Dissipativity and linear matrix inequalities *

Tobias Brüll†

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Abstract

In this paper we will consider first-order linear systems with quadratic cost or energy functionals. For such systems we will show that (under some controllability assumptions) dissipativity is equivalent to the solvability of a certain linear matrix inequality.

Key words. dissipativity, linear matrix inequality, available storage, required supply, storage function, linear quadratic optimal control

1 Introduction

Let $\mathcal{C}^q_\infty$ denote the set of infinitely often differentiable functions $z : \mathbb{R} \to \mathbb{C}^q$. In this paper we will consider systems of the form

$$F\dot{z}(t) + Gz(t) = 0, \tag{1}$$

where $F, G \in \mathbb{C}^{p,q}$ are the system matrices, $z \in \mathcal{C}^q_\infty$ is the trajectory, $\dot{z}$ denotes the derivative of z with respect to t, and the identity in (1) is assumed to hold for all $t \in \mathbb{R}$. We call equation (1) a first-order *linear system in behavior form*. We also refer to the tuple (F, G) as a linear system. In the following we will drop the dependence of z on t and simply write $F\dot{z} + Gz = 0$ instead of (1).

In addition to $\mathcal{C}^q_\infty$ we will need the following three classes of functions. By $\mathcal{C}^q_c$ we denote the elements of $\mathcal{C}^q_\infty$ which have compact support. Also, by $\mathcal{C}^q_+$ we denote all $z \in \mathcal{C}^q_\infty$ for which there exist $\alpha, \beta > 0$ such that $\|z(t)\| \leq \alpha e^{-\beta t}$ for all $t \in \mathbb{R}$, and similarly by $\mathcal{C}^q_-$ all $z \in \mathcal{C}^q_\infty$ for which there exist $\alpha, \beta > 0$ such that $\|z(t)\| \leq \alpha e^{\beta t}$ for all $t \in \mathbb{R}$. We say that the elements of $\mathcal{C}^q_+$ are *right exponentially decaying* and the elements of $\mathcal{C}^q_-$ are *left exponentially decaying*. Clearly, we have

$$\mathcal{C}^q_c \subset \mathcal{C}^q_\pm \subset \mathcal{C}^q_\infty.$$

*This research is supported by the DFG Research Center MATHEON in Berlin.

†Institut für Mathematik, TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany, E-mail: bruell@math.tu-berlin.de

Definition 1. Let $F, G \in \mathbb{C}^{p,q}$. Then we call

$$\mathfrak{B}(F,G) := \left\{ z \in \mathcal{C}_\infty^q \middle| F\dot{z} + Gz = 0 \right\}$$

the *behavior of* (F,G),

$$\mathfrak{B}_c(F,G) := \left\{ z \in \mathcal{C}_c^q \middle| F\dot{z} + Gz = 0 \right\}$$

the *compact behavior of* (F,G), and

$$\begin{aligned} \mathfrak{B}_+(F,G) &:= \left\{ z \in \mathcal{C}_+^q \middle| F\dot{z} + Gz = 0 \right\}, \\ \mathfrak{B}_-(F,G) &:= \left\{ z \in \mathcal{C}_-^q \middle| F\dot{z} + Gz = 0 \right\}, \end{aligned}$$

the *right and left exponentially decaying behavior of* (F,G), resp. We call the elements of $\mathfrak{B}(F,G)$, $\mathfrak{B}_c(F,G)$, $\mathfrak{B}_+(F,G)$, and $\mathfrak{B}_-(F,G)$ *trajectories of* (F,G).

Clearly, $\mathfrak{B}(F,G)$, $\mathfrak{B}_+(F,G)$, $\mathfrak{B}_-(F,G)$, and $\mathfrak{B}_c(F,G)$ are linear subspaces of $\mathcal{C}_\infty^q$, $\mathcal{C}_+^q$, $\mathcal{C}_-^q$, and $\mathcal{C}_c^q$, resp., and we have

$$\mathfrak{B}_c(F,G) \subset \mathfrak{B}_\pm(F,G) \subset \mathfrak{B}(F,G).$$

Let $H = H^* \in \mathbb{C}^{q,q}$ be a matrix which measures the energy which is supplied to the system (F,G) along a given trajectory $z \in \mathfrak{B}(F,G)$ in the time interval $[t_0, t_1]$ through the term

$$\int_{t_0}^{t_1} z^*(t) H z(t) dt. \tag{2}$$

One can also think of (2) to measure the cost that a given trajectory $z \in \mathfrak{B}(F,G)$ causes over the time interval $[t_0, t_1]$, implying that energy causes cost.

Definition 2. Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Then (F,G,H) is called *dissipative* if the dissipation inequality

$$0 \leq \int_{-\infty}^{\infty} z^*(t) H z(t) dt,$$

holds for all $z \in \mathfrak{B}_c(F,G)$, i.e., for all trajectories of (F,G) with compact support.

In an informal way Definition 2 states that energy cannot be extracted from the system (F,G) through a trajectory with compact support. One can also think of dissipativity as semi-definiteness of H on the linear subspace given by $\mathfrak{B}_c(F,G)$.

With this notation we are ready to vaguely state the main result of this paper. In the main result Corollary 28 we will see that under some controllability assumptions on the system (F,G) dissipativity of (F,G,H) is equivalent to the solvability of the linear matrix inequality

$$\begin{aligned} F^*Z &= Z^*F, \\ 0 &\leq G^*Z + Z^*G + H, \end{aligned} \tag{3}$$

where $Z \in \mathbb{C}^{p,q}$ is the unknown.

In the rest of this section we will see that solvability of the linear matrix inequality (3) implies dissipativity. For this, let $Z \in \mathbb{C}^{p,q}$ be a solution of (3) and define the function $\Theta : \mathbb{C}^q \to \mathbb{R}$ through

$$\Theta(\hat{z}) := \hat{z}^* F^* Z \hat{z}. \tag{4}$$

With this definition we observe that for any trajectory $z \in \mathfrak{B}(F, G)$ of the system we have that

$$\begin{aligned} \frac{d}{dt}\Theta(z(t)) &= \frac{d}{dt}\left(z^*(t)F^*Zz(t)\right) \\ &= \dot{z}^*(t)F^*Zz(t) + z^*(t)Z^*F\dot{z}(t) \\ &= -z^*(t)G^*Zz(t) - z^*(t)Z^*Gz(t) \\ &\leq z^*(t)Hz(t). \end{aligned}$$

Integrating this inequality from t_0 to t_1 gives

$$\Theta(z(t_1)) - \Theta(z(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt}\Theta(z(t))dt \leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt,$$

which leads to the following definition.

Definition 3. Let $F, G \in \mathbb{C}^{p,q}$, $H = H^* \in \mathbb{C}^{q,q}$. Then we say that a function $\Theta : \mathbb{C}^q \to \mathbb{R} \cup \{\pm\infty\}$ is a *storage function* of (F, G, H) if the following properties hold:

1. Θ is continuous.
2. $\Theta(0) = 0$.
3. $\Theta\left(z(t)\right) \in \mathbb{R}$ for all $t \in \mathbb{R}$ and all $z \in \mathfrak{B}_c(F, G)$.
4. The inequality
$$\Theta\left(z(t_1)\right) - \Theta\left(z(t_0)\right) \leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt \tag{5}$$
holds for all $t_0 \leq t_1$ and all $z \in \mathfrak{B}_c(F, G)$.

We have the following interpretation of Definition 3. A storage function Θ measures the amount of energy that is stored internally in the system. In particular, the expression on the left hand side of (5) measures the gain in internally stored energy which occurred from time point t_0 to t_1. On the other hand, the right hand side of (5) measures the amount of energy that has been supplied to the system. The existence of a storage function thus guarantees that one can measure the internally stored energy in a way such that never more energy is stored than the amount of energy supplied to the system.

We have already seen that a solution $Z \in \mathbb{C}^{p,q}$ of (3) induces a storage function through (4). To see that the existence of a storage function implies dissipativity, let Θ

be any storage function and let $z \in \mathfrak{B}_c(F,G)$ be a trajectory with compact support. Then there exists an $R > 0$ such that $z(t) = 0$ for all $|t| \geq R$ and thus we obtain that

$$\begin{aligned} 0 &\leq \Theta(z(R)) - \Theta(z(-R)) \\ &= \int_{-R}^{R} z^*(t) H z(t) dt \\ &= \int_{-\infty}^{\infty} z^*(t) H z(t) dt, \end{aligned}$$

which shows dissipativity since $z \in \mathfrak{B}_c(F,G)$ was arbitrary.

To recall, we have shown that if $Z \in \mathbb{C}^{p,q}$ is a solution of the linear matrix inequality (3), then $\Theta(\hat{z}) := \hat{z}^* F^* Z \hat{z}$ is a storage function and that the existence of a storage function implies dissipativity. Thus, the solvability of (3) implies dissipativity. The reverse direction is considerably harder to show (and needs additional controllability assumptions). The rest of the paper is devoted to obtain these conditions under which also the reverse direction holds.

To achieve this, we introduce in Section 2 the available storage and the required supply. Based on these definitions, relations to the notion of dissipativity will be drawn, which are vastly inspired by [12]. Note that while in [12] more general systems are considered, we will present the results here for linear systems with quadratic cost term, because this makes the presentation easier. Another difference to [12] is that we do not require storage functions to be positive semi-definite. Related with this we have to introduce the available storage and required supply in a somewhat different way.

In Section 3 we will see that the available storage and the required supply are quadratic functions by employing an idea of [1]. Although in the literature it is sometimes stated that this quadraticity is trivial (e.g., [2, p. 167]) the proof turns out to be quite difficult. Having obtained this quadraticity we introduce two types of linear matrix inequalities. The one type (called symmetrized linear matrix inequality in this paper) has already been introduced in [11], but there the additional assumption has been made, that the system is trim, i.e., that for every $\hat{z} \in \mathbb{C}^q$ there exists a trajectory $z \in \mathfrak{B}(F,G)$ such that $z(0) = \hat{z}$. For the other type (simply called linear matrix inequality in this paper) this assumption is not necessary.

In the third part (Sections 4 and 5) we will use the obtained results to derive statements about deflating subspaces of a special matrix pencil, the spectral factorization of a certain matrix function, and passivity of descriptor systems. The obtained results will be used to derive a result which is a modification of a result in [5]. The inequalities in [5] were the starting point of the work that lead to this paper.

For $f, g \in \mathcal{C}_+^q$ we denote by $\langle f, g \rangle_+$ the inner product on the positive half axis

$$\langle f, g \rangle_+ := \int_0^\infty g^*(t) f(t) dt$$

and

$$\|f\|_+ := \sqrt{\langle f, f \rangle_+} = \sqrt{\int_0^\infty \|f(t)\|_2^2 dt}.$$

Similar, for $f, g \in \mathcal{C}_-^q$ we denote by $\langle f, g\rangle_-$ the inner product on the negative half axis

$$\langle f, g\rangle_- := \int_{-\infty}^0 g^*(t) f(t) dt$$

and

$$\|f\|_- := \sqrt{\langle f, f\rangle_-} = \sqrt{\int_{-\infty}^0 \|f(t)\|_2^2 dt}.$$

2 Available storage and required supply

In this section we will introduce the concepts of available storage and required supply and see how they relate to dissipativity. To give an informal definition, the available storage is the maximum amount of energy that can be extracted from a system which is currently in the state $\hat{z}$ and the required supply is the minimum amount of energy which has to be supplied to the system to put the system into the state $\hat{z}$.

Definition 4. Let $F, G \in \mathbb{C}^{p,q}$ and let $H = H^* \in \mathbb{C}^{q,q}$. Then we call the function $\Theta_+ : \mathbb{C}^q \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\begin{aligned} \Theta_+(\hat{z}) &:= \sup_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=\hat{z}}} -\int_0^\infty z^*(t) H z(t) dt \\ &= -\inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=\hat{z}}} \int_0^\infty z^*(t) H z(t) dt, \end{aligned} \tag{6}$$

the *available storage* of (F, G, H) and the function $\Theta_- : \mathbb{C}^q \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\Theta_-(\hat{z}) := \inf_{\substack{z \in \mathfrak{B}_-(F,G) \\ z(0)=\hat{z}}} \int_{-\infty}^0 z^*(t) H z(t) dt, \tag{7}$$

the *required supply* of (F, G, H).

The available storage and the required supply can be determined via the solution of an optimal control problem as shown in [4].

Remark 5. In the following we will frequently state inequalities in which one or both sides are allowed to be ∞ or $-\infty$. Therefore we introduce the convention that the inequalities $\infty \leq \infty$, $-\infty \leq -\infty$, and $-\infty < \infty$ are considered to be true but not the inequalities $\infty < \infty$, $-\infty < -\infty$, and $\infty \leq -\infty$. Of course, the inequality $-\infty < a < \infty$ is considered to be true for all $a \in \mathbb{R}$. Also, $a \cdot \infty = \infty$ and $a \cdot (-\infty) = (-\infty)$ for all $a \in \mathbb{R} \setminus \{0\}$. The expressions $0 \cdot \infty$ and $0 \cdot (-\infty)$ will not be used in this paper and are considered to be undefined.

Remark 6. If $z_1 \in \mathfrak{B}_+(F,G)$, then the integral

$$\int_0^\infty z_1^*(t) H z_1(t) dt$$

exists. Thus we have $\Theta_+(z_1(0)) > -\infty$, since then in (6) the infimum over a non-empty set is taken.

If $z_2 \in \mathfrak{B}_-(F,G)$, then the integral

$$\int_{-\infty}^0 z_2^*(t) H z_2(t) dt$$

exists. Thus we have $\Theta_-(z_2(0)) < \infty$, since then in (7) the infimum over a non-empty set is taken.

Definition 7. Let $F, G \in \mathbb{C}^{p,q}$ and consider the associated behavior. Then we call

$$R(F,G) := \left\{ \hat{z} \in \mathbb{C}^q \middle| \exists z \in \mathfrak{B}(F,G) \text{ such that } \hat{z} = z(0) \right\}$$

the *reachable set of* (F,G),

$$R_c(F,G) := \left\{ \hat{z} \in \mathbb{C}^q \middle| \exists z \in \mathfrak{B}_c(F,G) \text{ such that } \hat{z} = z(0) \right\}$$

the *compact reachable set of* (F,G), and

$$\begin{aligned} R_+(F,G) &:= \left\{ \hat{z} \in \mathbb{C}^q \middle| \exists z \in \mathfrak{B}_+(F,G) \text{ such that } \hat{z} = z(0) \right\}, \\ R_-(F,G) &:= \left\{ \hat{z} \in \mathbb{C}^q \middle| \exists z \in \mathfrak{B}_-(F,G) \text{ such that } \hat{z} = z(0) \right\}, \end{aligned}$$

the *right and left exponentially decaying reachable sets of* (F,G), respectively.

Clearly $R(F,G)$, $R_c(F,G)$, $R_+(F,G)$, and $R_-(F,G)$ are linear subspaces of $\mathbb{C}^q$ and we have

$$R_c(F,G) \subset R_\pm(F,G) \subset R(F,G).$$

Lemma 8. *Let $F, G \in \mathbb{C}^{p,q}$ and let $z_1 \in \mathfrak{B}_-(F,G)$, $z_2 \in \mathfrak{B}_+(F,G)$ be such that*

$$z_1(0) = z_2(0).$$

Then for all $\epsilon > 0$ there exists a trajectory $\tilde{z} \in \mathfrak{B}_+(F,G) \cap \mathfrak{B}_-(F,G)$ such that

$$\|(\tilde{z} - z_1)\|_- + \|(\tilde{z} - z_2)\|_+ < \epsilon.$$

Proof. Employing the construction from [8, pp. 35-42 (esp. Corollary 2.4.12)] one can easily construct the desired $\tilde{z}$. □

With this we derive the first condition which is equivalent to dissipativity.

Lemma 9. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Then (F,G,H) is dissipative if and only if*

$$\Theta_+(\hat{z}) \leq \Theta_-(\hat{z}) \tag{8}$$

for all $\hat{z} \in R_c(F,G)$.

Proof. First, assume that (F, G, H) is dissipative. Assume to the contrary that there is a $\hat{z} \in R_c(F, G)$ such that $\Theta_+(\hat{z}) > \Theta_-(\hat{z})$. This implies that there exist $z_1 \in \mathfrak{B}_-(F, G)$ and $z_2 \in \mathfrak{B}_+(F, G)$ such that $z_1(0) = \hat{z} = z_2(0)$ and

$$-\int_0^\infty z_2^*(t)Hz_2(t)dt > \int_{-\infty}^0 z_1^*(t)Hz_1(t)dt. \tag{9}$$

Define η through

$$\eta := \int_{-\infty}^0 z_1^*(t)Hz_1(t)dt + \int_0^\infty z_2^*(t)Hz_2(t)dt < 0.$$

Then $-\frac{\eta}{2} > 0$ and thus there exists an $\epsilon > 0$ such that

$$3\epsilon^2\|H\|_+ + 2\epsilon\|H\|_+\|z_2\|_+ < -\frac{\eta}{2}.$$

With this ϵ, define the function $\tilde{z} : \mathbb{R} \to \mathbb{C}^q$ from z_1 and z_2 through Lemma 8. This implies that $\|z_2 - \tilde{z}\|_+ < \epsilon$. Since

$$\begin{aligned} & \langle H(z_2 - \tilde{z}), (z_2 - \tilde{z})\rangle_+ \\ = \; & \langle Hz_2, z_2\rangle_+ - 2\mathrm{Re}\left\{\langle Hz_2, \tilde{z}\rangle_+\right\} + \langle H\tilde{z}, \tilde{z}\rangle_+, \end{aligned}$$

with the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned} & \left|\langle Hz_2, z_2\rangle_+ - \langle H\tilde{z}, \tilde{z}\rangle_+\right| \\ = \; & \left|\langle H(z_2 - \tilde{z}), (z_2 - \tilde{z})\rangle_+ + 2\mathrm{Re}\left\{\langle Hz_2, \tilde{z}\rangle_+\right\}\right. \\ & \left. -2\langle H\tilde{z}, \tilde{z}\rangle_+\right| \\ \leq \; & \left|\langle H(z_2 - \tilde{z}), (z_2 - \tilde{z})\rangle_+\right| + \\ & 2\left|\mathrm{Re}\left\{\langle Hz_2, \tilde{z}\rangle_+ - \langle H\tilde{z}, \tilde{z}\rangle_+\right\}\right| \\ \leq \; & \left|\langle H(z_2 - \tilde{z}), (z_2 - \tilde{z})\rangle_+\right| + 2\left|\langle Hz_2, \tilde{z}\rangle_+ - \langle H\tilde{z}, \tilde{z}\rangle_+\right| \\ \leq \; & \|H\|_+\|z_2 - \tilde{z}\|_+^2 + 2\|H\|_+\|z_2 - \tilde{z}\|_+\|\tilde{z}\|_+ \\ < \; & \epsilon^2\|H\|_+ + 2\epsilon\|H\|_+\|(\tilde{z} - z_2) + z_2\|_+ \\ \leq \; & \epsilon^2\|H\|_+ + 2\epsilon\|H\|_+\left(\|\tilde{z} - z_2\|_+ + \|z_2\|_+\right) \\ < \; & 3\epsilon^2\|H\|_+ + 2\epsilon\|H\|_+\|z_2\|_+ < -\frac{\eta}{2}. \end{aligned}$$

From this we deduce that

$$\langle Hz_2, z_2\rangle_+ - \langle H\tilde{z}, \tilde{z}\rangle_+ > \frac{\eta}{2}.$$

Then from (9) and with the assumption of dissipativity we find that

$$\begin{aligned} \eta &= \langle Hz_1, z_1\rangle_- + \langle Hz_2, z_2\rangle_+ \\ &= \langle H\tilde{z}, \tilde{z}\rangle_- + \langle H\tilde{z}, \tilde{z}\rangle_+ + \langle Hz_2, z_2\rangle_+ - \langle H\tilde{z}, \tilde{z}\rangle_+ \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \tilde{z}^*(t) H \tilde{z}(t) dt + \langle H z_2, z_2 \rangle_+ - \langle H \tilde{z}, \tilde{z} \rangle_+ \\ &\geq \langle H z_2, z_2 \rangle_+ - \langle H \tilde{z}, \tilde{z} \rangle_+ > \frac{\eta}{2}, \end{aligned}$$

which is a contradiction since η is negative.

