Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles

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Abstract

We study the risk assessment of uncertain cash flows in terms of dynamic convex risk measures for processes as introduced in Cheridito, Delbaen, and Kupper [10]. These risk measures take into account not only the amounts but also the timing of a cash flow. We discuss their robust representation in terms of suitably penalized probability measures on the optional σ-field. This yields an explicit analysis both of model and discounting ambiguity. We focus on supermartingale criteria for time consistency. In particular we show how "bubbles" may appear in the dynamic penalization, and how they cause a breakdown of asymptotic safety of the risk assessment procedure.

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1 Introduction

The classical assessment of an uncertain cash flow takes the sum of the discounted future payments and computes its expectation with respect to a given probability measure. Both the probabilistic model and the discounting factors are assumed to be known. In reality, however, one is usually confronted both with model uncertainty and with uncertainty about the time value of money. The purpose of this paper is to deal with this problem by using concepts and methods from the theory of convex risk measures.

In a situation where financial positions are described by random variables on some probability space, a convex risk measure can usually be represented as the worst expected loss over a class of suitably

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penalized probabilistic models; see Artzner, Delbaen, Eber, and Heath [2, 3], Delbaen [12, 13] for the coherent case, and Föllmer and Schied [22, 23], Frittelli and Rosazza Gianin [24] for the general convex case. This can be seen as a robust method which deals explicitly with the problem of model uncertainty. In the dynamical setting of a filtered probability space, the risk assessment at a given time should depend on the available information. This is specified by a dynamic risk measure, i.e., by a sequence (ρ_t) of conditional convex risk measures adapted to the filtration. On the level of random variables, and under an additional requirement of time consistency, the structure of such dynamic risk measures is now well understood; cf., e.g, [4, 37, 16, 14, 40, 30, 6, 20, 11, 39, 34, 15, 1], and references therein.

There is also a growing literature on dynamic risk measures applied to cash flows that are described as adapted stochastic processes on the given filtered probability space; cf. Artzner, Delbaen, Eber, Heath, and Ku [4], Cheridito, Delbaen, and Kupper [10], and also [35, 8, 9, 11, 25, 28]. In this context, not only the amount of a payment matters, but also its timing. In particular, the risk is reduced by having positive payments earlier and negative ones later. This is expressed by the property of cash subadditivity, which was introduced by El Karoui and Ravanelli [18] in the context of risk measures for random variables in order to account for discounting ambiguity. Convex risk measures for processes have that property, and so they provide a natural framework to capture both model uncertainty and uncertainty about the time value of money.

In this paper we study dynamic convex risk measures for bounded adapted processes, as introduced in [10]. Any such process can be viewed as a bounded measurable function on the product space $\bar{\Omega} = \Omega \times \mathbb{T}$ endowed with the optional σ-field. It is thus natural to use results from the theory of risk measures for random variables and to apply them on product space. This idea already appears in [4] in a static setting, and even earlier in Epstein and Schneider [19] in the context of dynamic preferences. Here we use it for dynamic risk measures, and we take a more probabilistic approach. This involves a careful study of absolutely continuous probability measures $\bar{Q}$ on the optional σ-field. In particular, we derive a decomposition $\bar{Q} = Q \otimes D$, where Q is a locally absolutely continuous probability measure on the original space, and D is a predictable discounting process. The probabilistic approach has two advantages. In the first place, it allows us to make explicit the joint role of model uncertainty, as expressed by the measures Q, and of discounting uncertainty, as described by the discounting processes D, in the robust representation of conditional risk measures. Moreover, it is crucial for our analysis of the supermartingale aspects of time consistency.

Time consistency is a key issue in the analysis of dynamic risk measures; see [4, 14, 16, 30, 10, 6, 20, 11, 15] and references therein. We characterize time consistency by supermartingale properties of the discounted penalty and risk processes, in analogy to various results for random variables from [4, 14, 6, 20, 34, 7]. These characterizations allow us to apply martingale arguments to prove maximal inequalities and convergence results for the risk assessment procedure. In particular, we show that the appearance of a martingale component in the Riesz decomposition of the discounted penalty process amounts to a breakdown of asymptotic safety. Such a martingale can be seen as a “bubble”, which appears on the top of the “fundamental” penalization and thus causes an excessive neglect of the model under consideration.

The paper is organized as follows. In Section 3 we clarify the probabilistic structure of conditional convex risk measures for processes. To this end, we introduce the appropriate product space in Subsection 3.1 and state a decomposition theorem for probability measures on the optional σ-field; its proof is given in Appendix B. In Subsection 3.2 risk measures for processes are identified with risk measures for random variables on the product space. Under an assumption of global continuity from above, this allows us to obtain a robust representation of risk measures for processes in Subsection 3.3, which involves both model ambiguity and discounting ambiguity. Section 4 characterizes time consistency of dynamic risk measures, with special emphasis on the corresponding supermartingale properties. We focus on the strong notion of time consistency introduced in [4]. In Subsection 4.1 we state several equivalent criteria, and we use them in Subsection 4.2 to derive the Doob and the Riesz decomposition of the penalty processes. In Subsection 4.3 we discuss asymptotic properties such as asymptotic safety and asymptotic precision, and we relate them to the appearance of "bubbles" in the Riesz decomposition. Subsection 4.4 states a maximal inequality for the excess of the capital requirement over the penalized expected loss computed for a specific model. The coherent case is discussed in Subsection 4.5. In Section 5 we discuss cash subadditivity of risk measures for processes, and we characterize their calibration with respect to some numéraire following [18]. If a time consistent dynamic risk measure is calibrated to a term structure specified by the prices of zero coupon bonds, then discounting ambiguity is completely resolved, and we are only left with model ambiguity. In Section 6 our analysis is illustrated by some examples, including entropic risk measures and variants of Average Value at Risk for processes.

2 Preliminaries

We consider a discrete-time setting with time horizon $T \in \mathbb{N} \cup \{\infty\}$. We denote by $\mathbb{T}$ the set of time points, i.e., $\mathbb{T} := \{0, \dots, T\}$ if $T < \infty$, and in case $T = \infty$ we distinguish between the two cases $\mathbb{T} := \mathbb{N}_0$ and $\mathbb{T} := \mathbb{N}_0 \cup \{\infty\}$. We use the notation $\mathbb{T}_t := \{s \in \mathbb{T} \,|\, s \geq t\}$ for $t \in \mathbb{T}$.

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}, P)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F} = \mathcal{F}_\infty := \sigma(\cup_{t\in\mathbb{N}_0}\mathcal{F}_t)$ for $T = \infty$. For $t \in \mathbb{T}$, we use the notation

$$L_t^\infty := L^\infty(\Omega, \mathcal{F}_t, P), \qquad L_{t,+}^\infty := \{X \in L_t^\infty \bigm| X \geq 0\},$$

and $L^\infty := L^\infty(\Omega, \mathcal{F}_T, P)$. All equalities and inequalities between random variables and between sets are understood to hold P-almost surely, unless stated otherwise.

We denote by $\mathcal{M}(P)$ (resp. by $\mathcal{M}_{\mathrm{loc}}(P)$) the set of all probability measures Q on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to P (resp. locally absolutely continuous with respect to P in the sense that $Q \ll P$ on $\mathcal{F}_t$ for each $t \in \mathbb{T}\cap\mathbb{N}_0$), and by $\mathcal{M}^e(P)$ (resp. by $\mathcal{M}^e_{\mathrm{loc}}(P)$) the set of all probability measures on $(\Omega, \mathcal{F})$ which are equivalent (resp. locally equivalent) to P. Note that $\mathcal{M}(P)$ coincides with $\mathcal{M}_{\mathrm{loc}}(P)$ if $T < \infty$.

Let $\mathcal{R}^\infty$ denote the space of adapted stochastic processes $X = (X_t)_{t\in\mathbb{T}}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, P)$ such that

$$\|X\|_\infty := \inf\left\{x \in \mathbb{R} \bigm| \sup_{t\in\mathbb{T}} |X_t| \leq x\right\} < \infty. \tag{1}$$

For $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$ we also consider the subspace

$$\mathcal{X}^\infty := \left\{ X \in \mathcal{R}^\infty \;\middle|\; X_\infty = \lim_{t\to\infty} X_t \ P\text{-a.s.} \right\}.$$

For $0 \le t \le s \le T$, we define the projection $\pi_{t,s} : \mathcal{R}^\infty \to \mathcal{R}^\infty$ as

$$\pi_{t,s}(X)_r = 1_{\{t\le r\}} X_{r\wedge s}, \quad r \in \mathbb{T},$$

and use the notation $\mathcal{R}^\infty_{t,s} := \pi_{t,s}(\mathcal{R}^\infty)$ and $\mathcal{R}^\infty_t := \pi_{t,T}(\mathcal{R}^\infty)$. The spaces $\mathcal{X}^\infty_{t,s}$ and $\mathcal{X}^\infty_t$ are defined accordingly.

On the one hand, a process $X \in \mathcal{R}^\infty$ can be interpreted as a value process, which might model the evolution of some financial value such as the market value of a firm's equity or of an investment portfolio. On the other hand, X can be seen as a *cumulated cash flow*, as explained in Remark 1 and in Example 2.

Remark 1. *An adapted cash flow $C = (C_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ yielding an uncertain amount $C_t \in L^\infty_t$ at time t induces a cumulated cash flow $X = (X_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ with*

$$X_t = \sum_{s=0}^{t} C_t.$$

If $T < \infty$, or if $T = \infty$ and $\sum_{t\in\mathbb{T}\cap\mathbb{N}_0} \|C_t\|_\infty < \infty$, the process X belongs to $\mathcal{R}^\infty$, and even to $\mathcal{X}^\infty$, with $X_\infty := \sum_{s=0}^\infty C_t$. Conversely, each process $X \in \mathcal{R}^\infty$ induces an adapted cash flow

$$C_t := \Delta X_t := X_t - X_{t-1}, \qquad t \in \mathbb{T}\cap\mathbb{N}_0, \tag{2}$$

where we use the convention $X_{-1} := 0$.

Example 2. *Assume that there is a money market account $(B_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ of the form*

$$B_t = \prod_{s=1}^{t} (1 + r_s)$$

with some adapted (or even predictable) process $(r_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ of nonnegative short rates. For a given (undiscounted) adapted cash flow $(\tilde{C}_t)_{t\in\mathbb{T}\cap\mathbb{N}_0} \in \mathcal{R}^\infty$ consider the discounted cash flow $C = (C_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ defined by $C_t = B_t^{-1}\tilde{C}_t$. If $T = \infty$ and the short rates are bounded away from zero by some constant $\delta > 0$, then the cumulated discounted cash flow X with $X_\infty := \sum_{t=0}^\infty C_t$ belongs to $\mathcal{R}^\infty$, and even to $\mathcal{X}^\infty$, since

$$\sum_{t=0}^{\infty} \|C_t\|_\infty \le \frac{1}{\delta} \|\tilde{C}\|_\infty < \infty.$$

Here the norm in the first term is the usual essential supremum norm on random variables, and that in the second term is the norm on processes defined in (1).

In the preceding example, the value X_∞ arises naturally as the limiting value of a cumulated cash flow. More generally, for $X \in \mathcal{R}^\infty$ with $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, the value X_∞ can be seen as a terminal payment. In this way, dynamic risk measures for random variables with infinite time horizon as considered in [20] can be included into our framework; cf. Remark 39.

Considering the interpretation in terms of cash flows, our results will be formulated both for X and for the underlying cash flow C given by (2). On a technical level, however, our main focus will be on cumulated cash flows $X \in \mathcal{R}^\infty$. This will allow us to apply in a straightforward manner standard results for convex risk measures defined on bounded random variables.

3 Conditional risk measures

At each time the risk of a future cumulative cash flow will be assessed by a conditional risk measure based on the information available at that time. The following definition was introduced in [10].

Definition 3. *A map $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ for $t \in \mathbb{T} \cap \mathbb{N}_0$ is called a* conditional convex risk measure (for processes) *if it satisfies the following properties for all $X, Y \in \mathcal{R}_t^\infty$*

- *Conditional cash invariance: for all $m \in L_t^\infty$,*

$$\rho_t(X + m1_{\mathbb{T}_t}) = \rho_t(X) - m;$$

- *Monotonicity: $\rho_t(X) \geq \rho_t(Y)$ if $X \leq Y$ componentwise;*
- *Conditional convexity: for all $\lambda \in L_t^\infty$ with $0 \leq \lambda \leq 1$,*

$$\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda\rho_t(X) + (1-\lambda)\rho_t(Y);$$

- *Normalization: $\rho_t(0) = 0$.*

A conditional convex risk measure is called a conditional coherent risk measure (for processes) *if it has in addition the following property:*

- *Conditional positive homogeneity: for all $\lambda \in L_t^\infty$ with $\lambda \geq 0$,*

$$\rho_t(\lambda X) = \lambda\rho_t(X).$$

A sequence $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ is called a dynamic convex risk measure (for processes) *if, for each t, $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ is a conditional convex risk measure (for processes).*

Definition 3 is analogous to the definition of risk measures for random variables given in [16]. Note, however, that conditional cash invariance in the context of processes takes into account the timing of the cash payment; the consequences will be discussed in more detail in Section 5.

Conditional cash invariance and convexity could also be formulated in terms of cash flows C as in Remark 1 rather than in terms of cumulated cash flows X. Note, however, that monotonicity with respect to X is stronger than monotonicity with respect to C. This stronger condition is natural since it reflects the time value of money; cf. Section 5.

3.1 Optional filtration and predictable discounting

In their study of dynamic preferences for consumption processes, Epstein and Schneider [19] derive a numerical representation by applying results from Gilboa and Schmeidler [26] on the appropriate product space endowed with the optional filtration. In the same spirit, Artzner et al. [4] identify static risk measures for processes with risk measures for random variables on product space. Here we extend this idea to the dynamic setting, and we focus on the probabilistic structure of the resulting robust representation in terms of probability measures on the optional σ-field.

Consider the product space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ defined by

$$\bar{\Omega} = \Omega \times \mathbb{T}, \quad \bar{\mathcal{F}} = \sigma(\{A_t \times \{t\} \mid A_t \in \mathcal{F}_t,\, t \in \mathbb{T}), \quad \bar{P} = P \otimes \mu,$$

where $\mu = (\mu_t)_{t\in\mathbb{T}}$ is some adapted reference process such that $\sum_{t\in\mathbb{T}} \mu_t = 1$ and $\mu_t > 0\ \forall t \in \mathbb{T}$, and where

$$E_{P\otimes\mu}[X] := E_P\left[\sum_{t\in\mathbb{T}} X_t \mu_t\right]$$

for any bounded measurable function X on $(\bar{\Omega}, \bar{\mathcal{F}})$.

Note that $\bar{\mathcal{F}}$ coincides with the *optional* σ-field generated by all adapted processes. Every adapted process can be identified with a random variable on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, and in particular we have

$$\mathcal{R}^\infty = \bar{L}^\infty := L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}).$$

We also introduce the *optional filtration* $(\bar{\mathcal{F}}_t)_{t\in\mathbb{T}}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ given by

$$\bar{\mathcal{F}}_t = \sigma\left(\{A_j \times \{j\}, A_t \times \mathbb{T}_t \mid A_j \in \mathcal{F}_j,\, j < t,\, A_t \in \mathcal{F}_t\}\right), \quad t \in \mathbb{T}.$$

A random variable $X = (X_s)_{s\in\mathbb{T}}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ is $\bar{\mathcal{F}}_t$-measurable if and only if X_s is $\mathcal{F}_s$-measurable for all $s = 0, \ldots, t$ and $X_s = X_t\ \forall s > t$. In particular,

$$\mathcal{R}^\infty_{0,t} = \bar{L}^\infty_t := L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P}).$$

The set $\mathcal{R}^\infty_{0,0}$ of all constant processes will be identified with $\mathbb{R}$.

For $T = \infty$ we will use the Lebesgue decomposition of a measure $Q \in \mathcal{M}_{\mathrm{loc}}(P)$ with respect to P. Let $M = (M_t)_{t\in\mathbb{N}_0}$ denote the density process of Q with respect to P. The limit $M_\infty := \lim_{t\to\infty} M_t$ exists P-a.s., since M is a nonnegative P-martingale. By [38, Theorem VII.6.1] M_∞ exists also Q-a.s., and Q admits the Lebesgue decomposition

$$Q[A] = E_P[1_A M_\infty] + Q[A \cap \{M_\infty = \infty\}], \quad A \in \mathcal{F}_\infty \tag{3}$$

into the absolutely continuous and the singular part with respect to P on $(\Omega, \mathcal{F}_\infty)$.

