Representations for optimal stopping under dynamic monetary utility functionals

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Abstract

In this paper we consider the optimal stopping problem for general dynamic monetary utility functionals. Sufficient conditions for the Bellman principle and the existence of optimal stopping times are provided. Particular attention is paid to representations which allow for a numerical treatment in real situations. To this aim, generalizations of standard evaluation methods like policy iteration, dual and consumption based approaches are developed in the context of general dynamic monetary utility functionals. As a result, it turns out that the possibility of a particular generalization depends on specific properties of the utility functional under consideration.

1 Introduction

Dynamic monetary utility functionals, or DMU functionals for short, can be seen as generalizations of the ordinary conditional expectation, the usual functional which is to be maximized in standard stopping problems, which occur for instance in the theory of pricing of American (Bermudan) options in a complete market. It is well known that in an incomplete market the price of an American option is determined by the so called upper and lower Snell envelope which in turn are obtained via optimal stopping of the reward process with respect to two particular mutually conjugate DMU functionals (cf. e.g. [17]). From an economic point of view, dynamic monetary utility functionals may be seen as representations of dynamic preferences in terms of utilities of financial investors.
By changing sign, a DMU functionals becomes a dynamic risk measure (e.g. in [25]) which represents preferences in terms of losses instead of utilities in fact. Therefore, technically, the study of DMU functionals is basically equivalent to the study of dynamic risk measures which became an increasing research field in the last years. A realistic dynamic risk assessment of financial positions should allow for updating as time evolves, taking into account new information. The notion of dynamic risk measures has been established to provide a proper framework (cf. e.g. [3], [10], [12], [13], [16]). It is based on an axiomatic characterization extending the classical axioms for the concept of one-period risk measures in [2] to the dynamic multiperiod setting. From the very beginning one crucial issue was to find reasonable conditions

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of mutual relationships between the risk functionals, so-called dynamic consistency, leading to different concepts (cf. e.g. [3], [10], [12], [13], [34], [35], [38], [39]). The mostly used one is often called strong time consistency, and it is linked with a technical condition for dynamic risk measures known as recursiveness. This condition will play an important technical role in our investigations.

Recently, dynamic monetary utility functionals (as being dynamic risk measures with changed sign) have been incorporated into different topics such as, for example, the dynamics of indifference prices (see [25], [11]), and the pricing of derivatives in incomplete financial markets (cf. e.g. [34], [17], [32]). In this respect we want to emphasize the contributions in [17] and [32] as being the starting point of this paper. There the superhedging of American options is analyzed as solutions of optimal stopping problems in the context of coherent dynamic monetary utility functionals. We want to extend these considerations to more general monetary utility functionals. For instance, we will not necessarily assume translation invariance which has been recently questioned as a suitable condition for risk assessment since it tacitly supposes certainty on discounting factors by the investors (cf. [15]).

Within a time discrete setting we shall look for a set of conditions for dynamic monetary utility functionals which is as minimal as possible in a sense, while keeping the presentation compact and such that solutions for the related optimal stopping problems may be guaranteed. For classical stopping problems with respect to ordinary conditional expectations the starting point for any solution representation is the Bellman principle. This suggests to investigate when the Bellman principle holds for the general optimal stopping problems. The above mentioned condition of recursiveness in connection with a specific regularity condition will turn out to be sufficient.

Beyond the considerations of the general optimal stopping, the main contribution of this paper is the development of iterative methods and other representations for solving them. Based on these methods we naturally construct simulation based solution algorithms which allow for solving such stopping problems in practice. In contrast to meanwhile industrial standard approaches for Bermudan options, hence the ordinary stopping problem in discrete time (among others, [1], [8], [26], [28], [37]), we have not seen yet a comprehensive generic approach for treating generalized optimal stopping problems numerically. In this respect this paper intends to be a first step in this direction.

The paper is organized as follows. In Section 2 the concept of dynamic monetary utility functionals is introduced. In Section 3 we investigate the Bellman principle and the existence of optimal stopping strategies. In Section 4 a generalization of the policy iteration method of [26] is presented. Section 5, Section 6, and Section 7 generalize, respectively, the additive dual method of [33]-[20], the multiplicative dual of [23], and the consumption based approach in [4]-[5]. In Section 8 we shall provide a simulation setting to utilize the results of sections 4-7 to construct approximations of the optimal values of the investigated stopping problems. More technical proofs are given in Appendix A.

2 Dynamic monetary utility functionals

Let $\big(\Omega, (\mathcal{F}_t)_{t\in\{0,\ldots,T\}}, \mathcal{F}, \mathrm{P}\big)$ be a filtered probability space with $\{0,1\}-$valued $\mathrm{P}\,|\mathcal{F}_0$, and let $\mathfrak{X}$ be a real vector subspace of $L^0(\Omega,\mathcal{F},\mathrm{P})$ containing the indicator mappings 1_A of subsets $A\in\mathcal{F}$. It is assumed that for any $X\in\mathfrak{X}$ and $A\in\mathcal{F}$ it holds $1_A X\in\mathfrak{X}$. Moreover, $X\wedge Y\in\mathfrak{X}$ and $X\vee Y\in\mathfrak{X}$ is valid for any $X,Y\in\mathfrak{X}$. Hence in particular $\mathfrak{X}$ is a vector lattice.

A family of mappings $\Phi := (\Phi_t)_{t\in\{0,\ldots,T\}}$ with $\Phi_t : \mathfrak{X}\to\mathfrak{X}\cap L^0(\Omega,\mathcal{F}_t,\mathrm{P})$ being **monotone**, i.e. $\Phi_t(X)\leq\Phi_t(Y)$ for $X,Y\in\mathfrak{X}$ with $X\leq Y$ $\mathrm{P}-$a.s.. is called a **dynamic monetary utility functional** or shortly **DMU functional**.

We shall say that $(\Phi_t)_{t\in\{0,\ldots,T\}}$ is **recursively generated** if there is some family $(\Psi_t)_{t\in\{0,\ldots,T\}}$ of mappings

$\Psi_t : \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_{t+1}, \mathrm{P}) \to \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P})$ with $\mathcal{F}_{T+1} := \mathcal{F}$ such that

$$\Psi_T = \Phi_T, \text{ and } \Phi_t = \Psi_t \circ \Phi_{t+1} \text{ for } t = 0, ..., T-1.$$

In this case the mappings Ψ_t will be given the name **generators** of $(\Phi_t)_{t\in\{0,...,T\}}$.
Let us introduce some further notations. Henceforth $\mathcal{T}_t$ will stand for the set of the finite stopping times τ with $\tau \geq t$ P −a.s., whereas $\mathcal{H}$ will denote the set of adapted processes $Z := (Z_t)_{t\in\{0,...,T\}}$ such that $Z_t \in \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P})$ for $t \in \{0, ..., T\}$.
The following conditions on $(\Phi_t)_{t\in\{0,..,T\}}$ will play an important role in the context of optimal stopping of DMU functionals studied later on.

(C1) $\Phi_t(X) \leq \Phi_t(Y)$ P −a.s. for $t \in \{0, ..., T-1\}$, $X, Y \in \mathfrak{X}$ with $\Phi_{t+1}(X) \leq \Phi_{t+1}(Y)$ P −a.s. (**time consistency**).

(C2) $\Phi_t(1_A X) = 1_A \Phi_t(X)$ P −a.s. for $t \in \{0, ..., T\}$, $A \in \mathcal{F}_t$, and $X \in \mathfrak{X}$ (**regularity**).

(C3) $\Phi_t(X + Y) = \Phi_t(X) + Y$ P −a.s. for $t \in \{0, ...T\}$, and $X, Y \in \mathfrak{X}$ with Y being $\mathcal{F}_t$−measurable (**conditional translation invariance**).

(C4) $\Phi_t = \Phi_t \circ \Phi_{t+1}$ P −a.s. for $t \in \{0, ..., T-1\}$ (**recursiveness**).

(C5) $\Phi_t(0) = 0$ P −a.s. for $t \in \{0, ..., T\}$ (**normalization**).

(C6) $\Phi_t(YX) = Y\Phi_t(X)$ P −a.s. for $t \in \{0, ..., T\}, X \in \mathfrak{X}$ and $Y \in \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P})$ with $Y \geq 0$ P −a.s. as well as $XY \in \mathfrak{X}$ (**conditional positive homogeneity**)

(C7) For each $X \in \mathfrak{X}$ with $X \geq 0$ P−a.s. there exist a function $g : [0, \infty) \to \mathbb{R}_+$ such that $\lim_{\varepsilon\downarrow 0} g(\varepsilon) = 0$, and
$$\Phi_t (X \vee \varepsilon) \leq \Phi_t (X) + g(\varepsilon) \text{ for } t \in \{0, ..., T\}. \tag{1}$$

Remark 2.1 The construction of DMU functionals via generators opens up the possibility to obtain functionals with desired properties by just imposing them on the generators.
In this paper we frequently use one of the following implications. Their proofs are simple and therefore omitted.

- Recursiveness implies that $(\Phi_t)_{t\in\{0,..,T\}}$ is recursively generated, where the generators are the restrictions $\Phi_t | \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_{t+1}, \mathrm{P})$ for $t = 0, ..., T$.
- Let $(\Phi_t)_{t\in\{0,..,T\}}$ be recursively generated by $(\Psi_t)_{t\in\{0,..,T\}}$. Then,
  - If $\Phi_t(X) = X$ P −a.s. for $t \in \{0, ..., T\}$ and $X \in \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P})$, then $(\Phi_t)_{t\in\{0,..,T\}}$ is recursive.
  - If $\Psi_t(X) = X$ P −a.s. for $t \in \{0, ..., T\}$ and $X \in \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P})$, then $(\Phi_t)_{t\in\{0,..,T\}}$ is recursive.
  - If for any $X \in \mathfrak{X}$ and $A \in \mathcal{F}_t$ it holds $\Psi_t(1_A X) = 1_A \Psi_t(X)$, then Φ is regular.

Example 2.2 The functional Φ given by the conditional expectations $\Phi_t := E[\cdot \mid \mathcal{F}_t]$, defined on $\mathfrak{X} := L^1(\Omega, \mathcal{F}, \mathrm{P})$, is a basic example for a DMU functional. It satisfies all the conditions (C1)-(C7).

It is natural to generalize the usual martingale concept to the notion of "$\Phi-$martingale" for a given DMU functional Φ as defined below. The notion of $\Phi-$martingales will be used for different representations of optimal stopping problems in Sections 5,6.

