Approximation of spectral intervals and associated leading directions for linear differential-algebraic systems via smooth singular value decompositions *

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Abstract

This paper is devoted to the numerical approximation of Lyapunov and Sacker-Sell spectral intervals for linear differential-algebraic equations (DAEs). The spectral analysis for DAEs is improved and the concepts of leading directions and solution subspaces associated with spectral intervals are extended to DAEs. Numerical methods based on smooth singular value decompositions are introduced for computing all or only some spectral intervals and their associated leading directions. The numerical algorithms as well as implementation issues are discussed in detail and numerical examples are presented to illustrate the theoretical results.

Keywords: differential-algebraic equation, strangeness index, Lyapunov exponent, Bohl exponent, Sacker-Sell spectrum, exponential dichotomy, spectral interval, leading direction, smooth singular value decomposition.

AMS(MOS) subject classification: 65L07, 65L80, 34D08, 34D09

1 Introduction

In this paper we study the spectral analysis for linear differential-algebraic systems with variable coefficients (DAEs)

$$E(t)\dot{x} = A(t)x + f(t), \tag{1}$$

on the half-line $\mathbb{I} = [0, \infty)$, together with an initial condition $x(0) = x_0$. Here we assume that $E, A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$, and $f \in C(\mathbb{I}, \mathbb{R}^n)$ are sufficiently smooth. We use the notation $C(\mathbb{I}, \mathbb{R}^{n\times n})$ to denote the space of continuous functions from $\mathbb{I}$ to $\mathbb{R}^{n\times n}$. Linear systems of the form (1) arise when one linearizes a general implicit nonlinear system of DAEs

$$F(t, x, \dot{x}) = 0, \quad t \in \mathbb{I}, \tag{2}$$

along a particular solution [13].

DAEs are an important and convenient modeling concept in many different application areas, see [9, 26, 31, 32, 44] and the references therein. However, many numerical difficulties arise due to the fact that the dynamics is constrained to a manifold, which often is only given implicitly, see [32, 43, 44].

Similar to the situation of constant coefficient systems, where the spectral theory is based on eigenvalues and associated eigenvectors or invariant subspaces, in the variable coefficient case

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one is interested in the spectral intervals and associated leading directions, i.e., the initial vectors that lead to specific spectral intervals. We introduce these concepts for DAEs and develop numerical methods for computing this spectral information on the basis of smooth singular value decompositions associated with the homogenous version of (1).

The numerical approximation of Lyapunov exponents for ordinary differential equations (ODEs) has been investigated widely, see e.g. [3, 4, 6, 10, 15, 23, 24, 22, 29, 27] and the references therein. Recently, in [37, 38], the classical spectral theory for ODEs such as Lyapunov, Bohl and Sacker-Sell intervals, see [1] and the references therein, was extended to DAEs. It was shown that there are substantial differences in the theory and that most results for ODEs hold for DAEs only under further restrictions. In [37, 38] also the numerical methods (based on QR factorization) for computing spectral quantities of ODEs of [23, 25], were extended to DAEs.

In this paper, motivated by the results in [20, 21] for ODEs, we present a characterization for the leading directions and solution subspaces associated with the spectral intervals associated with (1). Using the approach of [38], we also discuss the extension of recent methods introduced in [20, 21] to DAEs. These methods compute the spectral intervals of ODEs and their associated leading directions via smooth singular value decompositions (SVDs). Under an integral separation condition, we show that these SVD based methods apply directly to DAEs. Most of the theoretical results as well as the numerical methods are direct generalizations of [20] but, furthermore, we also prove that the limit (as t tends to infinity) of the V-component in the smooth SVD of any fundamental solution provides not only a normal basis, but also an integrally separated fundamental solution matrix, see Theorem 38. This significantly improves Theorem 5.14 and Corollary 5.15 in [20].

The outline of the paper is as follows. In the following section, we revisit the spectral theory of differential-algebraic equations that was developed in [37]. In Section 3 we extend the concepts of leading directions and growth subspaces associated with spectral intervals to DAEs. In Section 4, we propose discrete and continuous SVD methods for approximating the spectral intervals and leading directions. Algorithmic details and comparisons of the methods are discussed as well. Finally, in Section 5 some numerical experiments are given to illustrate the theoretical results as well as the efficiency of the SVD methods.

2 Spectral theory for strangeness-free DAEs

2.1 Strangeness-free DAEs

General linear DAEs with variable coefficients have been studied in detail in the last twenty years, see [32] and the references therein. In order to understand the solution behavior and to obtain numerical solutions, the necessary information about derivatives of equations has to be used. This has led to the concept of the strangeness index, which under very mild assumptions allows to use the DAE and (some of) its derivatives to be reformulated as a system with the same solution that is strangeness-free, i.e., for which the algebraic and differential part of the system are easily separated.

In this paper for the discussion of spectral intervals, we restrict ourselves to *regular* DAEs, i.e., we require that (1) (or (2) locally) has a unique solution for sufficiently smooth E, A, f (F) and appropriately chosen (consistent) initial conditions, see again [32] for a discussion of existence and uniqueness of solution of more general nonregular DAEs.

With this theory and appropriate numerical methods available, then for regular DAEs we may assume that the homogeneous DAE in consideration is already strangeness-free and has the form

$$E(t)\dot{x} = A(t)x, \quad t \in \mathbb{I}, \tag{3}$$

where

$$E(t) = \begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}, \; A(t) = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix},$$

with $E_1 \in C(\mathbb{I}, \mathbb{R}^{d\times n})$ and $A_2 \in C(\mathbb{I}, \mathbb{R}^{(n-d)\times n})$, are such that the matrix function

$$\bar{E}(t) := \begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix} \tag{4}$$

is invertible for all t. As a direct consequence, then $E_1(t)$ and $A_2(t)$ are of full row-rank. For the numerical analysis, the solutions of (3) (and the coefficients E, A) are supposed to be sufficiently smooth so that the convergence results for the numerical methods [32] applied to (3) hold. It is then easy to see that an initial vector $x_0 \in \mathbb{R}^n$ is *consistent* for (3) if and only if $A_2(0)x_0 = 0$, i.e., if x_0 satisfies the algebraic equation.

The following lemma, which can be viewed as a generalized Schur form for matrix functions, is the key to the theory and numerical methods for the computation of spectral intervals for DAEs. It is a slight modification of [37, Lemma 7] using also different notation to avoid confusion with later sections.

Lemma 1 *Consider a strangeness-free DAE system of the form (3) with continuous coefficients E, A. Let $\hat{U} \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$ be an arbitrary orthonormal basis of the solution subspace of (3). Then there exists a matrix function $\hat{P} \in C(\mathbb{I}, \mathbb{R}^{n\times d})$ with pointwise orthonormal columns such that by the change of variables $x = \hat{U}z$ and multiplication of both sides of (3) from the left by $\hat{P}^T$, one obtains the system*

$$\mathcal{E}\dot{z} = \mathcal{A}z, \tag{5}$$

where $\mathcal{E} := \hat{P}^T E\hat{U}$, $\mathcal{A} := \hat{P}^T A\hat{U} - \hat{P}^T E\dot{\hat{U}}$ and $\mathcal{E}$ is upper triangular.