For a measure $Q \in \mathcal{M}_{\mathrm{loc}}(P)$ we introduce the set $\Gamma(Q)$ of *optional random measures* $\gamma = (\gamma_t)_{t\in\mathbb{T}}$ on $\mathbb{T}$ which are normalized with respect to Q. More precisely, $\gamma \in \Gamma(Q)$ is a nonnegative adapted process, such that

$$\sum_{t\in\mathbb{T}} \gamma_t = 1 \quad Q\text{-a.s.},$$

with the additional property that

$$\gamma_\infty = 0 \quad Q\text{-a.s. on} \quad \{M_\infty = \infty\}, \quad \text{if} \quad \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}.$$

We also consider the following set $\mathcal{D}(Q)$ of *predictable discounting processes*: $D = (D_t)_{t\in\mathbb{T}} \in \mathcal{D}(Q)$ is a predictable non-increasing process with $D_0 = 1$, and $D_\infty = \lim_{t\to\infty} D_t$ Q-a.s. for $T = \infty$, where

$$D_\infty = 0 \quad Q\text{-a.s.} \quad \text{for} \quad \mathbb{T} = \mathbb{N}_0,$$

and

$$D_\infty = 0 \quad Q\text{-a.s.} \quad \text{on} \quad \{M_\infty = \infty\} \quad \text{for} \quad \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}.$$

For $T < \infty$ we define $D_{T+1} := 0$.

Lemma 4. *For any probability measure $Q \in \mathcal{M}_{loc}(P)$, the set $\Gamma(Q)$ can be identified with $\mathcal{D}(Q)$. More precisely, to each γ in $\Gamma(Q)$ we can associate a process $D \in \mathcal{D}(Q)$ given by*

$$D_t := 1 - \sum_{s=0}^{t-1} \gamma_s, \quad t \in \mathbb{T} \cap \mathbb{N}_0, \quad \textit{and} \quad D_\infty := \gamma_\infty \quad \textit{for} \quad \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}. \tag{4}$$

In particular we have

$$D_t = \sum_{s\in\mathbb{T}_t} \gamma_s \quad Q\text{-a.s.} \quad \forall t \in \mathbb{T}. \tag{5}$$

Conversely, every process $D \in \mathcal{D}(Q)$ defines an optional random measure $\gamma \in \Gamma(Q)$ via

$$\gamma_t := D_t - D_{t+1}, \quad t \in \mathbb{T} \cap \mathbb{N}_0, \quad \textit{and} \quad \gamma_\infty := D_\infty \quad \textit{for} \quad \mathbb{T} = \mathbb{N}_0 \cup \{\infty\}. \tag{6}$$

Moreover, for any pair $\gamma \in \Gamma(Q)$ and $D \in \mathcal{D}(Q)$ related to each other via (5) *and* (6)*, the "integration by parts" formula*

$$\sum_{s\in\mathbb{T}_t} \gamma_s X_s = \sum_{s=t}^{T} D_s(X_s - X_{s-1}) \quad Q\text{-a.s.}, \quad t \in \mathbb{T}, \tag{7}$$

holds for any $X \in \mathcal{R}_t^\infty$ if $T < \infty$ or if $\mathbb{T} = \mathbb{N}_0$, and for $X \in \mathcal{X}_t^\infty$ if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$.

Proof. It is obvious that the process D defined by (4) belongs to $\mathcal{D}(Q)$ and satisfies (5), and that γ defined by (6) belongs to $\Gamma(Q)$. To prove (7), note that

$$\sum_{s=0}^{t} \gamma_s X_s = \sum_{s=0}^{t} D_s(X_s - X_{s-1}) - D_{t+1} X_t \tag{8}$$

for all $t \in \mathbb{T} \cap \mathbb{N}_0$. Thus (7) is obvious for $T < \infty$, and it also holds if $\mathbb{T} = \mathbb{N}_0$ for all $X \in \mathcal{R}_t^\infty$, since X is bounded and $D_t \searrow 0$ Q-a.s.. For $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$ and for any $X \in \mathcal{X}_t^\infty$, the limit $D_\infty X_\infty = \lim_{t\to\infty} D_{t+1} X_t$ exists Q-a.s., since $D_t \searrow 0$ Q-a.s. on the singular part of Q with respect to P, and so (7) follows again from (8). $\square$

From now on we use the following assumption which allows us to apply an extension result of Parthasarathy [33] for consistent sequences of measures. This will be needed in the proof of Theorem 6.

Assumption 5. *In the case $T = \infty$, we assume that for each $t \in \mathbb{T} \cap \mathbb{N}_0$ the σ-field $\mathcal{F}_t$ is σ-isomorphic to the Borel σ-field on some complete separable metric space, and that $\cap_n A_n \neq \emptyset$ for any decreasing sequence $(A_n)_{n\in\mathbb{N}_0}$ such that A_n is an atom of $\mathcal{F}_n$.*

We denote by $\mathcal{M}(\bar{P})$ the set of all probability measures on $(\bar{\Omega}, \bar{\mathcal{F}})$ which are absolutely continuous with respect to $\bar{P}$. The next theorem shows that each probability measure $\bar{Q}$ in $\mathcal{M}(\bar{P})$ admits a decomposition $\bar{Q}(d\omega, dt) = Q(dw) \otimes \gamma(w, dt)$ for some probability measure Q on $(\Omega, \mathcal{F}_T)$ and some optional random measure γ on $\mathbb{T}$ such that $Q \in \mathcal{M}_{\text{loc}}(P)$ and $\gamma \in \Gamma(Q)$.

Theorem 6. *For any probability measure $\bar{Q} \in \mathcal{M}(\bar{P})$ there exist a probability measure $Q \in \mathcal{M}_{loc}(P)$ and an optional random measure $\gamma \in \Gamma(Q)$ (resp. a predictable discounting factor $D \in \mathcal{D}(Q)$) such that*

$$E_{\bar{Q}}[X] = E_Q\left[\sum_{t\in\mathbb{T}} \gamma_t X_t\right] \tag{9}$$

$$= E_Q\left[\sum_{t=0}^{T} D_t(X_t - X_{t-1})\right], \tag{10}$$

where (9) *holds for all $X \in \mathcal{R}^\infty$, whereas* (10) *holds for all $X \in \mathcal{R}^\infty$ if $T < \infty$ or if $\mathbb{T} = \mathbb{N}_0$, and only for $X \in \mathcal{X}^\infty$ if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$.*

Conversely, any $Q \in \mathcal{M}_{loc}(P)$ and any $\gamma \in \Gamma(Q)$ (resp. any $D \in \mathcal{D}(Q)$) define a probability measure $\bar{Q} \in \mathcal{M}(\bar{P})$ such that (9) *and* (10) *hold.*

We write

$$\bar{Q} = Q \otimes \gamma = Q \otimes D$$

to denote the decomposition of Q in the sense of (9) *and* (10).

The proof is postponed to Appendix B.

Remark 7. *A continuous time analogue to Theorem 6 appears independently in Kardaras [29, Theorem 2.1]. While we make use of the Itô-Watanabe decomposition in discrete time (cf. Proposition 48) and of a measure theoretic extension, [29, Theorem 2.1] gives a direct construction of a discounting process and a local martingale, but without relating the latter to a probability measure Q in the general case.*

3.2 Conditional risk measures viewed on the optional filtration

In the previous section we have identified processes in $\mathcal{R}^\infty$ with random variables in $\bar{L}^\infty$. This induces a one-to-one correspondence between conditional risk measures for processes and conditional risk measures for random variables on the optional σ-field:

Proposition 8. *Any conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ for $t \in \mathbb{T} \cap \mathbb{N}_0$ defines a conditional convex risk measure for random variables $\bar{\rho}_t : \bar{L}^\infty \to \bar{L}_t^\infty$ via*

$$\bar{\rho}_t(X) = -X_0 1_{\{0\}} - \ldots - X_{t-1} 1_{\{t-1\}} + \rho_t(X) 1_{\mathbb{T}_t}, \quad X \in \mathcal{R}^\infty, \tag{11}$$

where we use the notation

$$\rho_t(X) := \rho_t \circ \pi_{t,T}(X) \quad \textit{for} \quad X \in \mathcal{R}^\infty.$$

Conversely, any conditional convex risk measure on random variables $\bar\rho_t : \bar L^\infty \to \bar L_t^\infty$ is of the form (11) with some conditional convex risk measure on processes $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$.

Proof. Clearly, $\bar\rho_t$ defined via (11) is a conditional convex risk measure in the sense of [16]. To see, e.g., conditional cash invariance, let $m \in \bar L_t^\infty$, i.e. $m = (m_0, \ldots, m_{t-1}, m_t, m_t, \ldots)$ with $m_i \in L_i^\infty$ for $i = 0, \ldots, t$. Then

$$\begin{aligned}\bar\rho_t(X+m) &= (-X_0 - m_0, \ldots, -X_{t-1} - m_{t-1}, \rho_t(X+m), \rho_t(X+m), \ldots)\\ &= \bar\rho_t(X) - m\end{aligned}$$

by conditional cash invariance of ρ_t.

To prove the converse implication, let $\bar\rho_t : \bar L^\infty \to \bar L_t^\infty$ be a conditional convex risk measure for random variables. Since $A_t := \Omega \times \{0, \ldots, t-1\} \in \bar{\mathcal{F}}_t$, the local property (cf., e.g., [16, Proposition 2]), conditional cash invariance and normalization of $\bar\rho_t$ imply

$$\begin{aligned}\bar\rho_t(X) &= 1_{A_t}\bar\rho_t(1_{A_t}X) + 1_{A_t^c}\bar\rho_t(1_{A_t^c}X)\\ &= -X_0 1_{\{0\}} - \cdots - X_{t-1}1_{\{t-1\}} + \bar\rho_t(X 1_{\mathbb{T}_t})1_{\mathbb{T}_t}.\end{aligned}$$

Finally, it is easy to see that $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ defined by $\rho_t(X) := (\bar\rho_t(X))_t$ is a conditional convex risk measure for processes in the sense of Definition 3. $\square$

Let $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ be a conditional convex risk measure for processes, and consider the corresponding acceptance set

$$\mathcal{A}_t = \{X \in \mathcal{R}_t^\infty \mid \rho_t(X) \le 0\}.$$

Then the acceptance set of $\bar\rho_t$ related to ρ_t via (11) is given by

$$\begin{aligned}\bar{\mathcal{A}}_t &= \left\{X \in \bar L^\infty \mid \bar\rho_t(X) \le 0 \ \bar P\text{-a.s.}\right\}\\ &= \left\{X \in \mathcal{R}^\infty \mid X_s \ge 0 \,\forall s = 0, \ldots, t-1, \ \rho_t(X) \le 0 \ P\text{-a.s.}\right\}\\ &= \mathcal{A}_t + (L_{0,+}^\infty \times \ldots \times L_{t-1,+}^\infty \times \{0\} \times \ldots). \end{aligned} \tag{12}$$

For each $\bar Q \in \mathcal{M}(\bar P)$, the minimal penalty function of $\bar\rho_t$ is given by

$$\bar\alpha_t(\bar Q) = \bar{\mathrm{Q}}\text{-}\operatorname*{ess\,sup}_{X \in \bar{\mathcal{A}}_t} E_{\bar Q}[-X \mid \bar{\mathcal{F}}_t].$$

Due to (12) and Corollary 52, this takes the form

$$\bar\alpha_t(\bar Q) = \alpha_t(\bar Q)1_{\mathbb{T}_t}, \tag{13}$$

where $\alpha_t(\bar Q)$ denotes the minimal penalty function of ρ_t and is given by

$$\begin{aligned}\alpha_t(Q\otimes\gamma) = \alpha_t(Q \otimes D) &= \mathrm{Q}\text{-}\operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q\left[-\sum_{s \in \mathbb{T}_t} \frac{\gamma_s}{D_t} X_s \;\Big|\; \mathcal{F}_t\right] \\ &= \mathrm{Q}\text{-}\operatorname*{ess\,sup}_{X \in \mathcal{R}^\infty} \left(E_Q\left[-\sum_{s \in \mathbb{T}_t} \frac{\gamma_s}{D_t} X_s \;\Big|\; \mathcal{F}_t\right] - \rho_t(X)\right).\end{aligned} \tag{14}$$

Here $Q \otimes D = Q \otimes \gamma$ denotes the decomposition of the measure $\bar Q$ in the sense of Theorem 6. Note that $\alpha_t(Q \otimes \gamma)$ is well defined Q-a.s. on $\{D_t > 0\}$; cf. Corollary 52.

3.3 Robust representations

In this section we derive a robust representation of a conditional convex risk measure for processes which expresses explicitly the combined role of model ambiguity and discounting ambiguity. Our proof will consist in combining the robust representation of risk measures for random variables as stated in [16], [5], [7], [30], [20], and [1], with our Decomposition Theorem 6 for measures on the optional σ-field.

The following continuity property was introduced in [10, Definition 3.15].

Definition 9. *A conditional convex risk measure $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ for processes is called continuous from above if*

$$\rho_t(X^n) \nearrow \rho_t(X) \quad P\text{-a.s with } n \to \infty$$

for any decreasing sequence $(X^n)_n \subseteq \mathcal{R}^\infty$ and $X \in \mathcal{R}^\infty$ such that $X_s^n \searrow X_s$ P-a.s. for all $s \in \mathbb{T}_t$.

Theorem 10. *A conditional convex risk measure for processes ρ_t is continuous from above if and only if it admits the following robust representation:*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{loc}} \operatorname*{ess\,sup}_{\gamma \in \Gamma_t(Q)} \left(E_Q\left[-\sum_{s \in \mathbb{T}_t} \gamma_s X_s \;\middle|\; \mathcal{F}_t \right] - \alpha_t(Q \otimes \gamma) \right), \qquad X \in \mathcal{R}_t^\infty \tag{15}$$

where α_t is defined in (14),

$$\mathcal{Q}_t^{loc} := \left\{ Q \in \mathcal{M}_{loc}(P) \;\middle|\; Q = P \text{ on } \mathcal{F}_t \right\},$$

and

$$\Gamma_t(Q) := \left\{ \gamma \in \Gamma(Q) \;\middle|\; \gamma_s = 0 \;\forall\; s < t \right\}.$$

Proof. It is easy to check that ρ_t is continuous from above if and only if the conditional risk measure $\bar{\rho}_t$ defined in (11) is continuous from above. By [16, Theorem 1], continuity from above of $\bar{\rho}_t$ is equivalent to the robust representation

$$\bar{\rho}_t(X) = \operatorname*{ess\,sup}_{\bar{Q} \in \bar{\mathcal{Q}}_t} \left(E_{\bar{Q}}\left[-X \;\middle|\; \bar{\mathcal{F}}_t \right] - \bar{\alpha}_t(\bar{Q}) \right),$$

where

$$\bar{\mathcal{Q}}_t := \left\{ \bar{Q} \in \mathcal{M}(\bar{P}) \;\middle|\; \bar{Q} = \bar{P} \text{ on } \bar{\mathcal{F}}_t \right\}. \tag{16}$$

Using Corollary 52, this takes the form

$$\begin{aligned} \bar{\rho}_t(X) = &- X_0 1_{\{0\}} - \ldots - X_{t-1} 1_{\{t-1\}} \\ &+ \operatorname*{ess\,sup}_{Q \otimes \gamma \in \bar{\mathcal{Q}}_t} \left(E_Q\left[-\sum_{s \in \mathbb{T}_t} \frac{\gamma_s}{D_t} X_s \;\middle|\; \mathcal{F}_t \right] - \alpha_t(Q \otimes \gamma) \right) 1_{\mathbb{T}_t}, \end{aligned} \tag{17}$$

where D is related to γ via (4). Lemma 54 implies that $Q \otimes \gamma \in \bar{\mathcal{Q}}_t$ if and only if $Q \in \mathcal{Q}_t^{\mathrm{loc}}$, and $\gamma_s = \mu_s$ for $s = 0, \ldots, t-1$; in particular $D_t = \sum_{s \in \mathbb{T}_t} \mu_s > 0$. For each $Q \in \mathcal{Q}_t^{\mathrm{loc}}$ we can identify the set $\{(\frac{\gamma_s}{D_t})_{s \in \mathbb{T}_t} \mid Q \otimes \gamma \in \bar{\mathcal{Q}}_t\}$ with $\Gamma_t(Q)$, and so the representation (15) follows from (17) due to (11). $\square$

Using the integration by parts formula (7) we can rewrite (15) as follows.

