Monetary valuation of cash flows under Knightian uncertainty.

Hans Föllmer* Irina Penner†

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Abstract

The classical valuation of an uncertain cash flow in discrete time consists in taking the expectation of the sum of the discounted future payoffs under a fixed probability measure, which is assumed to be known. Here we discuss the valuation problem in the context of Knightian uncertainty. Using results from the theory of convex risk measures, but without assuming the existence of a global reference measure, we derive a robust representation of concave valuations with an infinite time horizon, which specifies the interplay between model uncertainty and uncertainty about the time value of money.

1 Introduction

For an uncertain nonnegative cash flow $(\tilde{C}_t)_{t=0,1,\dots}$, for a risk free interest rate $r > 0$, and at any time t, the classical risk neutral valuation of the future cash flow takes the form

$$\tilde{V}_t := E_P\Big[\sum_{s=t}^{\infty} \frac{\tilde{C}_s}{(1+r)^{s-t}}\big|\mathcal{F}_t\Big]. \tag{1}$$

Here $\mathcal{F}_t$ denotes the information available at time t, and P is a given probability measure assumed to be known. Passing to the discounted quantities

$$V_t := \frac{\tilde{V}_t}{(1+r)^t} \quad \text{and} \quad C_t := \frac{\tilde{C}_t}{(1+r)^t},$$

the valuation formula (1) takes the simpler form

$$V_t = E_P\Big[\sum_{s=t}^{\infty} C_s\big|\mathcal{F}_t\Big]. \tag{2}$$

In this paper we discuss the valuation problem in a situation of "Knightian uncertainty", or model ambiguity, where there is no canonical choice of an underlying probability measure P. A systematic approach to such situations has been developed in the theory of coherent and, more generally, convex risk

*Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany. Email: foellmer@math.hu-berlin.de.

†Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany. Email: penner@math.hu-berlin.de. Supported by the DFG Research Center Matheon "Mathematics for key technologies".

measures; cf. [3, 4, 9, 10, 13, 14, 15, 11]. This suggests to focus on concave monetary valuations defined in terms of convex risk measure on the space of bounded adapted processes; cf. [7, 8, 5, 16, 18, 1]. From this general point of view, the valuation formula (2) turns out to be a special case of the robust valuation formula

$$V_t(C) = \min_{Q,D} \left(E_Q\Big[\sum_{s=t}^{\infty} C_s D_s | \mathcal{F}_t\Big] + \alpha_t(Q, D) \right) \tag{3}$$

in terms of a whole class of probability measures Q, a class of predictable discounting processes $D = (D_t)_{t=0,1,\ldots}$, and a penalty function α_t. Such a representation involves both model ambiguity, as reflected in the multiplicity of Q, and discounting ambiguity, as described by different discounting processes D, which come on top of the classical discounting by the money market account.

In Section 3 we focus on the unconditional case at the initial time 0. Here the valuation $V := V_0$ takes the form

$$V(C) = U(X) := -\rho(X),$$

where $X = (X_t)_{t=0,1,\ldots}$ is the cumulated cash flow associated to $C = (C_t)_{t=0,1,\ldots}$ by $X_t = \sum_{s=0}^{t} C_s$, and where ρ is a convex risk measure on the space of bounded processes which are adapted to the given filtration. Our aim is to derive the valuation formula (3) under full model uncertainty. This means that, in contrast to the closely related representation results in [1], we do *not* assume the existence of a global reference measure P which fixes the class of null sets, and thus allows one to use the standard techniques of risk measures on L^∞ as in [9, 10, 14, 15, 11, 6, 12, 8, 2, 7, 1]. Under a regularity assumption which is weaker than global continuity from below, we prove the representation (3) for the concave monetary valuation V, viewed as a functional on a suitable class of discounted cash flows. This involves a careful look at a class of finitely additive probability measures on the optional σ-field, and in particular a decomposition theorem for the corresponding σ-additive probability measures on an enlarged product space.

In Section 4 we sketch the extension to dynamic valuations. Using results from [1, 12, 2], we discuss the characterization of time consistency by supermartingale properties of the penalty processes arising in the robust representation (3). In this context we identify two potential sources of "bubbles". The first source is the penalization process $(\alpha_t(Q, D))_{t=0,1,\ldots}$. Time consistency implies that it can be decomposed into a potential and into an additional martingale which may be viewed as an excessive neglect of the model (Q, D), and which may introduce a component of "exuberance" in the valuation. The second source already appears in the classical case, where we have fixed a probability model P. It consists in taking into account the option to sell the cash flow at some future time, and this may create a bubble on top of the fundamental "buy and hold" valuation in (2) and (3).

2 Preliminaries

Consider a filtration $(\mathcal{F}_t)_{t=0,1,\ldots}$ on some measurable space $(\Omega, \mathcal{F})$, such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F} = \mathcal{F}_\infty := \sigma\left(\bigcup_{t=0}^{\infty} \mathcal{F}_t\right).$$

Note that we do not fix a reference measure P on $(\Omega, \mathcal{F})$.

