

On the orthogonal component of BSDEs in a Markovian setting

Anthony Réveillac

Humboldt-Universität zu Berlin

Institut für Mathematik

Unter den Linden 6

10099 Berlin Germany

areveill@mathematik.hu-berlin.de

August 3, 2010

Abstract

In this Note we consider a Lipschitz backward stochastic differential equation (BSDE) driven by a continuous martingale M . We prove (in Theorem 3.2) that if M is a strong Markov process and if the BSDE has the form (2.2) with regular data then the unique solution (Y, Z, N) of the BSDE is reduced to (Y, Z) , *i.e.* the orthogonal martingale N is equal to zero, showing that in a Markovian setting the "usual" solution (Y, Z) (of a BSDE with regular data) has not to be completed by a strongly orthogonal component even if M does not enjoy the martingale representation property.

1 Introduction

Since the seminal papers [1, 8, 4], a lot of interest has been given to the study of backward stochastic differential equations (BSDEs) especially for their use in applications like Finance or stochastic control theory. Roughly speaking, a BSDE can be seen as a generalization of the martingale representation property. Indeed, given a martingale M defined on a filtered

probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ enjoying the martingale representation property, a random variable ζ can be represented as

$$\zeta = \mathbb{E}[\zeta] + \int_0^T Z_s dM_s$$

where Z is a predictable process. More precisely, the martingale $Y_t := \mathbb{E}[\zeta | \mathcal{F}_t]$ can be written as

$$Y_t = \zeta - \int_t^T Z_s dM_s, \quad t \in [0, T]. \quad (1.1)$$

The pair of adapted processes (Y, Z) is said to satisfy the BSDE (1.1) with terminal condition $Y_T = \zeta$. The "usual" formulation of a BSDE consists in adding an additional component so that the equation (1.1) is replaced by

$$Y_t = \zeta - \int_t^T Z_s dM_s + \int_t^T f(r, Y_r, Z_r) d\langle M, M \rangle_r, \quad t \in [0, T] \quad (1.2)$$

where f is a given predictable function. The key ingredient for solving such an equation is once again the martingale representation property. If this property fails, one cannot *a priori* hope to find a solution to equation (1.2), and one has to consider the more general equation (1.3) given by

$$Y_t = \zeta - \int_t^T Z_s dM_s + \int_t^T f(r, Y_r, Z_r) d\langle M, M \rangle_r - \int_t^T dN_r, \quad t \in [0, T] \quad (1.3)$$

whose solution is a triple (Y, Z, N) where (Y, Z) is as before a pair of adapted processes and N is a martingale strongly orthogonal to M .

The surprising fact of this Note is to show that in a continuous Markovian setting (that is M is a continuous strong Markov martingale on a continuous filtration and (1.3) has the form of a forward-backward system (FBSDE) (2.1)-(2.2)) then the orthogonal component vanishes or more precisely $N_s = N_0$ for all s . Thus $dN = 0$ and so the component N

disappears in the equation (2.2). In this sense, by abuse of notation we will write that N is equal to zero meaning that N is constant equal to N_0 . After a draft version of this Note was finished we realized that this surprising phenomenon had been somehow observed in the paper [6] where the authors showed that a random variable of the form $F(M_T)$ can be represented without orthogonal component when M is a Markov process which is allowed to have jumps. However the conditions put by the authors on the function F do not really fit with the setting of BSDEs with non-zero driver. The reader will find a short discussion about this point in the Section 4.

We proceed as follows. First in the next section, we introduce our setting. Then in Section 3 we present our main result: Theorem 3.2. Finally in Section 4 we discuss the result and some extensions under consideration in a work in preparation.