Proof. Considering an arbitrary solution x and substituting $x = \hat{U}z$ into equation (3), we obtain

$$E\hat{U}\dot{z} = (A\hat{U} - E\dot{\hat{U}})z. \tag{6}$$

Since (3) is strangeness-free, and since $A_2\hat{U} = 0$, we have that the matrix $E\hat{U}$ must have full column-rank. Thus, see [19], there exists a smooth QR-decomposition

$$E\hat{U} = \hat{P}\mathcal{E},$$

where the columns of $\hat{P}$ form an orthonormal set and $\mathcal{E}$ is nonsingular and upper triangular. This decomposition is unique if the diagonal elements of $\mathcal{E}$ are chosen positive. Multiplying both sides of (6) by $\hat{P}^T$, we arrive at

$$\mathcal{E}\dot{z} = [\hat{P}^T A\hat{U} - \hat{P}^T E\dot{\hat{U}}]z.$$

Finally, setting $\mathcal{A} := \hat{P}^T A\hat{U} - \hat{P}^T E\dot{\hat{U}}$ completes the proof. □

System (5) is an implicitly given ODE, since $\mathcal{E}$ is nonsingular. It is called *essentially underlying implicit ODE system* (EUODE) of (3) and it can be made explicit by multiplying with $\mathcal{E}^{-1}$ from the left, see also [2] for constructing EUODEs of so-called properly-stated DAEs. In our numerical methods we will need to construct the coefficients of the EUODE pointwise. Note, however, that in (6) for a fixed given $\hat{U}$, the matrix function $\hat{P}$ is not unique. In fact, any $\hat{P}$ for which $\hat{P}^T E\hat{U}$ is invertible yields an implicit EUODE. However, obviously $\mathcal{E}^{-1}\mathcal{A}$ is unique, i.e., with a given basis, the explicit EUODE provided by Lemma 1 is unique. In the numerical methods, however, we need to choose the matrix function $\hat{P}$ appropriately.

For the theoretical analysis we will heavily use the fact that for a given basis $\hat{U}$, the correspondence between the solutions of (3) and those of (5) is one-to-one, i.e., x is a solution of (3) if and only if $z = \hat{U}^T x$ is a solution of (5). Different special choices of the basis $\hat{U}$ will, however, lead to different methods for approximating Lyapunov exponents. Note that $\hat{U}\hat{U}^T$ is just a projection onto the solution subspace of (3), hence $z = \hat{U}^T x$ implies $\hat{U}z = \hat{U}\hat{U}^T x = x$.

2.2 Lyapunov exponents and Lyapunov spectral intervals

In the following we briefly recall the basic concepts of the spectral theory for DAEs, see [37] for details.

Definition 2 *A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times k})$, $d \le k \le n$, is called* fundamental solution matrix of the strangeness-free DAE (3) *if each of its columns is a solution to (3) and* $\operatorname{rank} X(t) = d$*, for all $t \ge 0$. A fundamental solution matrix is said to be* minimal *if $k = d$.*

One may construct a minimal fundamental matrix solution by solving initial value problems for (3) with d linearly independent, consistent initial vectors. For example, let $Q_0 \in \mathbb{R}^{n\times n}$ be a nonsingular matrix such that $A_2(0)Q_0 = \begin{bmatrix} 0 & \tilde{A}_{22} \end{bmatrix}$, where $\tilde{A}_{22} \in \mathbb{R}^{a\times a}$ is a nonsingular matrix. Then, the d columns of the matrix

$$X_0 = Q_0 \begin{bmatrix} I_d \\ 0 \end{bmatrix} \tag{7}$$

form a set of linearly independent and consistent initial vectors for (3), see [33].

Definition 3 *Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a non-vanishing function. The quantities*

$$\chi^u(f) = \limsup_{t\to\infty} \frac{1}{t} \ln|f(t)|, \quad \chi^\ell(f) = \liminf_{t\to\infty} \frac{1}{t} \ln|f(t)|, \tag{8}$$

are called upper and lower, Lyapunov exponents of f, respectively*. In a similar way we define upper and lower Lyapunov exponents for vector valued functions, where the absolute values are replaced by norms.*

For a constant $c \neq 0$ and non-vanishing functions $f_1, \ldots, f_j$ Lyapunov exponents satisfy

$$\chi^u(cf_1) = c\chi^u(f_1), \quad \chi^\ell(cf_1) = c\chi^\ell(f_1) \tag{9}$$

and

$$\chi^u\left(\sum_{i=1}^{n} f_i\right) \le \max_{i=1,\ldots n} \chi^u(f_i), \tag{10}$$

where equality holds if the maximal Lyapunov exponent is attained by only one function.

Definition 4 *For a given fundamental solution matrix X of a strangeness-free DAE system of the form (3), and for $1 \le i \le d$, we introduce*

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln\|X(t)e_i\| \;\; \text{and} \;\; \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln\|X(t)e_i\|,$$

where e_i denotes the i-th unit vector and $\|\cdot\|$ denotes the Euclidean norm. The columns of a minimal fundamental solution matrix form a normal basis *if $\Sigma_{i=1}^d \lambda_i^u$ is minimal. The λ_i^u, $i = 1, 2, ..., d$ belonging to a normal basis are called* (upper) Lyapunov exponents *and the intervals $[\lambda_i^\ell, \lambda_i^u]$, $i = 1, 2, ..., d$, are called* Lyapunov spectral intervals*. The set of the Lyapunov spectral intervals is called the* Lyapunov spectrum *of (3).*

Similar as in the case of ODEs, a normal basis for (3) exists and it can be constructed from any (minimal) fundamental matrix solution.

Definition 5 *Suppose that $P \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $Q \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ are nonsingular matrix functions such that Q and Q^{-1} are bounded. Then the transformed DAE system*

$$\tilde{E}(t)\dot{\tilde{x}} = \tilde{A}(t)\tilde{x},$$

with $\tilde{E} = PEQ$, $\tilde{A} = PAQ - PE\dot{Q}$ and $x = Q\tilde{x}$ is called globally kinematically equivalent *to (3) and the transformation is called a* global kinematic equivalence transformation*. If $P \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ and, furthermore, also P and P^{-1} are bounded then we call this a* strong global kinematic equivalence transformation*.*

The Lyapunov exponents of a DAE system as well as the normality of a basis formed by the columns of a fundamental solution matrix are preserved under global kinematic equivalence transformations.

Proposition 6 *For any given minimal fundamental matrix X of (3), for which the Lyapunov exponents of the columns are ordered decreasingly, there exist a constant, nonsingular, and upper triangular matrix $C \in \mathbb{R}^{d\times d}$ such that the columns of XC form a normal basis for (3).*

Proof. Since orthonormal changes of basis keep the Euclidean norm invariant, the spectral analysis of (3) can be done via its EUODE. Thus, let Z be the corresponding fundamental matrix of (5), $X = \hat{U}Z$. Due to the existence result of a normal basis for ODEs [39] (see also [1, 23]), there exists a matrix C with the properties listed in the assertion such that ZC is a normal basis for (5). Thus $XC = \hat{U}ZC$ is a normal basis for (3). □

The fundamental solutions X and Z satisfy the following relation.

Theorem 7 *[37] Let X be a normal basis for (3). Then the Lyapunov spectrum of the DAE (3) and that of the ODE (5) are the same. If $\mathcal{E}, \mathcal{A}$ are as in (5) and if $\mathcal{E}^{-1}\mathcal{A}$ is bounded, then all the Lyapunov exponents of (3) are finite. Furthermore, the spectrum of (5) does not depend on the choice of the basis $\hat{U}$ and the matrix function $\hat{P}$.*

Similar to the regularity concept for DAEs introduced in [17], we have the following definition.

Definition 8 *The DAE system (3) is said to be* Lyapunov-regular *if its EUODE (5) is Lyapunov-regular, i.e., if*

$$\sum_{i}^{d} \lambda_i^u = \liminf_{t\to\infty} \frac{1}{t} \ln \|\det Z(t)\|,$$

where $Z(t)$ is a fundamental matrix solution of (5).

The Lyapunov-regularity of a strangeness-free DAE system (3) is well-defined, since it does not depend on the construction of (5), i.e., the choice of the pair $\hat{U}, \hat{P}$. Furthermore, the Lyapunov-regularity of (3) implies that for any nontrivial solution x, the limit $\lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|$ exists. Hence, we have $\lambda_i^l = \lambda_i^u$, i.e., the Lyapunov spectrum of (3) is a point spectrum.

In the following we consider the adjoint equation of (3), given by

$$E^T \dot{y} = -(A + \dot{E})^T y, \tag{11}$$

see e.g., [14, 34], and also a slightly different formulation in [2]. The following statement shows a relation between EUODEs of (3) and (11).

