t(\cdot, A)] \mu(dz, dt) + \int_{\mathbb{R}_0} \frac{\delta_t(z)}{P_t(\cdot, A)} \nu(dz) dA_t \\ &= \frac{\gamma_t}{P_t(\cdot, A)} dN_t^c + \int_{\mathbb{R}_0} \ln \left[ 1 + \frac{\delta_t(z)}{P_t(\cdot, A)} \right] \hat{\mu}(dz, dt) \\ &+ \frac{\gamma_t}{P_t(\cdot, A)} \left( \beta_t - \alpha_t - \frac{\gamma_t}{2P_t(\cdot, A)} \right) d\langle M^c \rangle_t \\ &+ \int_{\mathbb{R}_0} \left[ \frac{\delta_t(z)}{P_t(\cdot, A)} + (1 + (\beta_t - \alpha_t) H(t, z)) \ln \left( 1 + \frac{\delta_t(z)}{P_t(\cdot, A)} \right) \right] \nu(dz) dA_t. \end{aligned}$$

Since  $N^c$  and  $\hat{\mu}$  are local martingales under  $(\mathcal{G}_t)$  we have

$$\begin{aligned} E \left[ P_t(\cdot, A) \log \frac{P_t(\cdot, A)}{P_s(\cdot, A)} \right] &= E \left[ \int_s^t \mathbf{1}_A \frac{\gamma_t}{P_t(\cdot, A)} \left( \beta_t - \alpha_t - \frac{\gamma_t}{2P_t(\cdot, A)} \right) d\langle M^c \rangle_t \right. \\ &+ \left. \int_s^t \mathbf{1}_A \int_{\mathbb{R}_0} \left[ \frac{\delta_t(z)}{P_t(\cdot, A)} + (1 + (\beta_t - \alpha_t) H(t, z)) \ln \left( 1 + \frac{\delta_t(z)}{P_t(\cdot, A)} \right) \right] \nu(dz) dA_t \right]. \end{aligned}$$

From Theorem 5.3 we infer

$$\begin{aligned} E \left[ P_t(\cdot, A) \log \frac{P_t(\cdot, A)}{P_s(\cdot, A)} \right] &= E \left[ \int_s^t \mathbf{1}_A \frac{(\beta_t - \alpha_t)^2}{2} d\langle M^c \rangle_t \right. \\ &+ \int_s^t \mathbf{1}_A \int_{\mathbb{R}_0} (\beta_t - \alpha_t) H(t, z) \nu(dz) dA_t \\ &+ \left. (1 + (\beta_t - \alpha_t) H(t, z)) \ln(1 + (\beta_t - \alpha_t) H(t, z)) \nu(dz) dA_t \right]. \end{aligned}$$

Using the same steps as in the proof of Lemma 5.3 of [Ankirchner et al., 2006] we reach our result. •

In the case of a continuous market, as explained in [Ankirchner et al., 2006], the additional information  $H_{\mathcal{G}}(0, T)$  equals the expected logarithmic utility increment between the two filtrations, since  $u^{\mathcal{G}} - u^{\mathcal{F}} = E[\int_0^T \frac{(\beta_t - \alpha_t)^2}{2} dt]$ . However, in the case of a stochastic basis with both a continuous and a jump component this does not always hold, as is illustrated by the following example.

**Example 5.1 Minimal martingale measure.** We assume that under both filtrations (NFLVR) holds and that furthermore  $\alpha, \beta$  are in  $L^2(X)$ . Then the minimal martingale measure under  $\mathcal{F}$  and  $\mathcal{G}$  exist and are given by  $Z^{\mathcal{F}} = \mathcal{E}(-\alpha \cdot X)$  and  $Z^{\mathcal{G}} = \mathcal{E}(-\beta \cdot X)$  respectively. From Proposition 5.1 the extra utility difference of the two filtrations is given by

$$\begin{aligned} u^{\mathcal{G}} - u^{\mathcal{F}} &= E \left[ \int_0^T \left( \frac{1}{2}(\beta_t^2 - \alpha_t^2) \right) c_t dA_t \right] \\ &+ E \left[ \int_0^T \int_{\mathbb{R}_0} \left( (\beta_t - \alpha_t)H(t, z)(\alpha_t H(t, z) - 1) + \ln \frac{1 + \beta_t H(t, z)}{1 + \alpha_t H(t, z)} \right. \right. \\ &\left. \left. + (\beta_t - \alpha_t)H(t, z) \ln(1 + \beta_t H(t, z)) \right) \nu_t(dz) dA_t \right]. \end{aligned}$$