2 Preliminaries

Let $M := (M_t)_{t \in [0, T]}$ be a real-valued continuous square integrable martingale with respect to a continuous filtration $(\mathcal{F}_t)_{t \in [0, T]}$ both defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that M is a strong Markov process with respect to $(\mathcal{F}_t)_{t \in [0, T]}$. For (t, m) in $[0, T] \times \mathbb{R}$ we denote by $M^{t, m}$ the process M starting at m at time t defined as $M_s^{t, m} := m + M_s - M_t$, $s \in [t, T]$. On this filtered probability space we also consider a stochastic process $X^{t, x, m} := (X_s^{t, x, m})_{s \in [t, T]}$ defined as the unique strong solution of the following one-dimensional stochastic differential equation

$$X_s^{t, x, m} = x + \int_t^s \sigma(r, X_r^{t, x, m}, M_r^{t, m}) dM_r + \int_t^s b(r, X_r^{t, x, m}, M_r^{t, m}) d\langle M, M \rangle_r, \quad s \in [t, T], \quad t \in [0, T] \quad (2.1)$$

where $\sigma, b : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are deterministic functions of class $C^{0,2}([0, T] \times \mathbb{R}^2)$ with locally Lipschitz partial derivatives and such that there exists a positive constant k satisfying

$$|\sigma(t, x_1, m_1) - \sigma(t, x_2, m_2)| + |b(t, x_1, m_1) - b(t, x_2, m_2)|$$

$$\leq k(|x_1 - x_2| + |m_1 - m_2|), \quad \forall (t, x_1, x_2, m_1, m_2) \in [0, T] \times \mathbb{R}^4.$$

Let us finally introduce the object of interest of this Note that is the following backward stochastic differential equation (BSDE) coupled with the forward process $X^{t,x,m}$ as

$$\begin{aligned} Y_s^{t,x,m} = & F(X_T^{t,x,m}, M_T^{t,m}) - \int_t^T Z_r^{t,x,m} dM_r + \int_t^T f(r, X_r^{t,x,m}, M_r^{t,m}, Y_r^{t,x,m}, Z_r^{t,x,m}) d\langle M, M \rangle_r \\ & - \int_t^T dN_r^{t,x,m}, \end{aligned} \quad (2.2)$$

where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a bounded deterministic function regular enough and the generator $f : [0, T] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is assumed to be $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^4)$ -measurable where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -field on \mathbb{R} (so that $f(r, x, m, y, z)$ is deterministic for non-random (r, x, m, y, z) in $[0, T] \times \mathbb{R}^4$). In our main result we will make use of the assumptions below.

(L): There exists a constant $K > 0$ such that for all (x, m) in \mathbb{R}^2 and for all (y_i, z_i) in \mathbb{R}^2 ($i = 1, 2$) we have

$$|f(s, x, m, y_1, z_1) - f(s, x, m, y_2, z_2)| \leq K(|y_1 - y_2| + |z_1 - z_2|)$$

and there exists a positive constant a such that $\int_0^T |f(s, x, m, 0, 0)| d\langle M, M \rangle_s \leq a$ for all (x, m) in \mathbb{R}^2 .

(D1): The function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is twice differentiable with locally Lipschitz partial derivatives in x and m uniformly in time.

(D2): The driver f is differentiable in (x, m, y, z) and there exists a positive constant α such that

$$|\partial_i f(s, x_1, m_1, y_1, z_1) - \partial_i f(s, x_2, m_2, y_2, z_2)| \leq \alpha(|x_1 - x_2| + |m_1 - m_2| + |y_1 - y_2| + |z_1 - z_2|),$$

for all $(s, x, m, y, z) \in [0, T] \times \mathbb{R}^4$, and $i = 2, \dots, 5$.

Given an arbitrary non-negative progressively measurable process ψ , for t in $[0, T]$, we let $\Psi_t := \int_0^t \psi_s d\langle M, M \rangle_s$. We recall that under assumption (L), it is shown in [3] that there exist a positive constant β large enough and an unique triple $(Y^{t,x,m}, Z^{t,x,m}, N^{t,x,m}) \in \mathcal{S}_\beta^2 \times H_\beta^2 \times \mathcal{M}^2$ solution of (2.2) where \mathcal{S}_β^2 is the space of continuous adapted processes ϕ such that $\mathbb{E}[\sup_{t \in [0, T]} e^{\beta \Psi_t} |\phi_t|^2] < \infty$; H_β^2 denotes the space of predictable processes ϕ such that $\mathbb{E}[\int_0^T e^{\beta \Psi_t} |\phi_t|^2 d\langle M, M \rangle_t] < \infty$ and \mathcal{M}^2 the space of square integrable martingales $N^{t,x,m}$ strongly orthogonal to M (i.e. $\langle M, N^{t,x,m} \rangle = 0$).