Let $\mathcal{X}$ denote the space of all adapted processes $X = (X_t)_{t=0,1,\dots}$ such that

$$\|X\| := \sup_{\omega,t} |X_t(\omega)| < \infty.$$

We also consider the subspaces

$$\mathcal{X}_t := \left\{ X \in \mathcal{X} \mid X_s = X_t \ \forall\, s \geq t \right\}, \quad t = 0, 1, \dots$$

and

$$\mathcal{X}_\infty := \left\{ X \in \mathcal{X} \mid \exists\ X_\infty(\omega) := \lim_{t\to\infty} X_t(\omega)\ \forall\, \omega \in \Omega \right\}.$$

The processes in $\mathcal{X}$ are regarded as cumulated discounted cash flows. Via

$$C_t := X_t - X_{t-1}, \qquad X_t := \sum_{s=0}^{t} C_s,$$

the space $\mathcal{X}_\infty$ can be identified with the space

$$\mathcal{C} := \left\{ C = (C_t)_{t=0,1,\dots} \in \mathcal{X} \mid \exists\ \sum_{t=0}^{\infty} C_t(\omega)\ \forall\, \omega \in \Omega \right\}$$

of discounted cash flows whose sum converges pointwise. As explained in the following remark, any bounded adapted cash flow discounted by a suitable numéraire is in fact an element of $\mathcal{C}$.

Remark 1. *Assume that there is a money market account*

$$N_t := \prod_{s=1}^{t} (1 + r_s) \qquad t = 0, 1, \dots$$

generated by a predictable process $(r_t)_{t=0,1,\dots}$ of nonnegative short rates bounded away from zero by some constant $\delta > 0$. Consider an uncertain adapted cash flow $\tilde{C} = (\tilde{C}_t)_{t=0,1,\dots}$ such that $\|\tilde{C}\| < \infty$. Using $N = (N_t)_{t=0,1,\dots}$ as a numéraire, the discounted cash flow *$C = (C_t)_{t=0,1,\dots}$ defined by*

$$C_t = N_t^{-1} \tilde{C}_t$$

satisfies

$$\sum_{t=0}^{\infty} \|C_t\| \leq \frac{1}{\delta} \|\tilde{C}\| < \infty,$$

and hence belongs to the space $\mathcal{C}$.

We focus on the valuation of a cumulated cash flow at the initial time $t = 0$, using the notation $U(X) := V(C)$, where $X \in \mathcal{X}$ is the cumulated cash flow associated to C.

Definition 2. *A map $U : \mathcal{X} \to \mathbb{R}$ is called a* concave monetary valuation *if U is*

- *cash invariant, i.e.,*

  $$U(X + m 1_{\{0,1,\dots\}}) = U(X) + m$$

  for all $m \in \mathbb{R}$;

- *monotone, i.e.,*
$$X \le Y \quad \Rightarrow \quad U(X) \le U(Y);$$
- *concave, i.e.,*
$$U(\lambda X + (1-\lambda)Y) \ge \lambda U(X) + (1-\lambda)U(Y)$$
for $X, Y \in \mathcal{X}$ *and* $\lambda \in [0,1]$*;*
- *normalized, i.e.,* $U(0I_{\{0,1,\dots\}}) = 0$.

The concave set
$$\mathcal{A} := \left\{ X \in \mathcal{X} \,\middle|\, U(X) \ge 0 \right\}$$
is called the acceptance set *of* U*.*

Let us fix a concave monetary valuation U on $\mathcal{X}$. Note that $\mathcal{X}$ can be identified with the Banach space of all bounded measurable functions on the product space
$$\bar{\Omega} = \Omega \times \{0, 1, \dots\}$$
endowed with the optional σ-field
$$\bar{\mathcal{F}} = \sigma(\mathcal{X}) = \sigma(\{A_t \times \{t\} \mid A_t \in \mathcal{F}_t,\, t = 0, 1, \dots),$$
and that the functional $\rho := -U$ can be viewed as a convex risk measure on this Banach space; cf. [14, Sections 4.1, 4.2]. Applying the representation theorem [14, Theorem 4.15], and denoting by $\bar{\mathcal{M}}_{1,f}$ the class of finitely additive probability measures on $(\bar{\Omega}, \bar{\mathcal{F}})$, we obtain the following representation of U:

Proposition 3. *For any* $X \in \mathcal{X}$ *we have*
$$U(X) = \min_{\bar{Q} \in \bar{\mathcal{M}}_{1,f}} \left(E_{\bar{Q}}[X] + \alpha(\bar{Q}) \right), \tag{4}$$
where the penalty function $\alpha : \bar{\mathcal{M}}_{1,f} \to (-\infty, \infty]$ *is given by*
$$\alpha(\bar{Q}) = \sup_{X \in \mathcal{A}} E_{\bar{Q}}[-X] = \sup_{X \in \mathcal{X}} \left(E_{\bar{Q}}[-X] + U(X) \right). \tag{5}$$