For the converse assume that condition (8) is fulfilled and let $\tilde{z} \in \mathfrak{B}_c(F,G)$ be arbitrary. Then, using Remark 6, we see that both the available storage $\Theta_+(\tilde{z}(0)) \in \mathbb{R}$ and the required supply $\Theta_-(\tilde{z}(0)) \in \mathbb{R}$ are real numbers. Thus we obtain

$$\begin{aligned} 0 &\leq \Theta_-(\tilde{z}(0)) - \Theta_+(\tilde{z}(0)) \\ &= \inf_{\substack{z \in \mathfrak{B}_-(F,G) \\ z(0)=\tilde{z}(0)}} \int_{-\infty}^{0} z^*(t) H z(t) dt + \\ &\qquad \inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=\tilde{z}(0)}} \int_{0}^{\infty} z^*(t) H z(t) dt \\ &\leq \int_{-\infty}^{0} \tilde{z}^*(t) H \tilde{z}(t) dt + \int_{0}^{\infty} \tilde{z}^*(t) H \tilde{z}(t) dt \\ &= \int_{-\infty}^{\infty} \tilde{z}^*(t) H \tilde{z}(t) dt, \end{aligned}$$

and with this dissipativity follows. □

We recall the following result which is needed in the proof of the subsequent theorem.

Theorem 10. *Let $F, G \in \mathbb{C}^{p,q}$, $H = H^* \in \mathbb{C}^{q,q}$, and let (F, G, H) be dissipative. Then we have*

$$\begin{aligned} \inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(t)=\tilde{z}(t), t \leq 0}} \int_0^\infty z^*(t) H z(t) dt &= \inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=\tilde{z}(0)}} \int_0^\infty z^*(t) H z(t) dt \\ &= \inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ Fz(0)=F\tilde{z}(0)}} \int_0^\infty z^*(t) H z(t) dt. \end{aligned}$$

for all $\tilde{z} \in \mathfrak{B}_+(F,G)$.

Proof. The proof can be found in [4, Theorem 33]. □

The following lemma is a modification of [12, Theorem 1].

Lemma 11. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let $z_1 \in \mathfrak{B}_+(F,G)$, $z_2 \in \mathfrak{B}_-(F,G)$, and let $t_0, t_1 \in \mathbb{R}$ be such that $t_0 \leq t_1$. Then we have*

$$\Theta_+(z_1(t_1)) \leq \int_{t_0}^{t_1} z_1^*(t) H z_1(t) dt + \Theta_+(z_1(t_0)), \tag{10}$$

$$\Theta_-(z_2(t_1)) \leq \int_{t_0}^{t_1} z_2^*(t) H z_2(t) dt + \Theta_-(z_2(t_0)). \tag{11}$$

Proof. Using Theorem 10 we find that

$$\begin{aligned}
& \Theta_+(z_1(t_0)) \\
= & -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(0)=z_1(t_0)}} \int_0^\infty z^*(t)Hz(t)dt \\
= & -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(t_0)=z_1(t_0)}} \int_{t_0}^\infty z^*(t)Hz(t)dt \\
\geq & -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(t)=z_1(t),t\leq t_1}} \int_{t_0}^\infty z^*(t)Hz(t)dt \\
= & -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(t)=z_1(t),t\leq t_1}} \int_{t_0}^{t_1} z_1^*(t)Hz_1(t)dt + \int_{t_1}^\infty z^*(t)Hz(t)dt \\
= & -\int_{t_0}^{t_1} z_1^*(t)Hz_1(t)dt - \inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(t)=z_1(t),t\leq t_1}} \int_{t_1}^\infty z^*(t)Hz(t)dt \\
= & -\int_{t_0}^{t_1} z_1^*(t)Hz_1(t)dt + \Theta_+(z_1(t_1)),
\end{aligned}$$

from which (10) follows. Inequality (11) can be obtained analogously. □

The following theorem is a modification of [12, Theorem 2].

Theorem 12. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Then the following statements hold:*

1. *If there exists a storage function, then (F, G, H) is dissipative.*
2. *Every storage function Θ fulfills*

$$\Theta_+(\hat{z}) \leq \Theta(\hat{z}) \leq \Theta_-(\hat{z}),$$

   for all $\hat{z} \in R_c(F, G)$ and all $t \in \mathbb{R}$.

3. *If (F, G, H) is dissipative, then the available storage and the required supply both are storage functions.*

Proof. First assume that Θ is a storage function and let $\tilde{z} \in \mathfrak{B}_c(F, G)$ be arbitrary. Choose $R \in \mathbb{R}^+$ such that $\tilde{z}(t) = 0$ for all $|t| \geq R$. Then we have

$$\begin{aligned}
0 &= \Theta(0) - \Theta(0) = \Theta(\tilde{z}(R)) - \Theta(\tilde{z}(-R)) \\
&\leq \int_{-R}^{R} \tilde{z}^*(t)H\tilde{z}(t)dt = \int_{-\infty}^{\infty} \tilde{z}^*(t)H\tilde{z}(t)dt,
\end{aligned}$$

which means dissipativity and 1. is shown. To show 2., let $\hat{z} \in R_c(F, G)$ and $\tilde{z} \in \mathfrak{B}_c(F, G)$ be such that $\tilde{z}(0) = \hat{z}$, which is possible, since $\mathfrak{B}_c(F, G)$ is shift invariant.

Then for every $z \in \mathfrak{B}_+(F,G)$ with $z(0) = \tilde{z}(0)$, from the definition of the storage function, we have that

$$\begin{aligned} -\Theta(\tilde{z}(0)) &= \Theta(\lim_{\tilde{t}\to\infty} z(\tilde{t})) - \Theta(z(0)) = \lim_{\tilde{t}\to\infty} \Theta(z(\tilde{t})) - \Theta(z(0)) \\ &\leq \lim_{\tilde{t}\to\infty} \int_0^{\tilde{t}} z^*(s)Hz(s)ds \\ &= \int_0^{\infty} z^*(s)Hz(s)ds. \end{aligned}$$

Since $z \in \mathfrak{B}_+(F,G)$ was allowed to be arbitrary this implies that

$$-\Theta(\tilde{z}(0)) \leq \inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(t)=\tilde{z}(0)}} \int_0^{\infty} z^*(t)Hz(t)dt = -\Theta_+(\tilde{z}(0)).$$

Analogously, we obtain that $\Theta(\tilde{z}(0)) \leq \Theta_-(\tilde{z}(0))$.

For 3. assume that (F,G,H) is dissipative. Then from Remark 6 and Lemma 9 we see that for all trajectories $\tilde{z} \in \mathfrak{B}_c(F,G)$ we have that both $\Theta_-(\tilde{z}(0)) \in \mathbb{R}$ and $\Theta_+(\tilde{z}(0)) \in \mathbb{R}$ are real numbers. Thus the inequalities from Lemma 11 can be transformed to match the inequality from Definition 3. Properties 1. and 2. of Definition 3 follow since the available storage and the required supply are quadratic functions, see Lemma A.6 in the appendix. □

Corollary 13. *Dissipativity is equivalent to the existence of a storage function.*

Proof. This follows from parts 1. and 3. of Theorem 12. □

3 Linear matrix inequalities

In the previous section we have seen that dissipativity is equivalent to the existence of a storage function. To show that dissipativity implies the existence of a storage function, we have proved that the available storage Θ_+ and the required supply Θ_- of a dissipative system constitute storage functions. However, the functions Θ_+ and Θ_- have further interesting and useful properties which are summed up in the following Theorem.

Theorem 14. *Let $F,G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let (F,G,H) be dissipative. Then there exist matrices $Z_+ \in \mathbb{C}^{p,q}$ and $X_+ = X_+^* \in \mathbb{C}^{q,q}$ such that $F^*Z_+ = Z_+^*F$ and*

$$\begin{aligned} \Theta_+(\hat{z}) &= \hat{z}^*F^*Z_+\hat{z} \\ &= \hat{z}^*F^*X_+F\hat{z} \end{aligned}$$

for all $\hat{z} \in R_c(F,G)$ and there exist matrices $Z_- \in \mathbb{C}^{p,q}$ and $X_- = X_-^* \in \mathbb{C}^{q,q}$ such that $F^*Z_- = Z_-^*F$ and

$$\Theta_-(\hat{z}) = \hat{z}^*F^*Z_-\hat{z}$$

$$= \quad \hat{z}^* F^* X_- F \hat{z}$$

for all $\hat{z} \in R_c(F,G)$.

Proof. The proof can be found in Appendix A, see Theorem A.10. □

From Theorem 14 we conclude that dissipativity not only implies the existence of a storage function but also the existence of a quadratic storage function.

Theorem 15. *Let* $F, G \in \mathbb{C}^{p,q}$ *and* $H = H^* \in \mathbb{C}^{q,q}$. *Then the following statements are equivalent:*

1. (F,G,H) *is dissipative.*
2. *There exists a storage function of* (F,G,H).
3. *There exists* $Z \in \mathbb{C}^{p,q}$ *such that* $Z^*F = F^*Z$ *and*

$$0 \leq \hat{z}^* \left[H + Z^*G + G^*Z\right] \hat{z}, \tag{12}$$

   for all $\hat{z} \in R_c(F,G)$.

4. *There exists* $X = X^* \in \mathbb{C}^{p,p}$ *such that*

$$0 \leq \hat{z}^* \left[H + G^*XF + F^*XG\right] \hat{z}, \tag{13}$$

   for all $\hat{z} \in R_c(F,G)$.

Proof. The equivalence of 1. and 2. has already been shown in Corollary 13. To show that 1. implies 3. and 4. first note that the available storage Θ_+ is a storage function, due to Theorem 12. Using Theorem 14 we deduce the existence of a matrix $Z \in \mathbb{C}^{p,q}$ such that $\Theta_+(\hat{z}) = \hat{z}^*F^*Z\hat{z}$ for all $\hat{z} \in R_c(F,G)$. Let $z \in \mathfrak{B}_c(F,G)$ and let $t \in \mathbb{R}$ be arbitrary. Since Θ_+ is a storage function, we see that for all $h > 0$ we have

$$\Theta_+(z(t+h)) - \Theta_+(z(t)) \leq \int_t^{t+h} z^*(s)Hz(s)ds.$$

Dividing by h, using the mean-value theorem, and taking $h \to 0$ we obtain

$$\frac{d}{dt}\Theta_+(z(t)) \leq z^*(t)Hz(t),$$

for all $z \in \mathfrak{B}_c(F,G)$ and $t \in \mathbb{R}$. Looking closer at the term $\frac{d}{dt}\Theta(z(t))$ we find that

$$\begin{aligned} z^*(t)Hz(t) \quad &\geq \quad \frac{d}{dt}\Theta(z(t)) = \frac{d}{dt}\left(z^*(t)F^*Zz(t)\right) \\ &= \quad \dot{z}^*(t)F^*Zz(t) + z^*(t)Z^*F\dot{z}(t) = -z^*(t)\left(G^*Z + Z^*G\right)z(t), \end{aligned}$$

which implies that

$$0 \leq z^*(t)\left[H + Z^*G + G^*Z\right]z(t),$$

and thus (12) is shown. To show (13) the same argument can be applied since

$$\begin{aligned}\frac{d}{dt}\Theta(z(t)) &= \frac{d}{dt}\left(z^*(t)F^*XFz(t)\right)\\ &= \dot{z}^*(t)F^*XFz(t) + z^*(t)F^*XF\dot{z}(t)\\ &= -z^*(t)G^*XFz(t) - z^*(t)F^*XGz(t).\end{aligned}$$

Thus, we have shown points 3. and 4.

To show that 4. implies 3. set $Z := XF$. To show that 3. implies 2. let there exist a $Z \in \mathbb{C}^{p,q}$ such that (12) holds. Define Θ through $\Theta(\hat{z}) := \hat{z}^*Z^*F\hat{z}$ for all $\hat{z} \in R_c(F,G)$. Then we have

$$\begin{aligned}\frac{d}{dt}\Theta(z(t)) &= \frac{d}{dt}\left(z^*(t)Z^*Fz(t)\right)\\ &= \dot{z}^*(t)Z^*Fz(t) + z^*(t)Z^*F\dot{z}(t)\\ &= \dot{z}^*(t)F^*Zz(t) - z^*(t)Z^*Gz(t)\\ &= -z^*(t)\left(G^*Z + Z^*G\right)z(t)\\ &\leq z^*(t)Hz(t),\end{aligned}$$

where we have used the shift invariance (with respect to the time t) of $\mathfrak{B}_c(F,G)$. Integrating this inequality from t_0 to t_1 yields

$$\begin{aligned}\Theta(z(t_1)) - \Theta(z(t_0)) &= \int_{t_0}^{t_1}\frac{d}{dt}\Theta(z(t))dt\\ &\leq \int_{t_0}^{t_1} z^*(t)Hz(t)dt,\end{aligned}$$

which shows that Θ is a storage function. □

In the following we show that under certain conditions the inequalities (12) and (13) are equivalent to linear matrix inequalities which not only hold on the subspace $R_c(F,G)$. To be more precise we introduce the following terms.

Definition 16. Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Then we call the system of equations

$$\begin{aligned}F^*Z &= Z^*F\\ 0 &\leq H + Z^*G + G^*Z\end{aligned} \tag{14}$$

a *linear matrix inequality* (where $Z \in \mathbb{C}^{p,q}$ is the unknown) and the system of equations

$$\begin{aligned}X &= X^*\\ 0 &\leq H + G^*XF + F^*XG\end{aligned} \tag{15}$$

a *symmetrized linear matrix inequality* (where $X \in \mathbb{C}^{p,p}$ is the unknown).

From Theorem 15 it is immediately clear that if one of the linear matrix inequalities (14) or (15) has a solution, then there exists a storage function and thus the system is dissipative. The converse is not always true as one can see from the following example.

Example 17. With $z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T$ consider the system

$$F_{\mathcal{N}}\dot{z}(t) + G_{\mathcal{N}}z(t) := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = 0.$$

Clearly, $\mathfrak{B} = \mathfrak{B}_c = \{0\}$ and thus the system is dissipative with respect to any $H = H^* \in \mathbb{C}^{q,q}$ by definition. Then the following questions arise:

1. Can we find $Z \in \mathbb{C}^{2,2}$ such that $F_{\mathcal{N}}^* Z = Z^* F_{\mathcal{N}}$ and $0 \leq G_{\mathcal{N}}^* Z + Z^* G_{\mathcal{N}} + H$ for any given $H = H^* \in \mathbb{C}^{2,2}$?
2. Can we find $X = X^* \in \mathbb{C}^{2,2}$ such that $0 \leq G_{\mathcal{N}}^* X F_{\mathcal{N}} + F_{\mathcal{N}}^* X G_{\mathcal{N}} + H$ for any given $H = H^* \in \mathbb{C}^{2,2}$?

For 1. and $Z = [z_{ij}]$ we see that the identity $F_{\mathcal{N}}^* Z = Z^* F_{\mathcal{N}}$ is equivalent to

$$\begin{bmatrix} 0 & 0 \\ z_{11} & z_{12} \end{bmatrix} = \begin{bmatrix} 0 & \overline{z_{11}} \\ 0 & \overline{z_{12}} \end{bmatrix}.$$

Thus, Z has to take the form

$$Z = \begin{bmatrix} 0 & r \\ z_{21} & z_{22} \end{bmatrix},$$

where $r \in \mathbb{R}$ and $z_{21}, z_{22} \in \mathbb{C}$ are allowed to be arbitrary. With $H = [h_{ij}]$ we can rewrite the inequality $0 \leq G_{\mathcal{N}}^* Z + Z^* G_{\mathcal{N}} + H$ as

$$0 \leq \begin{bmatrix} 0 & r + \overline{z_{21}} \\ r + z_{21} & z_{22} + \overline{z_{22}} \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix},$$

which will not be possible if $h_{11} < 0$. For 2. and $X = [x_{ij}]$ we see that the identity $X = X^*$ implies that X has to take the form

$$X = \begin{bmatrix} r & x_{12} \\ \overline{x_{12}} & s \end{bmatrix},$$

with $r, s \in \mathbb{R}$ and $x_{12} \in \mathbb{C}$. The inequality $0 \leq G_{\mathcal{N}}^* X F_{\mathcal{N}} + F_{\mathcal{N}}^* X G_{\mathcal{N}} + H$ then becomes

$$\begin{aligned} 0 \quad &\leq \quad \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} + \begin{bmatrix} r & x_{12} \\ \overline{x_{12}} & s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r & x_{12} \\ \overline{x_{12}} & s \end{bmatrix} \\ &= \quad \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} + \begin{bmatrix} 0 & r \\ 0 & \overline{x_{12}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r & x_{12} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} + \begin{bmatrix} 0 & r \\ r & x_{12} + \overline{x_{12}} \end{bmatrix}, \end{aligned}$$

which will not be possible if $h_{11} < 0$.

The problem in Example 17 is that the pencil $\lambda F + G$ has an infinite eigenvalue with index 2. To explain what that means, we introduce the Kronecker canonical form in the following Theorem.

We denote by $\mathbb{C}[\lambda]$ the set of polynomials with coefficients in $\mathbb{C}$ and by $\mathbb{C}[\lambda]^{p,q}$ the p-by-q matrices with polynomial entries.

Theorem 18. *Let $F, G \in \mathbb{C}^{p,q}$. Then there exist $\epsilon, \rho, \sigma, \eta, p, r, s, q \in \mathbb{N}_0$ and nonsingular matrices $P \in \mathbb{C}^{p,p}$ and $Q \in \mathbb{C}^{q,q}$ such that*

$$P(\lambda F + G)Q \quad = \quad \operatorname{diag}\left(\mathcal{L}, \mathcal{J}, \mathcal{N}, \mathcal{M}\right), \tag{16}$$

where $\mathcal{L} \in \mathbb{C}[\lambda]^{\epsilon,\epsilon+p}$, $\mathcal{J} \in \mathbb{C}[\lambda]^{\rho,\rho}$, $\mathcal{N} \in \mathbb{C}[\lambda]^{\sigma,\sigma}$, and $\mathcal{M} \in \mathbb{C}[\lambda]^{\eta+q,\eta}$ are first-order matrix polynomials which can be further partitioned as

$$\begin{aligned} \mathcal{L} &=: \operatorname{diag}\left(\mathcal{L}_{\epsilon_1}, \ldots, \mathcal{L}_{\epsilon_p}\right) & \mathcal{J} &=: \operatorname{diag}\left(\mathcal{J}_{\rho_1}, \ldots, \mathcal{J}_{\rho_r}\right) \\ \mathcal{N} &=: \operatorname{diag}\left(\mathcal{N}_{\sigma_1}, \ldots, \mathcal{N}_{\sigma_s}\right) & \mathcal{M} &=: \operatorname{diag}\left(\mathcal{M}_{\eta_1}, \ldots, \mathcal{M}_{\eta_q}\right), \end{aligned}$$

where $\epsilon = \epsilon_1 + \ldots + \epsilon_p$, $\rho = \rho_1 + \ldots + \rho_r$, $\sigma = \sigma_1 + \ldots + \sigma_s$, and $\eta = \eta_1 + \ldots + \eta_q$ and the blocks $\mathcal{L}_{\epsilon_j}$, $\mathcal{J}_{\rho_j}$, $\mathcal{N}_{\sigma_j}$, and $\mathcal{M}_{\eta_j}$ have the following forms:

1. *Every entry $\mathcal{L}_{\epsilon_j}$ has the size $\epsilon_j \times (\epsilon_j + 1)$, $\epsilon_j \in \mathbb{N}_0$ and the form*

$$\mathcal{L}_{\epsilon_j}(\lambda) := \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}. \tag{17}$$

2. *Every entry $\mathcal{J}_{\rho_j}$ has the size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$ and the form*

$$\mathcal{J}_{\rho_j}(\lambda) := \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \tag{18}$$

   where $\lambda_j \in \mathbb{C}$.