Definition 2.3 *$M := (M_t)_{t\in\{0,...,T\}} \in \mathcal{H}$ is said to be a $\Phi-$**martingale** if $\Phi_t(M_{t+1}) = M_t$ $\mathrm{P}-a.s.$ for every $t \in \{0,...,T-1\}$. Note that for recursive Φ, $M \in \mathcal{H}$ is a $\Phi-$martingale if and only if $\Phi_t(M_s) = M_t$ $\mathrm{P}-a.s.$ for every $s,t \in \{0,...,T-1\}$ with $s > t$.*

Let us discuss some further examples of DMU functionals. First of all we want to consider the relationship with the so called dynamic risk measures.

Remark 2.4 DMU functionals may be viewed as generalizations of dynamic risk measures. Recall, a family $(\rho_t)_{t\in\{0,...,T\}}$ is a dynamic risk measure if and only if $(-\rho_t)_{t\in\{0,...,T\}}$ is a conditionally translation invariant monetary utility functional. The property of translation invariance suggests to restrict considerations to normalized functionals because for a translation invariant Φ we have $\Phi_t(X - \Phi_t(X)) = 0$. In the normalized case it then holds $\Phi_t(Y) = Y$ $\mathrm{P}-$a.s. for every $t \in \{0,...,T\}$ and any $Y \in \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P})$, and in view of Remark 2.1 Φ is recursively generated if and only if it is recursive.
We shall call a normalized conditional translation invariant Φ to be **convex**/**concave** if the mappings Φ_t $(t \in \{0,...,T\})$ are simultaneously convex/concave. If Φ is convex/concave, then

$$\overline{\Phi}_t : \mathfrak{X} \to \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P}),\ X \mapsto -\Phi_t(-X)$$

defines a concave/convex normalized conditional translation invariant DMU functional called the **conjugate of** Φ. The conditions of recursiveness and regularity are satisfied by Φ if and only its conjugate $\overline{\Phi}$ fulfills them. Conditional translation invariance of convex/concave Φ implies the regularity condition for the restriction of Φ to $\mathfrak{X}\cap L^\infty(\Omega, \mathcal{F}, \mathrm{P})$ (cf. [25], where this restriction is essential for the proof). Moreover, regularity is even valid on the entire space $\mathfrak{X}$ if $\lim_{n\to\infty} \Phi_t\left((X-n)^+\right) = 0$ $\mathrm{P}-$a.s. for every $t \in \{0,...,T\}$ and any nonnegative $X \in \mathfrak{X}$. Indeed, one may conclude from Lemma 6.5 along with Proposition 6.6 in [27] that

$$\Phi_t(X) = \operatorname*{ess\,inf}_{m\in\mathbb{N}} \operatorname*{ess\,sup}_{n\in\mathbb{N}} \Phi_t(X^+ \wedge n - X^- \wedge m)$$

holds for $t \in \{0,...,T\}$ and $X \in \mathfrak{X}$.
In the context of dynamic risk measures the property of recursiveness plays an important role. On the one hand it is intimately linked with the property of time consistency which has a specific meaning in expressing dynamic preferences of investors. For a thorough study the reader may consult e.g. [16] or [3]. On the other hand optimal stopping with dynamic risk measures may be related to specific financial applications.

In the following we shall give some examples of recursive regular DMU functionals. Let us start with DMU functionals related to time consistent coherent dynamic risk measures.

Example 2.5 (DMU functionals related to time consistent coherent dynamic risk measures)
A large class of convex, regular, and recursive conditional translation invariant DMU functionals is provided by

$$\Phi_t(X) := \operatorname*{ess\,sup}_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}[X|\mathcal{F}_t],$$

where $\mathcal{Q}$ denotes a set of probability measure which are equivalent with P, and $\mathcal{Q}$ should be stable in the sense of [17]. In order to keep the presentation compact we here assume for simplicity that such DMU

functional Φ is defined on $\mathfrak{X} := L^\infty(\Omega, \mathcal{F}, \mathrm{P})$ (cf. Theorem 6.53 in [17]). Extensions to functionals defined on suitable sets of unbounded random variables may be found in [32] or [17], and a combination of both results might yield further refinements.
Notice that in addition Φ fulfills conditional positive homogeneity, which means that $\rho(X) := \Phi_t(-X)$ defines a so called coherent dynamic risk measure $(\rho_t)_{t\in\{0,\dots,T\}}$, which is time consistent.

We may extend the class of time consistent coherent dynamic risk measures to convex ones.

Example 2.6 (DMU functionals related to time consistent convex dynamic risk measures)
Let $\mathcal{Q}$ be a set of probability measures which are equivalent with P, and let $(\alpha_t)_{t\in\{0,\dots,T\}}$ be a family of mappings α_t from $\mathcal{Q}$ into the space of $\mathcal{F}_t-$ measurable random variables with values in $\mathbb{R}\cup\{\infty\}$ fulfilling $\operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \alpha_t(\mathrm{Q}) = 0 \;\; \mathrm{P}-$a.s.. Then we may define on $\mathfrak{X} := L^\infty(\Omega, \mathcal{F}, \mathrm{P})$ a DMU functional Φ via the generators $(\Psi_t)_{t\in\{0,\dots,T\}}$ of the following form

$$\Psi_t(X) := \operatorname*{ess\,sup}_{\mathrm{Q}\in\mathcal{Q}} \left(E_{\mathrm{Q}}[X|\mathcal{F}_t] - \alpha_t(\mathrm{Q})\right). \tag{2}$$

The obtaining DMU functional is normalized, convex and conditionally translation invariant, therefore regular (see Remark 2.4) as well as recursive (cf. Remark 2.1). Moreover, it defines a time consistent convex dynamic risk measure $(\rho_t)_{t\in\{0,\dots,T\}}$ by $\rho_t(X) := \Phi_t(-X)$. Notice furthermore that by a standard argument (see the proof of Theorem 1 in [13]) every generator Ψ_t of the form (2) satisfies $\Psi_t(X_n) \nearrow \Psi_t(X)$ whenever $X_n \nearrow X \;\; \mathrm{P}-$a.s.. From this we may conclude by routine backward induction that every Φ_t fulfills $\Phi_t(X_n) \nearrow \Phi_t(X)$ for $X_n \nearrow X \;\; \mathrm{P}-$a.s., and then in view of Theorem 2.3 from [16] we have the following representation of Φ

$$\Phi_t(X) = \operatorname*{ess\,sup}_{\mathrm{Q}\in\mathcal{Q}_t} \left(E_{\mathrm{Q}}[X|\mathcal{F}_t] - \alpha_t^{\min}(\mathrm{Q})\right).$$

Here $\mathcal{Q}_t$ denotes the set of all probability measures on $\mathcal{F}$ which are absolutely continuous w.r.t. P and coincide with P on $\mathcal{F}_t$, and $\alpha_t^{\min}$ is a certain mapping from $\mathcal{Q}_t$ into the space of $\mathcal{F}_t-$ measurable random variables with values in $\mathbb{R}\cup\{\infty\}$, which may generally differ from the initial α_t in the representation of the generator (2). In particular the DMU functional Φ does not have the same representation as its generators in general.

A prominent special case is obtained by using conditional relative entropies as a choice for the mappings α_t, i.e. $\mathcal{Q}$ consists of all probability measures Q with some strictly positive $\mathrm{P}-$Radon Nikodym derivative $\frac{d\,\mathrm{Q}}{d\,\mathrm{P}}$ satisfying $E_{\mathrm{P}}\left[\frac{d\,\mathrm{Q}}{d\,\mathrm{P}}\;\ln\left(\frac{d\,\mathrm{Q}}{d\,\mathrm{P}}\right)\right] < \infty$, and

$$\alpha_t(\mathrm{Q}) := \begin{cases} \frac{1}{\gamma}\; E_{\mathrm{P}}\left[\frac{d\,\mathrm{Q}}{d\,\mathrm{P}} \ln\left(\frac{d\,\mathrm{Q}}{d\,\mathrm{P}}\right) |\mathcal{F}_t\;\right] & : \;\mathrm{Q}\,|\mathcal{F}_t = \mathrm{P}\,|\mathcal{F}_t \\ \infty & : \quad \text{otherwise} \end{cases} \quad \text{for some } \gamma > 0.$$

This leads to the so called **entropic** DMU functional Φ with

$$\Phi_t(X) := \frac{1}{\gamma}\;\ln\Big(\;E[\;\exp(\gamma X)\;|\;\mathcal{F}_t\;]\;\Big)$$

(see [13] along with Th. 4.5 in [16]). The entropic DMU functional is not conditionally positive homogeneous like the ones in Example 2.5.

The next large class of DMU functionals concerns the so called g-expectations. They are prominent examples of nonlinear functionals satisfying martingale type properties like recursiveness but need not to be conditionally translation invariant as the examples before.

Example 2.7 Let $(\mathcal{G}_s)_{s\geq 0}$ be the augmented filtration on Ω associated with the filtration generated by a standard $d-$dimensional Brownian motion $(B_s)_{s\geq 0}$ with $B_0 := 0$, and let for $S > 0$ the function $g : \Omega \times [0, S] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfy some certain technical conditions such that it can be used as *driver* of a backward stochastic differential equation (abbreviated: BSDE)

$$Y_s = X + \int_s^S g(\cdot, r, Y_r, Z_r)\ dr\ - \int_s^S Z_r\ dB_r \text{ for } s \in [0, S],$$

where $X \in L^2(\Omega, \mathcal{G}_S, \mathrm{P})$ (cf. [29]). Moreover, there always exists a unique couple $\left(Y_s^X\right)_{s\in[0,S]}$ and $\left(Z_s^X\right)_{s\in[0,S]}$ of adapted respectively $1-$ and $d-$dimensional processes with square integrable paths solving the BSDE ([29] again). Now it is natural to define the family $(\mathcal{E}_g[\cdot|\mathcal{G}_s])_{s\in[0,S]}$ via

$$\mathcal{E}_g[\cdot \mid \mathcal{G}_s] : L^2(\Omega, \mathcal{G}_S, \mathrm{P}) \to L^2(\Omega, \mathcal{G}_s, \mathrm{P}),\ X \mapsto Y_s^X,$$