Corollary 11. *In terms of discounting factors, the representation* (15) *takes the following form for* $X \in \mathcal{R}_t^\infty$ *if* $T < \infty$ *or if* $\mathbb{T} = \mathbb{N}_0$*, and for* $X \in \mathcal{X}_t^\infty$ *if* $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$*:*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{loc}} \operatorname*{ess\,sup}_{D \in \mathcal{D}_t(Q)} \left(E_Q \left[-\sum_{s=t}^{T} D_s \Delta X_s \,\middle|\, \mathcal{F}_t \right] - \alpha_t(Q \otimes D) \right), \tag{18}$$

where

$$\mathcal{D}_t(Q) = \left\{ D \in \mathcal{D}(Q) \,\middle|\, D_s = 1 \ \forall \ s \leq t \right\}.$$

Remark 12. *In [10] Cheridito, Delbaen, and Kupper consider the cases* $T < \infty$ *and* $\mathbb{T} = \mathbb{N}_0$*. They work on the space* $\mathcal{R}^\infty$ *equipped with the dual space*

$$\mathcal{A}^1 := \left\{ a = (a_t)_{t \in \mathbb{T}} \,\middle|\, a \text{ adapted, } E_P \left[\sum_{t \in \mathbb{T}} |a_t - a_{t-1}| \right] < \infty \right\},$$

where $a_{-1} := 0$*. The robust representation of conditional convex risk measures in [10] is formulated in terms of the set*

$$\mathcal{D}_{0,T} := \left\{ a \in \mathcal{A}^1 \,\middle|\, a_t \geq a_{t-1} \text{ for all } t \in \mathbb{T},\ E_P \left[\sum_{t \in \mathbb{T}} (a_t - a_{t-1}) \right] = 1 \right\};$$

cf. [10, Theorem 3.16]. Note that $\mathcal{D}_{0,T}$ *can be identified with the set* $\mathcal{M}(\bar{P})$*. Indeed, every* $a \in \mathcal{D}_{0,T}$ *defines a density* $\bar{Z}$ *of* $\bar{Q} \in \mathcal{M}(\bar{P})$ *via*

$$Z_t \mu_t = a_t - a_{t-1}, \qquad t \in \mathbb{T},$$

and vice versa. By emphasizing $\mathcal{M}(\bar{P})$ *rather than* $\mathcal{D}_{0,T}$ *we take a more probabilistic approach. In particular, we exploit the decomposition* $\bar{Q} = Q \otimes \gamma = Q \otimes D$ *of probability measures in* $\mathcal{M}(\bar{P})$*. We have*

$$E_P \left[\sum_{s \in \mathbb{T}_t} X_s \Delta a_s | \mathcal{F}_t \right] = E_Q \left[\sum_{s \in \mathbb{T}_t} X_s \gamma_s | \mathcal{F}_t \right] = E_Q \left[\sum_{s \in \mathbb{T}_t} \Delta X_s D_s | \mathcal{F}_t \right]$$

for all $t \in \mathbb{T}$ *and all* $X \in \mathcal{R}_t^\infty$*. The representation on the right hand side has two advantages. In the first place it allows us to make explicit the joint role of model uncertainty, as expressed by the measures* $Q \in \mathcal{M}_{loc}(P)$*, and of discounting uncertainty, as described by the discounting processes* $D \in \mathcal{D}(Q)$*. Moreover, the probabilistic approach allows us to discuss the case* $T = \infty$ *in terms of a measure theoretic extension problem, and it will be crucial for our analysis of the supermartingale aspects of time consistency.*

As a special case, our representation (18) *applied for* $T = 1$ *at* $t = 0$ *to the process* $(0, X_T)$ *with* $X_T \in L^\infty$*, yields the representation (4.5) in [18, Corollary 4.4] in the static context of cash subadditive risk measures for random variables; cf. also Remark 30.*

In analogy to the proof of Theorem 10, the results in [20, Corollary 2.4], [1, Corollary 11] and [20, Lemma 3.5] translate into robust representations in our context which use a smaller set of measures:

Corollary 13. *A conditional convex risk measure on processes* ρ_t *is continuous from above if and only if any of the following representations hold:*

1. ρ_t *is of the form* (15)*, where the essential supremum is taken over the set*

$$\left\{ Q \otimes \gamma \;\middle|\; Q \in \mathcal{Q}_t^{loc},\ \gamma \in \Gamma_t(Q),\ E_Q\Big[\big(\sum_{s \in \mathbb{T}_t} \mu_s\big)\alpha_t(Q \otimes \gamma)\Big] < \infty \right\}.$$

2. *for all* $\bar{Q} = Q \otimes D \in \mathcal{M}(\bar{P})$ *and* $X \in \mathcal{R}_t^\infty$ *we have*

$$\rho_t(X) = \operatorname*{Q\text{-ess\,sup}}_{R\otimes\xi \in \bar{\mathcal{Q}}_t(\bar{Q})} \left(\frac{1}{D_t} E_R \left[-\sum_{s \in \mathbb{T}_t} \xi_s X_s \;\middle|\; \mathcal{F}_t \right] - \alpha_t(R \otimes \xi) \right)$$

Q-a.s. on $\{D_t > 0\}$, where

$$\bar{\mathcal{Q}}_t(\bar{Q}) := \left\{ \bar{R} \in \mathcal{M}(\bar{P}) \;\middle|\; \bar{R} = \bar{Q}|_{\bar{\mathcal{F}}_t} \right\}.$$

Moreover, if there exists a probability measure $\bar{P}^* \approx \bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ such that $\alpha_t(\bar{P}^*) < \infty$, then continuity from above is also equivalent to a representation of the form (15) *as an essential supremum over the set*

$$\{Q \otimes \gamma \mid Q \in \mathcal{M}^e_{loc}(P),\ \gamma \in \Gamma^e(Q)\},$$

where

$$\Gamma^e(Q) := \left\{ \gamma \in \Gamma(Q) \;\middle|\; \gamma_t > 0 \ P\text{-a.s. for all}\, t \in \mathbb{T} \right\}.$$

4 Supermartingale criteria for time consistency

In this section we consider time consistency, derive corresponding criteria in terms of supermartingales, and discuss some of the consequences, in particular conditions for asymptotic safety.

4.1 Strong time consistency and its characterization

A strong notion of time consistency for risk measures for processes was introduced and characterized in [10] and [11]. Here we adopt the definition from [10], cf. [10, Definition 4.2, Proposition 4.4, Proposition 4.5].

Definition 14. *A dynamic convex risk measure for processes $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ on $\mathcal{R}^\infty$ is called* (strongly) time consistent *if for all t in $\mathbb{T}$ such that $t < T$ and for all $X, Y \in \mathcal{R}^\infty$*

$$X_t = Y_t \quad \textit{and} \quad \rho_{t+1}(X) \le \rho_{t+1}(Y) \quad \Longrightarrow \quad \rho_t(X) \le \rho_t(Y). \tag{19}$$

Note that a dynamic risk measure for processes $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ is time consistent if and only if the corresponding dynamic convex risk measure for random variables $(\bar{\rho}_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ on $\bar{L}^\infty$ defined by (11) is time consistent, that is, if $\bar{\rho}_{t+1}(X) \le \bar{\rho}_{t+1}(Y)$ implies $\bar{\rho}_t(X) \le \bar{\rho}_t(Y)$ for all $X, Y \in \bar{L}^\infty$ and all $t \in \mathbb{T}$, $t < T$. Criteria for time consistency of risk measures for random variables were studied intensively in the literature, see, e.g., [16], [30], [4], [20], [6], [7], [1] and the references therein. Using Proposition 8 we can translate these criteria into our present framework.

By [20, Proposition 4.2] applied to $\bar{\rho}$, time consistency (19) of ρ is equivalent to *recursiveness*, that is

$$\begin{aligned}\rho_t(X) &= \rho_t(X_t 1_{\{t\}} - \rho_{t+1}(X)1_{\mathbb{T}_{t+1}}) \\ &= -X_t + \rho_t(-\rho_{t+1}(X - X_t)1_{\mathbb{T}_{t+1}}).\end{aligned} \tag{20}$$

If we restrict the conditional convex risk measure $\bar{\rho}_t$ to the space $L^\infty(\bar{\mathcal{F}}_{t+1})$, the acceptance set is given by

$$\begin{aligned}\bar{\mathcal{A}}_{t,t+1} &:= \left\{ X \in L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_{t+1}, \bar{P}) \;\middle|\; \bar{\rho}_t(X) \leq 0 \;\; \bar{P}\text{-a.s.} \right\} \\ &= \mathcal{A}_{t,t+1} + (L^\infty_{0,+} \times \ldots \times L^\infty_{t-1,+} \times \{0\} \times \ldots),\end{aligned}$$

where

$$\mathcal{A}_{t,t+1} := \{ X \in \mathcal{R}^\infty_{t,t+1} \mid \rho_t(X) \leq 0 \}, \quad t \in \mathbb{T}, \quad t < T,$$

denotes the acceptance set of the risk measure for processes ρ_t restricted to $\mathcal{R}^\infty_{t,t+1}$. The corresponding one-step minimal penalty function for $\bar{\rho}_t$ takes the form

$$\bar{\alpha}_{t,t+1}(\bar{Q}) := \bar{\mathrm{Q}}\text{-}\operatorname*{ess\,sup}_{X \in \bar{\mathcal{A}}_{t,t+1}} E_{\bar{Q}}[-X \mid \bar{\mathcal{F}}_t] = \alpha_{t,t+1}(\bar{Q})\mathbb{T}_t, \qquad \bar{Q} \in \mathcal{M}(\bar{P}),$$

where the function $\alpha_{t,t+1}(\bar{Q})$ is given for $\bar{Q} = Q \otimes D = Q \otimes \gamma \in \mathcal{M}(\bar{P})$ by

$$\alpha_{t,t+1}(Q \otimes D) = \frac{1}{D_t} \operatorname*{Q\text{-}ess\,sup}_{X \in \mathcal{A}_{t,t+1}} E_Q\left[-\gamma_t X_t - D_{t+1}X_{t+1} \;\middle|\; \mathcal{F}_t\right], \quad t \in \mathbb{T}, \quad t < T,$$

due to Corollary 52. Note that the penalty functions $\alpha_t(Q \otimes D)$ and $\alpha_{t,t+1}(Q \otimes D)$ are only defined Q-a.s. on $\{D_t > 0\}$. In the following we *define* for $Q \otimes D \in \mathcal{M}(\bar{P})$

$$\alpha_t(Q \otimes D) := \infty, \quad \alpha_{t,t+s}(Q \otimes D) := \infty \quad Q\text{-a.s. on } \{D_t = 0\}$$

for all $t, s \geq 0$, and use henceforth the convention $0 \cdot \infty := 0$.

The following result characterizes time consistency in terms of a splitting property of the acceptance sets and in terms of supermartingale properties of the penalty process and the dynamic risk measure. It translates [20, Theorem 4.5] and [1, Theorem 20] to our present framework.

Theorem 15. *Let $(\rho_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ be a dynamic convex risk measure on $\mathcal{R}^\infty$ such that each ρ_t is continuous from above. Then the following conditions are equivalent:*

(i) $(\rho_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ *is time consistent;*

(ii) $\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$ *for all* $t \in \mathbb{T}$, $t < T$;

(iii) for all $t \in \mathbb{T}$, $t < T$ *and* $\bar{Q} = Q \otimes D \in \mathcal{M}(\bar{P})$

$$D_t \alpha_t(Q \otimes D) = D_t \alpha_{t,t+1}(Q \otimes D) + E_Q[D_{t+1}\alpha_{t+1}(Q \otimes D) \mid \mathcal{F}_t] \quad Q\text{-a.s.};$$

(iv) for all $X \in \mathcal{R}^\infty$, $t \in \mathbb{T}$, $t < T$, *and* $\bar{Q} = Q \otimes D \in \mathcal{M}(\bar{P})$

$$E_Q[D_{t+1}(X_t + \rho_{t+1}(X) + \alpha_{t+1}(Q \otimes D)) \mid \mathcal{F}_t] \leq D_t(X_t + \rho_t(X) + \alpha_t(Q \otimes D))$$

Q-a.s..

Moreover, if there exists a probability measure $\bar{P}^* \approx \bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ such that $\alpha_0(\bar{P}^*) < \infty$, condition (iv) stated only for the measures

$$\begin{aligned}\bar{\mathcal{Q}}^* &:= \left\{ \bar{Q} \in \mathcal{M}^e(\bar{P}) \;\middle|\; \alpha_0(\bar{Q}) < \infty \right\} \\ &= \left\{ Q \otimes \gamma \;\middle|\; Q \in \mathcal{M}^e_{loc}(P),\ \gamma \in \Gamma^e(Q),\ \alpha_0(Q \otimes \gamma) < \infty \right\}\end{aligned} \tag{21}$$

already implies time consistency, and the robust representation (15) *of ρ_t also holds if the essential supremum is taken only over the set $\bar{\mathcal{Q}}^*$.*

Proof. Follows from [1, Theorem 20] and [20, Theorem 4.5] applied to $\bar{\rho}_t$ defined in (11) using Corollary 52. ☐

Remark 16. *Equivalence of time consistency and (ii) of Theorem 15 holds without assuming continuity from above and was already proved in [10, Theorem 4.6]. Characterizations of time consistency in terms of penalty functions as in condition (iii) are given in [10, Theorem 4.19, Theorem 4.22]. However, the latter results use neither the decomposition of $\bar{Q}$ into a measure Q and a discounting factor D, nor the one-step penalty functions $\alpha_{t,t+1}$. The role of $\alpha_{t,t+1}$ in condition (iii) is analogous to the corresponding characterization of time consistency of risk measures for random variables in [6, Theorem 2.5] and [20, Theorem 4.5]. In the same way, the supermartingale characterization (iv) of time consistency translates the corresponding criterion from [20, Theorem 4.5] into our present framework.*

In the following we use the notation

$$\bar{\mathcal{Q}}_0 := \left\{ Q \otimes D \in \mathcal{M}(\bar{P}) \;\middle|\; \alpha_0(Q \otimes D) < \infty \right\}.$$

Corollary 17. *Let $(\rho_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ be a time consistent dynamic convex risk measure on $\mathcal{R}^\infty$ such that each ρ_t is continuous from above. Then*

1. *For any $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_0$, the discounted penalty process $(D_t \alpha_t(Q \otimes D))_{t \in \mathbb{T} \cap \mathbb{N}_0}$ is a nonnegative Q-supermartingale. Its Doob decomposition is given by the predictable process*

   $$A_t^{Q,D} := \sum_{k=0}^{t-1} D_k \alpha_{k,k+1}(Q \otimes D), \qquad t \in \mathbb{T} \cap \mathbb{N}_0,$$

   i.e.,

   $$M_t^{Q,D} := D_t \alpha_t(Q \otimes D) + A_t^{Q,D}, \qquad t \in \mathbb{T} \cap \mathbb{N}_0, \tag{22}$$

   is a Q-martingale.

2. *For all $X \in \mathcal{R}^\infty$ and all $\bar{Q} \in \bar{\mathcal{Q}}_0$, the process*

   $$W_t^{Q,D}(X) := D_t \rho_t(X - X_t 1_{\mathbb{T}_t}) + \sum_{s=0}^{t} D_s(-\Delta X_s) + D_t \alpha_t(Q \otimes D), \quad t \in \mathbb{T} \cap \mathbb{N}_0, \tag{23}$$

   is a Q-supermartingale.

Remark 18. *In the same way as in Theorem 15, we can translate the weaker concepts of time consistency from [40, 4, 39, 36, 34, 17, 1] into our present framework, and obtain results analogous to [1, Theorem 31, Proposition 33, Proposition 37].*

4.2 Riesz decomposition of the penalty process and the appearance of bubbles

The following proposition characterizes the martingale $M^{Q,D}$ in the Doob decomposition of the Q-supermartingale $(D_t\alpha_t(Q\otimes D))_{t\in\mathbb{T}\cap\mathbb{N}_0}$ from Corollary 17; it translates [1, Proposition 24] and [34, Proposition 2.3.2] into our present context.