Proposition 9 *Let the orthonormal columns of the matrix $\hat{U}$ form a basis of the solution subspace of (3). Then there exists $\hat{P} \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$ such that the columns of $\hat{P}$ form an orthonormal basis for the solution subspace of (11). Furthermore, by the change of variables $y = \hat{P}w$ and multiplication of both sides of (11) by $\hat{U}^T$, we obtain the EUODE for the adjoint system (11), given by*

$$\mathcal{E}^T \dot{w} = -(\mathcal{A} + \dot{\mathcal{E}})^T w, \tag{12}$$

which is exactly the adjoint of (5). If $\hat{U}$ is such the matrix $\mathcal{E}$ is upper triangular with positive diagonal elements, then the corresponding $\hat{P}$ is unique.

Proof. We first prove uniqueness. Suppose that there exist matrix functions $\tilde{P}$ and $\hat{P}$ with orthonormal columns, such that $\tilde{\mathcal{E}}\dot{z} = \tilde{\mathcal{A}}z$ and $\hat{\mathcal{E}}\dot{z} = \hat{\mathcal{A}}z$, respectively, where both $\tilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$ are upper triangular with positive diagonal elements. Since the columns of $\tilde{P}$ and $\hat{P}$ form bases of the same subspace, there exists a matrix function $S \in C(\mathbb{I}, \mathbb{R}^{n\times d})$, such that $\hat{P} = \tilde{P}S$. We have $S^T S = \hat{P}^T(\tilde{P}\tilde{P}^T\hat{P}) = \hat{P}^T\hat{P} = I_d$ (because $\tilde{P}\tilde{P}^T$ is a projection onto the solution subspace, i.e., range $\hat{P}$), and thus S is orthogonal. On the other hand, by the construction of the EUODE, we

have $S^T\hat{\mathcal{E}} = \tilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$ is invertible, which implies that S^T is upper triangular. Hence, S is a diagonal matrix with diagonal elements $+1$ or -1. But since $\tilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$ have positive diagonals, then $S = I$.

For the existence we give a constructive proof. From the proof of Lemma 1, $\hat{P}$ is determined via $E\hat{U} = \hat{P}\hat{\mathcal{E}}$. Due to the special form of E, we can first determine an auxiliary $\tilde{P}$ such that $E\hat{U} = \begin{bmatrix} E_1\hat{U} \\ 0 \end{bmatrix} = \tilde{P}\hat{\mathcal{E}}$, which implies that $\tilde{P} = \begin{bmatrix} \tilde{P}_1 \\ 0 \end{bmatrix}$. Here $\tilde{P}_1$ and $\hat{\mathcal{E}}$ are determined by, e.g., a smooth QR decomposition of $E_1\hat{U}$. Unfortunately, in general such a $\tilde{P}$ is not a basis of the solution subspace of (11) yet. We observe that we can replace the zero block by any $\tilde{P}_2$ and the relation $\tilde{P}^T E\hat{U} = \mathcal{E}$ still holds. Hence for the construction, we look for an appropriate $\tilde{P}_2$ block so that $\tilde{P} = \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix}$ satisfies the algebraic constraint of (11). If we then orthonormalize the columns of $\tilde{P}$, we are done. The adjoint DAE (11) has the form

$$\begin{bmatrix} E_{11}^T & 0 \\ E_{12}^T & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -(A_{11} + \dot{E}_{11})^T & -A_{21}^T \\ -(A_{12} + \dot{E}_{12})^T & -A_{22}^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and, since the adjoint of (3) is again strangeness-free, see [34], we can reorder the equations so that the left upper $d \times d$-block of the coefficient matrix on the left-hand side is nonsingular and then eliminate the left lower block. Then, we obtain an equivalent DAE of the form

$$\begin{bmatrix} \tilde{E}_{11}^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -\tilde{A}_{11}^T & -\tilde{A}_{21}^T \\ -\tilde{A}_{12}^T & -\tilde{A}_{22}^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

In this system, the algebraic constraint is given explicitly by $0 = \tilde{A}_{12}^T y_1 + \tilde{A}_{22}^T y_2$, where $\tilde{A}_{22}^T$ is nonsingular. Hence, let $\tilde{P}_2 = -\tilde{A}_{22}^{-T}\tilde{A}_{12}^T\tilde{P}_1$. Finally, applying a Gram-Schmidt orthogonalization to $\tilde{P}$, we obtain a basis of (11), denoted by $\hat{P}$, which also fulfils $\hat{P}^T E\hat{U} = \mathcal{E}$, where $\mathcal{E}$ is upper triangular. One easily verifies that the obtained EUODE of the adjoint equation (11) is exactly the adjoint of EUODE (5) of the original DAE (3). □

With these preparations we obtain a generalization of [37, Theorem 19] on the relation between the Lyapunov exponents of (3) and those of (11).

Theorem 10 *Suppose that the matrix function $\mathcal{E} = \hat{P}^T E\hat{U}$ and its inverse are bounded on $\mathbb{I}$, where the columns of $\hat{U}, \hat{P}$ form bases of the solution spaces in Proposition 9. If λ_i^l are the lower Lyapunov exponents of (3) and $-\mu_i^u$ are the upper Lyapunov exponents of the adjoint system (11), both in increasing order, then $\lambda_i^l = \mu_i^u, \quad i = 1, 2, ..., d$, Furthermore, system (3) is Lyapunov regular if and only if (11) is Lyapunov regular, and in this case we have the* Perron identity

$$\lambda_i = \mu_i, \quad i = 1, 2, ..., d. \tag{13}$$

Proof. Due to Proposition 9, it suffices to consider two implicit EUODEs which are adjoint of each other. The assertion then follows by the same argument as that in the proof of [37, Theorem 19], which is based on the Lagrange identity $W^T(t)\mathcal{E}(t)Z(t) = W^T(0)\mathcal{E}(0)Z(0)$, where Z and W are fundamental solutions of EUODE (5) and its adjoint (12), respectively. □

Since we have used the assumption that $\mathcal{E}$ and its inverse are bounded, it would be more practical to check this property in terms of the original data. This is easily possible for the class of so-called semi-implicit DAEs (i.e., those with $E_{12} = 0$), see [37, Theorem 19].

We stress that unlike the approach in [17], where certain inherent ODEs of the same size as the original DAE are used, our spectral analysis is based on the essentially underlying ODEs, which have reduced size and can be constructed numerically.

2.3 Stability of Lyapunov exponents

Lyapunov exponents may be very sensitive to small changes in the system. In order to study this sensitivity for DAEs, we consider the specially perturbed system

$$[E(t)+F(t)]\dot{x} = [A(t)+H(t)]x, \quad t \in \mathbb{I}, \tag{14}$$

where

$$F(t) = \begin{bmatrix} F_1(t) \\ 0 \end{bmatrix}, \; H(t) = \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix},$$

and where F_1 and H_1, H_2 are assumed to have the same order of smoothness as E_1 and A_1, A_2, respectively. Perturbations of this structure are called *admissible*, which is the generalization to the variable coefficient case of the concepts for the constant coefficient DAEs studied in [12]. The DAE (3) is said to be *robustly strangeness-free* if it stays strangeness-free under all sufficiently small admissible perturbations. It is easy to see that the DAE (3) is robustly strangeness-free under admissible perturbations if and only if the matrix function $\bar{E}$ defined by (4) is boundedly invertible. It should be noted that it is *an open problem* even in the constant coefficient case how to analyze the behavior of DAEs under non-admissible perturbations.

In the following we restrict ourselves to robustly strangeness-free DAE systems under admissible perturbations.

Definition 11 *The upper Lyapunov exponents $\lambda_1^u \geq ... \geq \lambda_d^u$ of (3) are said to be* stable *if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \|F(t)\| < \delta$, $\sup_t \|H(t)\| < \delta$, and $\sup_t \left\|\dot{H}_2(t)\right\| < \delta$ on the perturbations imply that the perturbed DAE system (14) is strangeness-free and*

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \quad \text{for all } i = 1, 2, ..., d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of the perturbed system (14).

It is clear that the stability of upper Lyapunov exponents is invariant under strong global kinematic equivalence transformations. Compared with the ODE case, the boundedness condition on $\dot{H}_2$ is an extra condition. The following simple example shows its necessity.