known as (a family of) **conditional g-expectations**, where $\mathcal{E}_g[\cdot \mid \mathcal{G}_0]$ is just called **g-expectation**. For $g \equiv 0$ we retrieve the usual (conditional) expectation of a square integrable random variable. For applications of conditional g-expectations in finance the reader is referred to [14] and [30].
Let us now pick some observation times $0 =: s_0 < s_1 < ... < s_T := S$, and define $\left(\Omega, (\mathcal{F}_t)_{t\in\{0,...,T\}}, \mathcal{F}, \mathrm{P}\right)$ and $\Phi := (\Phi_t)_{t\in\{0,...,T\}}$ by $\mathcal{F}_t := \mathcal{G}_{s_t}, \mathcal{F} := \mathcal{F}_T$, and $\Phi_t := \mathcal{E}_g[\cdot \mid \mathcal{G}_{s_t}]$. Drawing on basic properties of conditional g-expectation as derived by Peng in [29], Φ is always a regular recursive DMU functional fulfilling $\Phi_t(X) = X$ $\mathrm{P}-$a.s. for $t \in \{0, ..., T\}$ and $\mathcal{F}_t-$measurable X.
Furthermore Φ is conditional translation invariant if and only if $g(\omega, s, \cdot, z)$ is constant for every $\omega \in \Omega,\ s \in [0, S]$ and $z \in \mathbb{R}^d$ (for the if part see [29], for the only if part cf. [22]). In this case Φ is even a convex normalized conditionally translation invariant DMU functional if and only if in addition

$$g(\cdot, \cdot, \cdot, \lambda z_1 + (1 - \lambda)z_2) \leq \lambda g(\cdot, \cdot, \cdot, z_1) + (1 - \lambda)g(\cdot, \cdot, \cdot, z_2)\ \ \mathrm{P} \otimes dt - \text{a.s.}$$

for $z_1, z_2 \in \mathbb{R}^d$ and $\lambda \in [0, 1]$ (cf. [22]). Note that $\rho_t(X) := \Phi_t(-X)$ defines a convex dynamic risk measure $(\rho_t)_{t\in\{0,...,T\}}$. The relationship between $g-$expectations and convex dynamic risk measures has been observed in [36] too.

We shall finish the section with some simple nonstandard examples.

Examples 2.8 Let $\mathfrak{X} = L^\infty(\Omega, \mathcal{F}, \mathrm{P})$, and let $\mathrm{Q}_1, ..., \mathrm{Q}_K$ denote probability measures which are equivalent with P.

1. For strictly increasing $U_1, ..., U_K : \mathbb{R} \to \mathbb{R}$ with $U_1(0) = ... = U_K(0) = 0$ and positive $\alpha_1, ..., \alpha_K$, let Φ be recursively generated with generators $(\Psi_t)_{t\in\{0,...,T\}}$ defined by

$$\Psi_t(X) := \sum_{k=1}^{K} \alpha_k U_k^{-1}\left(E_{\mathrm{Q}_k}[U_k(X) \mid \mathcal{F}_t]\right) \text{ for } t \in \{0, ..., T\} \text{ and } X \in L^\infty(\Omega, \mathcal{F}_{t+1}, \mathrm{P}).$$

Obviously the functional Φ is regular. Moreover, if $\sum_{k=1}^{K} \alpha_k = 1$, it satisfies $\Phi_t(X) = X$ P −a.s. for $X \in L^\infty(\Omega, \mathcal{F}_t, \mathrm{P})$, $t \in \{0, ..., T\}$, hence Φ is recursive. In the case of $K = \alpha_1 = 1$, Φ_t is defined in literally the same way as its generator Ψ_t leading to a so called **conditional certainty equivalent** (cf. [18]) which may be viewed as a dynamic version of a premium principle for insurance contracts known as **mean value premium principle** (cf. e.g. [24]). We note that Φ is not conditionally translation invariant in general.

2. For nondecreasing $U_1, ..., U_K : \mathbb{R} \to \mathbb{R}$ with $U_1(0) = ... = U_K(0) = 0$ and positive $\alpha_1, ..., \alpha_K$ let Φ be recursively generated with generators $(\Psi_t)_{t\in\{0,...,T\}}$ defined by

$$\Psi_t(X) := \sum_{k=1}^{K} \alpha_k \ E_{\mathrm{Q}_k}[U_k(X) \mid \mathcal{F}_t] \text{ for } t \in \{0, ..., T\} \text{ and } X \in L^\infty(\Omega, \mathcal{F}_{t+1}, \mathrm{P}).$$

In general, this Φ will be neither recursive nor conditionally translation invariant, but still regular.

3 The optimal stopping problem

We will study the following stopping problem

$$Y_t^* := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau), \quad t \in \{0, ..., T\}, \tag{3}$$

for $Z \in \mathcal{H}$. We refer to the process Y^* as the ($\Phi-$)Snell envelope of Z. Below we consider two important aspects. Firstly, we investigate the existence of optimal stopping times and secondly, we try to find Bellman principles. The crucial step to guarantee optimal stopping times is provided by the following Lemma.

Lemma 3.1 *Let $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$, let for some fixed $t \in \{0, ..., T-1\}$ some $\tau_{t+1}^* \in \mathcal{T}_{t+1}$ exist such that $\Phi_{t+1}(Z_{\tau_{t+1}^*}) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t+1}} \Phi_{t+1}(Z_\tau)$. Defining the event $B_t := \left[\Phi_t(Z_t) - \Phi_t(Z_{\tau_{t+1}^*}) \geq 0\right]$ and $\tau_t^* := t1_{B_t} + \tau_{t+1}^* 1_{\Omega\setminus B_t}$, we obtain $B_t \in \mathcal{F}_t$, $\tau_t^* \in \mathcal{T}_t$, and under the conditions of time consistency and regularity*

$$\Phi_t(Z_{\tau_t^*}) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau) = \Phi_t(Z_t) \vee \Phi_t(Z_{\tau_{t+1}^*}).$$

Proof:
$B_t \in \mathcal{F}_t$, $\tau_t^* \in \mathcal{T}_t$ follows from $\mathcal{F}_t-$measurability of the outcomes of Φ_t. Furthermore we may observe $Z_{\tau_t^*} = 1_{B_t} Z_t + 1_{\Omega\setminus B_t} Z_{\tau_{i+1}^*}$. Then the application of $(C2)$ yields

$$\begin{aligned}\Phi_t(Z_{\tau_t^*}) = 1_{B_t}\Phi_t(Z_{\tau_t^*}) + 1_{\Omega\setminus B_t}\Phi_t(Z_{\tau_t^*}) \overset{(C2)}{=} \Phi(1_{B_t} Z_t) + \Phi(1_{\Omega\setminus B_t} Z_{\tau_{t+1}^*}) &\overset{(C2)}{=} 1_{B_t}\Phi_t(Z_t) + 1_{\Omega\setminus B_t}\Phi_t(Z_{\tau_{t+1}^*}) \\ &= \Phi_t(Z_t) \vee \Phi_t(Z_{\tau_{t+1}^*}).\end{aligned}$$

Next let us define the mapping $\sigma : \mathcal{T}_t \to \mathcal{T}_{t+1}$ by $\sigma(\tau) := (t+1)1_{[\tau=t]} + \tau 1_{[\tau>t]}$. Then we obtain for $\tau \in \mathcal{T}_t$

$$\Phi_t(Z_\tau) = \Phi_t(1_{[\tau=t]} Z_t + 1_{[\tau>t]} Z_{\sigma(\tau)}) \overset{(C2)}{=} 1_{[\tau=t]}\Phi_t(Z_t) + 1_{[\tau>t]}\Phi_t(Z_{\sigma(\tau)}) \leq \Phi_t(Z_t) \vee \Phi_t(Z_{\sigma(\tau)}).$$

By assumption $\Phi_{t+1}(Z_{\sigma(\tau)}) \leq \Phi_{t+1}(Z_{\tau^*_{t+1}})$ P $-$a.s. so that condition $(C1)$ implies

$$\Phi_t(Z_\tau) \leq \Phi_t(Z_t) \vee \Phi_t(Z_{\sigma(\tau)}) \overset{\mathrm{P}}{\leq} \Phi_t(Z_t) \vee \Phi_t(Z_{\tau^*_{t+1}}) = \Phi_t(Z_{\tau^*_t}),$$

which completes the proof. ■

Since $\tau :\equiv T$ is always the optimal stopping time in $\mathcal{F}_T$, we may apply sequentially Lemma 3.1 to obtain the following result concerning the existence of optimal stopping times.

Theorem 3.2 *Let $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$. Then under conditions of time consistency and regularity there exists for any $t \in \{0, ..., T\}$ some $\tau^*_t \in \mathcal{T}_t$ such that*

$$\Phi_t(Z_{\tau^*_t}) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau).$$

The sequence $(\tau^*_t)_{t\in\{0,...,T\}}$ of optimal stopping times may be chosen such that $\tau^*_T = T$, and

$$1_{[\tau^*_t>t]}\tau^*_t = 1_{[\tau^*_t>t]}\tau^*_{t+1} \text{ for any } t \in \{0, ..., T-1\}.$$

From now on we consider exclusively recursively generated DMU functionals. The following Theorem gathers the main results concerning optimal stopping w.r.t. such functionals for our purposes later on.

Theorem 3.3 *Let $(\Phi_t)_{t\in\{0,...,T\}}$ be regular and recursively generated with generators satisfying the property $\Psi_t(X) \leq \Psi_t(Y)$ P $-a.s.$ for $t \in \{0, ..., T-1\}$ and $X, Y \in \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_{t+1}, \mathrm{P})$ with $X \leq Y$ P $-a.s.$ Then Theorem 3.2 may be restated, and we have the* **Bellman principle** *which means that*

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau) = \Phi_t(Z_t) \vee \Psi_t\left(\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t+1}} \Phi_{t+1}(Z_\sigma)\right)$$

is valid for any $Z \in \mathcal{H}$ and every $t \in \{0, ..., T-1\}$. If moreover Φ is recursive, then it holds

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau) = \Phi_t(Z_t) \vee \Phi_t\left(\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t+1}} \Phi_{t+1}(Z_\sigma)\right)$$

for arbitrary $Z \in \mathcal{H}$ and $t \in \{0, ..., T-1\}$.