Proposition 19. *The martingale $M^{Q,D}$ in (22) is of the form*

$$M_t^{Q,D} = E_Q\left[\sum_{k=0}^{T-1} D_k\alpha_{k,k+1}(Q\otimes D)\,\Big|\,\mathcal{F}_t\right] + N_t^{Q,D} \quad Q\text{-}a.s., \quad t\in\mathbb{T}\cap\mathbb{N}_0,$$

where

$$N_t^{Q,D} := \begin{cases} 0 & \text{if } T<\infty \\ \lim\limits_{s\to\infty} E_Q\left[D_s\alpha_s(Q\otimes D)\,|\,\mathcal{F}_t\right] & \text{if } T=\infty \end{cases} \quad Q\text{-}a.s., \quad t\in\mathbb{T}\cap\mathbb{N}_0,$$

is a nonnegative Q-martingale. Thus the Riesz decomposition of the Q-supermartingale $(D_t\alpha_t(Q\otimes D))$ into a potential and a martingale takes the form

$$D_t\alpha_t(Q\otimes D) = E_Q\left[\sum_{k=t}^{T-1} D_k\alpha_{k,k+1}(Q\otimes D)\,\Big|\,\mathcal{F}_t\right] + N_t^{Q,D} \quad Q\text{-}a.s., \quad t\in\mathbb{T}\cap\mathbb{N}_0. \tag{24}$$

Proof. Property (iii) of Theorem 15 yields

$$D_t\alpha_t(\bar{Q}) = E_Q\left[\sum_{k=t}^{t+s-1} D_k\alpha_{k,k+1}(\bar{Q})\,\Big|\,\mathcal{F}_t\right] + E_Q[\,D_{t+s}\alpha_{t+s}(\bar{Q})\,|\,\mathcal{F}_t\,] \quad Q\text{-a.s.} \tag{25}$$

for all $t,s\in\mathbb{N}_0$ s.t. $t+s\in\mathbb{T}$ and all $\bar{Q}\in\mathcal{M}(\bar{P})$. For $T<\infty$ the claim is obvious, since $\alpha_T(\bar{Q})=0$ P-a.s.. For $T=\infty$, by monotonicity there exists the limit

$$\begin{aligned} S_t^{Q,D} &= \lim_{s\to\infty} E_Q\left[\sum_{k=t}^{s} D_k\alpha_{k,k+1}(\bar{Q})\,\Big|\,\mathcal{F}_t\right] \\ &= E_Q\left[\sum_{k=t}^{\infty} D_k\alpha_{k,k+1}(\bar{Q})\,\Big|\,\mathcal{F}_t\right] \quad Q\text{-a.s.} \end{aligned}$$

for all $t\in\mathbb{T}\cap\mathbb{N}_0$, where we have used the monotone convergence theorem for the second equality. Thus (25) implies existence of

$$N_t^{Q,D} = \lim_{s\to\infty} E_Q[\,D_{t+s}\alpha_{t+s}(\bar{Q})\,|\,\mathcal{F}_t\,] \quad Q\text{-a.s.}, \quad t\in\mathbb{T}\cap\mathbb{N}_0$$

and

$$D_t\alpha_t(\bar{Q}) = S_t^{Q,D} + N_t^{Q,D} \quad Q\text{-a.s.}, \quad t\in\mathbb{T}\cap\mathbb{N}_0.$$

The process $(S_t^{Q,D})$ is a Q-potential. Indeed,

$$E_Q[\,S_t^{Q,D}\,] \le E_Q\left[\sum_{k=0}^{\infty} D_k\alpha_{k,k+1}(\bar{Q})\,\Big|\,\mathcal{F}_t\right] \le \alpha_0(\bar{Q}) < \infty$$

and $E_Q[\,S^{Q,D}_{t+1}\,|\,\mathcal{F}_t\,] \le S^{Q,D}_t$ Q-a.s. for all $t \in \mathbb{T} \cap \mathbb{N}_0$ by definition. Moreover, monotone convergence implies

$$\lim_{t\to\infty} E_Q[\,S^{Q,D}_t\,] = E_Q\left[\lim_{t\to\infty}\sum_{k=t}^{\infty} D_k\alpha_{k,k+1}(\bar{Q})\right] = 0 \quad Q\text{-a.s..}$$

The process $(N^{Q,D}_t)$ is a nonnegative Q-martingale, since

$$E_Q[N^{Q,D}_{t+1} - N^{Q,D}_t|\mathcal{F}_t] = D_t\alpha_{t,t+1}(\bar{Q}) - D_t\alpha_{t,t+1}(\bar{Q}) = 0 \qquad Q\text{-a.s.}$$

for all $t \in \mathbb{T} \cap \mathbb{N}_0$ by property (iii) of Theorem 15 and the definition of $(S^{Q,D}_t)$. □

The nonnegative martingale $N^{Q,D}$, which may appear in the decomposition (24) of the penalty process for $T = \infty$, plays the role of a "bubble". Indeed, it appears on top of the "fundamental" component which is given by the potential $S^{Q,D}$ generated by the one-step penalties, and this additional penalization causes an excessive neglect of the model $Q \otimes D$ in assessing the risk. As a result, asymptotic safety breaks down under the model $Q \otimes D$, as explained in the next section.

4.3 Asymptotic safety and asymptotic precision

In this section we discuss the asymptotic properties of dynamic convex risk measures for processes. Throughout this section we consider the case $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$. In the case $\mathbb{T} = \mathbb{N}_0$ our assumption of global continuity from above implies that there is "no mass at infinity", i.e., $D_\infty = 0$ Q-a.s. for all $Q \otimes D \in \mathcal{M}(\bar{P})$, and the discussion below reduces to the trivial case. A systematic discussion of the case $\mathbb{T} = \mathbb{N}_0$, but without assuming the existence of a global reference measure P and global continuity from above, appears in Föllmer and Penner [21].

Consider a time consistent dynamic convex risk measure for processes $(\rho_t)_{t\in\mathbb{N}_0}$. As before, $(\bar{\rho}_t)_{t\in\mathbb{N}_0}$ denotes the corresponding time consistent dynamic convex risk measure for random variables on product space given by (11). Let $\bar{Q} = Q \otimes \gamma = Q \otimes D \in \bar{\mathcal{Q}}_0$, and let us focus on the behavior of $(\bar{\rho}_t)_{t\in\mathbb{N}_0}$ under $\bar{Q}$. The measure $\bar{Q}$ will now play the same role as the reference measure P in [20, Section 5]. In particular, the assumption $\mathcal{Q}^* \neq \emptyset$ from [20, Section 5] is satisfied for $\bar{Q}$, since $\bar{Q} \in \bar{\mathcal{Q}}_0$.

The results in [20] imply the existence of the limits

$$\bar{\alpha}_\infty(\bar{Q}) := \lim_{t\to\infty}\bar{\alpha}_t(\bar{Q}) \quad \text{and} \quad \bar{\rho}_\infty(X) := \lim_{t\to\infty}\bar{\rho}_t(X) \quad \bar{Q}\text{-a.s.}$$

for all $X \in \mathcal{R}^\infty$. Due to (11) and (13), we have

$$\bar{\rho}_\infty(X) = -X1_{\mathbb{N}_0} + \rho_\infty(X)1_{\{\infty\}} \quad \text{and} \quad \bar{\alpha}_\infty(\bar{Q}) = \alpha_\infty(\bar{Q})1_{\{\infty\}} \quad \bar{Q}\text{-a.s.,} \tag{26}$$

where

$$\rho_\infty(X) := \lim_{t\to\infty}\rho_t(X) \quad \text{and} \quad \alpha_\infty(\bar{Q}) = \lim_{t\to\infty}\alpha_t(\bar{Q}) \quad Q\text{-a.s. on } \{D_\infty > 0\}$$

by 3 of Remark 51.

Definition 20. *We call a dynamic convex risk measure for processes* $(\rho_t)_{t\in\mathbb{N}_0}$ asymptotically safe under the model $\bar{Q} = Q \otimes D$ *if the limiting capital requirement* $\rho_\infty(X)$ *covers the final loss* $-X_\infty$*, i.e.*

$$\rho_\infty(X) \geq -X_\infty \quad Q\text{-a.s. on } \{D_\infty > 0\}$$

for any $X \in \mathcal{R}^\infty$*.*

Note that due to (26) asymptotic safety of $(\rho_t)_{t\in\mathbb{N}_0}$ is equivalent to the condition

$$\bar{\rho}_\infty(X) \geq -X \quad \bar{Q}\text{-a.s.},$$

i.e., to asymptotic safety of $(\bar{\rho}_t)_{t\in\mathbb{N}_0}$ in the sense of [20, Definition 5.2].

The following result translates [20, Theorem 5.4] and [34, Corollary 3.1.5] to our present setting. It characterizes asymptotic safety by the absence of bubbles in the penalty process. This is plausible since, as we saw in Subsection 4.2, such bubbles reflect an excessive neglect of models which may be relevant for the risk assessment.

Theorem 21. *Let* $(\rho_t)_{t\in\mathbb{N}_0}$ *be a time consistent dynamic convex risk measure such that each* ρ_t *is continuous from above. Then for any model* $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_0$*, the following conditions are equivalent:*

1. (ρ_t) *is asymptotically safe under the model* $\bar{Q}$*;*
2. *the model* $\bar{Q}$ *has no bubble, i.e., the martingale* $N^{Q,D}$ *in the Riesz decomposition* (24) *of the discounted penalty process* $(D_t\alpha_t(\bar{Q}))_{t\in\mathbb{N}_0}$ *vanishes;*
3. *the discounted penalty process* $(D_t\alpha_t(\bar{Q}))_{t\in\mathbb{N}_0}$ *is a* Q*-potential;*
4. *no model* $\bar{R} \ll \bar{Q}$ *with* $\alpha_0(\bar{R}) < \infty$ *admits bubbles.*

Proof. Properties 2 and 3 are equivalent by (24), and obviously 4 implies 2.
To prove $1 \Leftrightarrow 2$ we use [20, Theorem 5.4]. There it was shown that $(\bar{\rho}_t)$ is asymptotically safe under $\bar{Q}$ if and only if $\bar{\alpha}_\infty(\bar{Q}) = 0$ $\bar{Q}$-a.s. and in $L^1(\bar{Q})$. By Corollary 52, (13), and (5) we have

$$E_{\bar{Q}}[\bar{\alpha}_t(\bar{Q})] = E_Q\left[\sum_{s\in\mathbb{T}_t} \gamma_s \alpha_t(\bar{Q})\right] = E_Q\left[D_t\alpha_t(\bar{Q})\right].$$

Thus $\bar{\alpha}_t(\bar{Q}) \to 0$ in $L^1(\bar{Q})$ if and only if $D_t\alpha_t(\bar{Q}) \to 0$ in $L^1(Q)$. This is equivalent to $N^{Q,D} \equiv 0$, since the bubble $N^{Q,D} = (N_t^{Q,D})_{t\in\mathbb{N}_0}$ is a nonnegative Q-martingale with $N_0^{Q,D} = \lim_{t\to\infty} E_Q\left[D_t\alpha_t(\bar{Q})\right]$. Due to (24), $N^{Q,D} \equiv 0$ also implies $\alpha_\infty(\bar{Q}) = 0$ Q-a.s. on $\{D_\infty > 0\}$, thus $\bar{\alpha}_\infty(\bar{Q}) = 0$ $\bar{Q}$-a.s. by (26).
To prove $2 \Rightarrow 4$ note that asymptotic safety under $\bar{Q}$ implies asymptotic safety under any model $\bar{R} \ll \bar{Q}$ with $\alpha_0(\bar{R}) < \infty$, thus no model $\bar{R}$ admits bubbles by the same reasoning as above. □

Definition 22. *We call a dynamic convex risk measure for processes* $(\rho_t)_{t\in\mathbb{N}_0}$ asymptotically precise under the model $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_0$ *if*

$$\rho_\infty(X) = -X_\infty \quad Q\text{-a.s. on } \{D_\infty > 0\}$$

for any $X \in \mathcal{R}^\infty$*.*

By (26), asymptotic precision of (ρ_t) is equivalent to asymptotic precision of $(\bar{\rho}_t)$ in the sense of [20, Definition 5.9]. The following corresponds to [32, Lemma 2.7].

Lemma 23. *A time consistent dynamic convex risk measure $(\rho_t)_{t\in\mathbb{N}_0}$ such that each ρ_t is continuous from above is asymptotically precise under the model $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_0$ if and only if*

$$\rho_\infty(X) \le -X_\infty \quad Q\text{-a.s on } \{D_\infty > 0\} \quad \text{for all} \quad X \in \mathcal{R}^\infty.$$

Proof. By [20, Lemma 5.1] the functional $\bar{\rho}_\infty$ is convex and normalized. This implies

$$\bar{\rho}_\infty(X) \ge -\bar{\rho}_\infty(-X) \quad \text{for all} \quad X \in \mathcal{R}^\infty.$$

Thus we obtain

$$-X \ge \bar{\rho}_\infty(X) \ge -\bar{\rho}_\infty(-X) \ge -X \quad \bar{Q}\text{-a.s} \quad \text{for all} \quad X \in \mathcal{R}^\infty,$$

which is equivalent to $\rho_\infty(X) = X_\infty$ Q-a.s. on $\{D_\infty > 0\}$ by (26). □

The following result translates [20, Proposition 5.11] to our present setting.

Proposition 24. *Let $(\rho_t)_{t\in\mathbb{N}_0}$ be a time consistent dynamic convex risk measure such that each ρ_t is continuous from above, and assume that for each $X \in \mathcal{R}^\infty$ the supremum in the robust representation (15) of $\rho_0(X)$ is attained by some "worst case" measure $Q^X \otimes \gamma^X = \bar{Q}^X$, such that $\bar{Q}^X \approx \bar{Q}$. Then $(\rho_t)_{t\in\mathbb{N}_0}$ is asymptotically precise under $\bar{Q}$.*

Proof. Since $\rho_0(X) = \bar{\rho}_0(X)$, $\bar{Q}^X$ is also a worst case measure for $\bar{\rho}_0(X)$. By [1, Proposition 21], the measure $\bar{Q}^X$ is then a worst case measure for X at all times $t \in \mathbb{N}_0$, i.e.,

$$\bar{\rho}_t(X) = E_{\bar{Q}^X}\left[-X|\bar{\mathcal{F}}_t\right] - \bar{\alpha}_t(\bar{Q}^X) \quad \bar{Q}\text{-a.s.} \quad \forall\, t \in \mathbb{N}_0,$$

and in particular $\bar{Q}^X \in \bar{\mathcal{Q}}_0$. By martingale convergence,

$$\bar{\rho}_\infty(X) = -X - \bar{\alpha}_\infty(\bar{Q}^X) \quad \bar{Q}\text{-a.s.},$$

which is equivalent to

$$\rho_\infty(X) = -X_\infty - \alpha_\infty(\bar{Q}^X) \quad Q\text{-a.s. on } \{D_\infty > 0\}$$

due to (26). Asymptotic precision of (ρ_t) now follows from Lemma 23, since $\alpha_\infty(\bar{Q}^X) \ge 0$ Q-a.s. on $\{D_\infty > 0\}$. □

4.4 A maximal inequality for the capital requirements

For $X \in \mathcal{R}^\infty$ and $Q \otimes D \in \mathcal{M}(\bar{P})$, we can interpret

$$F_t^{Q,D}(X) := E_Q\left[-\sum_{s\in\mathbb{T}_t} \frac{\gamma_s}{D_t} X_s \;\Big|\; \mathcal{F}_t\right] - \alpha_t(Q\otimes\gamma) \quad \text{on } \{D_t > 0\}$$

as a risk evaluation of the cash flow X at time $t \in \mathbb{T} \cap \mathbb{N}_0$, using the specific model Q and the specific discounting process D. The next proposition provides, from the point of view of the model Q, a maximal inequality for the excess of the required capital $\rho_t(X)$ over the risk evaluation $F_t^{Q,D}(X)$.

Proposition 25. *Let $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ be a time consistent dynamic convex risk measure such that each ρ_t is continuous from above. Then for $Q\otimes D\in\mathcal{M}(\bar P)$, $X\in\mathcal{R}^\infty$, and $c>0$ we have*

$$Q\left(\sup_{t\in\mathbb{T}\cap\mathbb{N}_0}\left\{D_t\left(\rho_t(X)-F_t^{Q,D}(X)\right)\right\}\geq c\right)\leq\frac{\rho_0(X)-F_0^{Q,D}(X)}{c}. \tag{27}$$

Proof. Fix $Q\otimes D\in\mathcal{M}(\bar P)$. If $\alpha_0(Q\otimes D)=\infty$, then the inequality (27) holds trivially. Assume that $\alpha_0(Q\otimes D)<\infty$. By 2) of Corollary 13 we have

$$\rho_t(X)\geq E_Q\left[-\sum_{s\in\mathbb{T}_t}\frac{\gamma_s}{D_t}X_s \mid \mathcal{F}_t\right]-\alpha_t(Q\otimes\gamma)=F_t^{Q,D}(X)\quad Q\text{-a.s. on }\{D_t>0\}.$$

Thus the Q-supermartingale $W^{Q,D}(X)$ defined in (23) satisfies

$$W_t^{Q,D}(X)\geq -E_Q\left[\sum_{s\in\mathbb{T}}\gamma_sX_s \mid \mathcal{F}_t\right]\qquad Q\text{-a.s.}\quad\text{on }\{D_t>0\}.$$

On $\{D_t=0\}=\{D_s=0\;\forall s\in\mathbb{T}_t\}$, we have $W_t^{Q,D}(X)=-\sum_{s=0}^{t-1}D_s\Delta X_s$. Therefore, the process

$$\begin{aligned}Y_t^{Q,D}(X)&:=D_t\left(\rho_t(X)-F_t^{Q,D}(X)\right)\\&=W_t^{Q,D}(X)+E_Q\left[\sum_{s\in\mathbb{T}}\gamma_sX_s \mid \mathcal{F}_t\right],\quad t\in\mathbb{T}\cap\mathbb{N}_0,\end{aligned}$$

is a nonnegative Q-supermartingale, and (27) follows by a classical maximal inequality; cf., e.g., [38, Theorem VII.3.1]. ☐

4.5 The coherent case

Due to positive homogeneity of a coherent risk measure, the penalty function can only take values 0 or ∞, and thus a coherent risk measure for processes ρ_t is continuous from above if and only if it admits the robust representation

$$\rho_t(X)=\operatorname*{ess\,sup}_{Q\otimes\gamma\in\mathcal{Q}_t^0}E_Q\left[-\sum_{s\in\mathbb{T}_t}\gamma_sX_s \mid \mathcal{F}_t\right],\qquad X\in\mathcal{R}_t^\infty, \tag{28}$$

where

$$\mathcal{Q}_t^0:=\left\{\bar Q\in\bar{\mathcal{Q}}_t \;\middle|\; \alpha_t(\bar Q)=0\right\}.$$

In this subsection we reformulate properties (iii) and (iv) of Theorem 15 in the coherent case. This involves a translation of the notions of *pasting* of measures and *stability* of sets as used in [4], [14], [20] in context of coherent risk measures for random variables to our present framework.