We finally denote by c a constant which can differ from one line to another. We recall below an important fact about Markov processes.

Theorem 2.1. (*[2, Theorem (8.11)] or [9, V. Theorem 35]*) *The process $(X_s^{t,x,m}, M_s^{t,m})_{s \in [t, T]}$ is a strong Markov process for the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. If in addition M is assumed to have independent increments then the stochastic process $(X_s^{t,x,m})_{s \in [t, T]}$ is a strong Markov process.*

The Markov property of the pair $(X_s^{t,x,m}, M_s^{t,m})_{s \in [t, T]}$ transfers to the solution of (2.2).

Theorem 2.2. (*[5, Proposition 3.2]*) *There exist two deterministic functions $u, v : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $\mathcal{B}([0, T]) \otimes \mathcal{B}_e(\mathbb{R}^2)$ such that $(Y^{t,x,m}, Z^{t,x,m})$ in (2.2) satisfy:*

$$Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad Z_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}), \quad s \in [t, T]$$

where $\mathcal{B}_e(\mathbb{R}^2)$ is the σ -field on \mathbb{R}^2 generated by the functions $(x, m) \mapsto \mathbb{E} \left[\phi(s, X_s^{t,x,m}, M_s^{t,m}) d\langle M, M \rangle_s \right]$ with $\phi : \Omega \times [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ a continuous bounded function.

3 Main result

We will use the following ¹property for the solution of the BSDE (2.2).

Proposition 3.1. *(Particular case of [5, Proposition 4.7]) Assume that the assumptions (L), (D1)-(D2) are in force. Then the function u given in Theorem 2.2 satisfy:*

- (i) *The map $(x, m) \mapsto u(t, x, m)$ is of class $\mathcal{C}^1(\mathbb{R}^2)$ for all $t \in [0, T]$.*
- (ii) *There exists two constants ζ_1 and ζ_2 depending only $\|F\|_\infty$ and on the Lipschitz constant K of f such that*

$$\zeta_1 \leq u(t, x, m) \leq \zeta_2, \quad \forall (t, x, m) \in [0, T] \times \mathbb{R}^2.$$

- (iii) *The maps $(t, x, m) \mapsto \partial_i u(t, x, m)$ ($i = 2, 3$) are continuous.*

We are now ready to state and prove the main result of this Note.

Theorem 3.2. *Assume that F and f satisfy the assumptions (L) and (D1)-(D2). Then $N^{t,x,m}$ in (2.2) is equal to zero and equation (2.2) becomes*

$$Y_s^{t,x,m} = F(X_T^{t,x,m}) - \int_t^T Z_r^{t,x,m} dM_r + \int_t^T f(r, Y_r^{t,x,m}, Z_r^{t,x,m}) d\langle M, M \rangle_r.$$

Proof. In [3], the existence of an unique solution $(Y^{t,x,m}, Z^{t,x,m}, N^{t,x,m})$ under our hypotheses was obtained. By Theorem 2.2 there exists a deterministic function u such that

$$Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad \forall s \in [t, T].$$

¹Note that this result has been proved under an additional technical assumption called (MRP) with f a quadratic generator and when the BSDE (2.2) is replaced by the more general BSDE

$$\begin{aligned} Y_s^{t,x,m} = & F(X_T^{t,x,m}, M_T^{t,m}) - \int_t^T Z_r^{t,x,m} dM_r + \int_t^T f(r, X_r^{t,x,m}, M_r^{t,m}, Y_r^{t,x,m}, Z_r^{t,x,m}) d\langle M, M \rangle_r - \int_t^T dN_r^{t,x,m} \\ & + \kappa \int_t^T d\langle N^{t,x,m}, N^{t,x,m} \rangle_r. \end{aligned} \quad (3.1)$$

However when κ is equal to zero the assumption (MRP) is not needed anymore. The full proof of this fact will be presented in a paper in preparation.