Proof:
The assumptions on the generators $(\Psi_t)_{t\in\{0,...,T\}}$ imply condition $(C1)$ so that we may apply Theorem 3.2 immediately, whereas we may conclude directly from Lemma 3.1 that the Bellman principle holds. The last statement of Theorem 3.3 is an easy consequence of the observation that in case of recursive Φ the generators are just the restrictions $\Phi_t|\mathfrak{X} \cap L^0(\Omega, \mathcal{F}_{t+1}, \mathrm{P})$ for $t \in \{0, ..., T-1\}$. ■

Example 3.4 Let us consider the optimal stopping problem for the regular and recursively generated DMU functionals from Remark 2.4. There so exist a family $(\tau^*_t)_{t\in\{0,...,T\}}$ of optimal stopping times as in Theorem 3.2, and the Bellman principle is fulfilled due to Theorem 3.3. In particular this applies to functionals from Example 2.6, as well as to the functional

$$\Phi_t : \mathfrak{X} \to \mathfrak{X} \cap L^0(\Omega, \mathcal{F}_t, \mathrm{P}),\ X \mapsto \operatorname*{ess\,sup}_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}[X \mid \mathcal{F}_t],$$

and its conjugate $\overline{\Phi}$

$$\overline{\Phi}_t(X) = \operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}[X \mid \mathcal{F}_t],$$

where $\mathcal{Q}$ is a stable set of probability measures which are equivalent w.r.t. P, and $\mathfrak{X} := L^\infty(\Omega, \mathcal{F}, \mathrm{P})$ (cf. Example 2.5).

As a special case let $\mathcal{Q}$ denote the set of equivalent martingale measures of an arbitrage free incomplete financial market with reference measure P . It is known that $\mathcal{Q}$ is stable (cf. [17], Theorem 6.45) so that Φ and $\overline{\Phi}$ are recursive. Both DMU functionals play a key role for the issue of pricing and hedging American contingent claims in the following sense: For any nonnegative $Z \in \mathcal{H}$ the stopping problems (3) according to Φ and $\overline{\Phi}$ correspond to the upper and lower Snell envelopes of Z w.r.t. $\mathcal{Q}$ respectively. Moreover, the initial value of the lower and upper snell envelope are just the lower and upper hedging price, respectively. Further, the optimal stopping time according to the lower hedging prices corresponds to an optimal exercise strategy for the buyer of the option. For details see for example [17], Theorems 7.13, 7.14.

Example 3.5 Let Φ be a finite subfamily of conditional g-expectations. Then in view of Example 2.7 combined with Theorem 3.3 we may find for any $Z \in \mathcal{H}$ some family $(\tau_t^*)_{t\in\{0,\dots,T\}}$ of stopping times $\tau_t^* \in \mathcal{T}_t$ satisfying $\tau_T^* = T$ as well as $1_{[\tau_t^*>t]}\tau_t^* = 1_{[\tau_t^*>t]}\tau_{t+1}^*$, and

$$\Phi_t(Z_{\tau_t^*}) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau) = \Phi_t(Z_t) \vee \Phi_t\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t+1}} \Phi_{t+1}(Z_\tau)\right) = \Phi_t(Z_t) \vee \Phi_t(Z_{\tau_{t+1}^*})$$

for $t \in \{0, ..., T-1\}$.

Example 3.6 The DMU functionals introduced in Examples 2.8 admit families of optimal stopping times as in Theorem 3.2 and satisfy the Bellman principle due to Theorem 3.3.

Remark 3.7 Stopping problems for certain specific DMU functionals has been already studied in the literature. Let us give a short review and comment.

Existence of optimal stopping families and the verification of the Bellman principle has been obtained for recursive DMU functionals Φ of the form $\Phi_t(X) := \operatorname*{ess\,sup}_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}[X \mid \mathcal{F}_t]$ or $\Phi_t(X) := \operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}[X \mid \mathcal{F}_t]$, where $\mathcal{Q}$ is some stable set of probability measures which are equivalent with P (e.g. [17], [32]). In view of Example 3.4 these results may be extended to a considerably larger class of conditionally translation invariant DMU functionals.

Recently, Bayraktar and Shao ([6]) solved the optimal stopping problem for recursive conditionally translation invariant DMU functionals within a time continuous setting w.r.t. a Brownian filtration. In [7] the studies have been detailed for DMU functionals which are additionally concave.

Theorem 3.3 applies to DMU functionals which are not necessarily conditionally translation invariant, or even not recursive, see Example 3.5 and Example 3.6. We may thus avoid conditions like conditional translation invariance and recursiveness, which are questioned in the literature anyway (see e.g. [15], and e.g. [16], [35],[38], [39], respectively).

4 Iterative solution of optimal stopping problems

In this section we develop an iterative procedure for solving the optimal stopping problem. In fact we shall generalize the policy iteration method in [26] for classical optimal stopping with conditional expectations to optimal stopping of regular recursively generated DMU functionals. As such we obtain a procedure for approximating the optimal stopping time, hence the optimal value from below.

Throughout we fix a recursively generated regular DMU functional $(\Phi_t)_{t\in\{0,\ldots,T\}}$ with generators $(\Psi_t)_{t\in\{0,\ldots,T\}}$ satisfying the conditions in Theorem 3.3. So, for any $Z \in \mathcal{H}$ there exists a family $(\tau_t^*)_{t\in\{0,\ldots,T\}}$ of optimal stopping times $\tau_t^* \in \mathcal{T}_t$ satisfying

$$\tau_T^* = T, \text{ and } 1_{[\tau_t^*>t]}\tau_t^* = 1_{[\tau_t^*>t]}\tau_{t+1}^* \text{ for any } t \in \{0,...,T-1\}, \tag{4}$$

and

$$Y_t^* = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau) = \Phi_t(Z_{\tau_t^*}) \text{ for every } t \in \{0,...,T\}. \tag{5}$$

Let us define $(\tau_t)_{t\in\{0,\ldots,T\}}$ to be a **time consistent stopping family** if

$$\tau_t \in \mathcal{T}_t, \quad \tau_T = T, \quad \text{and} \quad 1_{[\tau_t>t]}\tau_t = 1_{[\tau_t>t]}\tau_{t+1} \quad \text{for} \quad t \in \{0,...,T-1\}.$$

The policy iteration step starts with any time consistent stopping family $(\tau_t)_{t\in\{0,\ldots,T\}}$ and corresponding process $(Y_t)_{t\in\{0,\ldots,T\}}$ with $Y_t := \Phi_t(Z_{\tau_t})$, being an approximation of $(Y_t^*)_{t\in\{0,\ldots,T\}}$. In order to improve this approximation we consider the process $\left(\widetilde{Y}_t\right)_{t\in\{0,\ldots,T\}}$ defined by $\widetilde{Y}_t := \max_{t\le s\le T}\Phi_t(Z_{\tau_s})$, and the new stopping family

$$\widehat{\tau}_T := T, \quad \widehat{\tau}_t := \inf\{s \in \{t,...,T\} \mid \Phi_s(Z_s) \ge \max_{s+1\le u\le T} \Phi_s(Z_{\tau_u})\}, \quad 0 \le t \le T-1. \tag{6}$$

Obviously, the stopping family $(\hat{\tau}_t)_{t\in\{0,\ldots,T\}}$ is also time consistent . By the next theorem, a generalization of Theorem 3.1 in [26] in fact, the process $\left(\widehat{Y}_t\right)_{t\in\{0,\ldots,T\}}$, defined by $\widehat{Y}_t := \Phi_t(Z_{\hat{\tau}_t})$, improves the initial approximation $(Y_t)_{t\in\{0,\ldots,T\}}$ of (5).

Theorem 4.1 *We have the inequalities*

$$Y_t \le \widetilde{Y}_t \le \widehat{Y}_t \le Y_t^*, \quad t \in \{0,...,T\}.$$

The proof of Theorem 4.1 is similar to the proof in [26]. However, it has to be focussed that it is sufficient that the DMU functional under consideration is regular and recursively generated. For the convenience of the reader the proof is therefore provided in Appendix A (while also comprising the structure of argumentation in [26] slightly).

In view of Theorem 4.1 the idea is to construct recursively a sequence of pairs

$$\left((\tau_t^{(m)})_{t\in\{0,\ldots,T\}}, (Y_t^{(m)})_{t\in\{0,\ldots,T\}}\right)_{m\in\mathbb{N}_0}$$

where $(\tau_t^{(m)})_{t\in\{0,\ldots,T\}}$ is a time consistent stopping family for any $m \in \mathbb{N}_0$ such that $Y_t^{(m)} = \Phi_t(Z_{\tau_t^{(m)}})$, and $\tau_t^{(m+1)} = \inf\{s \in \{t,...,T\} \mid \Phi_s(Z_s) \ge \max_{s+1\le u\le T} \Phi_s(Z_{\tau_u^{(m)}})\}$ for $t \in \{0,...,T-1\}$.

Next we start with some time consistent stopping family $(\tau_t^{(0)})_{t\in\{0,...,T\}}$, for example, a canonical choice is $\tau_t^{(0)} := t$. Then due to Theorem 4.1, we have

$$Y_t^{(0)} \leq Y_t^{(m)} \leq \widetilde{Y}_t^{(m+1)} \leq Y_t^{(m+1)} \leq Y_t^* \text{ for } m \in \mathbb{N}_0,\ t \in \{0,...,T\}, \tag{7}$$

where $\widetilde{Y}_t^{(m+1)} := \max_{t\leq s\leq T} \Phi_t(Z_{\tau_s^{(m)}})$.
The iteration procedure may be stopped after at most T iterations, yielding an optimal stopping family.

Proposition 4.2 *For $t \in \{0,...,T\}$ we have*

$$Y_t^{(m)} = Y_t^* \textit{ if } m \geq T - t.$$

Hence $\tau_t^{(m)}$ is an optimal stopping time for the corresponding stopping problem at time t, if $m \geq T - t$, and in particular $(\tau_t^{(m)})_{t\in\{0,...,T\}}$ is an optimal stopping family for $m \geq T$.

Proof:
The proof may be done by adapting the proof of Proposition 4.4 in [26] in a similar way as is done for proving Theorem 4.1 and therefore omitted. Indeed, a closer inspection of the proof of Proposition 4.4 (in [26]) shows that only regularity, the fact that the DMU functional is recursively generated by a monotonic system (Ψ_t), and the Bellman principle (see Theorem 3.3) is essential. ■

Examples 4.3

1. Referring to Example 3.4, Proposition 4.2 guarantees that the proposed iteration method provides a scheme to calculate super hedging prices and optimal exercises of discounted American options.

2. In view of Example 2.7 and Examples 2.8 the associated stopping problems may be solved iteratively by the introduced method. In particular we have a numerical scheme for optimal stopping with g-expectations.

5 Additive dual upper bounds

In this section a method for approximating the optimal value of the stopping problem from above will be developed for DMU functionals Φ which are *regular, conditional translation invariant,* and *recursive.* For such a Φ we propose an additive dual representation for the stopping problem (3), in terms of $\Phi-$martingales introduced in Definition 2.3. As such this generalization may be seen as a generalization of the representation of [33], and [20] for the standard stopping problem. We first extend the classical additive Doob decomposition theorem.