For $\bar Q^1,\bar Q^2\in\mathcal{M}(\bar P)$ such that $\bar Q^1\ll\bar Q^2$ on $\bar{\mathcal{F}}_t$ and for $B\in\bar{\mathcal{F}}_t$ we denote by $\bar Q^1\oplus_B^t\bar Q^2$ the *pasting* of $\bar Q^1$ and $\bar Q^2$ in t via B, i.e., the probability measure on $(\bar\Omega,\bar{\mathcal{F}})$ defined by

$$\bar Q^1\oplus_B^t\bar Q^2(A)=E_{\bar Q^1}\left[E_{\bar Q^2}[1_A|\bar{\mathcal{F}}_t]1_B+1_{B^c}1_A\right],\qquad A\in\bar{\mathcal{F}}.$$

Theorem 6 yields the decomposition $\bar{Q}^i = Q^i \otimes D^i$, $i = 1, 2$ with $Q^1 \ll Q^2$ on $\mathcal{F}_t$. Then

$$\bar{Q}^1 \oplus_B^t \bar{Q}^2 = Q^0 \otimes D^0,$$

where $B_t = \{\omega | (\omega, t) \in B\} \in \mathcal{F}_t$, and $Q^0 = Q^1 \oplus_{B_t}^t Q^2$, i.e.

$$Q^0(A) = E_{Q^1}\left[E_{Q^2}[1_A|\mathcal{F}_t]1_{B_t} + 1_{B_t^c}1_A\right], \qquad A \in \mathcal{F}_T,$$

and

$$\gamma_u^0 = \begin{cases} \gamma_u^1 & u = 0, \dots, t-1 \\ D_t^1 \dfrac{\gamma_u^2}{D_t^2} 1_{\{D_t^2 > 0\}} 1_{B_t} + \gamma_u^1 1_{B_t^c} & u \in \mathbb{T}_t. \end{cases}$$

Here γ^i and D^i are related to each other via (4) and (6) for $i = 0, 1, 2$. Note that $Q^0 \in \mathcal{M}_{\mathrm{loc}}(P), D^0 \in \mathcal{D}(Q^0)$, in other words, the pasting of $Q^1 \otimes D^1$ with $Q^2 \otimes D^2$ admits a decomposition with the pasting of Q^1 with Q^2 and the pasting of D^1 with D^2.

Definition 26. *We call a set $\bar{\mathcal{Q}} \subseteq \mathcal{M}(\bar{P})$* stable *if, whenever $\bar{Q}^1, \bar{Q}^2 \in \bar{\mathcal{Q}}$ and $\bar{Q}^1 \ll \bar{Q}^2$ on $\bar{\mathcal{F}}_t$, the pasting of $\bar{Q}^1$ and $\bar{Q}^2$ in t via B belongs to $\bar{\mathcal{Q}}$ for every $B \in \bar{\mathcal{F}}_t$ and all $t \in \mathbb{T} \cap \mathbb{N}_0$.*

We associate to any $\bar{Q} \in \mathcal{M}(\bar{P})$ the sets

$$\mathcal{Q}_t^0(\bar{Q}) = \left\{ \bar{R} \in \mathcal{M}(\bar{P}) \;\middle|\; \bar{R} = \bar{Q}|_{\bar{\mathcal{F}}_t},\ \bar{\alpha}_t(\bar{R}) = 0\ \bar{Q}\text{-a.s.} \right\},$$

and

$$\mathcal{Q}_{t,t+s}^0(\bar{Q}) = \left\{ \bar{R} \ll \bar{P}|_{\bar{\mathcal{F}}_{t+s}} \;\middle|\; \bar{R} = \bar{Q}|_{\mathcal{F}_t},\ \bar{\alpha}_{t,t+s}(\bar{R}) = 0\ \bar{Q}\text{-a.s.} \right\}.$$

The notion of pasting corresponds to *concatenation* defined in [10, Definition 4.10] on $\mathcal{A}^1$, and the following corollary is corresponds to [10, Theorem 4.13, Corollary 4.14].

Theorem 27. *Suppose that the dynamic risk measure $(\rho_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ is coherent, and that each ρ_t is continuous from above. Then the following conditions are equivalent:*

1. *$(\rho_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ is time consistent*
2. *For all $t \in \mathbb{T}$, $t < T$ and $\bar{Q} \in \mathcal{M}(\bar{P})$,*

$$\mathcal{Q}_t^0(\bar{Q}) = \left\{ \bar{Q}^1 \oplus_\Omega^{t+1} \bar{Q}^2 \;\middle|\; \bar{Q}^1 \in \mathcal{Q}_{t,t+1}^0(\bar{Q}),\ \bar{Q}^2 \in \mathcal{Q}_{t+1}^0(\bar{Q}^1) \right\}.$$

3. *For all $t \in \mathbb{T}$, $t < T$, $X \in \mathcal{R}^\infty$ and $\bar{Q} = Q \otimes D \in \mathcal{M}(\bar{P})$ such that $\alpha_t(\bar{Q}) = 0$ Q-a.s. on $\{D_t > 0\}$,*

$$E_Q[D_{t+1}(X_t + \rho_{t+1}(X)) \,|\, \mathcal{F}_t] \le D_t(X_t + \rho_t(X))$$

and $\alpha_{t+1}(\bar{Q}) = 0$ Q-a.s. on $\{D_{t+1} > 0\}$.

Moreover, if the set $\bar{\mathcal{Q}}^*$ defined in (21) is not empty, then time consistency is equivalent to each of the following conditions:

4. *The set $\bar{\mathcal{Q}}^*$ is stable, and ρ_t has the representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\otimes D\gamma\in\bar{\mathcal{Q}}^*} \frac{1}{D_t} E_Q\left[-\sum_{s\in\mathbb{T}_t} \gamma_s X_s \mid \mathcal{F}_t\right] \tag{29}$$

for all $X \in \mathcal{R}^\infty$ and $t \in \mathbb{T}\cap\mathbb{N}_0$.

5. *The representation* (29) *holds for all $t \in \mathbb{T}\cap\mathbb{N}_0$ and all $X \in \mathcal{R}^\infty$, and the process*

$$D_t\rho_t(X - X_t 1_{\mathbb{T}_t}) - \sum_{s=0}^{t} D_s \Delta X_s, \qquad t \in \mathbb{T}\cap\mathbb{N}_0$$

is a Q-supermartingale for all $\bar{Q} = Q\otimes D \in \bar{\mathcal{Q}}^*$.

Proof. Follows by applying [1, Corollary 26] and [20, Corollary 4.12] to $\bar{\rho}$ defined in (11) and using Corollary 52. ☐

Coherence implies that the risk measure is asymptotically safe under any model $\bar{Q} = Q\otimes D \in \mathcal{Q}_0^0$. Indeed, by 1 of Corollary 17, $(D_t\alpha_t(\bar{Q}))_{t\in\mathbb{N}_0}$ is a nonnegative Q-supermartingale beginning at 0, and hence it vanishes. In particular, there are no bubbles in the coherent case, and so asymptotic safety follows from Theorem 21.

5 Cash subadditivity and calibration to numéraires

As noted after Definition 3, cash invariance of risk measures for processes differs from the corresponding property of risk measures for random variables, since it takes into account the timing of the payment. This aspect can be made precise using the notion of cash subadditivity. Cash subadditivity was introduced by El Karoui and Ravanelli [18] in the context of risk measures for random variables in order to account for discounting ambiguity. It will be shown in Proposition 29, and it also follows from the robust representation given in Subsection 3.3, that every risk measure for processes is cash subadditive. Thus risk measures for processes provide a natural framework to capture uncertainty about the time value of money, and a systematic approach to the issue of discounting ambiguity.

5.1 Cash subadditivity

Definition 28. *A conditional convex risk measure for processes ρ_t is called*

- cash subadditive *if*

$$\rho_t(X + m1_{\mathbb{T}_{t+s}}) \geq \rho_t(X) - m, \quad \forall\, s > 0,\ \forall\, m \in L_t^\infty,\ \ m \geq 0; \tag{30}$$

- cash additive *at time $t+s$, with $s > 0$ and $t+s \in \mathbb{T}$, if*

$$\rho_t(X + m1_{\mathbb{T}_{t+s}}) = \rho_t(X) - m, \quad \forall\, m \in L_t^\infty,$$

- cash additive *if it is cash additive at all times* $s \in \mathbb{T}_{t+1}$.

Note that (30) is equivalent to

$$\rho_t(X + m1_{\mathbb{T}_{t+s}}) \leq \rho_t(X) - m, \quad \forall\, s > 0,\ \forall\, m \in L_t^\infty,\ \ m \leq 0,$$

since $\rho_t(X) = \rho_t(X + m1_{\mathbb{T}_{t+s}} - m1_{\mathbb{T}_{t+s}})$.

Cash subadditive risk measures account for the timing of the payment in the sense that the risk is reduced by having positive inflows earlier and negative ones later. Other equivalent characterizations of cash subadditivity can be found in [18, Section 3.1].

As noted in [11] in the time consistent case, cash subadditivity is an immediate consequence of the basic properties of a conditional risk measure for processes.

Proposition 29. *Every conditional convex risk measure for processes* ρ_t *is cash subadditive.*

Proof. Cash subadditivity follows straightforward from monotonicity and cash invariance of ρ_t:

$$\rho_t(X) - m = \rho_t(X + m1_{\mathbb{T}_t}) \leq \rho_t(X + m1_{\mathbb{T}_{t+s}}), \quad \forall s > 0,\ \forall\, m \in L_t^\infty,\ \ m \geq 0.$$

□

Cash subadditivity of risk measures for processes is also apparent from the robust representation given in Subsection 3.3 due to the appearance of the discounting factors.

Remark 30. *In particular, for* $T < \infty$ *or* $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$*, every risk measure for processes restricted to the space* $\{X \in \mathcal{R}^\infty | X_t = 0,\ t < T\}$ *defines a cash subadditive risk measure on* L^∞ *in the sense of [18, Definition 3.1].*

Remark 31. *For* $\mathbb{T} = \mathbb{N}_0$*, a conditional convex risk measure for processes* ρ_t *that is continuous from above cannot be cash additive. Indeed, if* ρ_t *is cash additive at* $t + s$ *for all* $s > 0$*, continuity from above implies for* $X \in \mathcal{R}^\infty$ *and* $m \in L_t^\infty, m > 0$,

$$-m + \rho_t(X) = \rho_t(X + m1_{\mathbb{T}_{t+s}}) \nearrow \rho_t(X) \quad \textit{with} \quad s \to \infty,$$

which is absurd. The interpretation of this result is clear: If we are indifferent between having an amount of money today or tomorrow or at any future time, then any payment can be shifted from one date to the next, and so it would never appear.

The following proposition describes the interplay between time consistency and cash additivity.

Proposition 32. *Let* $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ *be a time consistent dynamic convex risk measure on* $\mathcal{R}^\infty$ *such that each* ρ_t *is cash additive at time* $t + 1$*. Then each* ρ_t *is cash additive.*

Proof. Follows by induction using one-step cash additivity and recursiveness (20). □

In view of Proposition 32 and Remark 31 we obtain the following result.

Corollary 33. *For $\mathbb{T} = \mathbb{N}_0$, a dynamic convex risk measure $(\rho_t)_{t\in\mathbb{N}_0}$ on $\mathcal{R}^\infty$ such that each ρ_t is continuous from above and cash additive at time $t+1$ cannot be time consistent.*

Remark 34. *Corollary 33 and Remark 31 heavily depend on the assumption of continuity from above, which was formulated as a global property. For $\mathbb{T} = \mathbb{N}_0$, the corollary in fact suggests to replace global continuity from above by a local version; this is done in [21].*

5.2 Calibration to numéraires

Cash additivity can be seen as additivity with respect to the numéraire 1. In this section we discuss additivity with respect to other possible numéraires. To this end we formulate conditional versions of some results from [18].

As usual, we denote by α_t the minimal penalty function of ρ_t, and for $t \in \mathbb{T} \cap \mathbb{N}_0$ we define

$$\mathcal{Q}_t^\alpha := \left\{ Q \in \mathcal{Q}_t \;\middle|\; \alpha_t(Q) < \infty \right\}, \quad \bar{\mathcal{Q}}_t^\alpha := \left\{ \bar{Q} \in \bar{\mathcal{Q}}_t \;\middle|\; \alpha_t(\bar{Q}) < \infty \right\},$$

where

$$\mathcal{Q}_t := \left\{ Q \in \mathcal{M}(P) \;\middle|\; Q = P \text{ on } \mathcal{F}_t \right\},$$

and $\bar{\mathcal{Q}}_t$ is defined in (16).

The following lemma is a conditional version of [18, Lemma 2.3].

Lemma 35. *Let $\rho_t : L^\infty \to L_t^\infty$ be a conditional convex risk measure for random variables that is continuous from above, and let $N \in L^\infty$. Then the following conditions are equivalent:*

(i) $\rho_t(\lambda_t N) = \lambda_t \rho_t(N)$ *for all* $\lambda_t \in L_t^\infty$*;*

(ii) $E_Q[-N \,|\, \mathcal{F}_t] = \rho_t(N)$ *for all* $Q \in \mathcal{Q}_t^\alpha$*;*

(iii) $\rho_t(X + \lambda_t N) = \rho_t(X) + \lambda_t \rho_t(N)$ *for all* $X \in L^\infty$ *and all* $\lambda_t \in L_t^\infty$*.*

Proof. $(i) \Rightarrow (ii)$. (i) and [20, Corollary 2.4] imply for each $\lambda_t \in L_t^\infty$ and $Q \in \mathcal{Q}_t$

$$\lambda_t \rho_t(N) = \rho_t(\lambda_t N) \geq \lambda_t E_Q[-N|\mathcal{F}_t] - \alpha_t(Q).$$

If $\alpha_t(Q) < \infty$, we have $\alpha_t(Q) \geq -\lambda_t(E_Q[N|\mathcal{F}_t] + \rho_t(N))$ for any $\lambda_t \in L_t^\infty$, thus $\rho_t(N) = E_Q[-N|\mathcal{F}_t]$. $(ii) \Rightarrow (iii)$ follows from [20, Corollary 2.4], and $(iii) \Rightarrow (i)$ from normalization. □

Due to (i) of Lemma 35, we can assume without loss of generality that the random variable N satisfies the condition $\rho_t(N) = -1$. Then condition (ii) of Lemma 35 means that the conditional expectation of the "numéraire" N is unique under all relevant probability measures, and condition (iii) can be viewed as *additivity* with respect to the numéraire N:

$$\rho_t(X + \lambda_t N) = \rho_t(X) - \lambda_t \quad \forall X \in L^\infty, \quad \forall \lambda_t \in L_t^\infty.$$

The following lemma translates Lemma 35 to the framework of risk measures for processes.