3. *Every entry $\mathcal{N}_{\sigma_j}$ has the size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$ and the form*

$$\mathcal{N}_{\sigma_j}(\lambda) := \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{19}$$

4. *Every entry $\mathcal{M}_{\eta_j}$ has the size $(\eta_j + 1) \times \eta_j$, $\eta_j \in \mathbb{N}_0$ and the form*

$$\mathcal{M}_{\eta_j}(\lambda) := \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}. \tag{20}$$

Proof. A proof can be found in [6, p. 37]. □

With the Kronecker canonical form we may introduce the notion of eigenvalues of a matrix pencil in the following way.

Definition 19. Let $F, G \in \mathbb{C}^{p,q}$ and consider the Kronecker canonical form (16) of $\lambda F + G$. Then $\lambda_0 \in \mathbb{C}$ is called a *finite eigenvalue* of $\lambda F + G$ if there exists a block of the form (18) with $\lambda_j = \lambda_0$. We say that the system (F, G) is *controllable* if $\lambda F + G$ has no finite eigenvalues. We say that $\lambda F + G$ has an *infinite eigenvalue* if there exists a block of the form (19). We say that the infinite eigenvalues of the system (F, G) have *index* 1 if all blocks of the form (19) have the size $\sigma_j = 1$.

Lemma 20. *Let $F, G \in \mathbb{C}^{p,q}$ and let the Kronecker canonical form of $\lambda F + G$ be given by (16). For $f \in \mathcal{C}^1_\infty$ and $\epsilon \in \mathbb{N}_0$ let $\Delta_\epsilon f(t)$ denote*

$$\Delta_\epsilon f(t) := \begin{bmatrix} f(t) \\ \frac{d}{dt} f(t) \\ \vdots \\ \left(\frac{d}{dt}\right)^\epsilon f(t) \end{bmatrix} \in \mathcal{C}^{\epsilon+1}_\infty.$$

Then the behavior of (F, G) is given by

$$\mathfrak{B}(F,G) := \left\{ Q^{-1} \begin{bmatrix} \Delta_{\epsilon_1} z_1(-t) \\ \vdots \\ \Delta_{\epsilon_p} z_p(-t) \\ e^{-\mathcal{J}(0)t}\hat{x} \\ 0_{\sigma+\eta} \end{bmatrix} \Big| z_1, \dots, z_p \in \mathcal{C}^\infty, \hat{x} \in \mathbb{C}^\rho \right\},$$

and the compact behavior of (F, G) is given by

$$\mathfrak{B}_c(F,G) = \left\{ Q^{-1} \begin{bmatrix} \Delta_{\epsilon_1} z_1(-t) \\ \vdots \\ \Delta_{\epsilon_p} z_p(-t) \\ 0_{\rho+\sigma+\eta} \end{bmatrix} \Big| z_1, \dots, z_p \in \mathcal{C}^\infty_c \right\},$$

where for $n \in \mathbb{N}$ the vector $0_n \in \mathbb{C}^n$ is the vector consisting of only zeros. Moreover, the reachable set of (F, G) is given by

$$R(F,G) = \left\{ Q^{-1} \begin{bmatrix} \hat{z} \\ 0_{\sigma+\eta} \end{bmatrix} \Big| \hat{z} \in \mathbb{C}^{\epsilon+p+\rho} \right\},$$

and the compact reachable set of (F, G) is given by

$$R_c(F,G) = \left\{ Q^{-1} \begin{bmatrix} \hat{z} \\ 0_{\rho+\sigma+\eta} \end{bmatrix} \Big| \hat{z} \in \mathbb{C}^{\epsilon+p} \right\}.$$

Proof. To show the representation for $\mathfrak{B}(F,G)$ and $\mathfrak{B}_c(F,G)$, it is sufficient to examine the (compact) behavior of each of the blocks in the Kronecker canonical from. We start with the blocks of type (17). Thus, for $j \in \{1,\dots,p\}$ let $x \in \mathcal{C}^\infty_{(c)}(\mathbb{R},\mathbb{C}^{\epsilon_j})$ be a solution of the system

$$0 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{\epsilon_j+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{\epsilon_j+1} \end{bmatrix}.$$

This is equivalent to the system of scalar equations $\dot{x}_{i+1} + x_i = 0$ for $i = 1,\dots,\epsilon_j$. Hence $x_i = (-1)^{i-1}x_1^{(i-1)}$ for $i = 1,\dots,\epsilon_j+1$ and thus one can write x in the form $\Delta_{\epsilon_j}(z_j(-t))$ by setting $z_j(t) := x_1(-t) \in \mathcal{C}^\infty_{(c)}$. For blocks of the type (18) we obtain that $\mathfrak{B}_c(\mathcal{J}_{\rho_j}) = \{0\}$ and

$$\mathfrak{B}(\mathcal{J}_{\rho_j}) = \left\{ e^{-\mathcal{J}_{\rho_j}(0)t}\hat{x} \middle| \hat{x} \in \mathbb{C}^{\rho_j} \right\}$$

from the standard theory of ordinary differential equations [6, p. 115]. That $\mathfrak{B}(\mathcal{N}_{\sigma_j}) = \mathfrak{B}(\mathcal{M}_{\sigma_j}) = \mathfrak{B}_c(\mathcal{N}_{\eta_j}) = \mathfrak{B}_c(\mathcal{M}_{\eta_j}) = \{0\}$ is trivial. For the statements about the reachable set we note that $e^0 = I$ and that for every $\hat{z} \in \mathbb{C}^{\epsilon_j+1}$ there exists a trajectory with compact support $z \in \mathcal{C}^\infty_c$ such that $\hat{z} = \Delta_{\epsilon_j}(z(0))$. □

By splitting up the blocks of type (18) into eigenvalues with negative real part, eigenvalues with positive real part, and eigenvalues with vanishing real part one can also determine $R_+(F,G)$ and $R_-(F,G)$ similar to Lemma 20.

Consider a matrix pencil in Kronecker canonical form where there is (for reasons of simplicity) only one block of each type (17)-(19), i.e., let

$$\lambda F + G = \lambda \begin{bmatrix} F_{\mathcal{L}} & & & \\ & F_{\mathcal{J}} & & \\ & & F_{\mathcal{N}} & \\ & & & F_{\mathcal{M}} \end{bmatrix} + \begin{bmatrix} G_{\mathcal{L}} & & & \\ & G_{\mathcal{J}} & & \\ & & G_{\mathcal{N}} & \\ & & & G_{\mathcal{M}} \end{bmatrix} \in \mathbb{C}^{p,q}. \tag{21}$$

Also assume that $\lambda F_{\mathcal{N}} + G_{\mathcal{N}}$ has index 1, i.e., let $F_{\mathcal{N}} = 0$ and $G_{\mathcal{N}} = I$. This makes sense, since we already saw in Example 17 that for higher index systems the solvability of the linear matrix inequality can fail despite dissipativity. Partitioning a matrix $Z \in \mathbb{C}^{p,q}$ according to the block structure in $\lambda F + G$ as

$$Z =: \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{bmatrix}, \tag{22}$$

shows that the equation $F^*Z = Z^*F$ can be written as

$$\begin{bmatrix} F_{\mathcal{L}}^*Z_{11} & F_{\mathcal{L}}^*Z_{12} & F_{\mathcal{L}}^*Z_{13} & F_{\mathcal{L}}^*Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ 0 & 0 & 0 & 0 \\ F_{\mathcal{M}}^*Z_{41} & F_{\mathcal{M}}^*Z_{42} & F_{\mathcal{M}}^*Z_{43} & F_{\mathcal{M}}^*Z_{44} \end{bmatrix} = \begin{bmatrix} Z_{11}^*F_{\mathcal{L}} & Z_{21}^* & 0 & Z_{41}^*F_{\mathcal{M}} \\ Z_{12}^*F_{\mathcal{L}} & Z_{22}^* & 0 & Z_{42}^*F_{\mathcal{M}} \\ Z_{13}^*F_{\mathcal{L}} & Z_{23}^* & 0 & Z_{43}^*F_{\mathcal{M}} \\ Z_{14}^*F_{\mathcal{L}} & Z_{24}^* & 0 & Z_{44}^*F_{\mathcal{M}} \end{bmatrix} \tag{23}$$

and we have

$$\begin{aligned} & G^*Z + Z^*G \qquad (24)\\ = & \begin{bmatrix} Z_{11}^*G_{\mathcal{L}} + G_{\mathcal{L}}^*Z_{11} & Z_{21}^*G_{\mathcal{J}} + G_{\mathcal{L}}^*Z_{12} & Z_{31}^* + G_{\mathcal{L}}^*Z_{13} & G_{\mathcal{L}}^*Z_{14} + Z_{41}^*G_{\mathcal{M}} \\ Z_{12}^*G_{\mathcal{L}} + G_{\mathcal{J}}^*Z_{21} & Z_{22}^*G_{\mathcal{J}} + G_{\mathcal{J}}^*Z_{22} & Z_{32}^* + G_{\mathcal{J}}^*Z_{23} & G_{\mathcal{J}}^*Z_{24} + Z_{42}^*G_{\mathcal{M}} \\ Z_{13}^*G_{\mathcal{L}} + Z_{31} & Z_{23}^*G_{\mathcal{J}} + Z_{32} & Z_{33}^* + Z_{33} & Z_{34} + Z_{43}^*G_{\mathcal{M}} \\ Z_{14}^*G_{\mathcal{L}} + G_{\mathcal{M}}^*Z_{41} & Z_{24}^*G_{\mathcal{J}} + G_{\mathcal{M}}^*Z_{42} & Z_{34}^* + G_{\mathcal{M}}^*Z_{43} & G_{\mathcal{M}}^*Z_{44} + Z_{44}^*G_{\mathcal{M}} \end{bmatrix}. \end{aligned}$$

Assume that for some $H = H^* \in \mathbb{C}$ and the F, G from (21) we have that (F, G, H) is dissipative. Partition $H = \left[H_{ij}\right]_{i,j=1,\dots,4}$ analogously to (24). Since the definition of dissipativity only makes a statement about all trajectories $z \in \mathfrak{B}_c(F, G)$ with compact support, we conclude from Lemma 20 and Theorem 15 that dissipativity of (F, G, H) implies the existence of a Z as in (22) such that (23) holds and that

$$0 \leq Z_{11}^*G_{\mathcal{L}} + G_{\mathcal{L}}^*Z_{11} + H_{11}.$$

From this the question arises if under the assumption of dissipativity we can construct Z_{ij} as in (24) such that $0 \leq G^*Z + Z^*G + H$ while (23) still holds. The same question can be asked for the symmetrized linear matrix inequality.

In the following we will see that blocks of the type (20) in the Kronecker canonical form cause no trouble. To be more specific we state the following Lemmata.

Lemma 21. *Consider matrices $F, G \in \mathbb{C}^{\eta+1,\eta}$ of the form (20), i.e., let*

$$F = \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}$$

and let a Hermitian matrix $H = H^* \in \mathbb{C}^{\eta,\eta}$ be given. Then there exists a Hermitian matrix $X = X^* \in \mathbb{C}^{\eta+1,\eta+1}$ such that

$$0 = F^*XG + G^*XF + H.$$

Further, by setting $Z := XF \in \mathbb{C}^{\eta+1,\eta}$ we obtain that

$$\begin{aligned} F^*Z &= Z^*F, \\ 0 = G^*Z &+ Z^*G + H. \end{aligned}$$

Proof. Let Hermitian matrix X be given in the form

$$X = \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta+1,1} & \dots & x_{\eta+1,\eta+1} \end{bmatrix} = \left[x_{i,j}\right]_{i,j=1,\dots,\eta+1} = \left[\overline{x_{j,i}}\right]_{i,j=1,\dots,\eta+1}$$

and the Hermitian matrix H in the form

$$H = \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta} \\ \vdots & & \vdots \\ h_{\eta,1} & \dots & h_{\eta,\eta} \end{bmatrix} = \left[h_{i,j}\right]_{i,j=1,\dots,\eta} = \left[\overline{h_{j,i}}\right]_{i,j=1,\dots,\eta}.$$

Then we see that we are looking for an X such that

$$\begin{aligned}
0 &= F^*XG + G^*XF + H \\
&= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta+1,1} & \dots & x_{\eta+1,\eta+1} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} + (F^*XG)^* + H \\
&= \begin{bmatrix} x_{1,2} & \dots & x_{1,\eta+1} \\ \vdots & & \vdots \\ x_{\eta,2} & \dots & x_{\eta,\eta+1} \end{bmatrix} + \begin{bmatrix} \overline{x_{1,2}} & \dots & \overline{x_{\eta,2}} \\ \vdots & & \vdots \\ \overline{x_{1,\eta+1}} & \dots & \overline{x_{\eta,\eta+1}} \end{bmatrix} + \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta} \\ \vdots & & \vdots \\ h_{\eta,1} & \dots & h_{\eta,\eta} \end{bmatrix} \\
&= \left[x_{i,j+1}\right]_{i,j=1,\dots,\eta} + \left[\overline{x_{j,i+1}}\right]_{i,j=1,\dots,\eta} + \left[h_{i,j}\right]_{i,j=1,\dots,\eta} \\
&= \left[x_{i,j+1} + \overline{x_{j,i+1}} + h_{i,j}\right]_{i,j=1,\dots,\eta} \\
&= \left[x_{i,j+1} + x_{i+1,j} + h_{i,j}\right]_{i,j=1,\dots,\eta}. \qquad (25)
\end{aligned}$$

We construct such an X in the following recursive way. First, choose $x_{i,i} = 0$ for $i = 1, \dots, \eta + 1$ and choose $x_{i,i+1} := x_{i+1,i} := \frac{h_{i,i}}{2} \in \mathbb{R}$ for all $i = 1, \dots, \eta$. With this choice all $x_{i,j}$ with $|i - j| \leq 1$ are fixed and all equations in (25) with $|i - j| \leq 0$ are fulfilled.

As induction hypothesis, assume that for some $k \in \{1, \dots, \eta - 1\}$ we have that all $x_{i,j}$ with $|i-j| \leq k$ are fixed and all equations in (25) with $|i-j| \leq k-1$ are fulfilled.

For the inductive step, note that all equations in (25) with $|i - j| = k$ are given by

$$0 = x_{j+k,j+1} + x_{j+k+1,j} + h_{j+k,j}$$

for $j = 1, \dots, \eta - k$ and their complex conjugate equations, which are not really additional equations. Since $|(j + k) - (j + 1)| = k - 1 \leq k$, we know that all $x_{j+k,j+1}$ are already fixed but not the $x_{j+k+1,j}$, since $|(j + k + 1) - j| = k + 1 > k$. Thus we define

$$\overline{x_{j,j+k+1}} := x_{j+k+1,j} := -x_{j+k,j+1} - h_{j+k,j},$$

for $j = 1, \dots, \eta - k$ and thus have fixed all $x_{i,j}$ with $|i - j| \leq k + 1$ while at the same time all equations in (25) with $|i - j| \leq k$ are fulfilled. Thus the inductive argument is finished and the claim is proved. □

Lemma 22. *Consider the pencil*

$$\lambda F + G = \lambda \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

where the sub-pencils $\lambda F_1 + G_1 \in \mathbb{C}^{\eta_1+1,\eta_1}$ *and* $\lambda F_2 + G_2 \in \mathbb{C}^{\eta_2+1,\eta_2}$ *are of the form (20). Let an arbitrary* $H = H^* \in \mathbb{C}^{\eta_1+\eta_2,\eta_1+\eta_2}$ *be given. Then there exists a Hermitian matrix* $X = X^* \in \mathbb{C}^{\eta_1+\eta_2+2,\eta_1+\eta_2+2}$ *such that*

$$0 = F^*XG + G^*XF + H.$$

Further, by setting $Z := XF \in \mathbb{C}^{\eta_1+\eta_2+2,\eta_1+\eta_2}$ *we obtain that*

$$\begin{aligned} F^*Z &= Z^*F, \\ 0 = G^*Z &+ Z^*G + H. \end{aligned}$$

Proof. Partition the matrix X according to the partition of F and G as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{matrix} \eta_1+1 \\ \eta_2+1 \end{matrix}$$
$$\begin{matrix} \eta_1+1 & \eta_2+1 \end{matrix}$$

and observe that from $X = X^*$ we obtain that $X_{12} = X_{21}^*$, $X_{11} = X_{11}^*$, and $X_{22} = X_{22}^*$. Then we see that we are looking for an X such that

$$\begin{aligned} 0 &= F^*XG + G^*XF + H \\ &= \begin{bmatrix} F_1^* & 0 \\ 0 & F_2^* \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} + (F^*XG)^* + H \\ &= \begin{bmatrix} F_1^*X_{11}G_1 & F_1^*X_{12}G_2 \\ F_2^*X_{12}^*G_1 & F_2^*X_{22}G_2 \end{bmatrix} + \begin{bmatrix} G_1^*X_{11}F_1 & G_1^*X_{12}F_2 \\ G_2^*X_{12}^*F_1 & G_2^*X_{22}F_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \\ &= \begin{bmatrix} F_1^*X_{11}G_1 + G_1^*X_{11}F_1 + H_{11} & F_1^*X_{12}G_2 + G_1^*X_{12}F_2 + H_{12} \\ F_2^*X_{12}^*G_1 + G_2^*X_{12}^*F_1 + H_{12}^* & F_2^*X_{22}G_2 + G_2^*X_{22}F_2 + H_{22} \end{bmatrix}. \end{aligned} \tag{26}$$

Using Lemma 21 we immediately obtain X_{11} and X_{22} such that $F_2^*X_{22}G_2 + G_2^*X_{22}F_2 + H_{22} = 0$ and $F_1^*X_{11}G_1 + G_1^*X_{11}F_1 + H_{11} = 0$. Thus, we only need to find an $X_{12} \in \mathbb{C}^{\eta_1+1,\eta_2+1}$ such that $F_1^*X_{12}G_2 + G_1^*X_{12}F_2 + H_{12} = 0$, since the equation in the (2,1)-block of (26) is the conjugate transpose and thus not really an additional equation. For the matrix X_{12} we introduce the notation

$$X_{12} = \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} = \begin{bmatrix} x_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1+1 \\ j=1,\dots,\eta_2+1}} \in \mathbb{C}^{\eta_1+1,\eta_2+1},$$

and for the matrix H_{12} analogously

$$H_{12} = \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta_2} \\ \vdots & & \vdots \\ h_{\eta_1,1} & \dots & h_{\eta_1,\eta_2} \end{bmatrix} = \begin{bmatrix} h_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \in \mathbb{C}^{\eta_1,\eta_2}.$$

Then we see that we are looking for an X_{12} such that

$$0 = F_1^*X_{12}G_2 + G_1^*X_{12}F_2 + H_{12}$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2+1} \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} + H_{12} \\
&= \begin{bmatrix} x_{1,2} & \dots & x_{1,\eta_2+1} \\ \vdots & & \vdots \\ x_{\eta_1,2} & \dots & x_{\eta_1,\eta_2+1} \end{bmatrix} + \begin{bmatrix} x_{2,1} & \dots & x_{2,\eta_2} \\ \vdots & & \vdots \\ x_{\eta_1+1,1} & \dots & x_{\eta_1+1,\eta_2} \end{bmatrix} + \begin{bmatrix} h_{1,1} & \dots & h_{1,\eta_2} \\ \vdots & & \vdots \\ h_{\eta_1,1} & \dots & h_{\eta_1,\eta_2} \end{bmatrix} \\
&= \begin{bmatrix} x_{i,j+1} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} + \begin{bmatrix} x_{i+1,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} + \begin{bmatrix} h_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \\
&= \begin{bmatrix} x_{i,j+1} + x_{i+1,j} + h_{i,j} \end{bmatrix}_{\substack{i=1,\dots,\eta_1 \\ j=1,\dots,\eta_2}} \qquad (27)
\end{aligned}
$$

We construct such an X_{12} in the following recursive way. First, choose $x_{i,1} = 0$ for $i = 1, \dots, \eta_1 + 1$, choose $x_{i,2} := h_{i,1}$ for $i = 1, \dots, \eta_1$, and choose $x_{\eta_1+1,2}$ arbitrary. Then all $x_{i,j}$ with $j \leq 2$ are fixed and all equations in (27) with $j \leq 1$ are fulfilled.

As induction hypothesis, assume that for some $k \in \{2, \dots, \eta_2\}$ we have that all $x_{i,j}$ with $j \leq k$ are fixed and all equations in (27) with $j \leq k-1$ are fulfilled.