Our aim is to clarify the probabilistic structure of this representation if U is regarded as a functional on the subspace $\mathcal{X}_\infty$, or, equivalently, as a functional V on the space $\mathcal{C}$. To this end we introduce the following

Assumption 4. *The filtration* $(\mathcal{F}_t)_{t=0,1,\dots}$ *is a* standard system *in the sense of Parthasarathy [20], i.e.,*

i) Each σ*-field* $\mathcal{F}_t$ *is* σ*-isomorphic to the Borel* σ*-field on some complete separable metric space,*

ii) Any decreasing sequence of atoms $A_t \in \mathcal{F}_t$*,* $t = 0, 1, \dots$ *has a non-void intersection.*

This assumption guarantees that any consistent sequence of probability measures Q_t on $(\Omega, \mathcal{F}_t)$, $t = 0, 1, \dots$, has a (unique) extension to a probability measure Q on $(\Omega, \mathcal{F})$; cf. Parthasarathy [20, Theorem 4.1].

Example 5. *Consider the path space $\Omega = S^{\{1,2,\dots\}}$, where S is a Polish state space, and where $\mathcal{F}_t$ is generated by the coordinate maps $\omega \to \omega(s)$, $s = 1, \dots, t$. Then $(\mathcal{F}_t)_{t=0,1,\dots}$ is indeed a standard system, and Parthasarathy's extension theorem reduces to the classical extension theorem of Kolmogorov.*

3 Robust representation of concave monetary valuations on $\mathcal{X}_\infty$

Let $\mathcal{M}_1$ denote the class of probability measures on $(\Omega, \mathcal{F})$. For any $Q \in \mathcal{M}_1$, we denote by $\Gamma(Q)$ the class of all *optional random measures* γ on the extended time axis $\{0, \dots, \infty\}$ which are normalized under Q, i.e., $\gamma = (\gamma_t)_{t=0,\dots,\infty}$ is an adapted process such that $\gamma_t \geq 0$ and

$$\sum_{t=0}^{\infty} \gamma_t + \gamma_\infty = 1 \quad Q\text{-a.s..}$$

Via

$$D_t := 1 - \sum_{s=0}^{t-1} \gamma_s, \quad \gamma_t := D_t - D_{t+1}, \quad t = 0, 1, \dots, \quad D_\infty := \gamma_\infty, \tag{6}$$

the class $\Gamma(Q)$ can be identified with the class $\mathcal{D}(Q)$ of *predictable discounting processes* $D = (D_t)_{t=0,\dots,\infty}$ under Q, i.e., D is a predictable process such that $D_0 = 1$ and $D_t \geq D_{t+1} \geq 0$ Q-a.s.. Note that

$$D_\infty = \lim_{t\to\infty} D_t = \gamma_\infty \quad Q\text{-a.s.},$$

and that the "integration by parts" formula

$$\sum_{s=0}^{\infty} \gamma_s X_s + \gamma_\infty X_\infty = \sum_{s=0}^{\infty} D_s (X_s - X_{s-1}) \quad Q\text{-a.s.} \tag{7}$$

holds, where we put $X_{-1} := 0$.

Let U be a concave monetary utility valuation on $\mathcal{X}$.

Definition 6. *Let us say that U is* regular on $\mathcal{X}_\infty$ *if*

$$U(X) = \lim_{n\to\infty} U(X^n),$$

whenever $(X^n)_{n=0,1,\dots}$ is a sequence in $\mathcal{X}_\infty$ which increases to $X \in \mathcal{X}_\infty$ uniformly in t in the sense that

$$\lim_{n\to\infty} \sup_t (X - X^n)_t(\omega) = 0 \ \forall\, \omega \in \Omega.$$

Remark 7. *Regularity on $\mathcal{X}_\infty$ is a relaxed version of continuity from below. Note that it implies local continuity from below, i.e., continuity from below on the subspace $\mathcal{X}_t$ for any finite t. On the other hand, it is weaker than global continuity from below on $\mathcal{X}$. By [14, Proposition 4.21], the latter condition would immediately allow us to replace the finitely additive measures in* (4) *by σ-additive probability measures on $(\bar{\Omega}, \bar{\mathcal{F}})$. It turns out, however, that this would be too restrictive. For example, as explained in [1, Example 48], global continuity from below (or even from above) would not allow to calibrate the valuation to a given term structure. The point is that finitely additive measures such as Banach limits play an important role by creating mass at infinity. As a result, our representation theorem will involve probability measures on the extended product space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ introduced below.*

The following representation of U, viewed as a functional on the subspace $\mathcal{X}_\infty$, is the main result of this paper. It describes the interplay of model ambiguity, as specified by the measures $Q \in \mathcal{M}_1$, and of discounting ambiguity, as specified by the discounting processes $D \in \mathcal{D}(Q)$. In contrast to [1, Theorem 3.8], we neither assume the existence of a global reference measure P on $(\Omega, \mathcal{F})$, nor a global continuity condition on $\mathcal{X}$.