In addition, Proposition 3.1 applied to (2.2) gives that the application $(x, m) \mapsto u(t, x, m)$ is of class $\mathcal{C}^1(\mathbb{R}^2)$ for every t . We mimic a technique given in [5] and compute $\langle Y^{t,x,m}, N^{t,x,m} \rangle_s$ for $s \geq t$. Let $\pi^{(n)} := \{t = t_0^{(n)} \leq t_1^{(n)} \leq \dots \leq t_N^{(n)} = T\}$ be a sequence of subdivisions of $[t, T]$ whose mesh $|\pi^{(n)}|$ tends to zero as n goes to the infinity such that

$$\lim_{n \rightarrow \infty} \sup_{t \leq s \leq T} \left| \langle Y^{t,x,m}, N^{t,x,m} \rangle_s - \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}^{(n)}, X_{t_{j+1}}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_j^{(n)}, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m} \right| = 0 \quad (3.2)$$

where the limit is understood in probability with respect to \mathbb{P} , $\Delta_j N^{t,x,m} := N_{t_{j+1}}^{t,x,m} - N_{t_j}^{t,x,m}$ and $\varphi_s = j^{(n)}$ such that $t_j^{(n)} \leq s < t_{j+1}^{(n)}$. For simplicity we will drop the superscripts (n) in the rest of the proof. In addition, by choosing a subsequence (also denoted $(\pi^{(n)})$), the previous limit holds \mathbb{P} -a.s.. We will show that $\langle Y^{t,x,m}, N^{t,x,m} \rangle_s = 0$ for all s , \mathbb{P} -a.s.. Hence, together with relation (2.2) we will have that $\langle Y^{t,x,m}, N^{t,x,m} \rangle_s = \langle N^{t,x,m}, N^{t,x,m} \rangle_s = 0$ for all s , \mathbb{P} -a.s. and thus $N_s^{t,x,m} = N_0^{t,x,m}$ for all s , which will conclude the proof. We have that

$$\begin{aligned} & \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m} \\ = & \sum_{j=0}^{\varphi_s-1} \left[(u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m}) - u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m} \right. \\ & \left. + \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m} \right] \\ =: & A_{s,1}^{(n)} + A_{s,2}^{(n)}. \end{aligned} \quad (3.3)$$

We consider the two summands above separately. Assume first that the following result holds true

$$\lim_{n \rightarrow \infty} \sup_{t \leq s \leq T} A_{s,2}^{(n)} = 0, \quad \mathbb{P} - a.s.. \quad (3.4)$$

Then (3.2) and (3.3) entail that

$$\lim_{n \rightarrow \infty} \sup_{t \leq s \leq T} |A_{s,1}^{(n)} - \langle Y^{t,x,m}, N^{t,x,m} \rangle_s| = 0, \quad \mathbb{P} - a.s.$$

and so

$$\lim_{n \rightarrow \infty} \sup_{t \leq s \leq T} |A_{s,1}^{(n)} - \langle N^{t,x,m}, N^{t,x,m} \rangle_s| = 0, \mathbb{P} - a.s..$$

We will prove that $P := \langle N^{t,x,m}, N^{t,x,m} \rangle$ is a martingale, since it is by definition of finite variation this will show that $\langle N^{t,x,m}, N^{t,x,m} \rangle_s = 0$ for all s in $[t, T]$. We know that

$$\mathbb{E}[|\langle N^{t,x,m}, N^{t,x,m} \rangle_s|] < \infty, \forall s \in [t, T].$$

Now fix $t \leq s_1 \leq s_2 \leq T$. For an element t_j of the subdivisions considered above we let $\delta_j u := u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m}) - u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})$ We have that

$$\begin{aligned} & \mathbb{E}[P_{s_2} | \mathcal{F}_{s_1}] \\ &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} | \mathcal{F}_{s_1} \right] \\ &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} + (N_{s_2}^{t,x,m} - N_{t_{\varphi_{s_2}}}^{t,x,m}) | \mathcal{F}_{s_1} \right] \end{aligned} \quad (3.5)$$