For the inductive step, note that all equations in (27) with $j = k$ are given by

$$x_{i,k+1} + x_{i+1,k} + h_{i,k} = 0$$

for $i = 1, \dots, \eta_1$. Because of the induction hypothesis all $x_{i+1,k}$ are already fixed but not the $x_{i,k+1}$. Thus we define

$$x_{i,k+1} := -x_{i+1,k} - h_{i,k},$$

for $i = 1, \dots, \eta_1$ and choose $x_{\eta_1+1,k+1}$ arbitrary. Then all $x_{i,j}$ with $j \leq k+1$ are fixed and all equations in (27) with $j \leq k$ are fulfilled. Thus the inductive argument is finished and the claim is proved. □

Lemma 23. *Let $\lambda F_{\mathcal{L}} + G_{\mathcal{L}} \in \mathbb{C}^{\epsilon,\epsilon+1}$ be of the form (17) and $\lambda F_{\mathcal{M}} + G_{\mathcal{M}} \in \mathbb{C}^{\eta+1,\eta}$ be of the form (20). Let $H_{14} \in \mathbb{C}^{\epsilon+1,\eta}$ be arbitrary. Then there exist $Z_{14} \in \mathbb{C}^{\epsilon,\eta}$ and $Z_{41} \in \mathbb{C}^{\eta+1,\epsilon+1}$ such that*

$$
\begin{aligned}
F_{\mathcal{L}}^* Z_{14} &= Z_{41}^* F_{\mathcal{M}} \\
0 = G_{\mathcal{L}}^* Z_{14} &+ Z_{41}^* G_{\mathcal{M}} + H_{14}.
\end{aligned}
$$

Proof. Let the matrices Z_{14} and Z_{41} be given in the form

$$Z_{14} = \begin{bmatrix} z_{1,1} & \dots & z_{1,\eta} \\ \vdots & & \vdots \\ z_{\epsilon,1} & \dots & z_{\epsilon,\eta} \end{bmatrix},$$

$$Z_{41} \;=\; \begin{bmatrix} \tilde{z}_{1,1} & \cdots & \tilde{z}_{1,\epsilon+1} \\ \vdots & & \vdots \\ \tilde{z}_{\eta+1,1} & \cdots & \tilde{z}_{\eta+1,\epsilon+1} \end{bmatrix}.$$

Then equation $F_{\mathcal{L}}^* Z_{14} = Z_{41}^* F_{\mathcal{M}}$ can also be written as

$$\begin{aligned} \begin{bmatrix} 0 \\ Z_{14} \end{bmatrix} \;&=\; \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} Z_{14} = \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \cdots & \overline{\tilde{z}_{1,\eta+1}} \\ \vdots & & \vdots \\ \overline{\tilde{z}_{\epsilon+1,1}} & \cdots & \overline{\tilde{z}_{\epsilon+1,\eta+1}} \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \\ &=\; \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \cdots & \overline{\tilde{z}_{1,\eta}} \\ \vdots & & \vdots \\ \overline{\tilde{z}_{\epsilon+1,1}} & \cdots & \overline{\tilde{z}_{\epsilon+1,\eta}} \end{bmatrix} \end{aligned}$$

or as

$$\begin{bmatrix} 0 & Z_{14}^* \end{bmatrix} = \begin{bmatrix} \tilde{z}_{1,1} & \cdots & \tilde{z}_{1,\epsilon+1} \\ \vdots & & \vdots \\ \tilde{z}_{\eta,1} & \cdots & \tilde{z}_{\eta,\epsilon+1} \end{bmatrix},$$

which shows that Z_{41} takes the form

$$Z_{41} = \begin{bmatrix} \tilde{z}_{1,1} & & & \\ \vdots & & Z_{14}^* & \\ \tilde{z}_{\eta,1} & & & \\ \tilde{z}_{\eta+1,1} & \tilde{z}_{\eta+1,2} & \cdots & \tilde{z}_{\eta+1,\epsilon+1} \end{bmatrix}.$$

With this notation at hand we can verify that

$$\begin{aligned} & G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}} \\ =\; & \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} Z_{14} + \begin{bmatrix} \overline{\tilde{z}_{1,1}} & \cdots & \overline{\tilde{z}_{\eta,1}} & \overline{\tilde{z}_{\eta+1,1}} \\ & & & \overline{\tilde{z}_{\eta+1,2}} \\ & Z_{14} & & \vdots \\ & & & \overline{\tilde{z}_{\eta+1,\epsilon+1}} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \\ =\; & \begin{bmatrix} z_{1,1} & \cdots & z_{1,\eta} \\ \vdots & & \vdots \\ z_{\epsilon,1} & \cdots & z_{\epsilon,\eta} \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} \overline{\tilde{z}_{2,1}} & \cdots & \overline{\tilde{z}_{\eta,1}} & \overline{\tilde{z}_{\eta+1,1}} \\ z_{1,2} & \cdots & z_{1,\eta} & \overline{\tilde{z}_{\eta+1,2}} \\ \vdots & & \vdots & \vdots \\ z_{\epsilon,2} & \cdots & z_{\epsilon,\eta} & \overline{\tilde{z}_{\eta+1,\epsilon+1}} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} z_{1,1} + \overline{\tilde{z}_{2,1}} & z_{1,2} + \overline{\tilde{z}_{3,1}} & \cdots & z_{1,\eta-1} + \overline{\tilde{z}_{\eta,1}} & z_{1,\eta} + \overline{\tilde{z}_{\eta+1,1}} \\ z_{2,1} + z_{1,2} & z_{2,2} + z_{1,3} & \cdots & z_{2,\eta-1} + z_{1,\eta} & z_{2,\eta} + \overline{\tilde{z}_{\eta+1,2}} \\ \vdots & \vdots & & \vdots & \vdots \\ z_{i,1} + z_{i-1,2} & z_{i,2} + z_{i-1,3} & \cdots & z_{i,\eta-1} + z_{i-1,\eta} & z_{i,\eta} + \overline{\tilde{z}_{\eta+1,i}} \\ \vdots & \vdots & & \vdots & \vdots \\ z_{\epsilon,1} + z_{\epsilon-1,2} & z_{\epsilon,2} + z_{\epsilon-1,3} & \cdots & z_{\epsilon,\eta-1} + z_{\epsilon-1,\eta} & z_{\epsilon,\eta} + \overline{\tilde{z}_{\eta+1,\epsilon}} \\ z_{\epsilon,2} & z_{\epsilon,2} & \cdots & z_{\epsilon,\eta} & \overline{\tilde{z}_{\eta+1,\epsilon+1}}. \end{bmatrix}$$

Finally using $H = [h_{i,j}]$ we construct the $z_{i,j}$ and $\tilde{z}_{i,j}$ in the following way. First, fix all $z_{i,1}$ for $i = 1, \ldots, \epsilon$ to any arbitrary value. This fixes the first column of Z_{14}. Next, for $i = 1, \ldots, \epsilon - 1$ choose all $z_{i,2}$ such that $z_{i+1,1} + z_{i,2} + h_{i+1,1} = 0$ and $z_{\epsilon,2} = -h_{\epsilon+1,1}$. This also fixes the second column of Z_{14}. Iteratively continuing this way, we can fix the complete matrix Z_{14} such that the lower-left ϵ-by-η block of equations in $G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}} + H = 0$ is fulfilled. Since the $\overline{\tilde{z}_{i,1}}$ for $i = 2, \ldots, \eta + 1$ and the $\overline{\tilde{z}_{\eta+1,j}}$ for $j = 2, \ldots, \epsilon + 1$ are still free we can choose them in a way that all equations in $G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}} + H = 0$ are fulfilled. □

Note that under the assumptions of Lemma 23 it is in general not always possible to find $X_{14} \in \mathbb{C}^{\epsilon,\eta+1}$, $X_{41} \in \mathbb{C}^{\eta+1,\epsilon}$ such that $F_{\mathcal{L}}^* X_{14} F_{\mathcal{M}} = F_{\mathcal{L}}^* X_{41}^* F_{\mathcal{M}}$ and $G_{\mathcal{L}}^* X_{14} F_{\mathcal{M}} + F_{\mathcal{L}}^* X_{41}^* G_{\mathcal{M}} + H_{14} = 0$ as one can see from the following example.

Example 24. In Lemma 23 choose $\epsilon = 1$ and $\eta = 2$. Then we consider the pencils

$$\lambda F_{\mathcal{L}} + G_{\mathcal{L}} = \lambda \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$\lambda F_{\mathcal{M}} + G_{\mathcal{M}} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices H_{14}, X_{14}, and X_{41}^* take the form

$$H_{14} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \; X_{14} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}, \text{ and } X_{41} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

and thus we are looking to solve the equation

$$\begin{aligned} F_{\mathcal{L}}^* X_{14} F_{\mathcal{M}} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x_1 & x_2 \end{bmatrix} \\ = F_{\mathcal{L}}^* X_{41}^* F_{\mathcal{M}} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \overline{y_1} & \overline{y_2} & \overline{y_3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \overline{y_1} & \overline{y_2} \end{bmatrix} \end{aligned}$$

which implies

$$X_{41}^* = \begin{bmatrix} x_1 & x_2 & \overline{y_3} \end{bmatrix}$$

together with

$$\begin{aligned} 0 &= G_{\mathcal{L}}^* X_{14} F_{\mathcal{M}} + F_{\mathcal{L}}^* X_{41}^* G_{\mathcal{M}} + H_{14} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \overline{y_3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + H_{14} \\ &= \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_2 & \overline{y_3} \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}, \end{aligned}$$

which is never possible, once $h_{12} \neq h_{21}$.

The previous example suggests that the linear matrix inequality (14) is preferable to the symmetrized linear matrix inequality (15).

Lemma 25. *Let $\lambda F_{\mathcal{J}} + G_{\mathcal{J}} \in \mathbb{C}^{\rho,\rho}$ be of the form (18) with $\lambda_j =: \mu$ and let $\lambda F_{\mathcal{M}} + G_{\mathcal{M}} \in \mathbb{C}^{\eta+1,\eta}$ be of the form (20). Let an arbitrary $H_{24} \in \mathbb{C}^{\rho,\eta}$ be given. Then there exist matrices $Z_{24} \in \mathbb{C}^{\rho,\eta}$ and $Z_{42} \in \mathbb{C}^{\eta+1,\rho}$ such that*

$$\begin{aligned} F_{\mathcal{J}}^* Z_{24} &= Z_{42}^* F_{\mathcal{M}} \\ 0 = G_{\mathcal{J}}^* Z_{24} &+ Z_{42}^* G_{\mathcal{M}} + H_{24}. \end{aligned}$$

Proof. Let the matrices Z_{24} and Z_{42} be given in the form

$$\begin{aligned} Z_{24} &= \begin{bmatrix} z_{1,1} & \cdots & z_{1,\eta} \\ \vdots & & \vdots \\ z_{\rho,1} & \cdots & z_{\rho,\eta} \end{bmatrix}, \\ Z_{42} &= \begin{bmatrix} \tilde{z}_{1,1} & \cdots & \tilde{z}_{1,\rho} \\ \vdots & & \vdots \\ \tilde{z}_{\eta+1,1} & \cdots & \tilde{z}_{\eta+1,\rho} \end{bmatrix}. \end{aligned}$$

Since $F_{\mathcal{J}} = I$ we see that

$$Z_{24} = F_{\mathcal{J}}^* Z_{24} = Z_{42}^* F_{\mathcal{M}} = Z_{42}^* \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix},$$

which means that we have to find Z_{24} and Z_{42} with

$$Z_{42} = \begin{bmatrix} & Z_{24}^* & \\ & & \\ \tilde{z}_{\eta+1,1} & \cdots & \tilde{z}_{\eta+1,\rho} \end{bmatrix}.$$

Consequently, with the notation $\overline{\tilde{z}_{\eta+1,i}} =: z_{i,\eta+1}$ we find that

$$G_{\mathcal{J}}^* Z_{24} + Z_{42}^* G_{\mathcal{M}} = G_{\mathcal{J}}^* Z_{24} + \begin{bmatrix} & \overline{\tilde{z}_{\eta+1,1}} \\ Z_{24} & \vdots \\ & \overline{\tilde{z}_{\eta+1,\rho}} \end{bmatrix} \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mu z_{1,1} & \cdots & \mu z_{1,\eta} \\ \mu z_{2,1} + z_{1,1} & \cdots & \mu z_{2,\eta} + z_{1,\eta} \\ \mu z_{3,1} + z_{2,1} & \cdots & \mu z_{3,\eta} + z_{2,\eta} \\ \vdots & & \vdots \\ \mu z_{\rho,1} + z_{\rho-1,1} & \cdots & \mu z_{\rho,\eta} + z_{\rho-1,\eta} \end{bmatrix} + \begin{bmatrix} z_{1,2} & \cdots & z_{1,\eta} & \overline{\tilde{z}_{\eta+1,1}} \\ z_{2,2} & \cdots & z_{2,\eta} & \overline{\tilde{z}_{\eta+1,2}} \\ z_{3,2} & \cdots & z_{3,\eta} & \overline{\tilde{z}_{\eta+1,3}} \\ \vdots & & \vdots & \vdots \\ z_{\rho,2} & \cdots & z_{\rho,\eta} & \overline{\tilde{z}_{\eta+1,\rho}} \end{bmatrix}$$

$$= \begin{bmatrix} \mu z_{1,1} + z_{1,2} & \cdots & \mu z_{1,\eta-1} + z_{1,\eta} & \mu z_{1,\eta} + z_{1,\eta+1} \\ z_{1,1} + \mu z_{2,1} + z_{2,2} & \cdots & z_{1,\eta-1} + \mu z_{2,\eta-1} + z_{2,\eta} & z_{1,\eta} + \mu z_{2,\eta} + z_{2,\eta+1} \\ z_{2,1} + \mu z_{3,1} + z_{3,2} & \cdots & z_{2,\eta-1} + \mu z_{3,\eta-1} + z_{3,\eta} & z_{2,\eta} + \mu z_{3,\eta} + z_{3,\eta+1} \\ \vdots & & \vdots & \vdots \\ z_{\rho-1,1} + \mu z_{\rho,1} + z_{\rho,2} & \cdots & z_{\rho-1,\eta-1} + \mu z_{\rho,\eta-1} + z_{\rho,\eta} & z_{\rho-1,\eta} + \mu z_{\rho,\eta} + z_{\rho,\eta+1} \end{bmatrix}.$$

Using $H_{24} = [h_{i,j}]$, we can choose $z_{1,1} := 0$ and $z_{1,2} := h_{1,1}$. Defining $z_{1,j} := -h_{1,j-1} - \mu z_{1,j-1}$ for $j = 3, \ldots, \eta+1$ recursively, we find that all $z_{1,i}$ with $i = 1, \ldots, \eta+1$ are fixed and that the first row of $H_{24} + Z_{42}^* G_{\mathcal{M}} + G_{\mathcal{J}}^* Z_{24}$ vanishes. For an inductive argument assume that the first k rows of $H_{24} + Z_{42}^* G_{\mathcal{M}} + G_{\mathcal{J}}^* Z_{24}$ vanish and that all $z_{j,i}$ with $j \leq k$ are already fixed. Then in row $k+1$ we find the equations $h_{k+1,i} + z_{k,i} + \mu z_{k+1,i} + z_{k+1,i+1} = 0$ for $i = 1, \ldots, \eta$. Again, set $z_{k+1,1} = 0$, $z_{k+1,2} = -h_{k+1,1} - z_{k,1}$, and then define $z_{k+1,i+1} := -h_{k+1,i} - z_{k,i} - \mu z_{k+1,i}$ recursively for $i = 2, \ldots, \eta$. This fixes all $z_{j,i}$ with $j \leq k+1$ and the first $k+1$ rows of $H_{24} + Z_{42}^* G_{\mathcal{M}} + G_{\mathcal{J}}^* Z_{24}$ vanish. □

The following example shows that given $\lambda F_{\mathcal{L}} + G_{\mathcal{L}} \in \mathbb{C}[\lambda]^{\epsilon,\epsilon+1}$, $\lambda F_{\mathcal{J}} + G_{\mathcal{J}} \in \mathbb{C}[\lambda]^{\rho,\rho}$, and $H_{12} \in \mathbb{C}^{\epsilon+1,\rho}$ in general one can not find matrices $Z_{12} \in \mathbb{C}^{\epsilon,\rho}$ and $Z_{21} \in \mathbb{C}^{\rho,\epsilon+1}$ such that $F_{\mathcal{J}}^* Z_{12} = Z_{21}^*$ and $0 = Z_{21}^* G_{\mathcal{J}} + G_{\mathcal{L}}^* Z_{12} + H_{12}$.

Example 26. Consider the pencil

$$\begin{bmatrix} 0 & \lambda - 1 \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{=:F} + \underbrace{\begin{bmatrix} 0 & -1 \end{bmatrix}}_{=:G}$$

together with the Hermitian matrix $H = H^* \in \mathbb{C}^{2,2}$ given by

$$H = \begin{bmatrix} h_{11} & h_{12} \\ \overline{h_{12}} & h_{22} \end{bmatrix}.$$

Let $Z = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \in \mathbb{C}^{2,1}$ be such that $F^* Z = Z^* F$, i.e., let

$$F^* Z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ z_1 & z_2 \end{bmatrix}$$

be Hermitian. This implies that $z_1 = 0$ and $z_2 = \overline{z_2} \in \mathbb{R}$ is a real number. Thus Z takes the form $Z = \begin{bmatrix} 0 & r \end{bmatrix}$ with $r \in \mathbb{R}$ and the equation $0 = H + G^*Z + Z^*G$ becomes

$$0 \;=\; H + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & r \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix} \begin{bmatrix} 0 & -1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ \overline{h_{12}} & h_{22} - 2r \end{bmatrix},$$

and the entry h_{12} can not be influenced by means of Z. Note, that (F, G) is dissipative with respect to H if and only if $h_{11} \geq 0$.

After all these preparations we obtain the main theorem of this section.