Lemma 5.1 *Let Φ be a conditional translation invariant DMU functional. Then for any stochastic process $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$ there exists a unique pair $(M, A) \in \mathcal{H}\times\mathcal{H}$ of a $\Phi-$martingale M and a predictable process A, such that $M_0 = A_0 = 0$, and*

$$Z_t = Z_0 + M_t + A_t \textit{ for } t \in \{0,...,T\}, \quad \mathrm{P}-a.s. \tag{8}$$

Proof:
Define A recursively by $A_0 := 0$, and $A_{t+1} := A_t + \Phi_t(Z_{t+1}) - Z_t$ for $t \in \{0, ..., T-1\}$. Then of course $A \in \mathcal{H}$ and A is predictable. Next define $M \in \mathcal{H}$ via $M_t := Z_t - Z_0 - A_t$ for $t \in \{0, ..., T\}$. Obviously $M_0 = 0$, and by conditional translation invariance (property (C3)),

$$\Phi_t(M_{t+1}) \overset{(C3)}{=} \Phi_t(Z_{t+1}) - Z_0 - A_{t+1} = \Phi_t(Z_{t+1}) - Z_0 - (A_t + \Phi_t(Z_{t+1}) - Z_t) = Z_t - Z_0 - A_t = M_t.$$

So M is a $\Phi-$martingale and (8) holds. Now let $(M', A') \in \mathcal{H} \times \mathcal{H}$ be another pair as stated. Then for $t \in \{0, ..., T-1\}$ we may conclude by conditional translation invariance,

$$0 = \Phi_t(M'_{t+1} - M'_t) = \Phi_t(Z_{t+1}) - Z_t + A'_t - A'_{t+1},$$

in particular $A'_{t+1} = A'_t + \Phi_t(Z_{t+1}) - Z_t$. Hence by induction $A' = A$, and so $M' = M$. ■

Remark 5.2 The additive Doob decomposition has already been shown for the functional $\overline{\Phi}$ from Example 3.4 (cf. [32]).

The next lemma may be regarded as a generalization of Doob's optional sampling theorem. It is proved in Appendix A.

Lemma 5.3 *Let Φ be a regular, conditional translation invariant, and recursive DMU functional, and let M be any $\Phi-$martingale. Then for every $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$, each $t \in \{0, ..., T\}$, and each stopping time $\tau \in \mathcal{T}_t$, we have*

$$\Phi_t(Z_\tau) = \Phi_t(Z_\tau + M_T - M_\tau).$$

Remark 5.4 Under the assumptions of Lemma 5.3 the statement

$$\Phi_t(Z_\tau) = \Phi_t(Z_\tau + M_\tau) - M_t,$$

that one might expect at a first glance, does *not* hold.

The Doob type Lemmas 5.1,5.3, and the Bellman principle Theorem 3.3, provide the ingredients to establish the following additive dual representation. recursive. In the case of $K = \alpha_1 = 1$, Φ_t is defined in literally the same way as its generator Ψ_t leading to

Theorem 5.5 *Let Φ be a regular, conditional translation invariant, and recursive DMU functional, and $\mathcal{M}_0^\Phi$ be the set of all $\Phi-$martingales M with $M_0 = 0$. For $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$ let $M^* \in \mathcal{M}_0^\Phi$ be the $\Phi-$martingale of the decomposition of Y^* in (3) according to Lemma 5.1. Then*

$$\begin{aligned} Y_t^* &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau) = \operatorname*{ess\,inf}_{M\in\mathcal{M}_0^\Phi} \Phi_t\left(\max_{t\leq j\leq T}(Z_j - M_j + M_T)\right) \\ &= \Phi_t\left(\max_{t\leq j\leq T}(Z_j - M_j^* + M_T^*)\right) \quad \textit{for } t \in \{0, ..., T\}. \end{aligned}$$

Proof:
Let $A^* := (A_t^*)_{t\in\{0,...,T\}}$ denote the predictable part of the decomposition of Y^* according to Lemma 5.1. Since M^* is a $\Phi-$martingale, we have for $t \in \{0, ..., T\}$

$$0 = \Phi_t(M_{t+1}^*) - M_t^* \overset{(C3)}{=} \Phi_t(M_{t+1}^* - M_t^*) \overset{(C3)}{=} \Phi_t(Y_{t+1}^*) - Y_t^* - \left(A_{t+1}^* - A_t^*\right).$$

This implies $A_{t+1}^* - A_t^* = \Phi_t(Y_{t+1}^*) - Y_t^* \leq 0$ due to the Bellman principle. Hence A^* has nonincreasing paths. Furthermore, by the Bellman principle $\Phi_t(Z_t) = Z_t \leq Y_t^*$ holds for every $t \in \{0, ..., T\}$. We thus have

$$Z_t - M_t^* + M_T^* = Z_t + Y_T^* - Y_t^* + A_t^* - A_T^* \leq Y_T^* + A_t^* - A_T^* \text{ for } t \in \{0, ...T\}.$$

Since A^* is nonincreasing, Φ is conditional translation invariant and recursive, and M^* is a $\Phi-$martingale, it follows that

$$\Phi_t\left(\max_{t\leq j\leq T}(Z_j - M_j^* + M_T^*)\right) \overset{(C3)}{\leq} \Phi_t\left(Y_T^* - A_T^*\right) + A_t^* \overset{(C3)}{=} Y_0^* + \Phi_t(M_T^*) + A_t^* = Y_0^* + M_t^* + A_t^* = Y_t^* \quad (9)$$

for $t \in \{0, ..., T\}$. Finally, using Lemma 5.3 and (9) we have for any $t \in \{0, ..., T\}$ and $M \in \mathcal{M}_0^\Phi$,

$$\begin{aligned} Y_t^* = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \Phi_t(Z_\tau + M_T - M_\tau) &\leq \operatorname*{ess\,inf}_{M\in\mathcal{M}_0^\Phi} \Phi_t\left(\max_{t\leq j\leq T}(Z_j - M_j + M_T)\right) \\ &\leq \Phi_t\left(\max_{t\leq j\leq T}(Z_j - M_j^* + M_T^*)\right) \leq Y_t^*. \end{aligned}$$

■

Remark 5.6 Let us emphasize that the structure of the generalized dual representation in Theorem 5.5 cannot be obtained by simply replacing conditional expectations with DMU functionals in the standard dual representation of [33], and [20]. This remark applies to the generalized sampling Lemma 5.1 as well.

Example 5.7 Let $\mathcal{Q}$ denote the set of equivalent martingale measures w.r.t. some arbitrage-free financial market, and let $\mathfrak{X} := L^\infty(\Omega, \mathcal{F}, \mathrm{P})$. Fix any nonnegative $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$. It may be viewed as a discounted American Option. Then the DMU functionals Φ and $\overline{\Phi}$ from Example 3.4 meet the requirements of Theorem 5.5, and thus the superhedging price and the lowest arbitrage-free price of Z may be represented by

$$\inf_{X\in\mathcal{X}_0} \Phi_0\left[\max_{t\in\{0,...,T\}}(Z_t - \Phi_t(X) + X)\right] \quad \text{and} \quad \inf_{X\in\mathcal{X}_0} \overline{\Phi}_0\left[\max_{t\in\{0,...,T\}}(Z_t - \overline{\Phi}_t(X) + X)\right],$$

respectively. Here $\mathcal{X}_0 := \{X \in \mathfrak{X} \mid \sup_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}[X] = 0\}$.

Examples 5.8 Theorem 5.5 may be applied immediately to the following regular, conditional translation invariant, and recursive functionals (see also Remark 2.1).

1. Any family of g-expectations as in Example 2.7 with driver $g : \Omega \times [0, S] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $g(\omega, s, \cdot, z)$ is constant for $(\omega, s, z) \in \Omega \times [0, S] \times \mathbb{R}^d$.

2. The DMU functional Φ recursively defined as in Example 2.6.

6 Multiplicative dual upper bounds

The additive dual representation for constructing upper bounds for the standard stopping problem has a multiplicative version which is due to [23]. We will develop in this section a multiplicative dual representation for the stopping problem (3) when the DMU functional Φ is recursive and positively homogeneous. Note that from any positively homogeneous recursively generated DMU functional we may obtain a recursive one, by multiplication with a constant.

To our aim we need an extension of the multiplicative Doob decomposition theorem. As we do not want to burden the presentation with too much technicalities, we restrict ourselves in this section to the case where $\mathfrak{X} = L^\infty(\Omega, \mathcal{F}, \mathrm{P})$.

Lemma 6.1 *Let $\Phi := (\Phi_t)_{t\in\{0,...,T\}}$ be a positively homogeneous recursive DMU functional. Let $\delta > 0$, and $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$ with $Z_t \geq \delta$ $\mathrm{P}-a.s.$ for any $t \in \{0, ..., T\}$. Then there exists a unique pair $(N, U) \in \mathcal{H} \times \mathcal{H}$ of some $\Phi-$martingale N and a predictable process U such that $N_0 = U_0 = 1$ and*

$$Z_t = Z_0 N_t U_t \;\; \mathrm{P}\; a.s.$$

for $t \in \{0, ..., T\}$.

Proof:
Define processes U and N recursively by $U_0 := N_0 := 1$ and

$$U_{t+1} := U_t \,\frac{\Phi_t(Z_{t+1})}{Z_t}, \quad N_{t+1} := N_t \,\frac{Z_{t+1}}{\Phi_t(Z_{t+1})} \text{ for } t \in \{0, ..., T-1\}.$$

Observe that U and N are well defined since by assumption $\Phi_t(Z_t) \geq \Phi_t(\delta) = \delta$ due to monotonicity of Φ. Obviously, U is predictable, N is a $\Phi-$martingale, and it follows easily by induction that $Z_t = Z_0 N_t U_t$ for all $t \in \{0, ..., T\}$.
Now let $(N', U') \in \mathcal{H} \times \mathcal{H}$ be another pair as stated. We will show that $N'_t = N_t, U'_t = U_t$ $\mathrm{P}-$a.s. for $t \in \{0, ..., T\}$ by induction. The case $t = 0$ is trivial. So let $t \in \{0, ..., T-1\}$ such that $N'_t = N_t, U'_t = U_t$ $\mathrm{P}-$a.s.. Firstly, $\Phi_t(N_{t+1}) = N_t = N'_t = \Phi_t(N'_{t+1})$ $\mathrm{P}-$a.s. since N, N' are $\Phi-$martingales. Therefore by conditional positive homogeneity (C6)

$$Z_0 U_{t+1} N_t = Z_0 U_{t+1} \Phi_t(N_{t+1}) \overset{(C6)}{=} \Phi_t(Z_{t+1}) \overset{(C6)}{=} Z_0 U'_{t+1} \Phi_t(N'_{t+1}) = Z_0 U'_{t+1} N_t.$$

Thus $U'_{t+1} = U_{t+1}$ $\mathrm{P}-$a.s. due to $Z_0 N_t > 0$ $\mathrm{P}-$a.s., and

$$Z_0 U_{t+1} N_{t+1} = Z_{t+1} = Z_0 U'_{t+1} N'_{t+1} = Z_0 U_{t+1} N'_{t+1} \;\; \mathrm{P}-\text{a.s.}.$$

Since $Z_0 U_{t+1} > 0$ $\mathrm{P}-$a.s. we have $N_{t+1} = N'_{t+1}$ $\mathrm{P}-$a.s. ■

The next Lemma is a multiplicative version of Lemma 5.3. For a proof see Appendix A.