From the representation of  $X$  we have

$$\begin{aligned} N &= M + (\alpha - \beta)\langle X \rangle \Leftrightarrow \\ \alpha N &= \alpha M + \alpha(\alpha - \beta)\langle X \rangle. \end{aligned}$$

Taking expectations on both sides, we obtain

$$\begin{aligned} E[\alpha N] &= E[\alpha M] + E[\alpha(\alpha - \beta)\langle X \rangle] \Leftrightarrow \\ E[\alpha(\alpha - \beta)\langle X \rangle] &= 0 \Leftrightarrow \\ E[(\alpha^2 - \alpha\beta)(c + [H^2]\nu)A] &= 0. \end{aligned}$$

Using this relationship we can rewrite the expected logarithmic utility increment as

$$\begin{aligned} u^{\mathcal{G}} - u^{\mathcal{F}} &= E \left[ \int_0^T \left( \frac{1}{2}(\beta_t - \alpha_t)^2 \right) c_t dA_t \right] \\ &+ E \left[ \int_0^T \left( (\beta_t - \alpha_t) \{ 1 + \ln(1 + \beta_t H(t, z)) \} H(t, z) + \ln \frac{1 + \beta_t H(t, z)}{1 + \alpha_t H(t, z)} \right) \nu_t(dz) dA_t \right]. \end{aligned}$$

Hence

$$\begin{aligned} &H_{\mathcal{G}}(0, T) - (u^{\mathcal{G}} - u^{\mathcal{F}}) \\ &= E \left[ \int_0^T \int_{\mathbb{R}_0} \left\{ (1 - (\beta_t - \alpha_t)H(t, z)) \ln \frac{1 + (\beta_t - \alpha_t)H(t, z)}{1 + \beta_t H(t, z)} + \ln \frac{1 + \beta_t H(t, z)}{1 + \alpha_t H(t, z)} \right\} \nu_t(dz) dA_t \right]. \end{aligned}$$

Under the Assumption 4 and assuming that  $(\beta_t - \alpha_t)\alpha_t > 0$   $\mathbb{P}$ -a.s for all  $t \in [0, T]$ , we have  $H_{\mathcal{G}}(0, T) - (u^{\mathcal{G}} - u^{\mathcal{F}}) > 0$ . Hence, we can conclude that the expected logarithmic utility increment does not necessarily equal the additional information.

## 5.4 Purely discontinuous semimartingales

The expected logarithmic utility increment, as was noted before, is not always equal to the entropy of the additional information. However, as the next theorem shows, in purely discontinuous markets in which the jumps are hedgeable, the equality holds.

**Theorem 5.4** Let  $X$  be a quasi-left continuous semimartingale under the filtration  $\mathcal{F}$ , with characteristic triplet  $(\alpha H^2 \cdot \nu, 0, H \cdot \nu)$ , such that  $1 - \alpha_t H(t, z) > 0$   $P - a.s.$  for all  $t \in [0, T]$  and  $H$  is a predictable process. Then the optimal portfolio strategy is  $\pi_t = \frac{\alpha_t}{1 - \alpha_t H(t, z)}$ . Furthermore, for a filtration  $\mathcal{G} \supseteq \mathcal{F}$ , where  $X$  has the characteristic triplet  $(\beta H^2 \cdot \nu, 0, H[1 + (\beta - \alpha)H] \cdot \nu)$ , the optimal portfolio strategy is given by  $\rho_t = \frac{\beta_t}{1 - \alpha_t H(t, z)}$ . If  $\alpha, \beta \in L(X)$ , then

$$u^{\mathcal{G}} - u^{\mathcal{F}} = H_{\mathcal{G}}(0, T).$$

### Proof

Given the characteristic triplet under  $\mathcal{F}$ , from section 4 we have

$$F_t(\pi_t) = E_t(\pi_t) = \int_E \left( \frac{\pi_t H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz).$$