Theorem 27. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Assume that (F, G, H) is dissipative, (F, G) is controllable, and the infinite eigenvalues of (F, G) have index 1. Then there exists a solution $Z \in \mathbb{C}^{p,q}$ of the linear matrix inequality*

$$\begin{array}{rcl} F^*Z & = & Z^*F, \\ 0 \leq G^*Z & + & Z^*G + H. \end{array} \tag{28}$$

Proof. Since (F, G) is assumed to be controllable we have that $R_c(F, G) = R(F, G)$ due to Lemma 20. Consider the Kronecker canonical form (16) of $\lambda F + G$ as in Theorem 18. Then, using $\tilde{z} := Q^{-1}\hat{z}$, we see that (12) is equivalent to

$$\begin{array}{rcl} 0 & \leq & \hat{z}^*Q^{-*}Q^*(H + G^*P^*P^{-*}Z + Z^*P^{-1}PG)QQ^{-1}\hat{z} \\ & = & \tilde{z}^*(H + \tilde{G}^*\tilde{Z} + \tilde{Z}^*\tilde{G})\tilde{z} \end{array}$$

for all $\hat{z} \in B(F, G)$, where $H := Q^*HQ$, $\tilde{Z} = P^{-*}ZQ$, and $\tilde{G} := PGQ$. Partition $\tilde{Z} = [Z_{i,j}]_{i,j=1,\ldots,4}$, $H = [H_{i,j}]_{i,j=1,\ldots,4}$, and $\tilde{z} = [z_i]_{i=1,\ldots,4}$ with $z_1 \in \mathbb{C}^{\epsilon+p}$, $z_2 \in \mathbb{C}^{\rho}$, $z_3 \in \mathbb{C}^{\sigma}$, and $z_4 \in \mathbb{C}^{\eta}$ according to the block structure of $P(\lambda F + G)Q = \operatorname{diag}(\mathcal{L}, \mathcal{J}, \mathcal{N}, \mathcal{M})$. Then we can rewrite (12) with $G_{\mathcal{N}} = I$ (since we have assumed that the infinite eigenvalues of (F, G) have index 1) as

$$\begin{array}{rcl} 0 & \leq & \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \end{bmatrix}^* \left(\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \right. \\ & + & \begin{bmatrix} G_{\mathcal{L}}^*Z_{11} & G_{\mathcal{L}}^*Z_{12} & G_{\mathcal{L}}^*Z_{13} & G_{\mathcal{L}}^*Z_{14} \\ G_{\mathcal{J}}^*Z_{21} & G_{\mathcal{J}}^*Z_{22} & G_{\mathcal{J}}^*Z_{23} & G_{\mathcal{J}}^*Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ G_{\mathcal{M}}^*Z_{41} & G_{\mathcal{M}}^*Z_{42} & G_{\mathcal{M}}^*Z_{43} & G_{\mathcal{M}}^*Z_{44} \end{bmatrix} \\ & + & \left. \begin{bmatrix} Z_{11}^*G_{\mathcal{L}} & Z_{21}^*G_{\mathcal{J}} & Z_{31}^* & Z_{41}^*G_{\mathcal{M}} \\ Z_{12}^*G_{\mathcal{L}} & Z_{22}^*G_{\mathcal{J}} & Z_{32}^* & Z_{42}^*G_{\mathcal{M}} \\ Z_{13}^*G_{\mathcal{L}} & Z_{23}^*G_{\mathcal{J}} & Z_{33}^* & Z_{43}^*G_{\mathcal{M}} \\ Z_{14}^*G_{\mathcal{L}} & Z_{24}^*G_{\mathcal{J}} & Z_{34}^* & Z_{44}^*G_{\mathcal{M}} \end{bmatrix} \right) \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \end{bmatrix}, \end{array}$$

for every $\tilde{z} = Q^{-1}\hat{z}$ with $\hat{z} \in B(F, G)$. Due to Lemma 20 we know that for every

$\tilde{z}_1 \in \mathbb{C}^{\epsilon+p}$ and every $\tilde{z}_2 \in \mathbb{C}^{\rho}$ there exists a $\hat{z} \in B(F,G)$ such that

$$Q^{-1}\hat{z} = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ 0 \\ 0 \end{bmatrix}.$$

Using the assumption of dissipativity together with Theorem 15 implies that

$$\begin{aligned} & \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}^* \begin{bmatrix} H_{11} + G_{\mathcal{L}}^* Z_{11} + Z_{11}^* G_{\mathcal{L}} & H_{12} + G_{\mathcal{L}}^* Z_{12} + Z_{21}^* G_{\mathcal{J}} \\ H_{21} + G_{\mathcal{J}}^* Z_{21} + Z_{12}^* G_{\mathcal{L}} & H_{22} + G_{\mathcal{J}}^* Z_{22} + Z_{22}^* G_{\mathcal{J}} \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \\ = \;& \hat{z}^* Q^{-*} \left(H + \tilde{G}^* \tilde{Z} + \tilde{Z}^* \tilde{G} \right) Q^{-1} \hat{z} \\ = \;& \hat{z}^* \left(H + G^* Z + Z^* G \right) \hat{z} \geq 0, \end{aligned}$$

for arbitrary $\tilde{z}_1$, $\tilde{z}_2$. This means that there exist Z_{11}, Z_{12}, Z_{21}, and Z_{22} such that

$$\begin{bmatrix} H_{11} + G_{\mathcal{L}}^* Z_{11} + Z_{11}^* G_{\mathcal{L}} & H_{12} + G_{\mathcal{L}}^* Z_{12} + Z_{21}^* G_{\mathcal{J}} \\ H_{21} + G_{\mathcal{J}}^* Z_{21} + Z_{12}^* G_{\mathcal{L}} & H_{22} + G_{\mathcal{J}}^* Z_{22} + Z_{22}^* G_{\mathcal{J}} \end{bmatrix} \geq 0. \tag{29}$$

To construct the remaining Z_{13}, Z_{14}, ... we first note that $F^*Z = Z^*F$ with $F_{\mathcal{N}} = 0$ (due to the assumption that the infinite eigenvalues have index 1) and $F_{\mathcal{J}} = I$ becomes

$$\begin{bmatrix} F_{\mathcal{L}}^* Z_{11} & F_{\mathcal{L}}^* Z_{12} & F_{\mathcal{L}}^* Z_{13} & F_{\mathcal{L}}^* Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ 0 & 0 & 0 & 0 \\ F_{\mathcal{M}}^* Z_{41} & F_{\mathcal{M}}^* Z_{42} & F_{\mathcal{M}}^* Z_{43} & F_{\mathcal{M}}^* Z_{44} \end{bmatrix} = \begin{bmatrix} Z_{11}^* F_{\mathcal{L}} & Z_{21}^* & 0 & Z_{41}^* F_{\mathcal{M}} \\ Z_{12}^* F_{\mathcal{L}} & Z_{22}^* & 0 & Z_{42}^* F_{\mathcal{M}} \\ Z_{13}^* F_{\mathcal{L}} & Z_{23}^* & 0 & Z_{43}^* F_{\mathcal{M}} \\ Z_{14}^* F_{\mathcal{L}} & Z_{24}^* & 0 & Z_{44}^* F_{\mathcal{M}} \end{bmatrix}.$$

For the (4,4)-block use Lemma 21 and Lemma 22 to construct a matrix Z_{44} (which itself is composed of several block matrices, according to the $\mathcal{N}$-blocks in the Kronecker canonical form) such that $Z_{44}^* F_{\mathcal{M}} = F_{\mathcal{M}}^* Z_{44}$ and $0 = H_{44} + Z_{44}^* G_{\mathcal{M}} + G_{\mathcal{M}}^* Z_{44}$.
For the (3,3)-block simply set $Z_{33} = -\frac{1}{2} H_{33}$.
For the (3,4)-block set $Z_{43} = 0$ and $Z_{34} = -H_{34}$.
For the (2,3)-block set $Z_{23} = 0$ and $Z_{32} = -H_{32}$.
For the (1,3)-block set $Z_{13} = 0$ and $Z_{31} = -H_{31}$.
For the (1,4)-block use Lemma 23 to construct matrices Z_{14} and Z_{41} such that $F_{\mathcal{L}}^* Z_{14} = Z_{41}^* F_{\mathcal{M}}$ and $0 = H_{14} + G_{\mathcal{L}}^* Z_{14} + Z_{41}^* G_{\mathcal{M}}$.
For the (2,4)-block use Lemma 25 to construct matrices Z_{24} and Z_{42} such that $Z_{24} = Z_{42}^* F_{\mathcal{M}}$ and $0 = H_{24} + G_{\mathcal{J}}^* Z_{24} + Z_{42}^* G_{\mathcal{M}}$.
Thus, all in all we have constructed a matrix $\tilde{Z}$ such that $\tilde{Z}^* F = F^* \tilde{Z}$ and

$$\begin{aligned} & H + \tilde{G}^* \tilde{Z} + \tilde{Z}^* \tilde{G} \\ = \;& \left(\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} + \begin{bmatrix} G_{\mathcal{L}}^* Z_{11} & G_{\mathcal{L}}^* Z_{12} & G_{\mathcal{L}}^* Z_{13} & G_{\mathcal{L}}^* Z_{14} \\ G_{\mathcal{J}}^* Z_{21} & G_{\mathcal{J}}^* Z_{22} & G_{\mathcal{J}}^* Z_{23} & G_{\mathcal{J}}^* Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ G_{\mathcal{M}}^* Z_{41} & G_{\mathcal{M}}^* Z_{42} & G_{\mathcal{M}}^* Z_{43} & G_{\mathcal{M}}^* Z_{44} \end{bmatrix} \right. \end{aligned}$$

$$
\begin{aligned}
&+ \begin{bmatrix} Z_{11}^* G_{\mathcal{L}} & Z_{21}^* G_{\mathcal{J}} & Z_{31}^* & Z_{41}^* G_{\mathcal{M}} \\ Z_{12}^* G_{\mathcal{L}} & Z_{22}^* G_{\mathcal{J}} & Z_{32}^* & Z_{42}^* G_{\mathcal{M}} \\ Z_{13}^* G_{\mathcal{L}} & Z_{23}^* G_{\mathcal{J}} & Z_{33}^* & Z_{43}^* G_{\mathcal{M}} \\ Z_{14}^* G_{\mathcal{L}} & Z_{24}^* G_{\mathcal{J}} & Z_{34}^* & Z_{44}^* G_{\mathcal{M}} \end{bmatrix} \Bigg) \\
&= \begin{bmatrix} H_{11} + G_{\mathcal{L}}^* Z_{11} + Z_{11}^* G_{\mathcal{L}} & H_{12} + G_{\mathcal{L}}^* Z_{12} + Z_{21}^* G_{\mathcal{J}} & 0 & 0 \\ H_{21} + G_{\mathcal{J}}^* Z_{21} + Z_{12}^* G_{\mathcal{L}} & H_{22} + G_{\mathcal{J}}^* Z_{22} + Z_{22}^* G_{\mathcal{J}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0,
\end{aligned}
$$

and the assertion is shown. □

Corollary 28. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Assume that (F, G) is controllable and that the infinite eigenvalues of (F, G) have index* 1. *Then the following statements are equivalent:*

1. *(F, G, H) is dissipative.*
2. *There exists a solution of the linear matrix inequality (28).*

Proof. That 1. implies 2. follows from Theorem 27 even without the controllability assumption. The other direction is a consequence of Theorem 15. □

The assumptions in Corollary 28 can be weakened as one can see, e.g., by considering Example 17 with $h_{11} \geq 0$.

4 Two applications

In this section we show two applications of the results that have been obtained in the previous section. We need further notation. We denote by $\mathbb{C}(\lambda)$ the set of rational functions with coefficients in $\mathbb{C}$ and by $\mathbb{C}(\lambda)^{p,q}$ the rational matrices of size p-by-q.

Definition 29. Let $R \in \mathbb{C}(\lambda)^{p,q}$ be a rational matrix. Then we call

$$R^{\sim}(\lambda) := R^*(-\overline{\lambda})$$

the *para-Hermitian* of R. Also, if $R^{\sim} = R$ we say that R is *para-Hermitian* or *even*.

4.1 Deflating subspaces

Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$ and form the matrix pencil

$$N(\lambda) := \lambda \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q,p+q}. \tag{30}$$

It is easy to see that $N = N^{\sim}$ is even. Even pencils are intimately connected to problems that involve dissipativity, see [3, 4].

We say that $V \in \mathbb{C}^{p+q,s}$ spans a deflating subspace of N with dimension s if $\operatorname{rank}(V) = s$ and there exists a matrix pencil $\tilde{N} = \lambda\tilde{N}_1 + \tilde{N}_0 \in \mathbb{C}[\lambda]^{s,s}$ and $W \in \mathbb{C}^{p+q,s}$ such that

$$N(\lambda)V = W\tilde{N}(\lambda).$$

The most simple case is that $s = 1$ and V is an eigenvector associated with the eigenvalue λ_0, in which case one can choose $\tilde{N}(\lambda) = \lambda - \lambda_0$. For further results about the application of deflating subspaces to linear quadratic systems theory, see [7].

Lemma 30. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let $Z \in \mathbb{C}^{p,q}$ be a solution of the linear matrix inequality (14). Denote the rank of the positive semi-definite matrix from (14) by $r := \operatorname{rank}(H + Z^*G + G^*Z)$. Then positive semi-definiteness implies the existence of a full rank Cholesky factor $L \in \mathbb{C}^{r,n}$ such that*

$$L^*L = H + Z^*G + G^*Z. \tag{31}$$

Assume that $r \leq q - p$ (this implies that $q \geq p$). Then

$$\left(\lambda \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix}\right) \begin{bmatrix} Z \\ I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -Z^* & L^* \end{bmatrix} \begin{bmatrix} \lambda F + G \\ L \end{bmatrix},$$

i.e., $\begin{bmatrix} Z \\ I \end{bmatrix}$ spans a deflating subspace of the para-Hermitian pencil (30).

Proof. We have

$$\begin{aligned} \left(\lambda \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix}\right) \begin{bmatrix} Z \\ I \end{bmatrix} &= \lambda \begin{bmatrix} F \\ -F^*Z \end{bmatrix} + \begin{bmatrix} G \\ G^*Z + H \end{bmatrix} \\ &= \begin{bmatrix} \lambda F + G \\ -\lambda Z^*F + L^*L - Z^*G \end{bmatrix} \\ &= \begin{bmatrix} \lambda F + G \\ -Z^*(\lambda F + G) + L^*L \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -Z^* & L^* \end{bmatrix} \begin{bmatrix} \lambda F + G \\ L \end{bmatrix}, \end{aligned}$$

which proves the claim. □

Lemma 30 emphasizes the importance of solutions of the linear matrix inequality (14) that minimize the rank r of (31). The linear matrix inequality (14) with this rank minimizing requirement is called *Lur'e equation*, compare [9].

4.2 Spectral factorization

For a given rational matrix $R \in \mathbb{C}(\lambda)^{p,q}$ by $\operatorname{rank}_{\mathbb{C}(\lambda)}(R)$ we denote the rank of R over the field of rational functions. Also, by $\mathfrak{D}(R)$ we denote the domain of definition of R by which we mean all $\lambda_0 \in \mathbb{C}$ such that $R(\lambda_0) \in \mathbb{C}^{p,q}$ is well-defined, i.e., no entry of R has a pole at λ_0. Note that $\mathfrak{D}(R)$ is a finite set since a polynomial can only have a finite number of zeros.

Definition 31. Let $F, G \in \mathbb{C}^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$. Then the rational matrix $U \in \mathbb{C}(\lambda)^{q,q-r}$ and the polynomial matrix $V \in \mathbb{C}[\lambda]^{q,r}$ are called *kernel matrix and co-kernel matrix* of $\lambda F + G$, if they fulfill the following properties

1. $(\lambda F + G)U(\lambda) = 0$,
2. $\operatorname{rank}_{\mathbb{C}(\lambda)}((\lambda F + G)V(\lambda)) = r$,
3. $\operatorname{rank}_{\mathbb{C}(\lambda)}(U) = \operatorname{rank}(U(\lambda)) = q - r$ for all $\lambda \in \mathfrak{D}(U)$,
4. $\operatorname{rank}_{\mathbb{C}(\lambda)}(V) = \operatorname{rank}(V(\lambda)) = r$ for all $\lambda \in \mathbb{C}$,
5. $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular, i.e., there exists a non-zero constant $c \in \mathbb{C} \setminus \{0\}$ such that
$$\det \begin{bmatrix} U(\lambda) & V(\lambda) \end{bmatrix} = c,$$
for all $\lambda \in \mathbb{C}$.

The kernel matrix is closely related to the compact behavior of a system (F, G). Assume that U is a polynomial kernel matrix. Then the compact behavior is given by

$$\begin{aligned} \mathfrak{B}_c(F,G) &= \operatorname{range}_{\mathcal{C}_c^\infty}\left(U\left(\frac{d}{dt}\right)\right) \\ &= \left\{ z \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q) \mid \exists \alpha \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r}) \text{ such that } z \equiv U\left(\frac{d}{dt}\right)\alpha \right\}, \end{aligned}$$

as shown in [8] or [3, Lemma 18]. This is the reason why in [8] the matrix U is also called image representation of the system (F, G).

Definition 32. Let $\Pi = \Pi^\sim \in \mathbb{C}(\lambda)^{n,n}$ be a para-Hermitian matrix. If there exist $m \in \mathbb{N}$ and $K \in \mathbb{C}(\lambda)^{m,n}$ such that

$$\Pi = K^\sim K,$$

then we say that K is a spectral factor of Π. Also we refer to the product $K^\sim K$ as a spectral factorization of Π.

In the following we discuss special spectral factorizations.

Definition 33. Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$ and $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix. Then the even rational matrix

$$\Pi := U^\sim H U$$

is called a *Popov function* or a *spectral density function* of (F, G, H).

The following Theorem 34 shows that a spectral factorization of a Popov function can be obtained with the results from the previous section.

Theorem 34. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of $\lambda F + G$. Let*

$$\Pi(\lambda) := U^{\sim}(\lambda) H U(\lambda).$$

be the associated Popov function. Then we have the following:

1. *If there exists a solution $Z \in \mathbb{C}^{p,q}$ of the linear matrix inequality (14) and the matrix L is a Cholesky factor of*

$$0 \leq G^* Z + Z^* G + H = L^* L,$$

   then with $K(\lambda) := LU(\lambda)$ we have $K^{\sim}K = \Pi$, i.e., K is a spectral factor of Π.

2. *If on the other hand (F, G) is controllable, the infinite eigenvalues of (F, G) have index 1, and there exists a spectral factor of Π, i.e., if there exists a $K \in \mathbb{C}(\lambda)^{m,q-r}$ such that $K^{\sim}(\lambda)K(\lambda) = \Pi(\lambda)$ for all $\lambda \in \mathfrak{D}(K) \cap \mathfrak{D}(U)$, then the linear matrix inequality (14) has a solution $Z \in \mathbb{C}^{p,q}$.*

Proof. For 1. note that we have

$$\begin{aligned}
& K^{\sim}(\lambda)K(\lambda) = K^*(-\overline{\lambda})K(\lambda) \\
= \;& U^*(-\overline{\lambda})L^*LU(\lambda) \\
= \;& U^*(-\overline{\lambda})\left[G^*Z + Z^*G + H\right]U(\lambda) \\
= \;& U^*(-\overline{\lambda})HU(\lambda) + U^*(-\overline{\lambda})G^*ZU(\lambda) + U^*(-\overline{\lambda})Z^*GU(\lambda) \\
= \;& U^{\sim}(\lambda)HU(\lambda) + \left(GU(-\overline{\lambda})\right)^* ZU(\lambda) + U^*(-\overline{\lambda})Z^*\left(GU(\lambda)\right) \\
= \;& \Pi(\lambda) + \left(\overline{\lambda}FU(-\overline{\lambda})\right)^* ZU(\lambda) + U^*(-\overline{\lambda})Z^*\left(-\lambda FU(\lambda)\right) \\
= \;& \Pi(\lambda) + \lambda U^*(-\overline{\lambda})F^*ZU(\lambda) - \lambda U^*(-\overline{\lambda})Z^*FU(\lambda) = \Pi(\lambda)
\end{aligned}$$

since $(\lambda F + G)U(\lambda) = 0$ implies $GU(\lambda) = -\lambda FU(\lambda)$ and also $GU(-\overline{\lambda}) = \overline{\lambda}FU(-\overline{\lambda})$. For part 2. first note that for all $\omega \in \mathbb{R}$ for which $K(i\omega)$ and $U(i\omega)$ is well defined we have

$$\begin{aligned}
\Pi(i\omega) &= K^{\sim}(i\omega)K(i\omega) \\
&= K^*(-\overline{i\omega})K(i\omega) = K^*(i\omega)K(i\omega) \geq 0.
\end{aligned}$$

Using the continuity of Π this implies that also $\Pi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ such that $U(i\omega)$ is well defined. Using, e.g., [3, Theorem 36] this proves that (F, G, H) is dissipative. Using Theorem 27 and the additional assumptions we deduce the existence of a solution of the linear matrix inequalities. □

Corollary 35. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Also, let (F, G) be controllable and let all infinite eigenvalues of (F, G) have index 1. Set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of $\lambda F + G$. Then the following are equivalent:*

1. *(F, G, H) is dissipative.*

2. *There exists a solution of the linear matrix inequality (14).*
3. *There exists a spectral factor of the Popov function* $\Pi := U^{\sim} H U$.

Proof. The proof follows directly from the proof of the previous Theorem 34. □

From Corollary 35 the question arises if condition (12) (i.e., dissipativity) is equivalent to the existence of a spectral factor of Π which has the form $LU(\lambda)$ without further controllability assumptions.

Remark 36. Closely related to Corollary 35 is the result of the Youla factorization. It states that for every para-Hermitian rational function which is positive semi-definite along the imaginary axis there exists a spectral factorization [15, Theorem 2]. Also, for every para-Hermitian polynomial function there even exists a polynomial spectral factor [15, Corollary 2].

Lemma 37. *Let* $F, G \in \mathbb{C}^{p,q}$ *and* $H = H^* \in \mathbb{C}^{q,q}$. *Set* $r := \operatorname{rank}_{\mathbb{C}(\lambda)} (\lambda F + G)$ *and let* $U \in \mathbb{C}(\lambda)^{q,q-r}$ *be a kernel matrix of* $\lambda F + G$. *Define the functions*

$$\begin{aligned} \Pi(\lambda) &:= U^{\sim}(\lambda) H U(\lambda) \\ \hat{\Pi}(\lambda) &:= U^{*}(\lambda) H U(\lambda). \end{aligned}$$

(Note that Π *is the Popov function but not* $\hat{\Pi}$*). Then for all* $\omega \in \mathbb{R}$ *with* $i\omega \in \mathfrak{D}(U)$ *we have*

$$\tilde{\Pi}(i\omega) = \Pi(i\omega).$$

Furthermore, if there exists a solution $Z \in \mathbb{C}^{p,q}$ *of the linear matrix inequality (14) such that* $F^* Z \geq 0$ *then* $\tilde{\Pi}(\lambda) \geq 0$ *for all* $\lambda \in \overline{\mathbb{C}^+}$.