Example 12 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1, \\ 0 &= x_2. \end{aligned}$$

It is easy to see that this trivial DAE is robustly strangeness-free, Lyapunov-regular, and it has only one Lyapunov exponent $\lambda = 1$. Now, consider the perturbed DAE

$$\begin{aligned} (1+\varepsilon^2 \sin 2mt)\,\dot{x}_1 - \varepsilon \cos mt\, \dot{x}_2 &= x_1, \\ 0 &= -2\varepsilon \sin mt\, x_1 + x_2, \end{aligned} \tag{15}$$

where ε is a small perturbation parameter and m is a given integer. From the second equation of (15), we obtain $x_2 = 2\varepsilon \sin mt\, x_1$. Differentiating this expression for x_2 and inserting the result into the first equation, after some elementary calculations, we obtain

$$\dot{x}_1 = (1 + m\varepsilon^2 + m\varepsilon^2 \cos 2mt)\, x_1.$$

Explicit integration yields $x_1(t) = e^{(1+m\varepsilon^2)t + \varepsilon^2 \sin 2mt/2}$, from which the only Lyapunov exponent $\hat{\lambda} = 1 + m\varepsilon^2$ is calculated. Clearly, even if ε, and hence the perturbation in the coefficient matrices is small, the difference between two Lyapunov exponents may be arbitrarily large by choosing sufficiently large m.

Another important concept is that of integral separation for DAEs.

Definition 13 *A minimal fundamental solution matrix X for (3) is called* integrally separated *if for $i = 1, 2, ..., d-1$ there exist constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq c_2 e^{c_1(t-s)},$$

for all t, s with $t \geq s \geq 0$.

If a DAE system has an integrally separated minimal fundamental solution matrix, then we say that it has the integral separation property*.*

The integral separation property is invariant under strong global kinematic equivalence transformations. Furthermore, if a fundamental solution X of (3) is integrally separated, then so is the corresponding fundamental solution Z of (5) and vice versa.

Note that by using a global kinematic equivalence transformation, see [37, Remark 13], (3) can be transformed to a special structured form, where the block A_{21} becomes zero, i.e., we may assume that (3) has the form

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{16}$$

The advantage of this form is that then the associated EUODE is

$$E_{11}\dot{x}_1 = A_{11}x_1.$$

Therefore, for the perturbation analysis, in the following we may assume that (3) is already given with $A_{21} = 0$, i.e., the perturbed DAE has the form

$$\begin{aligned} (E_{11} + F_{11})\dot{\tilde{x}}_1 + (E_{12} + F_{12})\dot{\tilde{x}}_2 &= (A_{11} + H_{11})\tilde{x}_1 + (A_{12} + H_{12})\tilde{x}_2, \\ 0 &= H_{21}\tilde{x}_1 + (A_{22} + H_{22})\tilde{x}_2. \end{aligned} \tag{17}$$

We then have the following sufficient conditions for the upper Lyapunov exponents of (3) to be stable.

Theorem 14 *Consider a strangeness-free DAE (3) in the form (16). Suppose that the matrix $\bar{E}$ in (4) is boundedly invertible and that $E_{11}^{-1}A_{11}$, $A_{12}A_{22}^{-1}$ and $\dot{A}_{22}$ are bounded on $\mathbb{I}$. Then, the upper Lyapunov exponents of (3) are distinct and stable if and only if the system has the integral separation property.*

Proof. First of all, it is easy to see that the inverse of $\bar{E}$ is bounded if and only if E_{11}^{-1}, A_{22}^{-1}, and $E_{11}^{-1}E_{12}A_{22}^{-1}$ are bounded. In the algebraic equation of (17), for sufficiently small H_{22}, we can solve for x_2 and obtain $x_2 = -(A_{22} + H_{22})^{-1}H_{21}$. Differentiating this expression and inserting the result into the first equation, we obtain the perturbed EUODE

$$(E_{11} + \tilde{F}_{11})\dot{\tilde{x}}_1 = (A_{11} + \tilde{H}_{11})\tilde{x}_1,$$

where

$$\begin{aligned} \tilde{F}_{11} &= F_{11} - (E_{12} + F_{12})(A_{22} + H_{22})^{-1}H_{12}, \\ \tilde{H}_{11} &= H_{11} + (E_{12} + F_{12})\frac{d}{dt}[(A_{22} + H_{22})^{-1}H_{21}] - (A_{12} + H_{12})(A_{22} + H_{22})^{-1}H_{21}. \end{aligned}$$

The last equation is equivalent to

$$(I + E_{11}^{-1}\tilde{F}_{11})\dot{\tilde{x}}_1 = (E_{11}^{-1}A_{11} + E_{11}^{-1}\tilde{H}_{11})\tilde{x}_1.$$

Under the given boundedness assumptions (see Definition 11), it is not difficult to check that the terms $E_{11}^{-1}\tilde{F}_{11}$ and $E_{11}^{-1}\tilde{H}_{11}$ are arbitrary small in norm, provided that the norms of the original perturbation F, H are sufficiently small and $\frac{d}{dt}H_{22}$ is bounded. Using the well-known result on the stability of Lyapunov exponents for ODEs, see [1] or [23], the assertion follows. □

Remark 15 Example 12 and Theorem 14 demonstrate that, unlike in the perturbation analysis of time-invariant DAEs [12], that of time-varying DAEs requires more restrictive conditions. However, for some classes of structured problems, see [16] and [37, Section 3.2], parts of these conditions can be relaxed.

The integral separation of a fundamental matrix solution can be equivalently expressed in term of integral separation of a sequence of functions. Two continuous and bounded functions f_1 and f_2 are said to be *integrally separated* if there exist $c_1 > 0$ and $c_2 \geq 0$ such that

$$\int_s^t (f_1(r) - f_2(r))\, dr \geq c_1(t-s) - c_2, \quad \forall t > s \geq 0.$$

In practice, the integral separation of two functions can be tested via their Steklov difference. Defining *Steklov averages* by

$$f_i^H(t) := \frac{1}{H}\int_t^{t+H} f_i(r)dr, \quad (i = 1, 2),$$

it follows from a result of [1] that the functions f_1, f_2 are integrally separated if and only if there exists scalar $H > 0$ such that their *Steklov difference* is positive, i.e., for H sufficiently large, there exists a constant $c_3 > 0$ such that

$$f_1^H(t) - f_2^H(t) \geq c_3 > 0, \qquad \text{for all } t \geq 0.$$

For further discussions on integral separation and its importance in the context of approximating Lyapunov exponents, see [23, 24, 25].

2.4 Bohl exponents and Sacker-Sell spectrum

Further concepts that are important to describe the qualitative behavior of solutions to ordinary differential equations are the exponential-dichotomy or Sacker-Sell spectra [45] and the Bohl exponents [7], see also [18]. The extension of these concepts to DAEs has been presented in [37].

Definition 16 *Let x be a nontrivial solution of (3). The* (upper) Bohl exponent $\kappa_B^u(x)$ *of this solution is the greatest lower bound of all those values ρ for which there exist constants $N_\rho > 0$ such that*

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|$$

for any $t \geq s \geq 0$. If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$. Similarly, the lower Bohl exponent $\kappa_B^\ell(x)$ *is the least upper bound of all those values ρ' for which there exist constants $N'_\rho > 0$ such that*

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

Lyapunov exponents and Bohl exponents are related via

$$\kappa_B^\ell(x) \leq \lambda^\ell(x) \leq \lambda^u(x) \leq \kappa_B^u(x).$$

Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$ and the formulas characterizing Bohl exponents for ODEs, see e.g. [18], immediately extend to DAEs, i.e.

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \quad \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}.$$

Moreover, under the boundedness conditions of Theorem 14, the Bohl exponents are stable with respect to admissible perturbations without the integral separation assumption, see [16, 37].

Definition 17 *The DAE (3) is said to have* exponential dichotomy *if for any minimal fundamental solution X there exist a projection $\Pi \in \mathbb{R}^{d\times d}$ and positive constants K and α such that*

$$\begin{array}{rcll} \|X(t)\Pi X^+(s)\| & \leq & Ke^{-\alpha(t-s)}, & t \geq s, \\ \|X(t)(I_d-\Pi)X^+(s)\| & \leq & Ke^{\alpha(t-s)}, & s > t, \end{array} \tag{18}$$

where X^+ denotes the generalized Moore-Penrose inverse of X.