Lemma 6.2 *Let $\Phi := (\Phi_t)_{t\in\{0,...,T\}}$ be a positively homogeneous recursive DMU functional, and let $Z := (Z_t)_{t\in\{0,...,T\}} \in \mathcal{H}$ with $Z_t \geq 0$ $\mathrm{P}-a.s.$ for any $t \in \{0, ..., T\}$. If $N := (N_t)_{t\in\{0,...,T\}}$ denotes any $\Phi-$martingale satisfying $N_t > 0$ $\mathrm{P}-a.s.$, then*

$$\Phi_t(Z_\tau) = \Phi_t\left(\frac{Z_\tau N_T}{N_\tau}\right) \text{ for } t \in \{0, ..., T\} \text{ and } \tau \in \mathcal{T}_t.$$

Obviously, under the assumptions of this section Φ satisfies the Bellman principle (see Theorem 3.3), which allows us to establish a multiplicative dual representation for the stopping problem (3).

Theorem 6.3 *Let the DMU functional Φ be as in Lemma 6.1, let $\mathcal{M}^{\Phi}_{+1}$ be the set of all $\Phi-$martingales N with $N > 0$ and $N_0 = 1$, and let $Z \in \mathcal{H}$ with $Z \geq 0$. We then may state for every $t \in \{0, ..., T\}$ the following:*

(i)

$$Y^*_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \Phi_t(Z_\tau) \leq \inf_{N \in \mathcal{M}^{\Phi}_{+1}} \Phi_t\left(\max_{t \leq j \leq T} \frac{Z_j N_T}{N_j}\right).$$

(ii) If Φ satisfies in addition condition (C7), we have

$$Y^*_t = \operatorname*{ess\,inf}_{N \in \mathcal{M}^{\Phi}_{+1}} \Phi_t\left(\max_{t \leq j \leq T} \frac{Z_j N_T}{N_j}\right).$$

(iii) If Z is as in Lemma 6.1 we have

$$Y^*_t \;=\; \operatorname*{ess\,inf}_{N \in \mathcal{M}^{\Phi}_{+1}} \Phi_t\left(\max_{t \leq j \leq T} \frac{Z_j N_T}{N_j}\right) = \Phi_t\left(\max_{t \leq j \leq T} \frac{Z_j N^*_T}{N^*_j}\right),$$

where $N^* \in \mathcal{M}^{\Phi}_0$ is the $\Phi-$martingale in the multiplicative decomposition of Y^* in Lemma 6.1.

Proof:
Statement (i) is an immediate consequence of Lemma 6.2.
For the proof of statement (ii) let us consider an arbitrary $\varepsilon > 0$. The process Z^ε, defined by $Z^\varepsilon_t := Z_t \vee \varepsilon$ induces the process $Y^{\varepsilon*}$ via $Y^{\varepsilon*}_t := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \Phi_t(Z^\varepsilon_\tau)$ which fulfills the assumptions of Lemma 6.1. Therefore we may find a pair $(U^\varepsilon, N^\varepsilon)$ consisting of a predictable process U^ε and a $\Phi-$martingale $N^\varepsilon \in \mathcal{M}^{\Phi}_{+1}$ satisfying

$$Y^{\varepsilon*}_t = Y^{\varepsilon*}_0 N^\varepsilon_t U^\varepsilon_t \;\; \text{P a.s. for } t \in \{0, ..., T\}.$$

Due to conditional positive homogeneity of Φ, the predictability of U^ε, and since N^ε is a $\Phi-$martingale we may conclude

$$1 = \Phi_t\left(\frac{N^\varepsilon_{t+1}}{N^\varepsilon_t}\right) = \Phi_t\left(\frac{Y^{\varepsilon*}_{t+1} U^\varepsilon_t}{Y^{\varepsilon*}_t U^\varepsilon_{t+1}}\right) = \frac{U^\varepsilon_t}{U^\varepsilon_{t+1}} \frac{\Phi_t(Y^{\varepsilon*}_{t+1})}{Y^{\varepsilon*}_t} \text{ for } t \in \{0, ..., T-1\}.$$

In view of the Bellman principle this implies

$$\frac{U^\varepsilon_{t+1}}{U^\varepsilon_t} = \frac{\Phi_t(Y^{\varepsilon*}_{t+1})}{Y^{\varepsilon*}_t} \leq 1 \text{ for } t \in \{0, ..., T-1\}.$$

Hence U^ε has nonincreasing paths. Furthermore $Z^\varepsilon_t = \Phi_t(Z^\varepsilon_t)$ and so in particular, $Z^\varepsilon_t \leq Y^{\varepsilon*}_t$ due to the Bellman principle. Combining, we obtain for $t \in \{0, ..., T\}$,

$$\begin{aligned}
\Phi_t\left(\max_{t \leq j \leq T} \frac{Z^\varepsilon_j N^\varepsilon_T}{N^\varepsilon_j}\right) &\leq \Phi_t\left(\max_{t \leq j \leq T} \frac{Y^{\varepsilon*}_j N^\varepsilon_T}{N^\varepsilon_j}\right) = \Phi_t\left(\max_{t \leq j \leq T} \frac{Y^{\varepsilon*}_T U^\varepsilon_j}{U^\varepsilon_T}\right) \\
&\leq U^\varepsilon_t \Phi_t\left(\frac{Y^{\varepsilon*}_T}{U^\varepsilon_T}\right) = U^\varepsilon_t \Phi_t\left(Y^{\varepsilon*}_0 N^\varepsilon_T\right) = U^\varepsilon_t Y^{\varepsilon*}_0 \Phi_t\left(N^\varepsilon_T\right) = Y^{\varepsilon*}_t. \qquad (10)
\end{aligned}$$

Now let a common function g satisfy (1) in condition (C7) for all Z_j^ε, $j = 0, ..., T$. By regularity and condition (C7) it then holds

$$\begin{aligned} Y_t^{\varepsilon *} &:= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \sum_{j=t}^{T} 1_{[\tau=j]} \Phi_t(Z_\tau^\varepsilon) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \sum_{j=t}^{T} 1_{[\tau=j]} \Phi_t(Z_j^\varepsilon) \\ &\leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \sum_{j=t}^{T} 1_{[\tau=j]} \left(\Phi_t(Z_j) + g(\varepsilon)\right) = Y_t^* + g(\varepsilon). \end{aligned}$$

Hence with (10) we obtain

$$Y_t^* + g(\varepsilon) \geq \operatorname*{ess\,inf}_{N \in \mathcal{M}_{+1}^{\Phi}} \Phi_t\left(\max_{t \leq j \leq T} \frac{Z_j N_T}{N_j}\right) \overset{(i)}{\geq} Y_t^* \text{ for every } t \in \{0, ..., T\}.$$

The proof of (ii) is completed by sending $\varepsilon \to 0$.
Now let Z and $\delta > 0$ be as in Lemma 6.1 and take ε such that $0 < \varepsilon < \delta$. We so have $Z^\varepsilon = Z$ and then statement (iii) follows from statement (i) and using (10) in the proof of (ii) (which holds independently of condition (C7)). ■

Examples 6.4 Theorem 6.3 may be applied in the following situations.

1. Let $\mathcal{Q}$ denote the set of equivalent martingale measures w.r.t. some arbitrage-free financial market, and let $Z := (Z_t)_{t \in \{0,...,T\}}$ be a nonnegative adapted process of P −essentially bounded random variables. The process Z may be viewed as a discounted American Option with respect to the recursive conditional positive homogeneous DMU functional $\Phi_t(\cdot) := \operatorname*{ess\,sup}_{\mathrm{Q} \in \mathcal{Q}} E_{\mathrm{Q}}[\cdot \mid \mathcal{F}_t]$ on $L^\infty(\Omega, \mathcal{F}, \mathrm{P})$. Furthermore, let us denote by $\mathcal{X}_{+1}$ the set of $X \in L^\infty(\Omega, \mathcal{F}_T, \mathrm{P})$ with $X > 0$ P −a.s. such that $\sup_{\mathrm{Q} \in \mathcal{Q}} E_{\mathrm{Q}}[X] = 1$. Then the superhedging price and the lowest arbitrage-free price of Z may be represented by

   $$\inf_{X \in \mathcal{X}_{+1}} \Phi_0\left(\max_{t \in \{0,...,T\}} \frac{Z_t X}{\Phi_t(X)}\right) \quad \text{and} \quad \inf_{X \in \mathcal{X}_{+1}} \overline{\Phi}_0\left(\max_{t \in \{0,...,T\}} \frac{Z_t X}{\overline{\Phi}_t(X)}\right),$$

   respectively (see also Example 5.7).

2. Any family of g-expectations as in Example 2.7 with driver $g : \Omega \times [0, S] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $g(\omega, s, \cdot, \cdot)$ is homogeneous of degree one for each $(\omega, s) \in \Omega \times [0, S]$ (cf. [22]).

7 Consumption based representation

We now propose a representation for the optimal stopping problem (3) which can be seen as a generalization of the consumption upper bound approach in [4] and [5]. By this method one may infer from lower approximations upper approximations, and vice versa. In this context, we assume that Φ is a regular conditional translation invariant recursive DMU functional.

Using the Bellman principle we may proof the following theorem.

Theorem 7.1 *For any $Z \in \mathcal{H}$ we have*

$$Y_t^* := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \Phi_t(Z_\tau) = \Phi_t\left(Z_T + \sum_{j=t}^{T-1}(Z_j - \Phi_j(Y_{j+1}^*))^+\right), \quad t \in \{0, ..., T\},$$

with empty sums being defined zero.