Proof. First we see that for $\omega \in \mathbb{R}$ we have

$$\begin{aligned} \Pi(i\omega) &= U^{\sim}(i\omega) H U(i\omega) = U^{*}(-\overline{i\omega}) H U(i\omega) \\ &= U^{*}(i\omega) H U(i\omega) = \tilde{\Pi}(i\omega). \end{aligned}$$

To show the positive semi-definiteness of $\tilde{\Pi}$ in the right half plane, let Z be a solution of the linear matrix inequality with $F^* Z \geq 0$. Perform a Cholesky factorization of $0 \leq H + G^* Z + Z^* G = L^* L$ and set $K(\lambda) := LU(\lambda)$. Then we have

$$\begin{aligned} K^*(\lambda) K(\lambda) &= U^*(\lambda) L^* L U(\lambda) \\ &= U^*(\lambda) \left[H + G^* Z + Z^* G \right] U(\lambda) \\ &= U^*(\lambda) H U(\lambda) + (GU(\lambda))^* ZU(\lambda) + (ZU(\lambda))^* GU(\lambda) \\ &= \tilde{\Pi}(\lambda) + (-\lambda F U(\lambda))^* ZU(\lambda) + (ZU(\lambda))^* (-\lambda F U(\lambda)) \\ &= \tilde{\Pi}(\lambda) - 2\mathrm{Re}\left\{\lambda\right\} U^*(\lambda) F^* Z U(\lambda), \end{aligned}$$

which implies that for all $\lambda \in \overline{\mathbb{C}^+}$ we have

$$\tilde{\Pi}(\lambda) = K^*(\lambda) K(\lambda) + 2\mathrm{Re}\left\{\lambda\right\} U^*(\lambda) \underbrace{F^* Z}_{\geq 0} U(\lambda) \geq 0,$$

and the claim is shown. □

The converse of Lemma 37 is not true in general as the example in [13] shows. For further work on the existence of a positive semi-definite solution of the linear matrix inequality see [10] and also [14, Theorem 6.4].

5 Specialization to descriptor systems

Consider the state-space descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{32}$$

where $E, A \in \mathbb{R}^{\rho,n}$, $B \in \mathbb{R}^{\rho,m}$, $C \in \mathbb{R}^{p,n}$, $D \in \mathbb{R}^{p,m}$, $x \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)$ is called the state, $u \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^m)$ is called the input, and $y \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^p)$ is called the output. In the literature, see e.g. [2, Section 5.9], for such systems a *supply function* is frequently introduced as a quadratic function $s : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ of the form

$$s(u, y) := \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{33}$$

where $Q = Q^T \in \mathbb{R}^{p,p}$, $S \in \mathbb{R}^{p,m}$, and $R = R^T \in \mathbb{R}^{m,m}$. Then system (32) is called dissipative (compare Definition 2) if we have that

$$0 \leq \int_{-\infty}^{\infty} s(u(t), y(t))dt,$$

for every $(u, x, y) \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^m) \times \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^n) \times \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^p)$ that fulfills (32).

Using the equation for y in (32) we can rewrite the supply to depend on the state variables (instead of the output variables) via

$$\begin{aligned} s(u(t), y(t)) &= \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^TQC & C^TQD + C^TS \\ D^TQC + S^TC & D^TQD + D^TS + S^TD + R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &=: \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} =: \tilde{s}(u(t), x(t)), \end{aligned} \tag{34}$$

where $\tilde{Q} = \tilde{Q}^T \in \mathbb{R}^{n,n}$, $\tilde{S} \in \mathbb{R}^{n,m}$, $R \in \mathbb{R}^{m,m}$, and $\tilde{s} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$. Clearly, with this we can equivalently say that (32) is dissipative if

$$0 \leq \int_{-\infty}^{\infty} \tilde{s}(u(t), x(t))dt,$$

for every $(u, x) \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^m) \times \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^n)$ that fulfills $E\dot{x} = Ax + Bu$.

For quadratic E and A we state the following definition from [5, Definition 2].

Definition 38. Let $E, A \in \mathbb{R}^{n,n}$ with $\lambda E - A$ regular (as a matrix over $\mathbb{C}(\lambda)$) and $B \in \mathbb{R}^{n,m}$. Then the triplet (E, A, B) is called *completely controllable* if

$$\operatorname{rank}\left(\begin{bmatrix}\alpha E - \beta A & B\end{bmatrix}\right) = n \quad \text{for all } (\alpha, \beta) \in \mathbb{C}^2 \setminus \{0, 0\}.$$

With the notation $q := n + m$,

$$F := \begin{bmatrix}E & 0\end{bmatrix}, \quad G := \begin{bmatrix}-A & -B\end{bmatrix}, \quad H := \begin{bmatrix}\tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R}\end{bmatrix}, \quad \text{and } z := \begin{bmatrix}x \\ u\end{bmatrix}, \tag{35}$$

we see that $F, G \in \mathbb{R}^{\rho,q}$, $H = H^* \in \mathbb{R}^{q,q}$, and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and we can rewrite system (32) as the behavior system $F\dot{z}(t) + Gz(t) = 0$. Also the supply can be viewed as a function of the form $s : \mathbb{R}^q \to \mathbb{R}$ with

$$s(\hat{z}) = \hat{z}^T H \hat{z}.$$

We have the following result.

Lemma 39. *Let $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $\lambda E - A$ be regular, and let the triplet (E, A, B) be completely controllable. Then, with F and G as in (35), (F, G) is controllable and the infinite eigenvalues of (F, G) have index* 1.

Proof. To show that (F, G) has no finite eigenvalues, note that we have

$$n = \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda E - A) \leq \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G) \leq n,$$

since $\lambda E - A$ is regular and thus $n = \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G)$. On the other hand we have

$$\begin{aligned}\operatorname{rank}(\lambda_0 F + G) &= \operatorname{rank}\left(\begin{bmatrix}\lambda_0 E - A & -B\end{bmatrix}\begin{bmatrix}I & \\ & -I\end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix}\lambda_0 E - A & B\end{bmatrix}\right) = n,\end{aligned}$$

which we can see by choosing $\alpha = \lambda_0$ and $\beta = 1$ in Definition 38. This shows that

$$\operatorname{rank}(\lambda_0 F + G) = n = \operatorname{rank}_{\mathbb{C}(\lambda)}(\lambda F + G),$$

for all $\lambda_0 \in \mathbb{C}$ and thus we have shown that (F, G) has no finite eigenvalues which by definition means that (F, G) is controllable.

To show that all infinite eigenvalues of (F, G) have index 1 assume to the contrary that the Kronecker canonical form of (F, G) has a block of type (19) with size $\sigma > 1$. In this case, there exist invertible matrices W and V such that

$$W\left(\lambda\begin{bmatrix}E & 0\end{bmatrix} + \begin{bmatrix}-A & -B\end{bmatrix}\right) = \left(\lambda\begin{bmatrix}N & \\ & R_1\end{bmatrix} + \begin{bmatrix}I & \\ & R_2\end{bmatrix}\right)V,$$

where

$$N = \begin{bmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{bmatrix} \in \mathbb{C}^{\sigma,\sigma}$$

is a nilpotent Jordan block of size $\sigma > 1$. Define the vectors $v_1^T := \begin{bmatrix} 0 & \dots & 0 & 0 & 1 \end{bmatrix}$ and $v_2^T := \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \end{bmatrix}$ and note that with this choice we have $v_1^T N = 0$ and $v_2^T N = v_1^T I$. Introduce

$$w_1^T := \begin{bmatrix} v_1^T & 0_{n-\sigma} \end{bmatrix} W \qquad \text{and} \qquad w_2^T := \begin{bmatrix} v_2^T & 0_{n-\sigma} \end{bmatrix} W,$$

where $0_{n-\sigma} \in \mathbb{C}^{1,n-\sigma}$ is a vector of only zeros, and observe that this implies

$$\begin{aligned} w_1^T \begin{bmatrix} E & 0 \end{bmatrix} &= \begin{bmatrix} v_1^T & 0_{n-\sigma} \end{bmatrix} W \begin{bmatrix} E & 0 \end{bmatrix} \\ &= \begin{bmatrix} v_1^T & 0_{n-\sigma} \end{bmatrix} \begin{bmatrix} N & \\ & R_1 \end{bmatrix} V = 0, \end{aligned}$$

and also

$$\begin{aligned} w_2^T \begin{bmatrix} E & 0 \end{bmatrix} &= \begin{bmatrix} v_2^T & 0_{n-\sigma} \end{bmatrix} W \begin{bmatrix} E & 0 \end{bmatrix} = \begin{bmatrix} v_2^T & 0_{n-\sigma} \end{bmatrix} \begin{bmatrix} N & \\ & R_1 \end{bmatrix} V \\ &= \begin{bmatrix} v_2^T N & 0_{n-\sigma} \end{bmatrix} V = \begin{bmatrix} v_1^T I & 0_{n-\sigma} \end{bmatrix} V = \begin{bmatrix} v_1^T & 0_{n-\sigma} \end{bmatrix} \begin{bmatrix} I & \\ & R_2 \end{bmatrix} V \\ &= \begin{bmatrix} v_1^T & 0_{n-\sigma} \end{bmatrix} W \begin{bmatrix} -A & -B \end{bmatrix} = w_1^T \begin{bmatrix} -A & -B \end{bmatrix}. \end{aligned}$$

All in all we have obtained that $w_1, w_2 \neq 0$ with $w_1^T E = 0$, $w_2^T E = -w_1^T A$, and $w_1^T B = 0$. This implies that also

$$w_1^T \begin{bmatrix} E & B \end{bmatrix} = 0,$$

from which we deduce that $\operatorname{rank}\left(\begin{bmatrix} E & B \end{bmatrix}\right) < n$, since $w_1 \neq 0$. This, however, is a contradiction to the assumption of controllability as one can see by choosing $\alpha = 1$ and $\beta = 0$ in Definition 38. □

Corollary 40. *Consider the system (32) with $\rho = n$ and let $\lambda E - A$ be regular. Assume that (E, A, B) is completely controllable. Then the following statements are equivalent:*

1. *The system (32) is dissipative.*
2. *The Popov function*

$$\Pi(\lambda) := \begin{bmatrix} (\lambda E - A)^{-1} B \\ I \end{bmatrix}^{\sim} \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I \end{bmatrix}$$

   is positive semi-definite along the imaginary axis.

3. *There exist $X \in \mathbb{R}^{n,n}$ and $Y \in \mathbb{R}^{n,m}$ such that*

$$\begin{aligned} \begin{bmatrix} A^T X + X^T A - \tilde{Q} & A^T Y + X^T B - \tilde{S} \\ B^T X + Y^T A - \tilde{S}^T & B^T Y + Y^T B - \tilde{R} \end{bmatrix} &\leq 0, \\ E^T X = X^T E \qquad\qquad E^T Y &= 0. \end{aligned} \tag{36}$$

Proof. That dissipativity is equivalent to the positive semi-definiteness of the Popov function along the imaginary axis has been shown, e.g., in [3, Theorem 36].

To show that 1. implies 3. let F and G be given by (35). By Lemma 39 we see that the controllability of (E, A, B) implies that (F, G) is controllable and all infinite eigenvalues of (F, G) have index 1. Thus we can use Theorem 27 to deduce the existence of a $Z \in \mathbb{C}^{n,n+m}$ such that (14) holds. Partitioning

$$Z =: \begin{bmatrix} X & Y \end{bmatrix},$$

with $X \in \mathbb{C}^{n,n}$ and $Y \in \mathbb{C}^{n,m}$, analogously to the partitioning of F and G as given in (35) we see than the linear matrix inequality reads

$$\begin{bmatrix} E^T X & E^T Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E^T \\ 0 \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} = F^T Z = Z^T F = \begin{bmatrix} X^T \\ Y^T \end{bmatrix} \begin{bmatrix} E & 0 \end{bmatrix} = \begin{bmatrix} X^T E & 0 \\ Y^T E & 0 \end{bmatrix}$$

which implies $E^T X = X^T E$ and $E^T Y = 0$ and

$$\begin{aligned} 0 &\leq G^T Z + Z^T G + H \\ &= \begin{bmatrix} -A^T \\ -B^T \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} + \begin{bmatrix} X^T \\ Y^T \end{bmatrix} \begin{bmatrix} -A & -B \end{bmatrix} + \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \\ &= \begin{bmatrix} -A^T X - X^T A + \tilde{Q} & -A^T Y - X^T B + \tilde{S} \\ -B^T X - Y^T A + \tilde{S}^T & -B^T Y - Y^T B + \tilde{R} \end{bmatrix}, \end{aligned}$$

which proves the claim. □

5.1 Specialization to passive systems

Definition 41. A system of the form (32) is called passive if $m = p$ and it is dissipative with respect to the supply function

$$s(u, y) := \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix},$$

compare (33).

We obtain the following result which is closely related to [5, Theorem 13 (i)].

Corollary 42. *Consider the system (32) with $\rho = n$, $m = p$, and let $\lambda E - A$ be regular. Assume that (E, A, B) is completely controllable. Then the following statements are equivalent:*

1. *The system (32) is passive.*
2. *The transfer function $G(\lambda) := C(\lambda E - A)^{-1} B + D$ fulfills*

$$G(i\omega) + G^*(i\omega) \geq 0 \tag{37}$$

for all $\omega \in \mathbb{R}$ such that $i\omega E - A$ is invertible.

3. There exist $X \in \mathbb{R}^{n,n}$ and $Y \in \mathbb{R}^{n,m}$ such that

$$\begin{aligned} \begin{bmatrix} A^TX + X^TA & A^TY + X^TB - C^T \\ B^TX + Y^TA - C & B^TY + Y^TB - D - D^T \end{bmatrix} &\leq 0, \\ E^TX = X^TE \qquad\qquad E^TY &= 0. \end{aligned} \tag{38}$$

Proof. In this case the Popov function is given by

$$\begin{aligned} \Pi(\lambda) &= \begin{bmatrix} (\lambda E - A)^{-1}B \\ I \end{bmatrix}^{\sim} \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I \end{bmatrix} \\ &= \begin{bmatrix} G(\lambda) \\ I \end{bmatrix}^{\sim} \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} G(\lambda) \\ I \end{bmatrix} = G^{\sim}(\lambda) + G(\lambda), \end{aligned}$$

and for imaginary values $i\omega$ we have

$$G^{\sim}(i\omega) = G^*(-\overline{i\omega}) = G^*(i\omega).$$

Also, note that for passivity we have $Q = 0$, $R = 0$, and $S = I$ we obtain that

$$\begin{aligned} \tilde{Q} &= C^TQC = 0, \\ \tilde{S} &= C^TQD + C^TS = C^T, \\ \tilde{R} &= D^TQD + D^TS + S^TD + R = D^T + D, \end{aligned}$$

which proves the claim. □

Note that Corollary 42 and [5, Theorem 13 (i)] differ in one major point. In [5, Theorem 13 (i)] positive realness of the transfer function G is considered, i.e., that $G(\lambda_0)$ is well defined and that $G(\lambda_0) + G^*(\lambda_0) \geq 0$ for all $\lambda_0 \in \overline{\mathbb{C}^+}$. Well definiteness of $G(\lambda_0)$ for all $\lambda_0 \in \overline{\mathbb{C}^+}$ implies that $\lambda E - A$ is stable. In Corollary 42 we do not assume that $\lambda E - A$ is stable and only consider positive semi-definiteness of $G^* + G$ on the imaginary axis. Using Lemma 37 we see that if $E^*X \geq 0$, we already obtain that (37) is positive semi-definite in the closed right half plane. The author was not able to generalize the result that positive semi-definiteness of (37) in the closed right half plane implies the existence of an X such that $E^*X \geq 0$ (unless using exactly the same kind of proof that was used in [5], which would not be a generalization).

5.2 Specialization to regular differential equation systems

Consider the differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{39}$$

with $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, and $D \in \mathbb{R}^{p,m}$.

Corollary 43. *Consider system (39). Assume that (I_n, A, B) is completely controllable and assume that in the supply function (34) the matrix $\tilde{R}$ is invertible.*

Then (39) is dissipative with respect to (34) if and only if there exists an $X = X^T \in \mathbb{C}^{n,n}$ such that the algebraic Riccati inequality

$$0 \leq \tilde{Q} - \tilde{S}\tilde{R}^{-1}\tilde{S}^T + X\left(B\tilde{R}^{-1}\tilde{S}^T - A\right) + \left(\tilde{S}\tilde{R}^{-1}B^T - A^T\right)X - XB\tilde{R}^{-1}B^TX$$

has a solution.

Proof. We have to set $E = I$ in the linear matrix inequality (36), from which we obtain that $Y = 0$ and $X = X^T$. Thus, the linear matrix inequality in (36) becomes

$$0 \quad \leq \quad \begin{bmatrix} \tilde{Q} - A^TX - XA & \tilde{S} - XB \\ \tilde{S}^T - B^TX & \tilde{R} \end{bmatrix} =: \begin{bmatrix} \tilde{F} & \tilde{G} \\ \tilde{G}^T & \tilde{R} \end{bmatrix}. \tag{40}$$

Since $\tilde{R}$ is invertible and positive semi-definite, it is also positive definite and there exists a Cholesky factorization $\tilde{R} = LL^T$ with $L \in \mathbb{C}^{n,n}$ also begin invertible. Thus, the matrix

$$K := \begin{bmatrix} I & 0 \\ -\tilde{R}^{-1}G^T & L^{-T} \end{bmatrix}$$

is invertible and (40) is equivalent to

$$\begin{aligned} 0 \quad &\leq \quad K^T \begin{bmatrix} \tilde{F} & \tilde{G} \\ \tilde{G}^T & \tilde{R} \end{bmatrix} K \\ &= \quad \begin{bmatrix} I & -\tilde{G}\tilde{R}^{-T} \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} \tilde{F} & \tilde{G} \\ \tilde{G}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\tilde{R}^{-1}\tilde{G}^T & L^{-T} \end{bmatrix} \\ &= \quad \begin{bmatrix} I & -\tilde{G}\tilde{R}^{-1} \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} \tilde{F} - \tilde{G}\tilde{R}^{-1}\tilde{G}^T & \tilde{G}L^{-T} \\ 0 & LL^TL^{-T} \end{bmatrix} \\ &= \quad \begin{bmatrix} \tilde{F} - \tilde{G}\tilde{R}^{-1}\tilde{G}^T & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

This shows that (40) is equivalent to

$$\begin{aligned} 0 \quad &\leq \quad \tilde{F} - \tilde{G}\tilde{R}^{-1}\tilde{G} \\ &= \quad \tilde{Q} - A^TX - XA - \left(\tilde{S} - XB\right)\tilde{R}^{-1}\left(\tilde{S}^T - B^TX\right) \\ &= \quad \tilde{Q} - \tilde{S}\tilde{R}^{-1}\tilde{S}^T + X\left(B\tilde{R}^{-1}\tilde{S}^T - A\right) + \left(\tilde{S}\tilde{R}^{-1}B^T - A^T\right)X - XB\tilde{R}^{-1}B^TX, \end{aligned}$$

and thus the claim is shown. □

Remark 44. To make a statement about the solvability of the algebraic Riccati *equation* (instead of inequality in Lemma 43) one would have to move from solutions of the linear matrix inequality to rank minimizing solutions, i.e., to solutions of the Lur'e equation, see [9], to guarantee that the rank of the right hand side in the algebraic Riccati inequality has rank 0.