Theorem 8. *Suppose that U is regular on $\mathcal{X}_\infty$. Then, for each $X \in \mathcal{X}_\infty$, the valuation U takes the form*

$$U(X) = \min_{Q\in\mathcal{M}_1} \min_{\gamma\in\Gamma(Q)} \left(E_Q\Big[\sum_{t=0}^{\infty} X_t\gamma_t + X_\infty\gamma_\infty\Big] + \alpha(Q,\gamma)\right), \tag{8}$$

where

$$\alpha(Q,\gamma) := \sup_{X\in\mathcal{X}_\infty} \left(E_Q\Big[-\sum_{t=0}^{\infty} X_t\gamma_t - X_\infty\gamma_\infty\Big] + U(X)\right). \tag{9}$$

Alternatively, replacing $X \in \mathcal{X}_\infty$ by the corresponding discounted cash flow $C \in \mathcal{C}$, the valuation $V(C) := U(X)$ takes the form

$$V(C) = \min_{Q\in\mathcal{M}_1} \min_{D\in\mathcal{D}(Q)} \left(E_Q\Big[\sum_{t=0}^{\infty} C_t D_t\Big] + \alpha(Q,D)\right), \tag{10}$$

where

$$\alpha(Q,D) := \sup_{X\in\mathcal{C}} \left(E_Q\Big[-\sum_{t=0}^{\infty} C_t D_t\Big] + V(C)\right).$$

The proof will be given in several steps. In the first step, we are going to show that each finitely additive measure $\bar{Q} \in \bar{\mathcal{M}}_{1,f}$, which is relevant for the representation (4), can be replaced by a σ-additive probability measure on the extended product space

$$\tilde{\Omega} := \Omega \times \{0,\ldots,\infty\},$$

endowed with the σ-field

$$\tilde{\mathcal{F}} := \sigma\left(\left\{ A_t \times \{t,\ldots,\infty\} \;\middle|\; A_t \in \mathcal{F}_t, t = 0,1,\ldots\right\}\right).$$

Let $\tilde{\mathcal{M}}_1$ denote the class of σ-additive probability measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$.

Proposition 9. *For each $\bar{Q} \in \bar{\mathcal{M}}_{1,f}$ such that $\alpha(\bar{Q}) < \infty$, there exists a probability measure $\tilde{Q} \in \tilde{\mathcal{M}}_1$, such that*

$$E_{\bar{Q}}[X] = E_{\tilde{Q}}[\tilde{X}],$$

for any $X \in \mathcal{X}_\infty$, where $\tilde{X}$ denotes the measurable bounded function on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ corresponding to X via $\tilde{X}(\omega,s) := X_s(\omega)$ for $s = 0,1,\ldots$, and $\tilde{X}(\omega,\infty) := X_\infty(\omega) := \lim_{s\to\infty} X_s(\omega)$.

Proof. 1. Our assumption $\alpha(\bar{Q}) < \infty$ clearly implies

$$\alpha^t(\bar{Q}) := \sup_{X\in\mathcal{X}_t} \left(E_{\bar{Q}}[-X] + U(X)\right) \le \alpha(\bar{Q}) < \infty.$$

Note that $\alpha^t(\bar{Q})$ is the minimal penalty function in the robust representation of the convex risk measure ρ^t, defined as the restriction of $-U$ to $\mathcal{X}_t$. Note also that $\mathcal{X}_t$ can be identified with the Banach space of bounded measurable functions on $(\bar{\Omega}, \bar{\mathcal{F}}_t)$, where

$$\bar{\mathcal{F}}_t := \sigma\left(\left\{ A_s \times \{s, s+1, \ldots\} \;\middle|\; A_s \in \mathcal{F}_s,\ s \leq t\right\}\right),$$

and that our assumption on U implies that ρ^t is continuous from below. By [14, Proposition 4.21], we can conclude that the restriction of $\bar{Q}$ to $\bar{\mathcal{F}}_t$ is σ-additive.