Lemma 36. *Let $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ be a conditional convex risk measure for processes such that each ρ_t is continuous from above, and let $N_s \in L_s^\infty$ for some $s \in \mathbb{T}_{t+1}$. Then the following conditions are equivalent:*

(i) $\rho_t(\lambda_t N_s 1_{\mathbb{T}_s}) = \lambda_t \rho_t(N_s 1_{\mathbb{T}_s})$ *for all* $\lambda_t \in L_t^\infty$*;*

(ii) $E_Q\left[-N_s \frac{D_s}{D_t} \,\middle|\, \mathcal{F}_t\right] = \rho_t(N_s 1_{\mathbb{T}_s})$ *for all* $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_t^\alpha$

(iii) for all $X \in \mathcal{R}_t^\infty$ *and* $\lambda_t \in L_t^\infty$

$$\rho_t(X + \lambda_t N_s 1_{\mathbb{T}_s}) = \rho_t(X) + \lambda_t \rho_t(N_s 1_{\mathbb{T}_s}).$$

Proof. Consider the conditional convex risk measure $\bar{\rho}_t : \bar{L}^\infty \to \bar{L}_t^\infty$ associated to ρ_t via (11). The linearity condition (i) for ρ_t is equivalent to

$$\bar{\rho}_t(\lambda_t N_s 1_{\mathbb{T}_s}) = \lambda_t \bar{\rho}_t(N_s 1_{\mathbb{T}_s}) \quad \forall \lambda_t \in L_t^\infty,$$

i.e., $\bar{\rho}_t$ is linear on $\{\Lambda_t N_s 1_{\mathbb{T}_s} \,|\, \Lambda_t \in \bar{L}_t^\infty\}$. By Lemma 35 and (11) this is equivalent to

$$E_{\bar{Q}}[-N_s 1_{\mathbb{T}_s} \,|\, \bar{\mathcal{F}}_t] = \rho_t(N_s 1_{\mathbb{T}_s}) 1_{\mathbb{T}_t} \quad \bar{Q}\text{-a.s.} \quad \forall \bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_t^\alpha,$$

and this is equivalent to (ii) by Corollary 52. In the same way, Lemma 35 and (11) imply that (i) is equivalent to (iii). $\square$

Since each $D \in \mathcal{D}_t(Q)$ is non-decreasing, Lemma 36 applied to $N_s = 1$ for some $s > t$ yields the following characterization of cash additivity:

Corollary 37. *A conditional convex risk measure for processes $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ such that each ρ_t is continuous from above, is cash additive at time $s \in \mathbb{T}_{t+1}$ if and only if*

$$D_t = D_{t+1} = \ldots = D_s \quad Q\text{-a.s.}$$

for all $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_t^\alpha$.

In other words, cash additivity at time $s > t$ means that there is no discounting between t and s in all the relevant models. In particular we have the following proposition.

Proposition 38. *A conditional convex risk measure for processes ρ_t is continuous from above and cash additive at time $s \in \mathbb{T}_{t+1}$ if and only if it admits the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{loc}} \operatorname*{ess\,sup}_{\gamma \in \Gamma_s(Q)} \left(E_Q\left[-\sum_{k \in \mathbb{T}_s} \gamma_k X_k \,\middle|\, \mathcal{F}_t \right] - \alpha_t(Q \otimes \gamma) \right), \quad X \in \mathcal{R}_t^\infty. \tag{31}$$

In this case ρ_t is cash additive up to s, i.e., at all times $t+1, \ldots, s$.

In particular, if $T < \infty$ or if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, a risk measure for processes ρ_t that is continuous from above is cash additive if and only if it reduces to a risk measure on L^∞:

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left(E_Q[-X_T | \mathcal{F}_t] - \beta_t(Q) \right), \tag{32}$$

where $\beta_t(Q) := \alpha_t(Q \otimes \delta_{\{T\}})$, and $\delta_{\{T\}}$ denotes the Dirac measure at T.

Proof. Obviously, representation (31) implies continuity from above and cash additivity up to time s. The converse follows from 1) of Corollary 13 and Corollary 37. To prove the last part of the assertion, note that $\Gamma_T(Q) = \{\delta_{\{T\}}\}$ if $T < \infty$ or $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$. Moreover, we have $Q \ll P$ for any $Q \in \mathcal{Q}_t^{\mathrm{loc}}$ such that $Q \otimes \delta_{\{T\}} \in \bar{\mathcal{Q}}_t^{\alpha}$. This is obvious for $T < \infty$, and it follows from Lemma 53 if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, since $\gamma_\infty = 1$ Q-a.s. in this case. Thus the representation (32) follows from (31). □

Remark 39. *In particular, in the cash additive case and for $T < \infty$ or $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, the results of Section 4 reduce to the corresponding results for risk measures for random variables from [20, 1].*

The following example extends [18, Proposition 2.4] to our present framework.

Example 40. *Let $\rho_t : \mathcal{R}_t^\infty \to L_t^\infty$ be a conditional convex risk measure for processes that is continuous from above. Assume that there is a money market account $(B_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ as in Example 2, and that zero coupon bonds for all maturities $k > t, k \in \mathbb{T} \cap \mathbb{N}_0$ are available at prices $B_{t,k}$, respectively.*

Suppose that ρ_t satisfies the following calibration condition*:*

$$\rho_t(\lambda_t \frac{B_t}{B_k} 1_{\mathbb{T}_k}) = -\lambda_t B_{t,k} \quad \forall \lambda_t \in L_t^\infty, \quad \forall k \in \mathbb{T}_t \cap \mathbb{N}_0. \tag{33}$$

Lemma 36 applied to $N_k = \dfrac{B_t}{B_k}$ implies that the calibration condition (33) *is equivalent to*

$$\rho_t\left(X + \lambda_t \frac{B_t}{B_k} 1_{\mathbb{T}_k}\right) = \rho_t(X) - \lambda_t B_{t,k} \quad \forall X \in \mathcal{R}_t^\infty,\ \forall \lambda_t \in L_t^\infty,\ \forall k \in \mathbb{T}_t \cap \mathbb{N}_0,$$

and also to

$$E_Q\left[\frac{B_t}{B_k}\frac{D_k}{D_t}\,\Big|\,\mathcal{F}_t\right] = B_{t,k} \quad \forall k \in \mathbb{T}_t \cap \mathbb{N}_0, \quad \forall \bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_t^{\alpha}. \tag{34}$$

Using (34)*, the robust representation from part 1 of Corollary 13, and monotone convergence for $T = \infty$, it can be seen that the calibration condition* (33) *is equivalent to the following one, that may seem stronger at first sight:*

$$\rho_t\left(\sum_{k=t+1}^{T} \lambda_k \frac{B_t}{B_k} 1_{\mathbb{T}_k}\right) = -\sum_{k=t+1}^{T} \lambda_k B_{t,k} \quad \forall \lambda_k \in L_t^\infty.$$

Moreover, if the short rate process (r_t), and hence also the money market account $(B_s)_{s \in \mathbb{T} \cap \mathbb{N}_0}$ is predictable, then (34) *implies*

$$\frac{B_t}{B_{t+1}}\frac{D_{t+1}}{D_t} = B_{t,t+1},$$

and thus $D_{t+1} = D_t$ for all $\bar{Q} = Q \otimes D \in \bar{\mathcal{Q}}_t^{\alpha}$, since $B_{t,t+1} = (1 + r_{t+1})^{-1}$ by a standard no arbitrage argument. Hence ρ_t is cash additive at time $t+1$ by Corollary 37. In particular, if a dynamic convex risk measure (ρ_t) is time consistent, and if each ρ_t is continuous from above and satisfies the calibration condition (33) *with a predictable money market account, then each ρ_t is cash additive by Proposition 32. In view of Remark 31, a time consistent dynamic convex risk measure that is continuous from above cannot satisfy condition* (33) *for all $t \in \mathbb{T}$ if $\mathbb{T} = \mathbb{N}_0$.*

6 Examples

In this section we illustrate our analysis by discussing some examples, in particular analogues to classical risk measures for random variables such as the entropic risk measure and Average Value at Risk. Another class of examples is obtained by separating model and discounting ambiguity in the robust representations of Subsection 3.3.

6.1 Entropic risk measures

In this section we introduce entropic risk measures for processes. As a first variant we simply take the usual conditional entropic risk measure on product space, that is the map $\bar{\rho}_t : \bar{L}^\infty \to \bar{L}^\infty_t$ defined by

$$\bar{\rho}_t(X) = \frac{1}{R_t} \log E_{\bar{P}} \left[e^{-R_t X} \mid \bar{\mathcal{F}}_t \right]$$

with risk aversion parameter $R_t = (r_0, \ldots, r_{t-1}, r_t, r_t, \ldots) \in \bar{L}^\infty_t$, where $r_s > 0$ and $r_s^{-1} \in L^\infty_s$ for all $s = 0, \ldots, t$, and $e^{-R_t X} = (e^{-r_s X_s})_{s \in \mathbb{T}}$.

For an optional probability measure $\nu = (\nu_s)_{s \in \mathbb{T}}$ on $\mathbb{T}$, we denote by ν^t the normalized restriction to $\mathbb{T}_t$, i.e.

$$\nu^t_s = \begin{cases} \dfrac{\nu_s}{\sum_{j \in \mathbb{T}_t} \nu_j}, & \text{on } \left\{ \sum_{j \in \mathbb{T}_t} \nu_j > 0 \right\}, \\ 0, & \text{otherwise} \end{cases}$$

for $s \in \mathbb{T}_t$.

Proposition 41. *The conditional entropic risk measure for processes $\rho_t : \mathcal{R}^\infty_t \to L^\infty_t$ associated to $\bar{\rho}_t$ via* (11) *takes the form*

$$\rho_t(X) = \rho_t^{P, r_t} \left(-\rho_t^{\mu(\omega), r_t(\omega)} (X_\cdot(\omega)) \right). \tag{35}$$

Here $\rho_t^{P,r_t} : L^\infty \to L^\infty_t$ denotes the usual conditional entropic risk measure for random variables with risk aversion parameter r_t:

$$\rho_t^{P, r_t}(Y) = \frac{1}{r_t} \log E_P \left[e^{-r_t Y} \mid \mathcal{F}_t \right], \quad Y \in L^\infty.$$

On the other hand, $\rho_t^{\nu, r} : \mathbb{R}^{\mathbb{T}}_b \to \mathbb{R}$ is the entropic risk measure "with respect to time", defined on the set of sequences $\mathbb{R}^{\mathbb{T}}_b = \{ x = (x_s)_{s \in \mathbb{T}} | x_s \in \mathbb{R} \; \forall \, s, \; \sup_{s \in \mathbb{T}} x_s < \infty \}$ by

$$\rho_t^{\nu, r}(x) = \frac{1}{r} \log \left(\sum_{s \in \mathbb{T}_t} e^{-r x_s} \nu^t_s \right)$$

for a given probability measure ν on $\mathbb{T}$ and a risk aversion parameter $r \in \mathbb{R}$, $r > 0$.

The minimal penalty function α_t of ρ_t is given for $Q \otimes \gamma \in \mathcal{M}(\bar{P})$ by

$$\alpha_t(Q \otimes \gamma) = \frac{1}{r_t} E_Q \left[\sum_{s \in \mathbb{T}_t} \gamma^t_s \log \frac{M_s}{M_t} \mid \mathcal{F}_t \right] + \frac{1}{r_t} E_Q [H(\gamma^t(\cdot) | \mu^t(\cdot)) | \mathcal{F}_t], \tag{36}$$

where $H(\cdot|\cdot)$ is the usual relative entropy for probability measures on $\mathbb{T}_t$, $M_s = \frac{dQ}{dP}|_{\mathcal{F}_s}$, $s \in \mathbb{T} \cap \mathbb{N}_0$, and $M_\infty = \lim_{t \to \infty} M_t$ P-a.s. if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$.

Proof. Using Corollary 52 we obtain

$$\begin{aligned}
\bar{\rho}_t(X) &= -X1_{\{0,\dots,t-1\}} + \frac{1}{r_t}\log E\left[\sum_{s\in\mathbb{T}_t} e^{-r_t X_s}\mu_s^t \mid \mathcal{F}_t\right]1_{\mathbb{T}_t}\\
&= -X1_{\{0,\dots,t-1\}} + \rho_t^{P,r_t}\left(-\frac{1}{r_t}\log\left(\sum_{s\in\mathbb{T}_t} e^{-r_t X_s}\mu_s^t\right)\right)1_{\mathbb{T}_t}\\
&= -X1_{\{0,\dots,t-1\}} + \rho_t^{P,r_t}\left(-\rho_t^{\mu(\omega),r_t(\omega)}\left(X_\cdot(\omega)\right)\right)1_{\mathbb{T}_t}.
\end{aligned}$$

To prove the second part of the claim, note that the minimal penalty function $\bar{\alpha}_t$ of $\bar{\rho}_t$ on $\mathcal{M}(\bar{P})$ takes the form

$$\bar{\alpha}_t(\bar{Q}) = \frac{1}{R_t}H_t(\bar{Q}|\bar{P}),$$

where $H_t(\bar{Q}|\bar{P}) = E_{\bar{Q}}[\log\frac{Z_T}{Z_t}|\bar{\mathcal{F}}_t]$ is the conditional relative entropy of $\bar{Q}$ with respect to $\bar{P}$, and Z_s denotes the density of $\bar{Q}$ with respect to $\bar{P}$ on $\bar{\mathcal{F}}_s$; see, e.g., [16, Proposition 4]. Using Theorem 6, (47), Corollary 52, and (48) we obtain for each $\bar{Q} = Q\otimes\gamma \in \mathcal{M}(\bar{P})$,

$$\bar{\alpha}_t(Q\otimes\gamma) = \frac{1}{r_t}E_Q\left[\sum_{s\in\mathbb{T}_t}\gamma_s^t\log\left(\frac{\gamma_s^t M_s}{\mu_s^t M_t}\right) \mid \mathcal{F}_t\right]1_{\mathbb{T}_t}.$$

Hence the minimal penalty function α_t of ρ_t on $\mathcal{M}(\bar{P})$ is given by

$$\begin{aligned}
\alpha_t(Q\otimes\gamma) &= \frac{1}{r_t}E_Q\left[\sum_{s\in\mathbb{T}_t}\gamma_s^t\log\frac{M_s}{M_t} \mid \mathcal{F}_t\right] + \frac{1}{r_t}E_Q\left[\sum_{s\in\mathbb{T}_t}\gamma_s^t\log\left(\frac{\gamma_s^t}{\mu_s^t}\right) \mid \mathcal{F}_t\right]\\
&= \frac{1}{r_t}E_Q\left[\sum_{s\in\mathbb{T}_t}\gamma_s^t\log\frac{M_s}{M_t} \mid \mathcal{F}_t\right] + \frac{1}{r_t}E_Q[H(\gamma^t(\cdot)|\mu^t(\cdot))|\mathcal{F}_t].
\end{aligned}$$

□

One can characterize time consistency properties of the dynamic entropic risk measure for processes $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$, where each ρ_t is given by (35), using the corresponding results for $(\bar{\rho}_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$. In particular, by [1, Proposition 43] (cf. also [34, Proposition 4.1.4]), the entropic risk measure $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ is time consistent if the risk aversion parameter is constant, i.e., $r_t = r_0$ for all $t\in\mathbb{T}\cap\mathbb{N}_0$, and $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ is rejection (resp. acceptance) consistent if $r_t \geq r_{t+1}$ (resp. $r_t \leq r_{t+1}$) for all $t\in\mathbb{T}\cap\mathbb{N}_0$.

Remark 42. *A time consistent dynamic entropic risk measure $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ is asymptotically precise under the reference measure $\bar{P}$, and hence under each $\bar{Q}\in\mathcal{M}(\bar{P})$, due to Proposition 24. Indeed, for each $X\in\mathcal{R}^\infty$ the supremum in the robust representation* (15) *of $\rho_0(X)$ is attained by a "worst case" measure $\bar{Q}^X \approx \bar{P}$ for each $X\in\mathcal{R}^\infty$; cf., e.g., [23, Example 4.33].*

Formula (36) for the entropic penalty suggests to introduce a simplified version of the entropic risk measure, where the interaction between Q and γ in the penalty is reduced as follows: For $u_t, v_t > 0$ such that $u_t, v_t, u_t^{-1}, v_t^{-1} \in L_t^\infty$, define

$$\hat{\alpha}_t(Q\otimes\gamma) := \begin{cases} \dfrac{1}{u_t}H_t(Q|P) + \dfrac{1}{v_t}E_Q[H(\gamma(\cdot)|\mu^t(\cdot))|\mathcal{F}_t], & \text{if } Q\in\mathcal{Q}_t, \gamma\in\Gamma_t(P),\\ \infty, & \text{otherwise.}\end{cases} \tag{37}$$

This induces a new conditional convex risk measure $\hat{\rho}_t : \mathcal{R}_t^\infty \to L_t^\infty$ via

$$\begin{aligned}\hat{\rho}_t(X) &:= \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t,\gamma\in\Gamma_t(P)} \left(E_Q\left[-\sum_{s\in\mathbb{T}_t} \gamma_s X_s \;\middle|\; \mathcal{F}_t \right] - \hat{\alpha}_t(Q\otimes\gamma) \right) \\ &= \operatorname*{ess\,sup}_{\gamma\in\Gamma_t(P)} \rho_t^{P,u_t}\left(\sum_{s\in\mathbb{T}_t} \gamma_s X_s + \frac{1}{v_t} H(\gamma(\cdot)|\mu^t(\cdot)) \right).\end{aligned}$$

Proposition 43. *The conditional convex risk measure $\hat{\rho}_t$ satisfies*

$$\hat{\rho}_t(X) \le \rho_t^{P,u_t}\left(-\rho_t^{\mu(\omega),v_t(\omega)}(X_\cdot(\omega)) \right). \tag{38}$$

In particular, for $u_t = v_t = r_t$ we have

$$\hat{\rho}_t(X) \le \rho_t(X) \quad \textit{for all } X \in \mathcal{R}_t^\infty, \tag{39}$$

i.e., $\hat{\rho}_t$ is less conservative than the entropic risk measure ρ_t in (35).