6 Conclusion

In this paper we have first explored some basic properties of the so called available storage and the required supply of linear systems with quadratic supply term and drawn connections to the problem of dissipativity. We have then shown that the available storage and the required supply are quadratic functions and as such are the solution of a linear matrix inequality (12) which is restricted to the reachable set. Using the Kronecker canonical form, we have derived conditions under which the solvability of the restricted linear matrix inequality implies the solution of an unrestricted/algebraic linear matrix inequality. The unrestricted linear matrix inequality seems to be much easier to handle computationally, since a parameterization of the reachable set will most often not be available. Finally, we have used the obtained results to make statements about deflating subspaces of a para-Hermitian matrix pencil, the spectral factorization of another para-Hermitian rational matrix (called the Popov function or spectral density function), and about state-space descriptor systems, which often occur in practice.

References

[1] B. Anderson and D. Clements. *Singular Optimal Control: The Linear-Quadratic Problem*. Springer-Verlag, 1978.

[2] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM Publications, Philadelphia, PA, 2005.

[3] T. Brüll. Checking dissipativity of behavioral systems using para-Hermitian matrix polynomials. Preprint 683, MATHEON, 2009. `http://www.matheon.de/research/list_preprints.asp`.

[4] T. Brüll. Optimal control of behavior systems. Preprint 684, MATHEON, 2009. `http://www.matheon.de/research/list_preprints.asp`.

[5] D. Chu and R. C. E. Tan. Algebraic characterizations for positive realness of descriptor systems. *SIAM J. Matrix Anal. Appl.*, 30:197–222, 2008.

[6] F.R. Gantmacher. *The Theory of Matrices II*. Chelsea Publishing Company, New York, NY, 1959.

[7] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem*. Springer-Verlag, Berlin, 1991.

[8] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach*. Springer, Berlin, 1998.

[9] T. Reis. Lur'e equations and even matrix pencils. Preprint 672, MATHEON Preprint, 2009. `http://www.matheon.de/research/list_preprints.asp`.

[10] H. L. Trentelman and P. Rapisarda. Pick matrix conditions for sign-definite solutions of the algebraic Riccati equation. *SIAM J. Cont. Optim.*, 40(3):969–991, 2001.

[11] R.A.B. van der Geest and H.L. Trentelman. The Kalman-Yakubovich-Popov lemma in a behavioral framework. *Sys. Control Lett.*, 32(5):283–290, 1997.

[12] J. C. Willems. Dissipative dynamical systems, Part I: General theory. *Arch. Ration. Mech. Anal.*, 45:321–351, 1972.

[13] J. C. Willems. On the existence of a nonpositive solution to the Riccati equation. *IEEE Trans. Automat. Control*, 19:592–593, 1972.

[14] J. C. Willems and H. L. Trentelman. On quadratic differential forms. *SIAM J. Cont. Optim.*, 36(5):1703–1749, 1998.

[15] D. C. Youla. On the factorization of rational matrices. *IRE Trans. Inform. Theory*, 7(3):172–189, 1961.

A Quadraticity of the available storage and the required supply

Looking back at Definition 4 we see that the available storage Θ_+ and the required supply Θ_- both are the solution of an optimal control problem with linear constraints. This suggests that Θ_+ and Θ_- themself might be quadratic functions. In this section we show that this is indeed the case.

Definition A.1. Let $W \subset \mathbb{C}^n$ be a vector space. Then the function

$$B : W \times W \to \mathbb{C}$$

is called a *sesquilinear form on* W if the conditions

$$\begin{aligned} B(x,y) &= \overline{B(y,x)} && (41)\\ B(x,y_1+y_2) &= B(x,y_1)+B(x,y_2) && (42)\\ B(x,\alpha y) &= \overline{\alpha}B(x,y) && (43) \end{aligned}$$

hold for all $x, y, y_1, y_2 \in W$ and all $\alpha \in \mathbb{C}$.

Proposition A.2. *Let $W \subset \mathbb{C}^n$ be a vector space and let $B : W \times W \to \mathbb{C}$ be a sesquilinear form on W. Then there exists a unique Hermitian matrix $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ such that*

$$\begin{aligned} y^*\tilde{X}x &= B(x,y) && \text{for all } x, y \in W, && (44)\\ \tilde{X}x &= 0 && \text{for all } x \perp W. && (45) \end{aligned}$$

In particular, the function $\Theta : W \to \mathbb{R}$ defined by $\Theta(x) := B(x,x)$ is quadratic.

Proof. To see the existence, let $v_1, \ldots, v_m$ be an orthonormal basis of W. Set $V := [v_1, \ldots, v_m]$ and define the matrix $X = [x_{i,j}] \in \mathbb{C}^{m,m}$ through $x_{i,j} = B(v_j, v_i)$. Let $x, y \in V$ be arbitrary. Then there exist coordinate vectors $\alpha, \beta \in \mathbb{C}^m$ such that $x = V\alpha = \sum_{i=1}^m \alpha_i v_i$ and $y = V\beta = \sum_{i=1}^m \beta_i v_i$. This implies that

$$\begin{aligned} B(x,y) &= \sum_{i,j=1}^m B(\alpha_j v_j, \beta_i v_i)\\ &= \sum_{i,j=1}^m \alpha_j \overline{\beta_i} B(v_j, v_i)\\ &= \sum_{i,j=1}^m \alpha_j \overline{\beta_i} x_{i,j}\\ &= \sum_{i=1}^m \overline{\beta_i} \sum_{j=1}^m \alpha_j x_{i,j} \end{aligned}$$

$$= \left[\overline{\beta_1}, \ldots, \overline{\beta_m}\right] \begin{bmatrix} \sum_{j=1}^m \alpha_j x_{1,j} \\ \vdots \\ \sum_{j=1}^m \alpha_j x_{m,j} \end{bmatrix} = \beta^* X \alpha.$$

Set $\tilde{X} := VXV^*$. Then, for the arbitrary $x, y \in W$ from above, we see that $y^*\tilde{X}x = \beta^* V^* V X V^* V \alpha = \beta^* X \alpha = B(x, y)$. Also, we see that for any $x \perp W$, i.e., any $x \in \mathbb{C}^n$ with $V^*x = 0$, we have that $\tilde{X}x = VXV^*x = 0$.
To see the uniqueness let $\tilde{X}_1$ and $\tilde{X}_2$ be two matrices satisfying the properties (44) and (45). Then, for $i = 1, \ldots, n$ the unit vector $e_i \in \mathbb{C}^n$ can be written as $e_i = v_i + w_i$, where $v_i \in W$ and $w_i \perp W$. Thus, for $i, j \in \{1, \ldots, n\}$ and $k = 1, 2$ we have

$$\begin{aligned} e_i^* \tilde{X}_k e_j &= v_i^* \tilde{X}_k v_j + w_i^* \tilde{X}_k v_j + v_i^* \tilde{X}_k w_j + w_i^* \tilde{X}_k w_j \\ &= v_i^* \tilde{X}_k v_j, \end{aligned}$$

due to (45). Because of (44) this implies $e_i^* \tilde{X}_1 e_j = v_i^* \tilde{X}_1 v_j = B(v_j, v_i) = v_i^* \tilde{X}_2 v_j = e_i^* \tilde{X}_2 e_j$, i.e., $\tilde{X}_1 = \tilde{X}_2$. □

The following Lemma is an extension of [1, Lemma II.2.2.] to the complex hermitian case.

Lemma A.3. *Let $W \subset \mathbb{C}^n$ be a subspace and consider a function $\Theta : W \to \mathbb{R}$. Then there exists a unique Hermitian matrix $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ such that*

$$\begin{aligned} x^* \tilde{X} x &= \Theta(x) && \text{for all } x \in W, \\ \tilde{X} x &= 0 && \text{for all } x \perp W, \end{aligned}$$

if and only if for all $\alpha \in \mathbb{C}$, $\mu \in \mathbb{R}^+ \setminus \{1\}$ and all vectors $x_1, x_2 \in W$ we have

$$\Theta(\alpha x_1) = |\alpha|^2 \Theta(x_1), \tag{46}$$

$$\Theta(x_1 + x_2) + \Theta(x_1 - x_2) = 2\Theta(x_1) + 2\Theta(x_2), \tag{47}$$

$$\Theta(x_1 + \mu x_2) - \Theta(x_1 - \mu x_2) = \mu\Theta(x_1 + x_2) - \mu\Theta(x_1 - x_2). \tag{48}$$

Proof. The "only if" part is trivial. For the "if" part let $x_1, x_2, x_3 \in W$ be arbitrary vectors. Then from (47) we obtain

$$\begin{aligned} \Theta(x_1 + x_2) + \Theta(x_1 + x_3) &= \tfrac{1}{2}\left[\Theta(2x_1 + x_2 + x_3) + \Theta(x_2 - x_3)\right], \\ \Theta(x_1 - x_3) + \Theta(x_1 - x_2) &= \tfrac{1}{2}\left[\Theta(2x_1 - x_2 - x_3) + \Theta(x_2 - x_3)\right]. \end{aligned}$$

Subtracting these two equations yields

$$\begin{aligned} &\Theta(x_1 + x_2) - \Theta(x_1 - x_3) + \Theta(x_1 + x_3) - \Theta(x_1 - x_2) \\ = \;& \tfrac{1}{2}\left[\Theta(2x_1 + x_2 + x_3) - \Theta(2x_1 - x_2 - x_3)\right], \end{aligned}$$

which with (46) ($\alpha = -1$) and (48) ($\mu = 2$) is equivalent to

$$\Theta(x_1 + x_2) - \Theta(x_1 - x_3) + \Theta(x_1 + x_3) - \Theta(x_1 - x_2)$$

$$\begin{aligned}
& +\Theta(x_1 - x_2 - x_3) - \Theta(x_1 + x_2 + x_3) \\
= \; & \tfrac{1}{2}\left[\Theta(2x_1 + x_2 + x_3) - 2\Theta(x_1 + x_2 + x_3)\right. \\
& \left. -\Theta(2x_1 - x_2 - x_3) + 2\Theta(x_1 - x_2 - x_3)\right] \\
= \; & \tfrac{1}{2}\left[\Theta(x_2 + x_3 + 2x_1) - \Theta(x_2 + x_3 - 2x_1)\right. \\
& \left. -2\Theta(x_2 + x_3 + x_1) + 2\Theta(x_2 + x_3 - x_1)\right] \\
= \; & 0.
\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}
\Theta(x_1 + x_2) - \Theta(x_1 - x_3) + \Theta(x_1 + x_3) - \Theta(x_1 - x_2) = \\
\Theta(x_1 + x_2 + x_3) - \Theta(x_1 - x_2 - x_3).
\end{aligned} \tag{49}$$

Define the function $B : W \times W \to \mathbb{C}$ by

$$B(x, y) := \Theta(x + y) - \Theta(x - y) + i\left[\Theta(x + iy) - \Theta(x - iy)\right].$$

Then, using (49) we see that for all $x, y, y_1, y_2 \in W$ and all $\alpha \in \mathbb{C}$ we have

$$\begin{aligned}
B(x, y_1 + y_2) &= \Theta(x + y_1 + y_2) - \Theta(x - y_1 - y_2) \\
&\quad + i\left[\Theta(x + iy_1 + iy_2) - \Theta(x - iy_1 - iy_2)\right] \\
&= \Theta(x + y_1) - \Theta(x - y_2) + \Theta(x + y_2) - \Theta(x - y_1) \\
&\quad + i\left[\Theta(x + iy_1) - \Theta(x - iy_2) + \Theta(x + iy_2) - \Theta(x - iy_1)\right] \\
&= B(x, y_1) + B(x, y_2).
\end{aligned}$$

Using (46) we see that

$$\begin{aligned}
B(x, y) &= \Theta(x + y) - \Theta(x - y) + i\left[\Theta(x + iy) - \Theta(x - iy)\right] \\
&= \Theta(y + x) - |-1|^2\,\Theta(y - x) + i\left[|i|^2\,\Theta(y - ix) - |-i|^2\,\Theta(y + ix)\right] \\
&= \Theta(y + x) - \Theta(y - x) - i\left[\Theta(y + ix) - \Theta(y - ix)\right] \\
&= \overline{B(y, x)}.
\end{aligned}$$

Also, using (49) again, we see that condition (48) is equivalent to

$$\begin{aligned}
& \Theta(x_1 + \beta x_2) - \Theta(x_1 - \beta x_2) \\
= \; & \Theta(x_1 + \mathrm{Re}\{\beta\}\, x_2 + i\mathrm{Im}\{\beta\}\, x_2) - \Theta(x_1 - \mathrm{Re}\{\beta\}\, x_2 - i\mathrm{Im}\{\beta\}\, x_2) \\
= \; & \Theta(x_1 + \mathrm{Re}\{\beta\}\, x_2) - \Theta(x_1 - i\mathrm{Im}\{\beta\}\, x_2) \\
& +\Theta(x_1 + i\mathrm{Im}\{\beta\}\, x_2) - \Theta(x_1 - \mathrm{Re}\{\beta\}\, x_2) \\
= \; & \Theta(x_1 + \mathrm{Re}\{\beta\}\, x_2) - \Theta(x_1 - \mathrm{Im}\{\beta\}\, ix_2) \\
& +\Theta(x_1 + \mathrm{Im}\{\beta\}\, ix_2) - \Theta(x_1 - \mathrm{Re}\{\beta\}\, x_2) \\
= \; & \Theta(x_1 + \mathrm{Re}\{\beta\}\, x_2) - \Theta(x_1 - \mathrm{Re}\{\beta\}\, x_2) \\
& +\Theta(x_1 + \mathrm{Im}\{\beta\}\, ix_2) - \Theta(x_1 - \mathrm{Im}\{\beta\}\, ix_2) \\
= \; & \mathrm{Re}\{\beta\}\,\Theta(x_1 + x_2) - \mathrm{Re}\{\beta\}\,\Theta(x_1 - x_2)
\end{aligned}$$

$$\begin{aligned} & +\mathrm{Im}\,\{\beta\}\,\Theta(x_1+ix_2) - \mathrm{Im}\,\{\beta\}\,\Theta(x_1-ix_2) \\ = \;& \mathrm{Re}\,\{\beta\}\,(\Theta(x_1+x_2)-\Theta(x_1-x_2)) \\ & +\mathrm{Im}\,\{\beta\}\,(\Theta(x_1+ix_2)-\Theta(x_1-ix_2)), \end{aligned}$$

for all $\beta \in \mathbb{C}$ and all $x_1, x_2 \in W$. Thus, we see that

$$\begin{aligned} B(x,\alpha y) &= \Theta(x+\alpha y) - \Theta(x-\alpha y) + i\left[\Theta(x+i\alpha y) - \Theta(x-i\alpha y)\right] \\ &= \mathrm{Re}\,\{\alpha\}\,(\Theta(x+y)-\Theta(x-y)) + \mathrm{Im}\,\{\alpha\}\,(\Theta(x+iy)-\Theta(x-iy)) \\ &+ i\mathrm{Re}\,\{\alpha\}\,(\Theta(x+iy)-\Theta(x-iy)) + i\mathrm{Im}\,\{\alpha\}\,(\Theta(x+iiy)-\Theta(x-iiy)) \\ &= \mathrm{Re}\,\{\alpha\}\,(\Theta(x+y)-\Theta(x-y)) - i\mathrm{Im}\,\{\alpha\}\,(\Theta(x+y)-\Theta(x-y)) \\ &+ i\mathrm{Re}\,\{\alpha\}\,(\Theta(x+iy)-\Theta(x-iy)) + \mathrm{Im}\,\{\alpha\}\,(\Theta(x+iy)-\Theta(x-iy)) \\ &= \overline{\alpha}(\Theta(x+y)-\Theta(x-y)) + i\overline{\alpha}(\Theta(x+iy)-\Theta(x-iy)) = \overline{\alpha}B(x,y), \end{aligned}$$

which shows that B is a sesquilinear form. With Proposition A.2 this proves the existence of a unique $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ such that $B(x,y) = 4y^*\tilde{X}x$ for all $x, y \in W$ and $\tilde{X}x = 0$ for all $x \perp W$. Thus for all $x \in W$ we have

$$\begin{aligned} x^*\tilde{X}x &= \tfrac{1}{4}B(x,x) = \tfrac{1}{4}\left(\Theta(2x) - \Theta(0) - i\left[\Theta(x+ix) - \Theta(x-ix)\right]\right) \\ &= \tfrac{1}{4}4\Theta(x) - 0 - i\left[\Theta(x+ix) - |i|^2\Theta(x-ix)\right] \\ &= \Theta(x) - i\left[\Theta(x+ix) - \Theta(i(x-ix))\right] \\ &= \Theta(x) - i\left[\Theta(x+ix) - \Theta(ix+x))\right] = \Theta(x), \end{aligned}$$

where we used (46) extensively. This proves the claim. □

Proposition A.4. *Let $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$ be a Hermitian matrix such that for all $x \in \mathbb{C}^n$ we have $x^*Xx = 0$. Then $X = 0$.*

Proof. With the notation $X = [x_{i,j}]_{i,j=1,\ldots,n}$ we first see by choosing x to be the i-th unit vector e_i that the diagonal elements vanish, i.e., that $x_{i,i} = 0$ for $i = 1, \ldots, n$. To show that the off diagonal elements vanish, let $x := e_k + e_l$ with $k, l = 1, \ldots, n$. Then we have

$$\begin{aligned} 0 &= x^*Xx = x_{k,k} + x_{k,l} + x_{l,k} + x_{l,l} \\ &= x_{k,l} + \overline{x_{k,l}} = \mathrm{Re}\,\{x_{k,l}\} \end{aligned}$$

and analogously, by choosing $x := e_k + ie_l$ that

$$\begin{aligned} 0 &= x^*Xx = x_{k,k} + ix_{k,l} - ix_{l,k} - x_{l,l} \\ &= ix_{k,l} - i\overline{x_{k,l}} = ix_{k,l} - \overline{ix_{k,l}} \\ &= \mathrm{Re}\,\{ix_{k,l}\} = -\mathrm{Im}\,\{x_{k,l}\}, \end{aligned}$$

which implies $x_{k,l} = 0$. This proves the claim. □

Lemma A.5. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Then the available storage and the required supply of (F, G, H) satisfy the following properties:*

1. *For all $z_1 \in \mathfrak{B}_+(F,G)$, $t \in \mathbb{R}$, and $\alpha \in \mathbb{C} \setminus \{0\}$ we have*

$$\Theta_+(\alpha z_1(t)) = |\alpha|^2 \Theta_+(z_1(t)),$$

while at the same time for all $z_2 \in \mathfrak{B}_-(F,G)$, $t \in \mathbb{R}$, and $\alpha \in \mathbb{C} \setminus \{0\}$ we have

$$\Theta_-(\alpha z_2(t)) = |\alpha|^2 \Theta_-(z_2(t)),$$

(the case $\alpha = 0$ has to be excluded, since $0 \cdot \infty$ is undefined).

2. *If (F,G) is dissipative then we have $\Theta_+(0) = 0$ and $\Theta_-(0) = 0$.*

Proof. First note that with $\alpha \neq 0$ we also have that $\alpha z_1 \in \mathfrak{B}_+(F,G)$ and thus

$$\begin{aligned}
\Theta_+(\alpha z_1(t)) &= -\inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=\alpha z_1(t)}} \int_0^\infty z^*(t) H z(t) dt \\
&= -\inf_{\substack{\alpha z \in \mathfrak{B}_+(F,G) \\ \alpha z(0)=\alpha z_1(t)}} \int_0^\infty (\alpha z(t))^* H (\alpha z(t)) \, dt \\
&= -\inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=z_1(t)}} |\alpha|^2 \int_0^\infty z^*(t) H z(t) dt \\
&= -|\alpha|^2 \inf_{\substack{z \in \mathfrak{B}_+(F,G) \\ z(0)=z_1(t)}} \int_0^\infty z^*(t) H z(t) dt \\
&= |\alpha|^2 \Theta_+(z_1(t))
\end{aligned}$$

and analogously we find that $\Theta_-(\alpha z_2(t)) = |\alpha|^2 \Theta_-(z_2(t))$.

To see the second part we first note that $\Theta_+(0) \geq 0$ since the trivial trajectory $0 \in \mathfrak{B}_+(F,G)$ is exponentially decaying. Also we see that $\Theta_-(0) \leq 0$ since $0 \in \mathfrak{B}_-(F,G)$. Using Lemma 9 and the dissipativity this leads to

$$0 \leq \Theta_+(0) \leq \Theta_-(0) \leq 0,$$

and thus the claim follows. □

The following Lemma A.6 is a modification of [1, Theorem II.2.1.].