2. We introduce the optional filtration $(\tilde{\mathcal{F}}_t)_{t=0,1,\ldots}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$, where

$$\tilde{\mathcal{F}}_t := \sigma\left(\left\{ A_s \times \{s, \ldots, \infty\} \;\middle|\; A_s \in \mathcal{F}_s,\ s \leq t\right\}\right).$$

Define $\tilde{Q}$ consistently on each σ-field $\tilde{\mathcal{F}}_t$ by

$$\tilde{Q}[A_s \times \{s\}] = \bar{Q}[A_s \times \{s\}] \quad \text{for } s < t, \text{ and} \quad \tilde{Q}[A_t \times \{t, \ldots, \infty\}] = \bar{Q}[A_t \times \{t, t+1, \ldots\}].$$

Due to 1), $\tilde{Q}$ is σ-additive on each $\tilde{\mathcal{F}}_t$. Since $(\mathcal{F}_t)_{t=0,1,\ldots}$ is assumed to be a standard system, it follows that $(\tilde{\mathcal{F}}_t)_{t=0,1,\ldots}$ is also a standard system. In particular, any decreasing sequence of atoms $\tilde{A}_t = A_t \times \{t, \ldots, \infty\} \in \tilde{\mathcal{F}}_t$, $t = 0, 1, \ldots$, has a non-void intersection of the form $\cap_t A_t \times \{\infty\}$. Due to Parthasarathy's extension theorem [20, Theorem 4.1], there exists exactly one extension of $\tilde{Q}$ to the σ-field

$$\tilde{\mathcal{F}} := \tilde{\mathcal{F}}_\infty = \sigma\left(\bigcup_{t=0}^{\infty} \tilde{\mathcal{F}}_t\right)$$

such that

$$E_{\tilde{Q}}[\tilde{X}] = E_{\bar{Q}}[X]$$

for any $X \in \mathcal{X}_t$. Now take $X \in \mathcal{X}_\infty$, and note that the functions $X^n \in \mathcal{X}_n$ defined by $X^n_t := X_t$ for $t < n$ and $X^n_t = X_n$ for $t \geq n$, converge to X uniformly in t. Applying Lebesgue's convergence theorem for $\tilde{Q}$ in the last step, we obtain

$$E_{\bar{Q}}[X] = \lim_{n\to\infty} E_{\bar{Q}}[X^n] = \lim_{n\to\infty} E_{\tilde{Q}}[\tilde{X}^n] = E_{\tilde{Q}}[\tilde{X}].$$

In order to justify the first step, define $Z^n := \sup_{k\geq n} |X^k - X| \in \mathcal{X}_\infty$. Since Z^n decreases to 0 uniformly in t, regularity of U implies that $U(-\lambda Z^n)$ converges to 0 for any $\lambda > 0$. But

$$|E_{\bar{Q}}[X^n] - E_{\bar{Q}}[X]| \leq E_{\bar{Q}}[Z^n] \leq \lambda^{-1}\left(\alpha(\bar{Q}) - U(-\lambda Z^n)\right)$$

due to Proposition 3. Passing to ∞ first with n and then with λ, we obtain the desired convergence of $E_{\bar{Q}}[X^n]$ to $E_{\bar{Q}}[X]$.

□

Our next step consists in representing any probability measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ in terms of a probability measure on $(\Omega, \mathcal{F})$ and a predictable discounting process; for a related decomposition in continuous time cf. Kardaras [19].

Proposition 10. *Any probability measure $\tilde{Q} \in \tilde{\mathcal{M}}_1$ admits a decomposition*

$$\tilde{Q} = Q \otimes \gamma = Q \otimes D,$$

where Q is a probability measure on $(\Omega, \mathcal{F})$, $\gamma \in \Gamma(Q)$, and $D \in \mathcal{D}(Q)$ corresponds to γ via (6). More precisely,

$$E_{\tilde{Q}}[\tilde{X}] = E_{Q\otimes\gamma}[\tilde{X}] := E_Q\Big[\sum_{t=0}^{\infty} \tilde{X}_t \gamma_t + \tilde{X}_\infty \gamma_\infty\Big] \tag{11}$$

for any bounded measurable function $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$, and we can also write

$$E_{\tilde{Q}}[\tilde{X}] = E_{Q\otimes D}[\tilde{X}] := E_Q\Big[\sum_{t=0}^{\infty} D_t(\tilde{X}_t - \tilde{X}_{t-1})\Big], \tag{12}$$

if $\tilde{X}_\infty = \lim_{t\to\infty} \tilde{X}_t$.

The proof is similar to the proof of [1, Theorem 3.4], but here we do not have a reference measure P as in [1]. For the convenience of the reader we sketch the argument.

Proof. For each $t \in \{0, \ldots, \infty\}$, the restriction of $\tilde{Q}$ to $\Omega \times \{t\}$ is of the form $Q_t \otimes \delta_t$ for some subprobability measure Q_t on $\mathcal{F}_t$ such that $\sum_{t=1}^{\infty} Q_t(\Omega) + Q_\infty(\Omega) = 1$. Choose some extension of Q_t to $\mathcal{F}$, take

$$R := \sum_{t=1}^{\infty} Q_t + Q_\infty$$

as a reference measure on $(\Omega, \mathcal{F})$, and denote by Z_t the density of Q_t with respect to R. Then

$$\sum_{t=0}^{\infty} Z_t + Z_\infty = 1 \qquad R\text{-a.s.},$$

and

$$S_t := E_R\Big[\sum_{s=t}^{\infty} Z_s + Z_\infty | \mathcal{F}_t\Big], \quad t = 0, \ldots, \infty,$$