Proof. Inequality (38) holds since for any probability measure ν on $\mathbb{T}$

$$\rho_t^{\nu,v_t}(x) = \sup_y \left\{ -\sum_{s\in\mathbb{T}_t} y_s x_s - \frac{1}{v_t} H(y|\nu^t) \right\},$$

where the supremum is taken over all probability measures $y = (y_s)_{s\in\mathbb{T}_t}$ on $\mathbb{T}_t$. □

Remark 44. *Inequality* (39) *implies the converse relation for the respective minimal penalty functions of $\hat{\rho}_t$ and ρ_t, and thus* (37) *and* (36) *yield*

$$H_t(Q|P) \ge E_Q\left[\sum_{s\in\mathbb{T}_t} \gamma_s \log M_s \;\middle|\; \mathcal{F}_t \right]$$

for all $Q \in \mathcal{Q}_t$ and $\gamma \in \Gamma_t(P)$.

6.2 Average Value at Risk

For a given level $\Lambda_t = (\lambda_0, \dots, \lambda_{t-1}, \lambda_t, \lambda_t, \dots) \in \bar{L}_t^\infty$ such that $\lambda_s \in (0,1]$ for all $s = 0, \dots, t$ we define the conditional Average Value at Risk $\bar{\rho}_t : \bar{L}^\infty \to \bar{L}_t^\infty$ on the product space in the usual way as

$$\bar{\rho}_t(X) = \operatorname*{ess\,sup}\{ E_{\bar{Q}}[-X|\bar{\mathcal{F}}_t] \mid \bar{Q} \in \bar{\mathcal{Q}}_t, d\bar{Q}/d\bar{P} \le \Lambda_t^{-1} \}.$$

Proposition 45. *The conditional coherent risk measure for processes associated to $\bar{\rho}_t$ via* (11) *depends only on λ_t, and is given by*

$$\begin{aligned}\rho_t^{\lambda_t}(X) = \operatorname*{ess\,sup}\Big\{ E_Q\left[-\sum_{s\in\mathbb{T}_t} X_s\gamma_s \;\middle|\; \mathcal{F}_t \right] \;\Big|\; Q \in \mathcal{Q}_t^{loc}, \gamma \in \Gamma_t(Q), \\ \frac{\gamma_s M_s}{\mu_s^t} \le \lambda_t^{-1},\; s \in \mathbb{T}_t \Big\},\end{aligned} \tag{40}$$

where $M_s = \frac{dQ}{dP}|_{\mathcal{F}_s}$, $s \in \mathbb{T} \cap \mathbb{N}_0$, and $M_\infty = \lim_{t\to\infty} M_t$ P-a.s. if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$.

Proof. This is an immediate consequence of (47) and Corollary 52. □

Note that a probability measure Q and an optional measure γ in the robust representation of ρ_t are penalized simultaneously. As a simpler alternative, we can consider a "decoupled" version of conditional Average Value at Risk, defined by

$$\rho_t^{\lambda_1,\lambda_2}(X) := \operatorname*{ess\,sup}_{\gamma\in\Gamma_t^{\lambda_1}} AV@R_t^{\lambda_2}\left(\sum_{s\in\mathbb{T}_t} X_s\gamma_s\right), \quad X\in\mathcal{R}_t^\infty.$$

Here, λ_1 and λ_2 are $\mathcal{F}_t$-measurable random variables with values in $(0,1]$,

$$AV@R_t^{\lambda_2}(X) = \operatorname*{ess\,sup}\left\{E_Q[-X|\mathcal{F}_t] \;\middle|\; Q\in\mathcal{Q}_t, \frac{dQ}{dP}\le\frac{1}{\lambda_2}\right\}, \quad X\in L^\infty,$$

is the usual Average Value at Risk for random variables, and

$$\Gamma_t^{\lambda_1} = \left\{\gamma\in\Gamma_t(P) \;\middle|\; \frac{\gamma_s}{\mu_s}\le\frac{1}{\lambda_1},\ s\in\mathbb{T}_t\right\}.$$

Note that $\rho_t^{\lambda_1,\lambda_2}$ is an example of a "decoupled" risk measure of the form (42), which will be discussed in Subsection 6.3.

Proposition 46. *The conditional coherent risk measure* $\rho_t^{\lambda_1,\lambda_2}$ *satisfies*

$$\rho_t^{\lambda_1,\lambda_2}(X)\le\rho_t^{\lambda_1\lambda_2}(X) \quad \forall\, X\in\mathcal{R}^\infty.$$

In other words, the decoupled version is less conservative than the conditional Average Value at Risk defined in (40) *with* $\lambda_t=\lambda_1\lambda_2$.

Proof. Follows immediately from the definition of $\rho_t^{\lambda_1,\lambda_2}$. □

Recall that the dynamic Average Value at Risk for random variables is not time consistent; cf. e.g. [4]. Thus neither the dynamic Average Value at Risk for processes $(\rho_t^{\lambda_t})_{t\in\mathbb{T}\cap\mathbb{N}_0}$ defined in (40), nor its decoupled version $(\rho_t^{\lambda_1,\lambda_2})_{t\in\mathbb{T}\cap\mathbb{N}_0}$ will be time consistent in general. However, if the time horizon is finite, backward recursive construction of time consistent dynamic risk measures introduced in [10, Section 4.2] (see also [11, Sections 3.1, 4.1], [1, Section 4.4]) can be applied in order to obtain time consistent versions of Average Value at Risk for processes and of its decoupled version. This can be done either on the product space using the construction from [11, Sections 3.1] or directly for risk measures for processes as in [11, Sections 4.1]. Indeed, it can be easily seen that if $(\bar\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ and $(\rho_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ are associated to each other via (11), the corresponding time consistent dynamic risk measures obtained by recursive construction will be also associated to each other via (11).

6.3 Separation of model and discounting uncertainty

If the time horizon T is finite, we can replace $\Gamma_t(Q)$ by $\Gamma_t(P)$ due to Remark 50, and the robust representation (15) in Theorem 10 can be rewritten in the following form:

$$\rho_t(X) = \operatorname*{ess\,sup}_{\gamma\in\Gamma_t(P)} \psi_t^\gamma\Big(\sum_{s=t}^T X_s\gamma_s\Big), \quad X\in\mathcal{R}_t^\infty. \tag{41}$$

Here

$$\psi_t^\gamma(Y) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left(E_Q[-Y|\mathcal{F}_t] - \alpha_t(Q \otimes \gamma) \right), \quad Y \in L^\infty$$

is a conditional convex risk measure for random variables (see, e.g., [16, Theorem 1]), that depends on the discounting factor γ through its penalty function $\beta_t^\gamma(Q) := \alpha_t(Q \otimes \gamma)$.

The representation (41) suggests to consider a simple class of conditional convex risk measures for processes, both for $T < \infty$ and $T = \infty$, where the dependence of Q and γ is separated in the following manner: One begins with some conditional convex risk measure for random variables $\psi_t : L^\infty \to L_t^\infty$, specifies some set of discounting factors $G_t \subseteq \Gamma_t(P)$, and defines

$$\rho_t(X) = \operatorname*{ess\,sup}_{\gamma \in G_t} \psi_t \left(\sum_{s \in \mathbb{T}_t} X_s \gamma_s \right), \quad X \in \mathcal{R}^\infty. \tag{42}$$

It is easy to see that (42) defines a conditional convex risk measure ρ_t for processes, and that ρ_t is continuous from above if and only if ψ_t is continuous from above.

For example, for $G_t = \{\delta_{\{s\}}\}$ for some $s \in \mathbb{T}_t$, formula (42) reduces to

$$\rho_t(X) = \psi_t(X_s), \quad X \in \mathcal{R}^\infty,$$

i.e., ρ_t is a conditional convex risk measure on L_s^∞. More generally, one can fix, as in [11, Example 4.3.2], an optional measure $\gamma \in \Gamma_t(P)$, and define $G_t = \{\gamma\}$. In this case there is no ambiguity regarding the discounting process. For $T < \infty$ and $X \in \mathcal{R}_t^\infty$, or for $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$ and $X \in \mathcal{X}_t^\infty$, we can switch to discounted terms by associating to X a process Y defined via

$$Y_0 := X_0, \ \ \Delta Y_s := D_s \Delta X_s, \ \ s \in \mathbb{T} \cap \mathbb{N}_0, \quad Y_\infty := \lim_{t \to \infty} Y_t \ \text{ for } \mathbb{T} = \mathbb{N}_0 \cup \{\infty\},$$

where D is related to γ via (4). Then the risk measure ρ_t defined by (42) reduces to a risk measure for random variables:

$$\rho_t(X) = \psi_t \Big(\sum_{s=t}^T D_s \Delta X_s \Big) = \psi_t \Big(\sum_{s=t}^T \Delta Y_s \Big) = \psi_t(Y_T).$$

Remark 47. *In the case of unambiguous exponential discounting $D_s = \beta^{s-t}$ for $s \geq t$ and some $\beta \in (0,1)$, the representation* (42) *reduces to*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{loc}} \left(E_Q \left[-\sum_{s=t}^T \beta^{s-t} \Delta X_s \;\Big|\; \mathcal{F}_t \right] - \alpha_t(Q) \right).$$

This corresponds to the numerical representation of dynamic variational preferences (4) in Maccheroni et al. [31] and, in the coherent case, to (3.6) in Epstein and Schneider [19]. In this context the characterization (iii) of time consistency in our Theorem 15 corresponds to condition (11) of [31, Theorem 1], and stability in our Theorem 27 coincides with rectangularity in [19, Theorem 3.2].

A further example of a risk measure of the form (42) is given in [11, Example 4.3.3]; cf. also [28, Example 4.2]. In that case ρ_t is the maximal risk which arises by stopping the process $(\psi_t(X_s))_{s \in \mathbb{T}_t}$ in the least favorable way, i.e.,

$$\rho_t(X) = \operatorname*{ess\,sup}_{\tau \in \Theta_t} \psi_t(X_\tau),$$

where Θ_t denotes the set of all stopping times with values in $\mathbb{T}_t$. In our context this amounts to the representation (42) with $G_t = \left\{ (1_{\{\tau=s\}})_{s\in\mathbb{T}_t} \,\middle|\, \tau \in \Theta_t \right\}$.

A Discrete Itô-Watanabe decomposition

In this section we state a discrete time version of the Itô-Watanabe factorization of a nonnegative supermartingale; cf. [27].

Proposition 48. *Let $U = (U_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ be a nonnegative P-supermartingale on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}, P)$ with $U_0 = 1$. Then there exist a nonnegative P-martingale $M = (M_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ and a predictable non-increasing process $D = (D_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ such that $M_0 = D_0 = 1$ and*

$$U_t = M_t D_t, \qquad t \in \mathbb{T} \cap \mathbb{N}_0. \tag{43}$$

Moreover such a decomposition is unique on $\{t < \tau_0\}$, where $\tau_0 := \inf\{t > 0 \,|\, U_t = 0\}$.

Proof. We first assume that there exists a decomposition of U as in (43) and prove its uniqueness on $\{t < \tau_0\}$. Indeed, on $\{t < \tau_0\} = \{U_t > 0\} = \{M_t > 0\} \cap \{D_t > 0\}$ we have

$$\frac{E_P[U_{t+1}|\mathcal{F}_t]}{U_t} = \frac{D_{t+1}}{D_t}\frac{E_P[M_{t+1}|\mathcal{F}_t]}{M_t} = \frac{D_{t+1}}{D_t},$$

and hence the process D in the decomposition (43) is uniquely determined on $\{t \le \tau_0\}$ by

$$D_t = \prod_{s=0}^{t-1} \frac{E_P[U_{s+1}|\mathcal{F}_s]}{U_s}, \quad 0 \le t \le \tau_0.$$

Moreover, on $\{D_t > 0\}$,

$$M_t = \frac{U_t}{D_t},$$

and thus also the process M in the decomposition (43) is uniquely determined on $\{D_t > 0\} \supseteq \{t < \tau_0\}$.

To prove the existence of a decomposition as in (43), define the processes D and M via

$$D_t = \begin{cases} \displaystyle\prod_{s=0}^{t-1} \frac{E_P[U_{s+1}|\mathcal{F}_s]}{U_s}, & \text{for } 0 \le t \le \tau_0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$M_t = \begin{cases} \dfrac{U_t}{D_t}, & \text{on } \{D_t > 0\}, \\ M_{t-1}, & \text{on } \{D_t = 0\}. \end{cases}$$

Clearly, D is predictable and non-increasing with $D_0 = 1$ and $D_t \ge 0$ for all t, and M is adapted with

$M_0 = 1$ and $M_t \geq 0$ for all t. It remains to show that M is a martingale. Indeed,

$$\begin{aligned} E_P[M_{t+1}|\mathcal{F}_t] &= E_P[M_{t+1}1_{\{D_{t+1}=0\}}|\mathcal{F}_t] + E_P[M_{t+1}1_{\{D_{t+1}>0\}}|\mathcal{F}_t] \\ &= M_t 1_{\{D_{t+1}=0\}} + \frac{1}{D_{t+1}} E_P[M_{t+1}D_{t+1}|\mathcal{F}_t]1_{\{D_{t+1}>0\}} \\ &= M_t 1_{\{D_{t+1}=0\}} + \frac{1}{D_{t+1}} \frac{E_P[U_{t+1}|\mathcal{F}_t]}{U_t} U_t 1_{\{D_{t+1}>0\}} \\ &= M_t 1_{\{D_{t+1}=0\}} + \frac{1}{D_{t+1}} \frac{D_{t+1}}{D_t} U_t 1_{\{D_{t+1}>0\}} \\ &= M_t, \end{aligned}$$

where we have used that $U_t > 0$ on $\{D_{t+1} > 0\}$. $\square$

Remark 49. *Since U is a nonnegative supermartingale, the following equivalence holds on $\{\tau_0 = t\}$:*

$$D_t = 0 \iff E_P[U_t|\mathcal{F}_{t-1}] = 0 \iff P[U_t = 0|\mathcal{F}_{t-1}] = 1.$$

Thus $D_t = 0$ on the event $\{\tau_0 = t\}$ if this event is sure at time $t-1$. On the other hand, we have $M_t = 0$ on $\{D_t > 0\} \cap \{\tau_0 = t\} = \{E_P[U_t|\mathcal{F}_{t-1}] > 0, U_t = 0\} = \{P[U_t = 0|\mathcal{F}_{t-1}] < 1, U_t = 0\}$, i.e., M is uniquely determined also at time τ_0 if τ_0 is is not predicted one step ahead.