Let X be a fundamental solution matrix of (3) and let the columns of $\hat{U}$ form an orthonormal basis of the solution subspace, then we have $X = \hat{U}Z$, where Z is the fundamental solution matrix of the corresponding EUODE (5) and hence invertible. Observing that $X^+ = Z^{-1}\hat{U}^T$, we have

$$\|X(t)\Pi X^+(s)\| = \|Z(t)\Pi Z^{-1}\|, \text{ and } \|X(t)(I_d-\Pi)X^+(s)\| = \|Z(t)(I_d-\Pi)Z^{-1}\|.$$

Thus, the following statement is obvious.

Proposition 18 *The DAE (3) has exponential dichotomy if and only if its corresponding EUODE (5) has exponential dichotomy.*

Furthermore, as it has been remarked in [20, 23], the projector Π can be chosen to be orthogonal, i.e., $\Pi = \Pi^T$. The projector Π projects to a subspace of the complete solution subspace in which all the solutions are uniformly exponentially decreasing, while the solutions belonging to the complementary subspace are uniformly exponentially increasing.

In order to extend the concept of exponential dichotomy spectrum to DAEs, we need *shifted DAE systems*

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \in \mathbb{I}, \tag{19}$$

where $\lambda \in \mathbb{R}$. By using the transformation as in Lemma 1, we obtain the corresponding shifted EUODE for (19)

$$\mathcal{E}\dot{z} = (\mathcal{A} - \lambda\mathcal{E})z.$$

Definition 19 *The* Sacker-Sell (or exponential dichotomy) spectrum *of the DAE system (3) is defined by*

$$\Sigma_S := \{\lambda \in \mathbb{R}, \text{ the shifted DAE (19) does not have an exponential dichotomy}\}. \tag{20}$$

The complement of Σ_S is called the resolvent set *for the DAE system (3), denoted by $\rho(E, A)$.*

Then, using Proposition 18 and the result for the ODE case [45], we have the following result.

Theorem 20 *[37] The Sacker-Sell spectrum of (3) is exactly the Sacker-Sell spectrum of its EUODE (5). Furthermore, the Sacker-Sell spectrum of (3) consists of at most d closed intervals.*

Using the same arguments as in [37, Section 3.4], one can show that under the boundedness conditions of Theorem 14, unlike the Lyapunov spectrum, the Sacker-Sell spectrum of the DAE (3) is stable with respect to admissible perturbations. Theorem 50 in [37] also states that if X is an integrally separated fundamental matrix of (3), then the Sacker-Sell spectrum of the system is exactly given by the d (not necessarily disjoint) Bohl intervals associated with the columns of X. In the remainder of the paper, we assume that Σ_S consists of $p \leq d$ pairwise disjoint spectral intervals, i.e., $\Sigma_S = \cup_{i=1}^{p}[a_i, b_i]$, and $b_i < a_{i+1}$ for all $1 \leq i \leq p$. This assumption can be easily achieved by combining possibly overlapping spectral intervals to larger intervals.

3 Leading directions and subspaces

As we have noted before, initial vectors of (3) must be chosen consistent and they form a d-dimensional subspace in $\mathbb{R}^n$. Furthermore, the solutions of (3) also form a d-dimensional subspace of functions in $C^1(\mathbb{I}, \mathbb{R}^n)$. Let us denote these spaces by $\mathbb{S}_0$ and $\mathbb{S}(t)$, respectively. Furthermore, for $x_0 \in \mathbb{S}_0$ let us denote by $x(t; x_0)$ the (unique) solution of (3) that satisfies $x(0; x_0) = x_0$. In

order to obtain geometrical information about the subspaces of solutions which have a specific growth, we extend the analysis for ODEs given in [20] to DAEs.

For $j = 1, \ldots d$, define the set W_j of all consistent initial conditions w such that the upper Lyapunov exponent of the solution $x(t; w)$ of (3) satisfies $\chi^u(x(\cdot; w)) \leq \lambda_j^u$, i.e.,

$$W_j = \left\{ w \in \mathbb{S}_0 : \chi^u \left(x(\cdot; w) \right) \leq \lambda_j^u \right\}, \quad j = 1, \ldots, d.$$

Let the columns of $\hat{U}(\cdot)$ form a smoothly varying basis of the solution subspace $\mathbb{S}(\cdot)$ of (3) and consider an associated EUODE (5). Then we can consider (5) and, instead of W_j, the corresponding set of all initial conditions for (5) that lead to Lyapunov exponents not greater than λ_j^u. In this way it is obvious that all for ODEs in [20] (Propositions 2.8–2.10) apply to EUODEs of the form (5) and, as a consequence of Theorem 7, we obtain several analogous statements for (3). First, we state a result on the subspaces W_j.

Proposition 21 *Let d_j be the largest number of linearly independent solutions x of (3) such that $\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t)\| = \lambda_j^u$. Then W_j is a d_j dimensional linear subspace of $\mathbb{S}_0$. Furthermore, the spaces W_j, $j = 1, 2, \ldots$, form a* filtration *of $\mathbb{S}_0$, i.e., if p is the number of distinct upper Lyapunov exponents of the system, then we have*

$$\mathbb{S}_0 = W_1 \supset W_2 \supset \ldots \supset W_p \supset W_{p+1} = \{0\}.$$

It follows that $\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t; w)\| = \lambda_j^u$ if and only if $w \in W_j \backslash W_{j+1}$. Moreover, if we have d distinct upper Lyapunov exponents, then the dimension of W_j is $d - j + 1$.

If Y_j is defined as the orthogonal complement of W_{j+1} in W_j, i.e.,

$$W_j = W_{j+1} \oplus Y_j, \quad Y_j \perp W_{j+1},$$

then $\mathbb{S}_0 = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_p$, and

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t; w)\| = \lambda_j^u \text{ if and only if } w \in Y_j.$$

It follows that if we have $p = d$ distinct Lyapunov exponents, then $\dim(Y_j) = 1$ for all $j = 1, \ldots, d$. In the next section, similar to [20, 21], we will approximate the spaces Y_j by using smooth singular value decompositions (SVDs), see [11, 19], of fundamental solutions.

If the DAE system (3) is integrally separated, then it can be shown that the sets W_j, Y_j can be also used to characterize the set of initial solutions leading to lower Lyapunov exponents, see [20, Proposition 2.10] for details.

Consider now the resolvent set $\rho(E, A)$. For $\mu \in \rho(E, A)$, let us define the following *stable and unstable* sets, respectively.

$$\begin{aligned} \mathcal{S}_\mu &= \{ w \in \mathbb{S}_0 : \lim_{t\to\infty} e^{-\mu t} \|x(t; w)\| = 0 \}, \\ \mathcal{U}_\mu &= \{ w \in \mathbb{S}_0 : \lim_{t\to\infty} e^{-\mu t} \|x(t; w)\| = +\infty \} \cup \{0\}. \end{aligned}$$

Note that the definition of the unstable set $\mathcal{U}_\mu$ is different from that for ODEs in [20, 45], see also the formula (21) below. By definition, it is clear that $\mathcal{S}_\mu \cap \mathcal{U}_\mu = \emptyset$ for $\mu \in \rho(E, A)$. Furthermore, for $\mu_1, \mu_2 \in \rho(E, A)$, $\mu_1 < \mu_2$, we have $\mathcal{S}_{\mu_1} \subseteq \mathcal{S}_{\mu_2}$ and $\mathcal{U}_{\mu_1} \supseteq \mathcal{U}_{\mu_2}$.