where the last equality is a consequence of the continuity of the martingale $N^{t,x,m}$. In addition the sequence of random variables $\left(\sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} + (N_{s_2}^{t,x,m} - N_{t_{\varphi_{s_2}}}^{t,x,m}) \right)_n$ is uniformly integrable. Indeed, since the function u is bounded (by Proposition 3.1 (ii)) we have that

$$\begin{aligned} & \mathbb{E} \left[\left| \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} + (N_{s_2}^{t,x,m} - N_{t_{\varphi_{s_2}}}^{t,x,m}) \right|^2 \right] \\ &= \sum_{j=0}^{\varphi_{s_2}-1} \mathbb{E} \left[|\delta_j u|^2 |\Delta_j N^{t,x,m}|^2 + |(N_{s_2}^{t,x,m} - N_{t_{\varphi_{s_2}}}^{t,x,m})|^2 \right] \\ &\leq c \left(\sum_{j=0}^{\varphi_{s_2}-1} \mathbb{E} \left[|\Delta_j N^{t,x,m}|^2 + |(N_{s_2}^{t,x,m} - N_{t_{\varphi_{s_2}}}^{t,x,m})|^2 \right] \right) \\ &\leq c \left(\sum_{j=0}^{\varphi_{s_2}-1} \mathbb{E} \left[|N_{t_{j+1}}^{t,x,m}|^2 - |N_{t_j}^{t,x,m}|^2 + |N_{s_2}^{t,x,m}|^2 - |N_{t_{\varphi_{s_2}}}^{t,x,m}|^2 \right] \right) \end{aligned}$$

$$= c(\mathbb{E}[|N_{s_2}^{t,x,m}|^2] - |N_t^{t,x,m}|^2),$$

thus $\sup_n \mathbb{E} \left[\left| \sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} + (N_{s_2}^{t,x,m} - N_{\varphi_{s_2}}^{t,x,m}) \right|^2 \right] \leq c(\mathbb{E}[|N_{s_2}^{t,x,m}|^2] - |N_t^{t,x,m}|^2) < \infty$. Applying Lebesgue's dominated convergence Theorem in (3.5) we get

$$\begin{aligned} & \mathbb{E}[P_{s_2} | \mathcal{F}_{s_1}] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{j=0}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} + (N_{s_2}^{t,x,m} - N_{\varphi_{s_2}}^{t,x,m}) \middle| \mathcal{F}_{s_1} \right] \\ &= \lim_{n \rightarrow \infty} \left(\sum_{j=0}^{\varphi_{s_1}-1} \delta_j u \Delta_j N^{t,x,m} + \mathbb{E}[(\delta_{\varphi_{s_1}} u) \Delta_{\varphi_{s_1}} N^{t,x,m} | \mathcal{F}_{s_1}] \right. \\ & \quad \left. + \mathbb{E} \left[\sum_{j=\varphi_{s_1}+1}^{\varphi_{s_2}-1} \delta_j u \Delta_j N^{t,x,m} + (N_{s_2}^{t,x,m} - N_{\varphi_{s_2}}^{t,x,m}) \middle| \mathcal{F}_{s_1} \right] \right) \\ &= P_{s_1} + \lim_{n \rightarrow \infty} \left((\delta_{\varphi_{s_1}} u) (N_{s_1}^{t,x,m} - N_{\varphi_{s_1}}^{t,x,m}) \right) \\ &= P_{s_1} \end{aligned}$$

where for the last equality we have used the fact that u is bounded and that the martingale $N^{t,x,m}$ is continuous. Thus P is a martingale which by definition has finite variations so

$$\langle N^{t,x,m}, N^{t,x,m} \rangle_s = \langle N^{t,x,m}, N^{t,x,m} \rangle_t = 0, \quad \text{for all } s \in [t, T], \mathbb{P} - a.s.$$

which entails that $N_s^{t,x,m} = N_t^{t,x,m}$, for all $s \in [t, T]$, \mathbb{P} -a.s..