defines an R-supermartingale $S = (S_t)_{t=0,\ldots,\infty}$ such that $S_0 = 1$ and

$$S_\infty = \lim_{t\to\infty} S_t = Z_\infty \qquad R\text{-a.s.}.$$

Now consider the Itô-Watanabe factorization

$$S_t = M_t D_t, \quad t = 0, 1, \ldots,$$

where $M = (M_t)$ is a nonnegative R-martingale with $M_0 = 1$, and $D = (D_t)$ is a nonnegative predictable decreasing process with $D_0 = 1$.; cf. [1, Proposition A.1]. The martingale M induces a unique probability measure Q on $(\Omega, \mathcal{F}_\infty)$, due to Parthasarathy's extension theorem [20, Theorem 4.1]. The limits $M_\infty :=$

$\lim_{t\to\infty} M_t$ and $D_\infty := \lim_{t\to\infty} D_t$ exist and satisfy $S_\infty = M_\infty D_\infty$, both R- and Q-a.s.. We define the process $\gamma = (\gamma_t)_{t\in\mathbb{T}}$ via (6). Now take $X \in \mathcal{X}_\infty$ with $X \geq 0$. By monotone convergence,

$$\begin{aligned} E_{\bar{Q}}[X] &= E_R\Big[\sum_{t=0}^{\infty} X_t Z_t + Z_\infty X_\infty\Big] = \sum_{t=0}^{\infty} E_R\Big[X_t E_R[S_t - S_{t+1}|\mathcal{F}_t]\Big] + E_R[S_\infty X_\infty] \\ &= \sum_{t=0}^{\infty} E_R\Big[X_t(M_t D_t - M_{t+1}D_{t+1})\Big] + E_R[M_\infty D_\infty X_\infty] \\ &= E_Q\Big[\sum_{t=0}^{\infty} X_t\gamma_t\Big] + E_R[M_\infty D_\infty X_\infty]. \end{aligned} \tag{13}$$

Using the Lebesgue decomposition

$$Q[A] = E_R[I_A M_\infty] + Q[A \cap \{M_\infty = \infty\}], \quad A \in \mathcal{F}_\infty$$

of Q with respect to R on $(\Omega, \mathcal{F}_\infty)$ (cf.,e.g., [21, Theorem VII.6.1]), (13) takes the form

$$E_{\bar{Q}}[X] = E_Q\Big[\sum_{t=0}^{\infty} X_t\gamma_t\Big] + E_Q[X_\infty\gamma_\infty] - E_Q[\gamma_\infty X_\infty I_{\{M_\infty=\infty\}}]. \tag{14}$$

For $X = 1$ this yields

$$\begin{aligned} 1 = E_{\bar{Q}}[1] &= E_Q\Big[\sum_{t=0}^{\infty} \gamma_t + \gamma_\infty\Big] - E_Q[\gamma_\infty I_{\{M_\infty=\infty\}}] \\ &= 1 - E_Q[\gamma_\infty I_{\{M_\infty=\infty\}}]. \end{aligned}$$

Thus $\gamma_\infty = 0$ Q-a.s. on $\{M_\infty = \infty\}$, and (14) reduces to (11). □

We are now ready to conclude the proof of Theorem 8.

Proof of Theorem 8. Recall the representation (4), and take any $\bar{Q} \in \bar{\mathcal{M}}_{1,f}$ such that $\alpha(\bar{Q}) < \infty$. Let $\tilde{Q} \in \tilde{\mathcal{M}}_1$ be the probability measure on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ associated to $\bar{Q}$ via Proposition 9, and consider the decomposition $\tilde{Q} = Q \otimes \gamma$ with $Q \in \mathcal{M}_1$ and $\gamma \in \Gamma(Q)$ as in Proposition 10. For any $X \in \mathcal{X}_\infty$, we can thus write

$$E_{\bar{Q}}[X] = E_{\tilde{Q}}[\tilde{X}] = E_Q\Big[\sum_{t=0}^{\infty} X_t\gamma_t + X_\infty\gamma_\infty\Big], \tag{15}$$

Note that, in view of (5) and (9),

$$\alpha(Q,\gamma) = \sup_{X\in\mathcal{X}_\infty} \big(E_{\bar{Q}}[-X] + U(X)\big) \leq \alpha(\bar{Q}). \tag{16}$$

Now fix $X \in \mathcal{X}_\infty$. Since

$$\alpha(Q,\gamma) \geq E_{\tilde{Q}}[-X] + U(X)$$

for any $\tilde{Q} = Q \otimes \gamma \in \tilde{\mathcal{M}}_1$, we obtain

$$U(X) \leq \inf_{\tilde{Q}=Q\otimes\gamma\in\tilde{\mathcal{M}}_1} \Big(E_{\tilde{Q}}[\tilde{X}] + \alpha(Q,\gamma)\Big). \tag{17}$$