Clearly  $E_t \left( \frac{\alpha_t}{1 - \alpha_t H(t, z)} \right) = 0$   $P - a.s.$  for all  $t \in [0, T]$ , so we are in case 2. of the analysis. The optimal portfolio strategy is then given by  $\pi = \frac{\alpha}{1 - \alpha H}$ , and  $W^\pi$  satisfies

$$d \frac{1}{W^\pi} = \frac{- \int_{\mathbb{R}_0} \alpha_t H(t, z) d\tilde{\mu}}{W^\pi}.$$

If  $\alpha \in L(X)$ , then  $W^\pi$  is a local martingale and is both the numéraire portfolio and the density of the minimal martingale measure. The expected logarithmic utility of  $W^\pi$  is given by

$$\begin{aligned} u^{\mathcal{F}} = E[\ln W_T^\pi] &= E[-\ln(1 + \pi H)] * \tilde{\mu} + E[[\pi \alpha H^2 - \pi H + \ln(1 + \pi H)] * \nu] \\ &= -E \left[ \int_0^T \int_{\mathbb{R}_0} [\alpha_t H(t, z) + \ln(1 - \alpha_t H(t, z))] \nu(dz) dA_t \right]. \end{aligned}$$

For a larger filtration  $\mathcal{G}$  we have

$$F_t^*(\pi_t) = E_t^*(\pi_t) = \int_E \left( \frac{(\pi_t + \alpha_t - \beta_t) H^2(t, z)}{1 + \pi_t H(t, z)} - \alpha_t H^2(t, z) \right) \nu_t(dz).$$

For  $\rho_t = \frac{\beta}{1 - \alpha H}$  we get  $E_t^*(\rho_t) = 0$  and

$$d \frac{1}{W_t^\rho} = \frac{1}{W_t^\rho} \int_{\mathbb{R}_0} \frac{\beta_t H(t, z)}{1 + (\beta_t - \alpha_t) H(t, z)} d\bar{\mu}.$$

If  $\beta \in L(X)$  the solution of the previous equation is a local martingale, hence  $\frac{1}{W^\rho}$  is the density of martingale measure, and logarithmic utility given by

$$u^{\mathcal{G}} = E[\ln W_T^\rho] = E[[ (1 + (\beta - \alpha)H)(\ln(1 + (\beta - \alpha)H) - \ln(1 - \alpha H)) - \beta H ] * \nu].$$

Note that

$$\int_0^T \int_{\mathbb{R}_0} \ln(1 - \alpha_t H(t, z)) \bar{\mu} - \int_0^T \int_{\mathbb{R}_0} \ln(1 - \alpha_t H(t, z)) \tilde{\mu} = \int_0^T \int_{\mathbb{R}_0} (\beta_t - \alpha_t) \ln(1 - \alpha_t) H(t, z) \nu(dz) dA_t.$$

Hence

$$E \left[ \int_0^T \int_{\mathbb{R}_0} (\beta_t - \alpha_t) \ln(1 - \alpha_t) H(t, z) \nu(dz) dA_t \right] = 0.$$

We have

$$\begin{aligned} u^{\mathcal{G}} - u^{\mathcal{F}} &= E \left[ \int_0^T \int_{\mathbb{R}_0} (1 + (\beta - \alpha)H(t, z)) [\ln(1 + (\beta - \alpha)H(t, z)) - \ln(1 - \alpha H(t, z))] \nu(dz) dA_t \right] \\ &+ E \left[ \int_0^T \int_{\mathbb{R}_0} [-(\beta_t - \alpha_t)H(t, z) + \ln(1 - \alpha_t H(t, z))] \nu(dz) dA_t \right] \\ &= E \left[ \int_0^T \int_{\mathbb{R}_0} \left\{ (1 + (\beta - \alpha)H(t, z)) \ln \left( 1 + (\beta - \alpha)H(t, z) \right) - (\beta_t - \alpha_t)H(t, z) \right\} \nu(dz) dA_t \right]. \end{aligned}$$

Hence under these assumptions we recover the result of the continuous market, namely that the expected logarithmic utility increment is equal to the Shannon entropy of the additional information.

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**Remark 5.1** From 5.3 onwards we have assumed that the filtration  $\mathcal{G}$  is an initial enlargement of  $\mathcal{F}$ . This assumption can be relaxed to include progressive enlargements, as is shown in [Ankirchner et al., 2006]. However, this exceeds the scope of this paper.

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