Proof:
We shall proceed by backward induction over t. The case $t = T$ is trivial. So let us assume for any $t \in \{1, ..., T\}$ that $Y_t^* = \Phi_t\left(Z_T + \sum_{j=t}^{T-1}(Z_j - \Phi_j(Y_{j+1}^*))^+\right)$ is valid. Then due to Bellman principle $Y_{t-1}^* = (Z_{t-1} - \Phi_{t-1}(Y_t^*))^+ + \Phi_{t-1}(Y_t^*)$, which implies by assumption and recursiveness property (C4)

$$\begin{aligned}
Y_{t-1}^* &= (Z_{t-1} - \Phi_{t-1}(Y_t^*))^+ + \Phi_{t-1}\left(\Phi_t\left(Z_T + \sum_{j=t}^{T-1}(Z_j - \Phi_j(Y_{j+1}^*))^+\right)\right) \\
&= (Z_{t-1} - \Phi_{t-1}(Y_t^*))^+ + \Phi_{t-1}\left(Z_T + \sum_{j=t}^{T-1}(Z_j - \Phi_j(Y_{j+1}^*))^+\right)
\end{aligned}$$

Then the application of conditional translation invariance yields

$$\begin{aligned}
Y_{t-1}^* &= \Phi_{t-1}\left((Z_{t-1} - \Phi_{t-1}(Y_t^*))^+ + Z_T + \sum_{j=t}^{T-1}(Z_j - \Phi_j(Y_{j+1}^*))^+\right) \\
&= \Phi_{t-1}\left(Z_T + \sum_{j=t-1}^{T-1}(Z_j - \Phi_j(Y_{j+1}^*))^+\right).
\end{aligned}$$

■

The interesting feature of the representation in Theorem 7.1 is that if we replace Y^* on the right-hand-side by a lower (upper) approximation we obtain an upper (lower) bound for Y^* on the left-hand-side.

8 Numerical approaches for optimal stopping of some specific DMU functionals

In this section we sketch how the different representations developed in Sections 4-7 may be utilized for constructing (upper and/or lower) approximations of the of the optimal value of stopping problem (3). In order to enable a feasible algorithm or simulation procedure for optimal stopping of a particular DMU functional we naturally presume that we have a feasible algorithm or simulation procedure for the functional itself at hand. In this respect we underline that numerical (simulation) methods for specific DMU functionals is an interesting issue in it's own right but considered to be beyond the scope of this article. Another natural assumption is that we have some underlying process with some kind of Markovian structure which can be simulated straightforwardly. More specifically, we assume that we are in the following setting.

Setting for solving general optimal stopping problems by simulation

i) The filtration $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$ is generated by some underlying stochastic process $S := (S_t)_{t\in\{0,\dots,T\}}$ in some multi-dimensional state space, e.g. $\mathbb{R}^d$.

ii) The process $Z := (Z_t)_{t\in\{0,\dots,T\}}$ under consideration satisfies $Z_t = h(t, S_t)$ for some known nonnegative measurable function h. For ease of exposition, h is assumed to be bounded.

iii) The DMU functional $\Phi = (\Phi_t)_{t\in\{0,\dots,T\}}$ is regular, recursively generated by $(\Psi_t)_{t\in\{0,\dots,T\}}$ with generators satisfying $\Psi_t(X) = X$ if $X \in \mathcal{F}_t$, for any $t \in \{0, ..., T\}$. Hence in particular Φ is recursive with $\Phi_t(X) = X$ of $X \in \mathcal{F}_t$ for $t \in \{0, ..., T\}$.

iv) For any $t \in \{0, ..., T\}$ we have $\Phi_t(X)$ being $\sigma\{S_t\}-$measurable if X is $\sigma\{S_t, ...S_T\}-$measurable (We might think of S being Markovian w.r.t. the functional Φ). This condition is e.g. guaranteed in the case that for any $u, t \in \{0, ..., T\}$ with $u \le t$ we have $\Psi_t(X)$ is $\sigma\{S_u, ..., S_t\}-$measurable whenever X is $\sigma\{S_u, ..., S_{t+1}\}-$measurable.

v) For any $t \in \{0, ..., T\}$, we may compute $\Phi_t(X) \in \sigma\{S_t\}$ if $X \in \sigma\{S_t, ...S_T\}$ by some kind of simulation method.

In the standard case, where Φ represents the ordinary conditional expectation and S is Markovian in the ordinary sense, iii), iv), and v) are obviously fulfilled. A canonical way of evaluating conditional expectations is (Monte Carlo) simulation from a particular state (t, S_t) (particularly in higher dimensions). In general there are many interesting examples, for instance, within the class of $g-$expectations:

Example 8.1 Let Φ be a family of $g-$expectations as in example 2.7 with Brownian motion $B = (B_s)_{s\ge0}$ and driver $g : \Omega\times[0, S]\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ being of the form $g(\omega, s, y, z) := f(S_s, y, z)$. Here $f : \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is any Lipschitz function with $f(\cdot, 0) \equiv 0$, and $(S_s)_{s\ge0}$ is an $n-$dimensional diffusion process with dynamics given by the SDE,

$$dS_s = \mu(S_s)\ dt + \sigma(S_s)\ dB_s.$$

Under some further conditions of regularity for μ, σ and f, it may be verified that Φ satisfies assumption iv) (cf. [21], Theorem 6.2). Furthermore simulation algorithms as required in assumption v) are already available (see e.g. [19], [31]).
Moreover, if f does not depend on y, and is sublinear in z, then there is some set $\mathcal{Q}$ of probability measures which are absolutely continuous w.r.t. P such that Φ admits the following robust representation

$$\Phi_t(X) = \operatorname*{ess\,sup}_{\mathrm{Q}\in\mathcal{Q}} E_{\mathrm{Q}}\left[X \mid \sigma\{S_1, ..., S_t\}\right],$$

where the essential supremum is attained (see [9], proof of Theorem 3.1).

Below we will outline the implementation of the above simulation setting for different solution representations proposed in Sections 4-7.

Policy iteration

The policy iteration method in Section 4 may be readily applied if the time consistent stopping family (τ_t) we start with is such that $\{\tau_t = t\} \in \sigma\{S_t\}$. For example we just take the trivial family $\tau_t = t$, $t \in \{0, ..., T\}$. Then the iteration procedure will be analogue to the one spelled out in [26]. In short, given an input stopping family (τ_t), simulate a set of N (outer) trajectories $S^{(n)}$, $n = 1, ..., N$, from $t = 0$ to T. Determine on each outer trajectory $S^{(n)}$, the improved stopping time $\widehat{\tau}_0$. For this, one needs to simulate for each time $s = 0, 1, ..$ a set of M (inner) trajectories $({}_mS_u^{(n)})_{u=s,...,T}$, $m = 1, ..., M$ to check by simulation whether the event

$$\left\{\Phi_s(Z_s) \geq \max_{s+1 \leq u \leq T} \Phi_s(Z_{\tau_u})\right\}$$

in (6) is true. If $s^{(n)}$ is the first time where (5) is valid, we put $\widehat{\tau}_0^{(n)} = s^{(n)}$ on trajectory n. Finally we compute $\Phi_0(Z_{\widehat{\tau}_0})$ from the sample $Z_{\widehat{\tau}_0}^{(n)}$, $n = 1, ..., N$.

Dual upper bounds

We consider the construction of an additive dual upper bound for a regular, recursive DMU functional, which is translation invariant. Let us assume that we are given a proxy $Y_t = U(t, S_t)$ of the Snell envelope $Y_t^* = U^*(t, S_t)$. Note that the Snell envelope is indeed of this form due to assumptions i), ii), and iv). For instance, for the DMU functional in Example 3.4, a proxy may be constructed by approximating the Snell envelope with respect to a more simple functional, replacing the representing set $\mathcal{Q}$ of probability measures by a smaller subset or even a singleton. Let M^Y be the Doob Φ-martingale of Y and consider the upper bound

$$\begin{aligned}
Y_0^{up} &= \Phi_0\big(\max_{0 \leq t \leq T} \left(Z_t + M_T^Y - M_t^Y\right)\big) \\
&= \Phi_0\left(\max_{0 \leq t \leq T}\left(h(t, S_t) + \sum_{s=t}^{T-1}\left[U(s+1, S_{s+1}) - \Phi_s\left(U(s+1, S_{s+1})\right)\right]\right)\right).
\end{aligned}$$

Similar as in [1] we are going to construct an approximation of this upper bound by a nested simulation. We simulate N (outer) trajectories $S^{(n)}$, $n = 1, ..., N$, from $t = 0$ to T, and for each outer trajectory n, and time s, $s < T$, a set of M (inner) two step trajectories $({}_mS_u^{(n)})_{u=s,s+1}$, $m = 1, ..., M$. On a fixed outer trajectory $S^{(n)}$ we then construct for each s an approximation of $\Phi_s^{(n)}\left(U(s+1, S_{s+1})\right)$ by the inner sample $U(s+1, {}_1S_{s+1}^{(n)}), ..., U(s+1, {}_MS_{s+1}^{(n)})$, and next determine

$$\zeta^{(n)} := \max_{0 \leq t \leq T}\left(h(t, S_t^{(n)}) + \sum_{s=t}^{T-1}\left[U(s+1, S_{s+1}^{(n)}) - \Phi_s^{(n)}\left(U(s+1, S_{s+1})\right)\right]\right).$$

We thus end up with the sample $\zeta^{(1)}, ..., \zeta^{(N)}$ of the random variable $\zeta := \max_{0 \leq t \leq T}\left(Z_t + M_T^Y - M_t^Y\right)$, from which finally $Y_0^{up} = \Phi_0(\zeta)$ may be estimated.

Multiplicative and consumption upper bounds

From the simulation methods sketched above it will be clear in principle how to construct a multiplicative upper bound for a positively homogeneous DMU functional, and how to construct an upper (lower) bound due to the consumption representation in Theorem 7.1 for a translation invariant functional when a lower (upper) bound of the Snell envelope is given.

Concluding remark

In this article different representations for the optimal stopping problem with respect to general DMU functionals are presented. It is shown that these representations allow for a numerical treatment of the generalized stopping problem. A detailed analysis of the numerical algorithms sketched in Section8, which will depend on particular properties of the functional under consideration, remains to be done in future work.