On the other hand, take $\bar{Q}^X$ such that the minimum in (4) is attained. The corresponding probability measure $\tilde{Q}^X = Q^X \otimes \gamma^X$ satisfies

$$U(X) = E_{\tilde{Q}^X}[\tilde{X}] + \alpha(\bar{Q}) \geq E_{\tilde{Q}^X}[\tilde{X}] + \alpha(Q^X, \gamma^X),$$

due to (16). In view of (15) and (17), this implies the representation (8). Alternatively, using the predictable discounting process $D = (D_t) \in \mathcal{D}(Q)$ induced by γ and the integration by parts formula (7), we can write

$$E_{\tilde{Q}}[\tilde{X}] = E_Q\Big[\sum_{t=0}^{\infty} D_t(X_t - X_{t-1})\Big]$$

for any $X \in \mathcal{X}_\infty$, and this yields the valuation formula (10) for the cash flow $C = (C_t)$ associated to X via $C_t := X_t - X_{t-1}$. $\square$

4 Time consistency and the appearance of bubbles

In this section we sketch the extension to dynamic valuations, adapting results from [1, 12, 2] to our present context of Knightian uncertainty. In order to simplify the discussion we assume that we are in the situation of Example 5 with a countable state space S; a more thorough analysis will appear elsewhere.

At a given time t, the valuation of a future cash flow should be based on the information available at that time. Using the obvious conditional formulation of the properties in Section 2, we obtain the notion of a *conditional concave monetary valuation* U_t at time t, defined as a map from $\mathcal{X}$ to the space of bounded measurable functions on $(\Omega, \mathcal{F}_t)$. Adding the condition of *regularity on* $\mathcal{X}_\infty$, and repeating the construction of Section 3 on each atom of $\mathcal{F}_t$, we obtain the following conditional valuation formula for a cumulated cash flow $X \in \mathcal{X}_\infty$:

$$U_t(X) = \min_{Q \in \mathcal{M}_1} \min_{\gamma \in \Gamma(Q)} \left(E_Q\Big[\sum_{s=t}^{\infty} X_s \frac{\gamma_s}{D_t} + X_\infty \frac{\gamma_\infty}{D_t} \mid \mathcal{F}_t\Big] + \alpha_t(Q, \gamma) \right), \tag{18}$$

where

$$\alpha_t(Q, \gamma) = \sup_{X \in \mathcal{X}_\infty} \left(E_Q\Big[-\sum_{s=t}^{\infty} X_s \frac{\gamma_s}{D_t} - X_\infty \frac{\gamma_\infty}{D_t} \mid \mathcal{F}_t\Big] + U_t(X) \right),$$

and $D_t = 1 - \sum_{s=0}^{t-1} \gamma_s$; cf. [1, Theorem 3.8], where the formula is derived for the conditional risk measure $\rho_t := -U_t$, but under different assumptions.

Translating the conditional valuation formula for $U_t(X)$ to the level of cash flows $C \in \mathcal{C}$ and using the integration by parts formula (7), we obtain the following result:

Theorem 11. *For any $C \in \mathcal{C}$, the conditional valuation at time t of the future cash flow defined by*

$$V_t(C) := U_t(X) - X_{t-1},$$

where $X \in \mathcal{X}_\infty$ is the cumulated cash flow induced by C, takes the form

$$V_t(C) = \min_{Q \in \mathcal{M}_1} \min_{D \in \mathcal{D}(Q)} \left(E_Q\Big[\sum_{s=t}^{\infty} C_s \frac{D_s}{D_t} \mid \mathcal{F}_t\Big] + \alpha_t(Q, D) \right), \tag{19}$$

where

$$\alpha_t(Q,D) = \sup_{C\in\mathcal{C}} \left(E_Q\Big[-\sum_{s=t}^{\infty} C_s \frac{D_s}{D_t} \mid \mathcal{F}_t \Big] + V_t(C) \right),$$

The sequence $(U_t)_{t=0,1,\ldots}$, and also the corresponding sequence $(V_t)_{t=0,1,\ldots}$ on the level of cash flows, will be called a *dynamic concave valuation.*

Definition 12. *A dynamic concave valuation* $(V_t)_{t=0,1,\ldots}$ *is called (strongly)* time consistent *if it satisfies the recursion*

$$V_t(C) = C_t + V_t\left(V_{t+1}(C) I_{\{t+1\}}\right), \quad t = 0,1,\ldots \tag{20}$$

for any $C \in \mathcal{C}$.

From now on we assume that (V_t) is time consistent. In order to focus on the supermartingale aspects of time consistency, let us introduce the subspace

$$\mathcal{C}_{t,t+1} := \left\{ C \in \mathcal{C} \;\middle|\; C_s = 0 \text{ for } s \notin \{t,t+1\} \right\}$$

and the corresponding one-step penalty function

$$\alpha_{t,t+1}(Q,D) = \sup_{C\in\mathcal{C}_{t,t+1}} \left(E_Q\Big[-C_t - \frac{D_{t+1}}{D_t} C_{t+1} \mid \mathcal{F}_t \Big] + V_t(C) \right). \tag{21}$$

A straightforward translation of [1, Theorem 4.2] yields the following result.