It remains to prove (3.4). For this we also follow a technique used in [5]. Let \bar{M}_j (respectively \bar{X}_j) below a random point between $M_{t_j}^{t,x,m}$ and $M_{t_{j+1}}^{t,x,m}$ (resp. $X_{t_j}^{t,x,m}$ and $X_{t_{j+1}}^{t,x,m}$) in the computations below. We have

$$A_{s,2}^{(n)} = \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m}$$

$$\begin{aligned}
&= \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_{j+1}}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_{j+1}}^{t,x,m})) \Delta_j N^{t,x,m} \\
&\quad + \sum_{j=0}^{\varphi_s-1} (u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_{j+1}, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m} \\
&= \sum_{j=0}^{\varphi_s-1} \left[\partial_2 u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m}) \Delta_j X \Delta_j N^{t,x,m} + \partial_3 u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m}) \Delta_j M \Delta_j N^{t,x,m} \right. \\
&\quad \left. + R_{s,j,n} \right] \tag{3.6}
\end{aligned}$$

where $R_{s,j,n}$ is defined as

$$\begin{aligned}
R_{s,j,n} &:= (\partial_2 u(t_{j+1}, \bar{X}_j, M_{t_{j+1}}^{t,x,m}) - \partial_2 u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j X \Delta_j N^{t,x,m} \\
&\quad + (\partial_3 u(t_{j+1}, X_{t_j}^{t,x,m}, \bar{M}_j) - \partial_3 u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j M \Delta_j N^{t,x,m}.
\end{aligned}$$

Property (iii) of Proposition 3.1 implies that the remainder term $\sum_{j=0}^n R_{s,j,n}$ vanishes as n goes to infinity (we refer to [5, Proof of (5.13)] for the complete justifications). Then it follows using (3.6) that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \sum_{j=0}^{\varphi_s-1} (u(t_j, X_{t_{j+1}}^{t,x,m}, M_{t_{j+1}}^{t,x,m}) - u(t_j, X_{t_j}^{t,x,m}, M_{t_j}^{t,x,m})) \Delta_j N^{t,x,m} \\
&= \int_t^s (\partial_2 u(r, X_r^{t,x,m}, M_r^{t,x,m}) \sigma(r, X_r^{t,x,m}, M_r^{t,x,m}) + \partial_3 u(r, X_r^{t,x,m}, M_r^{t,x,m})) d\langle M, N^{t,x,m} \rangle_s = 0
\end{aligned}$$

by strong orthogonality between M and $N^{t,x,m}$ which shows (3.4). \square

Remark 3.3. *The assumptions (D1) and (D2) are not necessary in Theorem 3.2. In fact only the conclusion of Proposition 3.1 is used in the proof of Theorem 3.2. In other words if one can prove that the function u satisfy (i)-(iii) of Proposition 3.1 then the conclusion of Theorem 3.2 holds true.*

4 Comments

In this section we make a comment about the relation between our main result and the literature and we discuss possible extension under consideration in a work in preparation.

As mentioned in the introduction, our main result shares some similarities with the [6, Theorem 2.4]. This result corresponds to the case where our driver $f = 0$ and M is a strong Markov process which is a semimartingale eventually with jumps. One of the assumption in [6, Theorem 2.4] is that the deterministic function $t \mapsto u(t, x, m)$ in Proposition 3.1 is differentiable for all (x, m) in \mathbb{R}^2 . As shown in Proposition 3.1 the function u is quite regular in space and one can expect it to be continuous in time but it seems to much to ask to be differentiable in time especially in the case where the quadratic variation $\langle M, M \rangle$ is not absolutely continuous with respect to the Lebesgue measure.

In this Note we have considered a quite simple BSDE just to exhibit that the orthogonal component of the solution of the BSDE vanishes. In particular, the assumptions that we have made on our BSDE are quite restrictive especially the condition (D2). However we believe that Theorem 3.2 can be extended to a BSDE of the form (3.1) (see the footnote in Section 3) with quadratic driver (the case of a quadratic driver with $\kappa = 0$ in (3.1) is a straightforward extension of the results presented in this Note). The setting of (3.1) is very interesting since it relates to the utility maximization problem for exponential utility function in incomplete market (see [5, Section 6]). This program is under consideration in a paper in preparation. Finally, we have considered the case where all the processes are in dimension one. The multi-dimensional case can be obtained in the same way.

Acknowledgments

The author is grateful to DFG research center Matheon project E2 for financial support.

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