A Appendix

Proof of Theorem 4.1:
The inequalities $Y_t \leq \widetilde{Y}_t$ and $\widehat{Y}_t \leq Y_t^*$ are obvious for any $t \in \{0, ..., T\}$. So inequality $\widetilde{Y}_t \leq \widehat{Y}_t$ is left to show. We shall use backward use induction.
Due to the definition of $\widetilde{Y}$ and $\widehat{Y}$, we have $\widetilde{Y}_T = \widehat{Y}_T = \Phi_T(Z_T)$. Suppose that $\widetilde{Y}_t \leq \widehat{Y}_t$ holds for any $t \in \{1, ..., T\}$. We then have to show that $\widetilde{Y}_{t-1} \leq \widehat{Y}_{t-1}$. For this we first show sequentially

(1) $1_{[\hat{\tau}_{t-1}=t-1]}\widehat{Y}_{t-1} = 1_{[\hat{\tau}_{t-1}=t-1]}\Phi_{t-1}(Z_{t-1})$.

(2) $1_{[\hat{\tau}_{t-1}>t-1]}\widehat{Y}_{t-1} \geq 1_{[\hat{\tau}_{t-1}>t-1]} \max\limits_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s})$.

(3) $\Phi_{t-1}(Z_{\tau_{t-1}}) \leq \max\left\{\Phi_{t-1}(Z_{t-1}), \max\limits_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s})\right\}$.

Due to the definition of $\hat{\tau}_{t-1}$ we have on the set $\{\hat{\tau}_{t-1} = t-1\}$, $\Phi_{t-1}(Z_{t-1}) \geq \max\limits_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s})$, and on the set $\{\hat{\tau}_{t-1} > t-1\}$, $\Phi_{t-1}(Z_{t-1}) < \max\limits_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s})$. Thus we may conclude immediately from (1)-(3)

$$\widehat{Y}_{t-1} \geq \max\left\{\Phi_{t-1}(Z_{t-1}), \max_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s})\right\} \geq \max\left\{\Phi_{t-1}(Z_{\tau_{t-1}}), \max_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s})\right\} = \widetilde{Y}_{t-1},$$

as required.

proof of (1):
By regularity condition (C2) we may find sequentially

$$1_{[\hat{\tau}_{t-1}=t-1]}\widehat{Y}_{t-1} = \Phi_{t-1}(1_{[\hat{\tau}_{t-1}=t-1]}Z_{\hat{\tau}_{t-1}}) = \Phi_{t-1}(1_{[\hat{\tau}_{t-1}=t-1]}Z_{t-1}) = 1_{[\hat{\tau}_{t-1}=t-1]}\Phi_{t-1}(Z_{t-1}),$$

which proves (1).

proof of (2):
$1_{[\hat{\tau}_{t-1}>t-1]}Z_{\hat{\tau}_{t-1}} = 1_{[\hat{\tau}_{t-1}>t-1]}Z_{\hat{\tau}_t}$ due to time consistency of $(\hat{\tau}_t)_{t\in\{0,...,T\}}$. Hence the regularity condition implies

$$\begin{aligned} 1_{[\hat{\tau}_{t-1}>t-1]}\widehat{Y}_{t-1} &= \Phi_{t-1}(1_{[\hat{\tau}_{t-1}>t-1]}Z_{\hat{\tau}_{t-1}}) = \Phi_{t-1}(1_{[\hat{\tau}_{t-1}>t-1]}Z_{\hat{\tau}_t}) = 1_{[\hat{\tau}_{t-1}>t-1]}\Phi_{t-1}(Z_{\hat{\tau}_t}) \\ &= 1_{[\hat{\tau}_{t-1}>t-1]}\Psi_{t-1}(\widehat{Y}_t). \end{aligned}$$

By the induction hypothesis we have $\widehat{Y}_t \geq \widetilde{Y}_t$, so we may conclude by monotonicity of Ψ_{t-1},

$$1_{[\hat{\tau}_{t-1}>t-1]}\widehat{Y}_{t-1} \geq 1_{[\hat{\tau}_{t-1}>t-1]}\Psi_{t-1}(\widetilde{Y}_t) \geq 1_{[\hat{\tau}_{t-1}>t-1]} \max_{t\leq s\leq T} \Psi_{t-1}(\Phi_t(Z_{\tau_s})) = 1_{[\hat{\tau}_{t-1}>t-1]} \max_{t\leq s\leq T} \Phi_{t-1}(Z_{\tau_s}).$$

Thus (2) is shown.

proof of (3):
Using regularity condition (C2) we obtain

$$\begin{aligned}\Phi_{t-1}(Z_{\tau_{t-1}}) &= 1_{[\tau_{t-1}=t-1]}\Phi_{t-1}(Z_{\tau_{t-1}}) + 1_{[\tau_{t-1}>t-1]}\Phi_{t-1}(Z_{\tau_{t-1}})\\ &= \Phi_{t-1}(1_{[\tau_{t-1}=t-1]}Z_{t-1}) + \Phi_{t-1}(1_{[\tau_{t-1}>t-1]}Z_{\tau_{t-1}}).\end{aligned}$$

$1_{[\tau_{t-1}>t-1]}Z_{\tau_{t-1}} = 1_{[\tau_{t-1}>t-1]}Z_{\tau_t}$ due to time consistency of $(\tau_t)_{t\in\{0,\ldots,T\}}$. Hence the application of regularity again yields

$$\begin{aligned}\Phi_{t-1}(Z_{\tau_{t-1}}) &= sdarticle\Phi_{t-1}(1_{[\tau_{t-1}=t-1]}Z_{t-1}) + \Phi_{t-1}(1_{[\tau_{t-1}>t-1]}Z_{\tau_t})\\ &= 1_{[\tau_{t-1}=t-1]}\Phi_{t-1}(Z_{t-1}) + 1_{[\tau_{t-1}>t-1]}\Phi_{t-1}(Z_{\tau_t}),\end{aligned}$$

obviously implying (3), and hence completing the proof. ■

Proof of Lemma 5.3:
We shall show the statement of Lemma 5.3 via backward induction. The case $t = T$ is trivial since $\mathcal{T}_T = \{T\}$. So let us assume that for any $t \in \{1, ..., T\}$, we have $\Phi_t(Z_\sigma) = \Phi_t(Z_\sigma + M_T - M_\sigma)$ for every $\sigma \in \mathcal{T}_t$. Let us fix an arbitrary $\tau \in \mathcal{T}_{t-1}$, and define $\sigma(\tau) := t1_{[\tau=t-1]} + \tau 1_{[\tau>t-1]} \in \mathcal{T}_t$. Then by assumption $\Phi_t(Z_{\sigma(\tau)}) = \Phi_t(Z_{\sigma(\tau)} + M_T - M_{\sigma(\tau)})$, which implies via regularity and recursiveness,

$$\begin{aligned}1_{[\tau>t-1]}\Phi_{t-1}(Z_\tau) &= 1_{[\tau>t-1]}\Phi_{t-1}(Z_{\sigma(\tau)}) = 1_{[\tau>t-1]}\Phi_{t-1}\circ\Phi_t(Z_{\sigma(\tau)})\\ &= 1_{[\tau>t-1]}\Phi_{t-1}\left(\Phi_t(Z_{\sigma(\tau)} + M_T - M_{\sigma(\tau)})\right) = 1_{[\tau>t-1]}\Phi_{t-1}(Z_{\sigma(\tau)} + M_T - M_{\sigma(\tau)})\\ &\overset{(C2)}{=} 1_{[\tau>t-1]}\Phi_{t-1}(Z_\tau + M_T - M_\tau).\end{aligned}$$

Moreover, by regularity, conditional translation invariance, and the $\Phi-$martingale property of M we have,

$$\begin{aligned}1_{[\tau=t-1]}\Phi_{t-1}(Z_\tau + M_T - M_\tau) &\overset{(C2)}{=} 1_{[\tau=t-1]}\Phi_{t-1}(Z_{t-1} + M_T - M_{t-1})\\ &\overset{(C3)}{=} 1_{[\tau=t-1]}\left(Z_{t-1} - M_{t-1} + \Phi_{t-1}(M_T)\right)\\ &= 1_{[\tau=t-1]}Z_{t-1} = 1_{[\tau=t-1]}\Phi_{t-1}(Z_{t-1}).\end{aligned}$$

which completes the proof. ■

Proof of Lemma 6.2:
We shall show the statement of the lemma by backward induction. The case $t = T$ is trivial since $\mathcal{T}_T = \{T\}$. Let us assume that for $t \in \{1, ..., T\}$ the equality $\Phi_t(Z_\tau) = \Phi_t\left(\frac{Z_\tau N_T}{N_\tau}\right)$ is valid for every $\tau \in \mathcal{T}_t$.
Consider an arbitrary $\tau \in \mathcal{T}_{t-1}$, and define $\sigma(\tau) := 1_{\tau=t-1}t + 1_{\tau>t-1}\tau \in \mathcal{T}_t$. By the induction assumption we have $\Phi_t\left(\frac{Z_{\sigma(\tau)}N_T}{N_{\sigma(\tau)}}\right) = \Phi_t(Z_{\sigma(\tau)})$, so that regularity condition (C2) and recursiveness imply

$$\begin{aligned}1_{[\tau>t-1]}\Phi_{t-1}\left(\frac{Z_\tau N_T}{N_\tau}\right) &\overset{(C2)}{=} 1_{[\tau>t-1]}\Phi_{t-1}\left(\frac{Z_{\sigma(\tau)}N_T}{N_{\sigma(\tau)}}\right) = 1_{[\tau>t-1]}\Phi_{t-1}\left(\Phi_t\left(\frac{Z_{\sigma(\tau)}N_T}{N_{\sigma(\tau)}}\right)\right)\\ &= 1_{[\tau>t-1]}\Phi_{t-1}\left(\Phi_t\left(Z_{\sigma(\tau)}\right)\right) = 1_{[\tau>t-1]}\Phi_{t-1}\left(Z_{\sigma(\tau)}\right) \overset{(C2)}{=} 1_{[\tau>t-1]}\Phi_{t-1}\left(Z_\tau\right).\end{aligned}$$

Moreover, by regularity (C2), conditional positive homogeneity (C6), and the fact that N is a $\Phi-$martingale, it holds

$$\begin{aligned} 1_{[\tau=t-1]}\Phi_{t-1}\left(\frac{Z_\tau N_T}{N_\tau}\right) &\overset{(C2)}{=} 1_{[\tau=t-1]}\Phi_{t-1}\left(\frac{Z_{t-1}N_T}{N_{t-1}}\right) \\ &\overset{(C6)}{=} 1_{[\tau=t-1]}\frac{Z_{t-1}}{N_{t-1}}\Phi_{t-1}(N_T) = 1_{[\tau=t-1]}Z_{t-1} = 1_{[\tau=t-1]}\Phi_{t-1}(Z_\tau). \end{aligned}$$

■

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