Proposition 13. *Time consistency of* $(V_t)_{t=0,1,\ldots}$ *implies, for any* $Q \in \mathcal{M}_1$ *and any* $D \in \mathcal{D}(Q)$,

$$D_t\alpha_t(Q,D) = D_t\alpha_{t,t+1}(Q,D) + E_Q[D_{t+1}\alpha_{t+1}(Q,D) \mid \mathcal{F}_t] \quad Q\text{-a.s.} \tag{22}$$

and

$$E_Q[D_{t+1}(V_t(C) - \alpha_{t+1}(Q,D)) \mid \mathcal{F}_t] \geq D_t(V_t(C) - C_t - \alpha_t(Q,D)) \quad Q\text{-a.s.}.$$

for $t = 0,1,\ldots$.

For any $Q \otimes D$ such that $\alpha_0(Q,D) < \infty$, Proposition 13 shows that the process

$$D_tV_t(C) + \sum_{s=0}^{t-1} D_sC_s - D_t\alpha_t(Q,D), \qquad t = 0,1,\ldots$$

is a Q-submartingale, and that the process $(D_t\alpha_t(Q,D))_{t=0,1,\ldots}$ is a nonnegative Q-supermartingale. Moreover, (22) yields the *Doob decomposition*

$$D_t\alpha_t(Q,D) = M_t^{Q,D} - A_t^{Q,D}, \qquad t = 0,1,\ldots$$

into a nonnegative Q-martingale $(M_t^{Q,D})$ and the predictable increasing process

$$A_t^{Q,D} := \sum_{k=0}^{t-1} D_k\alpha_{k,k+1}(Q,D), \qquad t = 0,1,\ldots.$$

Splitting the martingale

$$M_t^{Q,D} = E_Q\left[A_\infty^{Q,D}|\mathcal{F}_t\right] + B_t^{Q,D}$$

into the "fundamental" component generated by the one-step penalties and into an additional nonnegative martingale $B^{Q,D}$, we obtain the *Riesz decomposition*

$$D_t\alpha_t(Q,D) = E_Q\left[A_\infty^{Q,D} - A_t^{Q,D}|\mathcal{F}_t\right] + B_t^{Q,D}, \qquad t = 0,1,\ldots.$$

into the potential generated by the process $A^{Q,D}$ and into a non-negative martingale. The martingale $(B_t^{Q,D})$ may be viewed as a "bubble" in the penalization of the model (Q,D): It comes on top of the fundamental component in the Riesz decomposition of the penalty process, and may thus lead to an excessive neglect of that model; cf. the discussion in [1, Section 4.3], and in particular [1, Theorem 4.8], where it is shown that the appearance of a bubble $(B_t^{Q,D})$ amounts to a breakdown of asymptotic safety of the valuation procedure under the model (Q,D).

There is, of course, an additional source of "bubbles" which appears already in the classical case under a fixed probability model P. So far, we have discussed the valuation of the future cash flow under the assumption that the cash flow will be held indefinitely ("buy and hold"). If we take into account the option to sell the cash flow at some future time, it is plausible to replace the classical valuation

$$V_t(C) = E_P\Big[\sum_{k=t}^{\infty} C_k|\mathcal{F}_t\Big]$$

by

$$\bar{V}_t(C) = \sup_{\tau\geq t} E_P\Big[\sum_{k=t}^{\tau-1} C_k + \pi_\tau(C) I_{\{\tau<\infty\}} \bigm| \mathcal{F}_t\Big], \tag{23}$$

where the supremum is taken over all stopping times $\tau \geq t$, and where $\pi_\tau(C)$ is the uncertain price of the future cash flow at time τ (usually assumed to satisfy the equilibrium condition $\pi_\tau(C) = \bar{V}_\tau(C)$; cf., e.g., [17]). The difference

$$B_t(C) := \bar{V}_t(C) - V_t(C), \qquad t = 0,1,\ldots,$$

is often called a bubble. Clearly, the same kind of bubble may appear in our setting of Knightian uncertainty if we replace (23) by

$$\bar{V}_t(C) = \sup_{\tau\geq t} \min_{Q\in\mathcal{M}_1} \min_{D\in\mathcal{D}(Q)} \left(E_Q\Big[\sum_{k=t}^{\tau-1} C_k \frac{D_k}{D_t} + \pi_\tau(C) I_{\{\tau<\infty\}} \bigm| \mathcal{F}_t\Big] - \alpha_t(Q,D) \right).$$

A detailed analysis of these two sources of bubbles and of their interplay will appear elsewhere.

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