B Disintegration of measures on the optional σ-field

In this section we prove Theorem 6. Recall that we use Assumption 5. It guarantees that any consistent sequence of probability measures Q_t on $\mathcal{F}_t$, $t \in \mathbb{T} \cap \mathbb{N}_0$, admits a unique extension to a probability measure on $\mathcal{F}_\infty = \sigma(\cup_{t \in \mathbb{T} \cap \mathbb{N}_0} \mathcal{F}_t)$, cf. [33, Theorem 4.1]. In particular, any martingale $(M_t)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ with $M_0 = 1$ induces a unique probability measure Q on $(\Omega, \mathcal{F}_\infty)$ such that

$$M_t = \frac{dQ}{dP}\Big|_{\mathcal{F}_t}, \qquad t \in \mathbb{T}. \tag{44}$$

Proof of Theorem 6. Let $\bar{Q} \in \mathcal{M}(\bar{P})$ with the density $\frac{d\bar{Q}}{d\bar{P}} =: \bar{Z} = (Z_t)_{t \in \mathbb{T}}$. We first prove (9) for $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$. To this end, consider the supermartingale $U = (U_t)_{t \in \mathbb{T}}$ defined by

$$U_t := E_P\left[\sum_{s \in \mathbb{T}_t} \mu_s Z_s |\mathcal{F}_t\right] \geq 0, \quad t \in \mathbb{T}. \tag{45}$$

By Proposition 48, U admits a decomposition

$$U_t = M_t D_t, \quad t \in \mathbb{N}_0,$$

where $M = (M_t)_{t \in \mathbb{N}_0}$ is a nonnegative P-martingale with $M_0 = 1$, and $D = (D_t)_{t \in \mathbb{N}_0}$ is a nonnegative predictable non-increasing process with $D_0 = 1$. The martingale M induces a unique probability measure Q on $(\Omega, \mathcal{F}_\infty)$ via (44), with $Q \in \mathcal{M}_{\text{loc}}(P)$. Let $M_\infty := \lim_{t \to \infty} M_t$ P-a.s., $D_\infty := \lim_{t \to \infty} D_t$ P- and

Q-a.s., and note that $Z_\infty \mu_\infty = U_\infty = \lim_{t\to\infty} U_t = M_\infty D_\infty$ P-a.s.. We define the process $\gamma = (\gamma_t)_{t\in\mathbb{T}}$ via (6). Then for $X \in \mathcal{R}^\infty$ with $X \geq 0$ we have by monotone convergence and (45)

$$\begin{aligned}
E_{\bar{Q}}[X] = E_P\left[\sum_{t\in\mathbb{T}} X_t \mu_t Z_t\right] &= \sum_{t=0}^{\infty} E_P\left[X_t E_P[U_t - U_{t+1}|\mathcal{F}_t]\right] + E_P[M_\infty D_\infty X_\infty] \\
&= \sum_{t=0}^{\infty} E_P\left[X_t(M_t D_t - M_{t+1} D_{t+1})\right] + E_P[M_\infty D_\infty X_\infty] \\
&= \sum_{t=0}^{\infty} E_Q[X_t \gamma_t] + E_P[M_\infty D_\infty X_\infty] \\
&= E_Q\left[\sum_{t=0}^{\infty} X_t \gamma_t\right] + E_P[M_\infty D_\infty X_\infty].
\end{aligned}$$

Using (3) this takes the form

$$E_{\bar{Q}}[X] = E_Q\left[\sum_{t=0}^{\infty} X_t \gamma_t\right] + E_Q[X_\infty \gamma_\infty] - E_Q[\gamma_\infty X_\infty 1_{\{M_\infty = \infty\}}]. \tag{46}$$

Plugging $X = 1$ into (46) yields

$$\begin{aligned}
1 = E_{\bar{Q}}[1] &= E_Q\left[\sum_{t=0}^{\infty} \gamma_t + \gamma_\infty\right] - E_Q[\gamma_\infty 1_{\{M_\infty = \infty\}}] \\
&= 1 - E_Q[\gamma_\infty 1_{\{M_\infty = \infty\}}].
\end{aligned}$$

Thus $\gamma_\infty = 0$ Q-a.s. on $\{M_\infty = \infty\}$, i.e., $\gamma \in \Gamma(Q)$, and (46) reduces to (9).

To prove (9) for $\mathbb{T} = \mathbb{N}_0$, note that every measure $\bar{Q}$ on $(\Omega \times \mathbb{N}_0, \bar{\mathcal{F}})$ can be extended to a measure $\tilde{Q}$ on $(\Omega \times (\mathbb{N}_0 \cup \{\infty\}), \bar{\mathcal{F}})$ by setting $\tilde{Q}[\Omega \times \{\infty\}] = 0$. Thus the first part of the proof yields

$$E_{\tilde{Q}}[X] = E_Q\left[\sum_{t=0}^{\infty} X_t \gamma_t\right] + E_Q[X_\infty \gamma_\infty]$$

with some probability measure $Q \in \mathcal{M}_{\mathrm{loc}}(P)$ and some optional measure γ such that $\sum_{t=0}^{\infty} \gamma_t + \gamma_\infty = 1$ Q-a.s.. Moreover, since

$$E_Q[\gamma_\infty] = E_{\tilde{Q}}[1_{\{\infty\}}] = 0,$$

we have $\gamma_\infty = 0$ Q-a.s., i.e., $\gamma \in \Gamma(Q)$ for $\mathbb{T} = \mathbb{N}_0$, and (9) holds.

Similarly, we can embed the case $\mathbb{T} = \{0, \ldots, T\}$ into the setting of $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, by setting $\mathcal{F}_t := \mathcal{F}_T$ for all $t > T$, and extending any measure $\bar{Q}$ on $(\Omega \times \{0, \ldots, T\}, \bar{\mathcal{F}})$ to a measure $\tilde{Q}$ on $(\Omega \times (\mathbb{N}_0 \cup \{\infty\}), \bar{\mathcal{F}})$ by setting $\tilde{Q}[\Omega \times \mathbb{T}_{T+1}] = 0$. The same reasoning as above yields a probability measure $Q \in \mathcal{M}_{\mathrm{loc}}(P)$, in particular $Q \ll P$ on $\mathcal{F}_T$, and an optional measure γ such that $\gamma_s = 0$ Q-a.s. for all $s > t$, i.e., $\gamma \in \Gamma(Q)$ for $\mathbb{T} = \{0, \ldots, T\}$, and (9) holds.

The equality (10) follows from (9) due to integration by parts formula (7).

To prove the converse implication of the theorem, note that each pair (Q, γ), with $Q \in \mathcal{M}_{\mathrm{loc}}(P)$ and $\gamma \in \Gamma(Q)$, defines a density $\bar{Z} = (Z_t)_{t\in\mathbb{T}}$ of a probability measure $\bar{Q} \in \mathcal{M}(\bar{P})$ via

$$Z_t = \frac{M_t \gamma_t}{\mu_t}, \qquad t \in \mathbb{T}, \tag{47}$$

where M_t denotes the density of Q with respect to P on $\mathcal{F}_t$ for each $t \in \mathbb{T} \cap \mathbb{N}_0$, and, if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, $M_\infty = \lim_{t\to\infty} M_t$ P-a.s.. Clearly, (9) and (10) hold for $\bar{Q}$. $\square$

Remark 50. *For $T < \infty$, and for $\mathbb{T} = \mathbb{N}_0$, one can also prove Theorem 6 directly, defining the supermartingale U via* (45) *and using the Itô-Watanabe decomposition of U as above. For $T < \infty$, one obtains in this way the additional property $\gamma \in \Gamma(P)$ in the decomposition $\bar{Q} = Q \otimes \gamma$ of any $\bar{Q} \in \mathcal{M}(\bar{P})$, and so we can replace the set $\Gamma(Q)$ by $\Gamma(P)$ in the representation* (15) *and in all further results.*

Remark 51. *Let $\bar{Q} \in \mathcal{M}(\bar{P})$ with decomposition $Q \otimes \gamma = Q \otimes D$ in the sense of* (9) *and* (10)*, let $\bar{Z} = (Z_t)_{t\in\mathbb{T}}$ denote the density of $\bar{Q}$ with respect to $\bar{P}$, $M = (M_t)_{t\in\mathbb{T}\cap\mathbb{N}_0}$ the density process of Q with respect to P, and $M_\infty = \lim_{t\to\infty} M_t$ P-a.s. for $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$.*

1. *The density $\bar{Z}$ takes the form* (47)*. Indeed, for all $X \in \mathcal{R}^\infty$, $X \geq 0$ we have*

$$E_{\bar{Q}}[X] = E_Q\left[\sum_{t\in\mathbb{T}} X_t\gamma_t\right] = E_P\left[\sum_{t\in\mathbb{T}} X_t\gamma_t M_t\right],$$

   where, for $T = \infty$, the last equality holds due to monotone convergence, and, for $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, we use (3) *and $\gamma_\infty = 0$ Q-a.s. on $\{M_\infty = \infty\}$.*

2. *In order to clarify to which extent the decomposition* (9) *is unique, we note that the Itô-Watanabe decomposition of the supermartingale U defined in* (45) *is determined by the density process M and the discounting process D. Indeed,*

$$\begin{aligned} U_t &= \sum_{s\in\mathbb{T}_t} E_P[\gamma_s M_s|\mathcal{F}_t] = \sum_{s\in\mathbb{T}_t} E_Q[\gamma_s|\mathcal{F}_t]\, M_t 1_{\{M_t>0\}} \\ &= M_t E_Q\left[\sum_{s\in\mathbb{T}_t}\gamma_s \mid \mathcal{F}_t\right] 1_{\{M_t>0\}} = M_t D_t, \qquad t \in \mathbb{T}, \end{aligned}$$

   where we have used (47), (5)*, and monotone convergence for $T = \infty$. In particular, if $\bar{Q} \in \mathcal{M}(\bar{P})$ admits two decompositions $\bar{Q} = Q^1 \otimes D^1 = Q^2 \otimes D^2$, the uniqueness stated in Proposition 48 yields*

$$M_t^1 = M_t^2 \quad \text{and} \quad D_t^1 = D_t^2 \quad \text{on } \{t < \tau_0\},$$

   where $\tau_0 = \inf\{t > 0 \,|\, U_t = 0\}$. Moreover, since $\bar{Z} > 0$ $\bar{Q}$-a.s., we have $\bar{Q}[\{(\omega, t)|t \geq \tau_0(\omega)\}] = 0$, and hence the processes M and D are uniquely determined and strictly positive $\bar{Q}$-a.s..

3. *Equality $\bar{Q}$-almost surely between two processes $X, Y \in \mathcal{R}^\infty$ can be characterized as follows in terms of Q and γ:*

$$\begin{aligned} X = Y \ \bar{Q}\text{-a.s.} \quad &\Longleftrightarrow \quad 1 = E_{\bar{Q}}[1_{\{X=Y\}}] = E_Q\left[\sum_{t\in\mathbb{T}}\gamma_t 1_{\{X_t=Y_t\}}\right] \\ &\Longleftrightarrow \quad X_t = Y_t \ Q\text{-a.s. on } \{\gamma_t > 0\} \qquad \forall\, t \in \mathbb{T}, \end{aligned}$$

   where the last equivalence follows since $\sum_{t\in\mathbb{T}}\gamma_t = 1$ Q-a.s.. In particular, an $\bar{\mathcal{F}}_t$-measurable random variable $X = (X_t)_{t\in\mathbb{T}}$ is well defined $\bar{Q}$-a.s. if and only if X_i is well defined Q-a.s. on $\{\gamma_i > 0\}$ for $i = 0, \ldots, t-1$, and X_t is well defined Q-a.s. on $\{\sum_{s\in\mathbb{T}_t}\gamma_s > 0\} = \{D_t > 0\}$.

Corollary 52. *For $\bar{Q} \in \mathcal{M}(\bar{P})$ with decomposition $\bar{Q} = Q \otimes \gamma = Q \otimes D$, the conditional expectation given $\bar{\mathcal{F}}_t$ takes the form*

$$E_{\bar{Q}}[X \mid \bar{\mathcal{F}}_t] = X_0 1_{\{0\}} + \ldots + X_{t-1} 1_{\{t-1\}} + E_Q\left[\sum_{s \in \mathbb{T}_t} \frac{\gamma_s}{D_t} X_s \mid \mathcal{F}_t\right] 1_{\mathbb{T}_t}, \qquad X \in \mathcal{R}^\infty,$$

where the last term on the right-hand-side is well defined Q-a.s. on $\{D_t > 0\}$.

Lemma 53. *Let $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$. For $\bar{Q} = \mathcal{Q} \otimes \gamma \in \mathcal{M}(\bar{P})$ with the density process $(M_t)_{t \in \mathbb{N}_0}$ of Q with respect to P, and $M_\infty = \lim_{t\to\infty} M_t$ P-a.s., we have*

$$\gamma_\infty > 0 \quad Q\text{-a.s.} \quad \Leftrightarrow \quad Q \in \mathcal{M}(P) \quad \text{and} \quad \gamma_\infty > 0 \quad P\text{-a.s. on } \{M_\infty > 0\}.$$

Proof. We have

$$\begin{aligned} Q[\gamma_\infty > 0] &= E_Q\left[\frac{\gamma_\infty}{\gamma_\infty} 1_{\{\gamma_\infty > 0\}}\right] = E_{\bar{Q}}\left[\frac{1}{\gamma_\infty} 1_{\{\gamma_\infty > 0\}} 1_{\{\infty\}}\right] = E_{\bar{P}}\left[\frac{Z_\infty}{\gamma_\infty} 1_{\{\gamma_\infty > 0\}} 1_{\{\infty\}}\right] \\ &= E_P\left[\frac{Z_\infty \mu_\infty}{\gamma_\infty} 1_{\{\gamma_\infty > 0\}}\right] = E_P\left[M_\infty 1_{\{\gamma_\infty > 0\}}\right], \end{aligned}$$

where we have used (9) and (47). The claim follows by noting that $Q \ll P$ if and only if $E_P[M_\infty] = 1$ for $Q \in \mathcal{M}_{\mathrm{loc}}(P)$ due to (3). ☐

Our robust representation of a conditional convex risk measure ρ_t involves probability measures $\bar{Q} = Q \otimes \gamma$ which coincide on the σ-field $\bar{\mathcal{F}}_t$. This can be characterized as follows in terms of Q and γ.

Lemma 54. *Let $\bar{Q}^1, \bar{Q}^2 \in \mathcal{M}(\bar{P})$ with the decompositions $\bar{Q}^i = Q^i \otimes \gamma^i = Q^i \otimes D^i$, $i = 1, 2$. Then the following relation holds for all $t \in \mathbb{T}$*

$$\bar{Q}^1 = \bar{Q}^2 \text{ on } \bar{\mathcal{F}}_t \quad \Longleftrightarrow \quad Q^1 = Q^2 \text{ on } \mathcal{F}_t \cap \{D_t^1 > 0\} \text{ and } \gamma_s^1 = \gamma_s^2 \ Q^1\text{-a.s. } \forall\, s < t.$$

Proof. We denote by $\bar{Z}^i = (Z_t^i)_{t \in \mathbb{T}}$ the density of $\bar{Q}^i$ with respect to $\bar{P}$, by $(M_t^i)_{t \in \mathbb{T} \cap \mathbb{N}_0}$ the density process of Q^i with respect to P, and $M_\infty^i = \lim_{t\to\infty} M_t^i$ P-a.s. if $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$, $i = 1, 2$. Assume that $\bar{Q}^1 = \bar{Q}^2$ on $\bar{\mathcal{F}}_t$ for some $t \in \mathbb{T}$, i.e., $E_{\bar{P}}[\bar{Z}^1 | \bar{\mathcal{F}}_t] = E_{\bar{P}}[\bar{Z}^2 | \bar{\mathcal{F}}_t]$, where

$$\begin{aligned} E_{\bar{P}}[\bar{Z}^i | \bar{\mathcal{F}}_t] &= Z_0^i 1_{\{0\}} + \ldots + Z_{t-1}^i 1_{\{t-1\}} + \frac{1}{\sum_{s \in \mathbb{T}_t} \mu_s} E_P\left[\sum_{s \in \mathbb{T}_t} Z_s^i \mu_s \mid \mathcal{F}_t\right] 1_{\mathbb{T}_t} \\ &= \frac{\gamma_0^i M_0^i}{\mu_0} 1_{\{0\}} + \ldots + \frac{\gamma_{t-1}^i M_{t-1}^i}{\mu_{t-1}} 1_{\{t-1\}} + \frac{D_t^i M_t^i}{\sum_{s \in \mathbb{T}_t} \mu_s} 1_{\mathbb{T}_t}, \qquad i = 1, 2 \end{aligned} \tag{48}$$

by (47) and Corollary 52. This implies

$$M_s^1 \gamma_s^1 = M_s^2 \gamma_s^2 \quad \forall\, s < t \quad \text{and} \quad M_t^1 D_t^1 = M_t^2 D_t^2. \tag{49}$$

Hence for any $A \in \mathcal{F}_{t-1}$ we obtain

$$\begin{aligned} E_P\left[\sum_{s=0}^{t-1} \gamma_s^1 M_s^1 1_A\right] &= E_P\left[M_{t-1}^1 \sum_{s=0}^{t-1} \gamma_s^1 1_A\right] = E_P\left[M_{t-1}^1 (1 - D_t^1) 1_A\right] \\ &= Q^1(A) - \bar{Q}^1(A \times \mathbb{T}_t) \\ &= Q^1(A) - \bar{Q}^2(A \times \mathbb{T}_t), \end{aligned}$$

where the last equality follows since $A \times \mathbb{T}_t \in \bar{\mathcal{F}}_t$ and $\bar{Q}^1 = \bar{Q}^2$ on $\bar{\mathcal{F}}_t$. In the same way we get

$$E\left[\sum_{s=0}^{t-1} \gamma_s^2 M_s^2 1_A\right] = Q^2(A) - \bar{Q}^2(A \times \mathbb{T}_t).$$

Therefore $Q^1 = Q^2$ on $\mathcal{F}_{t-1}$, and by (49) $\gamma_s^1 = \gamma_s^2$ Q^1-a.s. for all $s < t$. In particular $D_t^1 = D_t^2$ Q^1- and Q^2-a.s., which in turn implies $Q^1 = Q^2$ on $\mathcal{F}_t \cap \{D_t^1 > 0\}$ due to (49).
The proof of the inverse implication works in the same way. □

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