In the following we study the EUODE (5) associated with (3). For simplicity, we assume that Z is the *principal matrix solution*, i.e., $Z(0) = I_d$. This can always achieved by an appropriate kinematic equivalence transformation. Following the construction for ODEs in [20], we characterize the sets

$$\begin{aligned} \mathcal{S}_\mu^d &= \left\{ v \in \mathbb{R}^d : \lim_{t\to\infty} e^{-\mu t} \|Z(t)v\| = 0 \right\}, \\ \mathcal{U}_\mu^d &= \left\{ v \in \mathbb{R}^d : \lim_{t\to\infty} e^{\mu t} \left\|Z(t)^{-T} v\right\| = 0 \right\} \end{aligned} \tag{21}$$

associated with (5). Recalling that p is the number of disjoint spectral intervals, let us now choose a set of values $\mu_0 < \mu_1 < \ldots < \mu_p$, such that $\mu_j \in \rho(E,A)$ and $\Sigma_S \cap (\mu_{j-1}, \mu_j) = [a_j, b_j]$ for $j = 1, \ldots, p$. In other words, we have

$$\mu_0 < a_1 \leq b_1 < \mu_1 < \cdots < \mu_{j-1} < a_j \leq b_j < \mu_j < \cdots < \mu_{p-1} < a_p \leq b_p < \mu_p.$$

The following two theorems which are easily adopted from [20, 45] describe the relation between the stable and unstable sets and the Lyapunov spectral intervals.

Theorem 22 *Consider the EUODE (5) associated with (3), the corresponding sets $\mathcal{S}^d_{\mu_j}$ and $\mathcal{U}^d_{\mu_j}$, $j = 0, \ldots, p$ defined in (21), and the intersections*

$$\mathcal{N}^d_j = \mathcal{S}^d_{\mu_j} \cap \mathcal{U}^d_{\mu_{j-1}}, \ j = 1, \ldots, p. \tag{22}$$

Then every $\mathcal{N}^d_j$ is a linear subspace of dimension $\dim(\mathcal{N}^d_j) \geq 1$ with the following properties:

(i) $\mathcal{N}^d_k \cap \mathcal{N}^d_l = \{0\}$, for $k \neq l$,

(ii) $\mathbb{R}^d = \mathcal{N}^d_1 \oplus \mathcal{N}^d_2 \oplus \ldots \mathcal{N}^d_p$.

Theorem 23 *Consider the EUODE (5) associated with (3), and the sets $\mathcal{N}^d_j$ defined in (22), $j = 1, \ldots, p$. If $v \in \mathcal{N}^d_j$ and*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|Z(t)v\| = \chi^u, \quad \liminf_{t\to\infty} \frac{1}{t} \ln \|Z(t)v\| = \chi^\ell,$$

then $\chi^\ell, \chi^u \in [a_j, b_j]$.

Let $\hat{U}$ be an orthonormal basis of the solution subspace for (3) and introduce the sets

$$\mathcal{N}_j = \hat{U}(0)\mathcal{N}^d_j = \left\{ w \in \mathbb{S}_0 : w = \hat{U}(0)v, \ v \in \mathcal{N}^d_j \right\}, \ j = 1, \ldots, p, \tag{23}$$

then solutions of the DAE (3) with initial condition from $\mathcal{N}_j$ can be characterized as follows.

Corollary 24 *Consider the EUODE (5) associated with (3), and the sets $\mathcal{N}_j$ defined in (23), $j = 1, \ldots, p$. If $w \in \mathcal{N}_j$ and*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t;w)\| = \chi^u, \quad \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t;w)\| = \chi^\ell,$$

then $\chi^\ell, \chi^u \in [a_j, b_j]$.

This means that $\mathcal{N}_j$ is the subspace of initial conditions associated with solutions of (3) whose upper and lower Lyapunov exponents are located inside $[a_j, b_j]$.

The next theorem characterizes the uniform exponential growth of the solutions of (3).

Theorem 25 *Consider the EUODE (5) associated with (3), and the sets $\mathcal{N}_j$ defined in (23), $j = 1, \ldots, p$. Then $w \in \mathcal{N}_j \setminus \{0\}$ if and only if*

$$\frac{1}{K_{j-1}} e^{a_j(t-s)} \leq \frac{\|x(t;w)\|}{\|x(s;w)\|} \leq K_j e^{b_j(t-s)}, \quad \text{for all } t \geq s \geq 0, \tag{24}$$

and some positive constants K_{j-1}, K_j.

Proof. Due to the construction of the EUODE (5), see Lemma 1, we have $x(t;w) = \hat{U}(t)Z(t)v$, where $v = \hat{U}(0)^T w$, and thus $\|x(t;w)\| = \|Z(t)v\|$. Theorem 3.9 and Remark 3.10 of [20] state that $v \in \mathcal{N}^d_j$ if and only if

$$\frac{1}{K_{j-1}} e^{a_j(t-s)} \leq \frac{\|Z(t)v\|}{\|Z(s)v\|} \leq K_j e^{b_j(t-s)}, \quad \text{for all } t \geq s \geq 0,$$

and some positive constants K_{j-1}, K_j. Hence, the inequalities (24) follow immediately. □

We can also characterize the relationship of the sets N_j and the Bohl exponents.

Corollary 26 *Consider the EUODE (5) associated with (3) and the sets $\mathcal{N}_j$ defined in (23). Then for all $j = 1, \ldots, p$, we have*

(i) $w \in \mathcal{N}_j \setminus \{0\}$ if and only if $a_j \le \kappa^\ell(x(\cdot; w)) \le \kappa^u(x(\cdot; w)) \le b_j$, where κ^ℓ, κ^u are the Bohl exponents.

(ii) $\mathcal{S}_{\mu_j} = \hat{U}(0)\mathcal{S}^d_{\mu_j}$, $\mathcal{U}_{\mu_j} = \hat{U}(0)\mathcal{U}^d_{\mu_j}$, and $\mathcal{N}_j = \mathcal{S}_{\mu_j} \cap \mathcal{U}_{\mu_{j-1}}$, $j = 1, \ldots, p$.

Proof. (i) The proof of the first part follows from Theorem 25 and the definition of Bohl exponents, see Definition 16.

(ii) First we prove

$$\hat{U}(0)\mathcal{S}^d_{\mu_j} \subset \mathcal{S}_{\mu_j}, \quad \hat{U}(0)\mathcal{U}^d_{\mu_j} \subset \mathcal{U}_{\mu_j}.$$

To this end, take an arbitrary $w \in \hat{U}(0)\mathcal{S}^d_{\mu_j}$. Then v clearly belongs to $\mathcal{S}^d_{\mu_j}$, and consequently,

$$\frac{\|x(t; w)\|}{\|x(s; w)\|} = \frac{\|Z(t)v\|}{\|Z(s)v\|} \le K_j e^{b_j(t-s)},$$

where v is defined as in the proof of Theorem 25. It follows that $w \in \mathcal{S}_{\mu_j}$. The relation $\hat{U}(0)\mathcal{U}^d_{\mu_j} \subset \mathcal{U}_{\mu_j}$ is proved analogously. As a consequence of Theorem 3.4 in [20], we have $\dim \mathcal{S}^d_{\mu_j} + \dim \mathcal{U}^d_{\mu_j} = d$. Therefore,

$$d = \dim \hat{U}(0)\mathcal{S}^d_{\mu_j} + \dim \hat{U}(0)\mathcal{U}^d_{\mu_j} \le \dim \mathcal{S}_{\mu_j} + \dim \mathcal{U}_{\mu_j}.$$

On the other hand, $\mathcal{S}_{\mu_j} \cap \mathcal{U}_{\mu_j} = \{0\}$. Thus, the identities

$$\hat{U}(0)\mathcal{S}^d_{\mu_j} = \mathcal{S}_{\mu_j}, \quad \hat{U}(0)\mathcal{U}^d_{\mu_j} = \mathcal{U}_{\mu_j}.$$

hold and the proof is complete. □

Remark 27 An alternative way to characterize the unstable set $\mathcal{U}^d_\mu$ is as $\{v \in \mathbb{R}^d : \lim_{t\to\infty} e^{\mu t} \|Z(t)v\| = +\infty\} \cup \{0\}$ for $\mu \in \rho(E, A)$. Furthermore, we have

$$\begin{aligned} \mathcal{S}_{\mu_j} &= \{w \in \mathbb{S}_0 : \kappa^u(x(\cdot; w)) \le b_j\} \cup \{0\}, \\ \mathcal{U}_{\mu_{j-1}} &= \{w \in \mathbb{S}_0 : \kappa^\ell(x(\cdot; w)) \ge a_j\} \cup \{0\}, \end{aligned}$$

as well. For further properties of $\mathcal{S}^d_{\mu_j}$ and $\mathcal{U}^d_{\mu_j}$, see [20, Theorem 3.11] in the case when the system is integrally separated.