Lemma A.6. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let W be a linear subspace of $\mathbb{C}^q$ such that the available storage or the required supply is finite on W, i.e., assume that for all $\hat{z} \in W$ we have $\Theta_+(\hat{z}) \in \mathbb{R}$ or $\Theta_-(\hat{z}) \in \mathbb{R}$. Then, there exists a matrix $\tilde{X}_a = \tilde{X}_a^* \in \mathbb{C}^{q,q}$ such that*

$$\Theta_+(\hat{z}) = \hat{z}^* \tilde{X}_a \hat{z},$$

for all $\hat{z} \in W$ or a matrix $\tilde{X}_r = \tilde{X}_r^* \in \mathbb{C}^{q,q}$ such that

$$\Theta_-(\hat{z}) = \hat{z}^* \tilde{X}_r \hat{z},$$

for all $\hat{z} \in W$, respectively.

Proof. We show the three conditions from Lemma A.3. Condition (46) is fulfilled due to Lemma A.5. To see condition (48) let $\hat{z}_1, \hat{z}_2 \in W$ and $\mu \in \mathbb{R}^+$ with $\mu \neq 1$ be arbitrary. Define the function $\theta_+ : \mathfrak{B}_+ \to \mathbb{R}$ through

$$\theta_+(z) := \int_0^\infty z^*(t) H z(t) dt.$$

For all $z_1, z_2 \in \mathfrak{B}_+$ we have

$$\begin{aligned}
&\theta_+(z_1 + \mu z_2) + \mu\theta_+(z_1 - z_2) \\
=&\int_0^\infty (z_1(t) + \mu z_2(t))^* H (z_1(t) + \mu z_2(t)) dt + \\
&\mu \int_0^\infty (z_1(t) - z_2(t))^* H (z_1(t) - z_2(t)) dt \\
=&(1+\mu) \int_0^\infty z_1^*(t) H z_1(t) dt + (|\mu|^2 + \mu) \int_0^\infty z_2^*(t) H z_2(t) dt + \\
&\mathrm{Re}\left\{ \int_0^\infty \mu z_2^*(t) H z_1(t) dt \right\} - \mu \mathrm{Re}\left\{ \int_0^\infty z_2^*(t) H z_1(t) dt \right\} \\
=&(1+\mu) \int_0^\infty z_1^*(t) H z_1(t) dt + (|\mu|^2 + \mu) \int_0^\infty z_2^*(t) H z_2(t) dt
\end{aligned}$$

since μ is assumed to be real. Analogously we obtain that

$$\begin{aligned}
&\mu\theta_+(z_1 + z_2) + \theta_+(z_1 - \mu z_2) \\
=&\mu \int_0^\infty (z_1(t) + z_2(t))^* H (z_1(t) + z_2(t)) dt + \\
&\int_0^\infty (z_1(t) - \mu z_2(t))^* H (z_1(t) - \mu z_2(t)) dt \\
=&(\mu+1) \int_0^\infty z_1^*(t) H z_1(t) dt + (\mu + |\mu|^2) \int_0^\infty z_2^*(t) H z_2(t) dt
\end{aligned}$$

This shows that we have

$$\begin{array}{rcl} \theta_+(z_1 + \mu z_2) & + & \mu\theta_+(z_1 - z_2) \\ & = & \mu\theta_+(z_1 + z_2) + \theta_+(z_1 - \mu z_2), \end{array} \tag{50}$$

for all $z_1, z_2 \in \mathfrak{B}_+$. Let $\epsilon > 0$ be arbitrary and choose $z_3, z_4 \in \mathfrak{B}_+$ such that $z_3(0) = \hat{z}_1 + \hat{z}_2$, $z_4(0) = \hat{z}_1 - \mu\hat{z}_2$, and

$$\theta_+(z_3) \leq -\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon, \tag{51}$$

$$\theta_+(z_4) \leq -\Theta_+(\hat{z}_1 - \mu\hat{z}_2) + \epsilon. \tag{52}$$

Set $\tilde{z}_1 := \frac{1}{1+\mu}(\mu z_3 + z_4)$ and $\tilde{z}_2 := \frac{1}{1+\mu}(z_3 - z_4)$. Then we have

$$\tilde{z}_1 + \tilde{z}_2 = \frac{1}{1+\mu}(\mu z_3 + z_4) + \frac{1}{1+\mu}(z_3 - z_4) = z_3,$$

$$\tilde{z}_1 - \mu\tilde{z}_2 = \frac{1}{1+\mu}(\mu z_3 + z_4) - \frac{\mu}{1+\mu}(z_3 - z_4) = z_4,$$

and also

$$\begin{aligned}\tilde{z}_1(0) &= \left(\frac{1}{1+\mu}(\mu z_3 + z_4)\right)(0) = \frac{\mu}{1+\mu}z_3(0) + \frac{1}{1+\mu}z_4(0)\\ &= \frac{\mu}{1+\mu}(\hat{z}_1 + \hat{z}_2) + \frac{1}{1+\mu}(\hat{z}_1 - \mu\hat{z}_2) = \hat{z}_1\\ \tilde{z}_2(0) &= \ldots = \hat{z}_2.\end{aligned}$$

This shows that using (51), (52) and (50) we can obtain

$$\begin{aligned}-\Theta_+(\hat{z}_1 + \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 - \hat{z}_2) &\leq \theta_+(\tilde{z}_1 + \mu\tilde{z}_2) + \mu\theta_+(\tilde{z}_1 - \tilde{z}_2)\\ &= \mu\theta_+(\tilde{z}_1 + \tilde{z}_2) + \theta_+(\tilde{z}_1 - \mu\tilde{z}_2)\\ &= \mu\theta_+(z_3) + \theta_+(z_4)\\ &\leq \mu(-\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon) - \Theta_+(\hat{z}_1 - \mu\hat{z}_2) + \epsilon\\ &= -\mu\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \mu\hat{z}_2) + (1+\mu)\epsilon.\end{aligned}$$

For $\epsilon \to 0$ this gives

$$-\Theta_+(\hat{z}_1 + \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 - \hat{z}_2) \leq -\mu\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \mu\hat{z}_2). \tag{53}$$

Let $\epsilon > 0$ again be arbitrary and choose $z_5, z_6 \in \mathfrak{B}_+$ such that $z_5(0) = \hat{z}_1 + \mu\hat{z}_2$, $z_6(0) = \hat{z}_1 - \hat{z}_2$, and

$$\theta_+(z_5) \leq -\Theta_+(\hat{z}_1 + \mu\hat{z}_2) + \epsilon, \tag{54}$$

$$\theta_+(z_6) \leq -\Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon. \tag{55}$$

Setting $\tilde{z}_1 := \frac{1}{1+\mu}(z_5 + \mu z_6)$ and $\tilde{z}_2 := \frac{1}{1+\mu}(z_5 - z_6)$, we have

$$\begin{aligned}\tilde{z}_1 + \mu\tilde{z}_2 &= \frac{1}{1+\mu}(z_5 + \mu z_6) + \frac{\mu}{1+\mu}(z_5 - z_6) = z_5,\\ \tilde{z}_1 - \tilde{z}_2 &= \frac{1}{1+\mu}(z_5 + \mu z_6) - \frac{1}{1+\mu}(z_5 - z_6) = z_6.\end{aligned}$$

and also

$$\begin{aligned}\tilde{z}_1(0) &= \left(\frac{1}{1+\mu}(z_5 + \mu z_6)\right)(0) = \frac{1}{1+\mu}z_5(0) + \frac{\mu}{1+\mu}z_6(0)\\ &= \frac{1}{1+\mu}(\hat{z}_1 + \mu\hat{z}_2) + \frac{\mu}{1+\mu}(\hat{z}_1 - \hat{z}_2) = \hat{z}_1,\\ \tilde{z}_2(0) &= \ldots = \hat{z}_2.\end{aligned}$$

This shows that using (54), (55) and (50) we can obtain

$$-\Theta_+(\hat{z}_1 - \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 + \hat{z}_2) \leq \theta_+(\tilde{z}_1 - \mu\tilde{z}_2) + \mu\theta_+(\tilde{z}_1 + \tilde{z}_2)$$

$$\begin{aligned} &= \mu\theta_+(\tilde{z}_1 - \tilde{z}_2) + \theta_+(\tilde{z}_1 + \mu\tilde{z}_2) \\ &= \mu\theta_+(z_6) + \theta_+(z_5) \\ &\leq \mu(-\Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2) + \epsilon \\ &= -\mu\Theta_+(\hat{z}_1 - \hat{z}_2) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2) + (1+\mu)\epsilon. \end{aligned}$$

For $\epsilon \to 0$ this gives

$$-\Theta_+(\hat{z}_1 - \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 + \hat{z}_2) \leq -\mu\Theta_+(\hat{z}_1 - \hat{z}_2) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2). \tag{56}$$

Combining (53) and (56) proves that

$$-\Theta_+(\hat{z}_1 - \mu\hat{z}_2) - \mu\Theta_+(\hat{z}_1 + \hat{z}_2) = -\mu\Theta_+(\hat{z}_1 - \hat{z}_2) - \Theta_+(\hat{z}_1 + \mu\hat{z}_2),$$

and thus condition (48) is fulfilled. To see condition (47) let $\hat{z}_1, \hat{z}_2 \in W$ be arbitrary. Then the identity

$$\theta_+(z_1 + z_2) + \theta_+(z_1 - z_2) = 2\theta_+(z_1) + 2\theta_+(z_2), \tag{57}$$

can be shown for all $z_1, z_2 \in \mathfrak{B}_+$ as in (50). With this let $\epsilon > 0$ be arbitrary and let $z_3, z_4 \in \mathfrak{B}_+$ be such that $z_3(0) = \hat{z}_1 + \hat{z}_2$, $z_4(0) = \hat{z}_1 - \hat{z}_2$, and

$$\theta_+(z_3) \leq -\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon, \tag{58}$$
$$\theta_+(z_4) \leq -\Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon. \tag{59}$$

Set $\tilde{z}_1 := \frac{z_3 + z_4}{2}$ and $\tilde{z}_2 := \frac{z_3 - z_4}{2}$. Then we have

$$\begin{aligned} \tilde{z}_1 + \tilde{z}_2 &= z_3, \\ \tilde{z}_1 - \tilde{z}_2 &= z_4, \end{aligned}$$

as well as

$$\begin{aligned} \tilde{z}_1(0) &= \tfrac{1}{2}z_3(0) + z_4(0) = \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_2 + \tfrac{1}{2}\hat{z}_1 - \tfrac{1}{2}\hat{z}_2 = \hat{z}_1 \\ \tilde{z}_2(0) &= \tfrac{1}{2}z_3(0) - z_4(0) = \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_2 - \tfrac{1}{2}\hat{z}_1 + \tfrac{1}{2}\hat{z}_2 = \hat{z}_2. \end{aligned}$$

This shows that using (58), (59) and (57) we have that

$$\begin{aligned} -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2) &\leq 2\theta_+(\tilde{z}_1) + 2\theta_+(\tilde{z}_2) \\ &= \theta_+(\tilde{z}_1 + \tilde{z}_2) + \theta_+(\tilde{z}_1 - \tilde{z}_2) \\ &= \theta_+(z_3) + \theta_+(z_4) \\ &\leq -\Theta_+(\hat{z}_1 + \hat{z}_2) + \epsilon - \Theta_+(\hat{z}_1 - \hat{z}_2) + \epsilon \\ &= -\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) + 2\epsilon. \end{aligned}$$

For $\epsilon \to 0$ this gives

$$-2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2) \leq -\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2). \tag{60}$$

Let $\epsilon > 0$ again be arbitrary and let z_5, z_6 be such that $z_5(0) = \hat{z}_1$, $z_6(0) = \hat{z}_2$, and

$$\begin{aligned} \theta_+(z_5) &\leq -\Theta_+(\hat{z}_1) + \epsilon, & (61)\\ \theta_+(z_6) &\leq -\Theta_+(\hat{z}_2) + \epsilon. & (62) \end{aligned}$$

Using (61), (62) and (50) we see that

$$\begin{aligned} -\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) &\leq \theta_+(z_5 + z_6) + \theta_+(z_5 - z_6)\\ &= 2\theta_+(z_5) + 2\theta_+(z_6)\\ &\leq -2\Theta_+(\hat{z}_1) + 2\epsilon - 2\Theta_+(\hat{z}_2) + 2\epsilon\\ &\leq -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2) + 4\epsilon. \end{aligned}$$

For $\epsilon \to 0$ this gives

$$-\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) \leq -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2). \quad (63)$$

Combining (60) and (63) proves that

$$-\Theta_+(\hat{z}_1 + \hat{z}_2) - \Theta_+(\hat{z}_1 - \hat{z}_2) = -2\Theta_+(\hat{z}_1) - 2\Theta_+(\hat{z}_2),$$

and thus condition (47) is fulfilled. The proof for the required supply works analogously. □

The above results show that the available storage and the required supply are quadratic functions. In the rest of the section we will show that the quadratic matrix $\tilde{X}$ has a special form.

Definition A.7. Let $F \in \mathbb{C}^{m,n}$ and let $W \subset \mathbb{C}^n$ be a subspace. Then a function $\Theta : W \to \mathbb{R}$ is called F-neutral on W if it has the property

$$\Theta(x + y) = \Theta(x), \quad (64)$$

for all $x, y \in W$ with $y \in \text{kernel}\,(F)$.

Lemma A.8. *Let $\tilde{X} = \tilde{X}^* \in \mathbb{C}^{n,n}$, $F \in \mathbb{C}^{m,n}$, and let $W \subset \mathbb{C}^n$ be a linear subspace. Consider the quadratic function $\Theta(x) := x^*\tilde{X}x$. Assume that Θ is F-neutral on W and assume that $\tilde{X}x = 0$ for all $x \perp W$. Then, there exist matrices $Z \in \mathbb{C}^{m,n}$ and $X = X^* \in \mathbb{C}^{m,m}$ such that*

$$\begin{aligned} \tilde{X} &= F^*Z = Z^*F\\ &= F^*XF = F^*X^*F. \end{aligned}$$

Proof. Let the columns of $V_1 \in \mathbb{C}^{n,r}$ form an orthonormal basis of the linear vector space $\mathcal{V}_1 := \text{cokernel}\,(F) \cap W$, let the columns of $V_2 \in \mathbb{C}^{n,s}$ form an orthonormal basis of the linear vector space $\mathcal{V}_2 := \text{kernel}\,(F) \cap W$, and let $V_3 \in \mathbb{C}^{n,t}$ form an orthonormal basis of the linear vector space $\mathcal{V}_3 := W^\perp$. It is easy to see that $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are orthogonal to each other. Also one may verify that $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 = \mathbb{C}^n$. Thus the

matrix $V := \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \in \mathbb{C}^{n,n}$ is unitary and $r+s+t=n$. Since Θ is F-neutral on W we obtain that

$$\Theta(V_1\alpha_1 + V_2\alpha_2) = \Theta(V_1\alpha_1),$$

for all $\alpha_1 \in \mathbb{C}^r$ and $\alpha_2 \in \mathbb{C}^s$. Since $\tilde{X}V_3 = 0$ we also have

$$\begin{aligned} \Theta(v + V_3\alpha_3) &= (v + V_3\alpha_3)^* \tilde{X} (v + V_3\alpha_3) \\ &= v^*\tilde{X}v + v^*\tilde{X}V_3\alpha_3 + \alpha_3^*V_3^*\tilde{X}v + \alpha_3^*V_3^*\tilde{X}V_3\alpha_3 \\ &= v^*\tilde{X}v = \Theta(v) \end{aligned}$$

for all $\alpha_3 \in \mathbb{C}^t$ and $v \in \mathbb{C}^n$. This especially implies that

$$\begin{aligned} \Theta\left(V \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}\right) &= \Theta\left(V \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} + V_3\alpha_3\right) = \Theta\left(V \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}\right) \\ &= \Theta\left(V_1\alpha_1 + V_2\alpha_2\right) = \Theta\left(V_1\alpha_1\right) = \Theta\left(V \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix}\right). \end{aligned}$$

Introducing the notation

$$V^*\tilde{X}V =: \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix},$$

with partitioning analogous to the partitioning of V, we deduce that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}^* \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_1^*X_{11}\alpha_1 = \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix}^* \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix},$$

which, with Proposition A.4, shows that

$$V^*\tilde{X}V = \begin{bmatrix} \tilde{X}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the vectors of V_1 span part of cokernel (F) there exists a matrix $G \in \mathbb{C}^{m,r}$ such that $V_1 = F^*G$. With this we have

$$\begin{aligned} \tilde{X} &= V \begin{bmatrix} \tilde{X}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^* = \begin{bmatrix} V_1X_{11} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} \\ &= V_1X_{11}V_1^* = F^* \underbrace{GX_{11}V_1^*}_{=:Z} = F^*Z, \end{aligned}$$

and also $F^*Z = Z^*F$ since $\tilde{X} = \tilde{X}^*$. Finally, observe that with $X := GX_{11}G^*$ we have $\tilde{X} = F^*XF$ and $X = X^*$ which finishes the proof. □

Lemma A.9. *Let $F, G \in \mathbb{C}^{p,q}$ be a first-order matrix polynomial and let $H = H^* \in \mathbb{C}^{q,q}$. Then the available storage and the required supply are F-neutral on every subspace where they are finite.*

Proof. Let $W \in \mathbb{C}^q$ be a subspace where the available storage of the required supply is finite. Using Theorem 10 we find that for all $\hat{z}, \hat{y} \in W$ with $\hat{y} \in \text{kernel}\,(F)$ we have $F\hat{y} = 0$ and thus

$$\begin{aligned}
\Theta_+(\hat{z}+\hat{y}) &= -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(0)=\hat{z}+\hat{y}}} \int_0^\infty z^*(t)Hz(t)dt \\
&= -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ Fz(0)=F\hat{z}+F\hat{y}}} \int_0^\infty z^*(t)Hz(t)dt \\
&= -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ Fz(0)=F\hat{z}}} \int_0^\infty z^*(t)Hz(t)dt \\
&= -\inf_{\substack{z\in\mathfrak{B}_+(F,G)\\ z(0)=\hat{z}}} \int_0^\infty z^*(t)Hz(t)dt = \Theta_+(\hat{z}),
\end{aligned}$$

which shows that Θ_+ is F-neutral. The proof for Θ_- works analogously. □

Theorem A.10. *Let $F, G \in \mathbb{C}^{p,q}$ and $H = H^* \in \mathbb{C}^{q,q}$. Let (F, G, H) be dissipative. Then there exist matrices $Z_+ \in \mathbb{C}^{p,q}$ and $X_+ = X_+^* \in \mathbb{C}^{q,q}$ such that $F^*Z_+ = Z_+^*F$ and*

$$\begin{aligned}
\Theta_+(\hat{z}_1) &= \hat{z}_1^*F^*Z_+\hat{z}_1 \\
&= \hat{z}_1^*F^*X_+F\hat{z}_1
\end{aligned}$$

for all $\hat{z}_1 \in R_c(F, G)$ and there exist matrices $Z_- \in \mathbb{C}^{p,q}$ and $X_- = X_-^* \in \mathbb{C}^{q,q}$ such that $F^*Z_- = Z_-^*F$ and

$$\begin{aligned}
\Theta_-(\hat{z}_2) &= \hat{z}_2^*F^*Z_-\hat{z}_2 \\
&= \hat{z}_2^*F^*X_-F\hat{z}_2
\end{aligned}$$

for all $\hat{z}_2 \in R_c(F, G)$.

Proof. With Remark 6 and 9 we find that the available storage Θ_+ and the required supply Θ_- are both finite on the compact reachable set $R_c(F, G)$. Using Lemma A.6 we conclude that there exist matrices $\tilde{X}_+, \tilde{X}_- \in \mathbb{C}^{q,q}$ such that

$$\begin{aligned}
\Theta_+(\hat{z}) &= \hat{z}^*\tilde{X}_+\hat{z}, \\
\Theta_-(\hat{z}) &= \hat{z}^*\tilde{X}_-\hat{z},
\end{aligned}$$

for all $\hat{z} \in R_c(F, G)$. With Lemma A.9 and Lemma A.8 we deduce the existence of $Z_+, Z_- \in \mathbb{C}^{p,q}$ as in the assertion. □