In this section we have adapted and extended several results on spectral intervals, leading directions and stability sets to strangeness-free DAEs. In the next section we present the main results of the paper, the extension of smooth singular value decomposition based methods to DAEs.

4 SVD-based methods for DAEs

In this section we extend the approach in [20, 21] of using smooth SVDs for the computation of spectral intervals of ODEs to DAEs. We assume again that the DAE system is given in the strangeness-free form (3), i.e., whenever the value of $E(t), A(t)$ is needed, this has to be computed from the derivative array as described in [32]. This can be done for example with the `FORTRAN` code `GELDA` [35] or the corresponding `MATLAB` version [36].

Let X be an (arbitrary) minimal fundamental matrix solution of (3), in particular, assume that $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$. Suppose that we are able to compute a *smooth SVD*

$$X(t) = U(t)\Sigma(t)V^T(t), \quad t \ge 0, \tag{25}$$

where $U \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$, $V, \Sigma \in C^1(\mathbb{I}, \mathbb{R}^{d\times d})$, $U^T(t)U(t) = V^T(t)V(t) = I_d$ for all $t \in \mathbb{I}$, and $\Sigma(t) = \mathrm{diag}(\sigma_1(t), \ldots, \sigma_d(t))$ is diagonal. We assume that U, Σ, and V possess the same smoothness as X. This holds, e.g., if $X(t)$ is analytic, see [11] or if the singular values of $X(t)$ are distinct for all t, see [19, 42]. The explicit construction of smooth SVDs is rather computationally expensive, see [11] and the proof of [32, Theorem 3.9].

Remark 28 Note that U in the smooth SVD of X and $\hat{U}$ as in the construction of the EUODE, play the same role, they form orthonormal bases of the corresponding solution subspace $\mathbb{S}$, so we are in fact in the special case of the analysis in Section 2. If we set $Z = \Sigma V^T$, then this is a fundamental solution of the resulting EUODE of the form (5). Furthermore, the factorization $Z = \Sigma V^T$ is the SVD of the specially chosen fundamental solution Z.

We will also demonstrate how to modify the methods of [20, 21] to approximate only a few (dominant) spectral intervals. For this we need to select ℓ ($\ell < d$) columns of a fundamental solution, i.e., ℓ linearly independent solutions of the DAE (3), and proceed to work with them.

4.1 Discrete SVD method

We first discuss a discrete SVD method, where the fundamental matrix solution X is indirectly evaluated by solving (3) on subintervals and to reduce the accumulation of errors, the numerical integration is performed with a reorthogonalization. For T sufficiently large, we choose a set of N grid points $0 = t_0 < t_1 < \ldots < t_N = T$. Suppose that a skinny QR factorization, see e.g., [28], $X(0) = Q_0R_0$ is given, where $Q_0 \in \mathbb{R}^{n\times d}$ and $R_0 \in \mathbb{R}^{d\times d}$. We can proceed in the same way, when only $\ell < d$ (dominant) spectral intervals are of interest. Then we successively solve the initial value problems

$$\begin{cases} E(t)\dot{X}_k(t) = A(t)X_k(t), & t_{k-1} \le t \le t_k, \\ X_k(t_{k-1}) = Q_{k-1}, \quad k = 1, 2, \ldots, \end{cases} \tag{26}$$

followed by QR decompositions $X_k(t_k) = Q_kR_k$ which are determined using modified Gram-Schmidt orthogonalization or Householder transformations, see [28].

Then we have

$$X(t_N) = Q_NR_NR_{N-1}\cdots R_1R_0. \tag{27}$$

Up to this step the method is the same as the discrete QR method for approximating spectral intervals, see [23, 37]. We then proceed by using the product SVD algorithm [8, 41, 46] to compute the (approximate) SVD of the matrix product on the right-hand side of (27) without evaluating the product itself. As a result of applying the product SVD algorithm to the product $R_NR_{N-1}\cdots R_1R_0$ in (27), we then obtain the desired factorization of $X(t_N)$ in the form

$$Q_NU^{(N)}\Sigma^{(N)}\Sigma^{(N-1)}\ldots\Sigma^{(1)}\Sigma^{(0)}(V^{(N)})^T \tag{28}$$

with diagonal matrices $\Sigma^{(0)}, \ldots, \Sigma^{(N)}$. To extract the spectral information from the obtained factorization, one proceeds as in the discrete QR methods [37, 38], where only the diagonal components are used. In particular, assuming that T is sufficiently large, approximate upper and lower Lyapunov exponents are obtained by computing a truncated lim sup and lim inf in

$$\lambda_i^u = \limsup_{t_N\to\infty} \frac{1}{t_N}\sum_{j=0}^{N} \ln\left(\Sigma^{(j)}\right)_{i,i}, \quad \lambda_i^\ell = \liminf_{t_N\to\infty} \frac{1}{t_N}\sum_{j=0}^{N} \ln\left(\Sigma^{(j)}\right)_{i,i}, \tag{29}$$

for $i = 1, 2, \ldots, d$. Theorem 32 given below for the continuous SVD method justifies the use of (29) for approximating Lyapunov exponents.

The approximation of the Sacker-Sell spectrum is obtained analogously, see Theorem 34 below. Concretely, with $N > M > 0$, we have

$$\kappa_i^u = \limsup_{t_M, t_N - t_M\to\infty} \frac{1}{t_N - t_M}\sum_{j=M+1}^{N} \ln\left(\Sigma^{(j)}\right)_{i,i}, \quad \kappa_i^\ell = \liminf_{t_M, t_N - t_M\to\infty} \frac{1}{t_N - t_M}\sum_{j=M+1}^{N} \ln\left(\Sigma^{(j)}\right)_{i,i}, \tag{30}$$

for $i = 1, 2, \ldots, d$.

We summarize the procedure in the following algorithm.

Algorithm 1 (Discrete SVD algorithm for computing the Lyapunov spectrum of a linear DAE with variable coefficients)

Input: *A pair of sufficiently smooth matrix functions (E, A) in the form of the strangeness-free DAE (3) (if they are not available directly they must be obtained pointwise as output of a routine such as* `GELDA`*), the time interval $[0, T]$, $\tau \in (0, T)$, and a mesh $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$.*
Output: *Approximations to the endpoints of the spectral intervals $\{\lambda_i^l, \lambda_i^u\}_{i=1}^d$.*

Initialization

1. Set $t_0 := 0, X_0 = Q_0 R_0$.
2. Set $\lambda_i(t_0) := 0$ and $s_i := 0$ for $i = 1, \ldots, d$ (for computing the sum s_i of the logarithms).

Computing the fundamental solution in product form

For $j = 1, \ldots, N$

1. Solve the initial value problem (26) for $X_j(t_j)$ on $[t_{j-1}, t_j]$.
2. Compute the QR factorization $X_j(t_j) = Q_j R_j$.

Product SVD

Apply the product SVD Algorithm to $\prod_{j=N}^{0} R_j$ to compute the factors $\Sigma^{(j)}$, $j = 0, 1, 2, \ldots, N$ in (28) and the factor V^N.

Calculation of the Lyapunov exponents

For $j = 1, \ldots, N$

1. Update $s_i := s_i + \ln[(\Sigma^{(j)})_{ii}]$ and $\lambda_i(t_j) = \frac{1}{t_j} s_i, \quad i = 1, 2, \ldots, d$.
2. If desired, test the integral separation property by using $\{s_i\}_{i=1}^d$.
3. Update $\min_{\tau \le t \le t_j} \lambda_i(t)$ and $\max_{\tau \le t \le t_j} \lambda_i(t)$, $i = 1, 2, \ldots, d$.

For the approximation of Sacker-Sell spectral intervals only the last step has to be modified, by using (30) instead of (29).

4.2 Continuous SVD method

In the continuous SVD method one derives differential-algebraic equations for the factors U, Σ, and V and solves the corresponding initial value problems via numerical integration.

If we differentiate the expression for X in (25) with respect to t and substitute the result into (3), we obtain

$$E\dot{U}\Sigma V^T + EU\dot{\Sigma}V^T + EU\Sigma\dot{V}^T = AU\Sigma V^T,$$

or equivalently, using the orthogonality of V,

$$E\dot{U}\Sigma + EU\dot{\Sigma} + EU\Sigma\dot{V}^T V = AU\Sigma.$$

Since the second block-row of E is zero and Σ is invertible, it follows that

$$A_2 U = 0, \tag{31}$$

because the columns of U form an orthonormal basis of a subspace of the solution subspace. If we then differentiate (31) and insert this, we obtain

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} U\dot{\Sigma} + \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} U\Sigma\dot{V}^T V = \begin{bmatrix} A_1 \\ -\dot{A}_2 \end{bmatrix} U\Sigma. \tag{32}$$

We define the matrix function

$$\bar{A} = \begin{bmatrix} A_1 \\ -\dot{A}_2 \end{bmatrix},$$

and the skew-symmetric matrix functions

$$H = U^T\dot{U}, \quad K = V^T\dot{V}.$$

The latter two matrix functions are of size $d \times d$ (or $\ell \times \ell$ in the reduced case). We can use an idea suggested in [37, 38] to determine a matrix function $P \in C(\mathbb{I}, \mathbb{R}^{n\times d})$ such that

$$P^T\bar{E} = \mathcal{E}U^T, \tag{33}$$

where $\mathcal{E}$ is nonsingular and upper triangular with positive diagonal entries. Due to [38, Lemma 12], this defines $P, \mathcal{E}$ uniquely. The numerical computation of this pair will be discussed later. Note that the identity $P^TEU = P^T\bar{E}U = \mathcal{E}$ holds, so that P can also be used to produce an EUODE of the form (5) via Lemma 1.

The following property of $\mathcal{E}$ is important in the proof of numerical stability for the SVD method.

Proposition 29 *Consider the matrix function P defined via (33). Then*

$$\|\mathcal{E}\| \le \|\bar{E}\|, \quad \|\mathcal{E}^{-1}\| \le \|\bar{E}^{-1}\|.$$

Proof. The estimate for $\|\mathcal{E}\|$ follows immediately from the identity $P^T\bar{E}U = \mathcal{E}$. For the second inequality one observes that (33) is equivalent to

$$\bar{E}^{-T}U = P\mathcal{E}^{-T},$$

and hence $P^TE^{-T}U = \mathcal{E}^{-T}$. Thus, the estimate for $\|\mathcal{E}^{-1}\|$ is obtained analogously. □

Denoting by $\mathrm{cond}(M)$ the normwise condition number of a matrix M with respect to inversion, as a consequence of Proposition 29, we have that the $\mathrm{cond}\,\mathcal{E} \le \mathrm{cond}\,\bar{E}$, and thus the sensitivity of the implicit EUODE (5) that we are using to compute the spectral intervals is not larger than that of the original DAE.

Multiplying both sides of (32) with P^T from the left, we obtain

$$\mathcal{E}H\Sigma + \mathcal{E}\dot{\Sigma} + \mathcal{E}\Sigma K^T = P^T\bar{A}U\Sigma.$$

With

$$G = P^T\bar{A}U \text{ and } C = \mathcal{E}^{-1}G, \tag{34}$$

we then arrive at

$$H\Sigma + \dot{\Sigma} + \Sigma K^T = C\Sigma,$$

which is almost the same differential equation as in the ODE case, see [20, 21], there is just a different formula for $C = [c_{i,j}]$. Using the skew-symmetry of $H = [h_{i,j}], K = [k_{i,j}]$ and that $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$ is diagonal, we obtain the expressions

$$\begin{aligned} h_{i,j} &= \frac{c_{i,j}\sigma_j^2 + c_{j,i}\sigma_i^2}{\sigma_j^2 - \sigma_i^2}, \text{ for } i > j, \text{ and } \quad h_{i,j} = -h_{j,i} \text{ for } i < j; \\ k_{i,j} &= \frac{(c_{i,j} + c_{j,i})\sigma_i\sigma_j}{\sigma_j^2 - \sigma_i^2}, \text{ for } i > j, \text{ and } \quad k_{i,j} = -k_{j,i} \text{ for } i < j. \end{aligned} \tag{35}$$

We also get immediately the differential equation for the diagonal elements of Σ

$$\dot{\sigma}_i = c_{i,i}\sigma_i, \quad i = 1, \ldots, d, \tag{36}$$

and that for the V-factor,

$$\dot{V} = VK. \tag{37}$$

By some further elementary calculations, we also obtain the equation for the U-factor as

$$E\dot{U} = EU(H - C) + AU. \tag{38}$$

It is easy to see that (38) is a strangeness-free (non-linear) matrix DAE, that is furthermore linear with respect to the derivative. Furthermore, the algebraic constraint is also linear and the same as that of (3). We will discuss the efficient integration of this particular matrix DAE (38) below.

To proceed further, we have to assume that the matrix function C in (34) is uniformly bounded on $\mathbb{I}$. Furthermore, in order for the Lyapunov exponents to be stable, we will assume that the functions σ_i are integrally separated, i.e., there exist constants $k_1 > 0$ and $k_2,\ 0 < k_2 \leq 1$, such that

$$\frac{\sigma_j(t)}{\sigma_j(s)}\frac{\sigma_{j+1}(s)}{\sigma_{j+1}(t)} \geq k_2 e^{k_1(t-s)}, \quad t \geq s \geq 0,\ j = 1, 2, \ldots, d-1. \tag{39}$$

Condition (39) is equivalent to the integral separation of the diagonal of C. This property again can be easily checked in practice via the computation of Steklov differences, see Section 2.3.

The following results are then obtained in the same way as the corresponding results for ODEs in [20].

Proposition 30 *Consider the differential equations (37), (38) and (36) and suppose that the diagonal of C is integrally separated. Then, the following statements hold.*

(a) There exists $\bar{t} \in \mathbb{I}$, such that for all $t \geq \bar{t}$, we have

$$\sigma_j(t) > \sigma_{j+1}(t), \quad j = 1, 2, \ldots, d-1.$$

(b) The skew-symmetric matrix function $K(t)$ converges exponentially to 0 as $t \to \infty$.

(c) The orthogonal matrix function $V(t)$ converges exponentially to a constant orthogonal matrix $\bar{V}$ as $t \to \infty$.

Proof. The proofs of (a), (b), and the convergence of V follow in the same way as the corresponding results in in [20, 23]. Further, one can show that the convergence rate of K is not worse than $-k_1$, where k_1 is the constant in (39), see [23, Lemma 7.3]. Then, invoking the argument of [18, Lemma 2.4], we obtain

$$\|V(t) - \bar{V}\| \leq \left(e^{\int_t^\infty \|K(s)\| ds} - 1\right) \|\bar{V}\|,$$

where $\bar{V} = \lim_{t\to\infty} V(t)$. By elementary calculations, it is easy to show that the convergence rates for $t \to +\infty$ of V and K are the same. □

Remark 31 In [20, Theorem 5.4] it is shown that the exponential rate of convergence of V is given by

$$\alpha = \gamma \max_{1\leq j\leq d-1}(\lambda^u_{j+1} - \lambda^u_j),$$

where γ is a constant, $0 < \gamma \leq 1$. Furthermore, if the system (5) is regular, then $\gamma = 1$ [20, Corollary 5.5].

We can also characterize the relationship between the stability of Lyapunov exponents and integral separation of the singular values.

Theorem 32 *System (3) has distinct and stable Lyapunov exponents if and only if for any fundamental matrix solution X, the singular values of X are integrally separated. Moreover, if X is a fundamental solution, then*

$$\lambda_j^u = \limsup_{t\to\infty} \frac{1}{t} \ln \sigma_j(t), \quad \lambda_j^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \sigma_j(t), \quad j = 1, 2, \ldots, d. \tag{40}$$

Proof. For the proof, we apply [20, Theorem 4.2] to the EUODE (5) and consider the corresponding fundamental solution $Z = \hat{U}^T X$ for a fundamental solution X, where $\hat{U}$ is a fixed orthonormal basis determined by the continuous SVD algorithm. Note that (3) is integrally separated if and only if the associated EUODE (5) is integrally separated. Invoking Theorem 7 and the fact that the singular values of X and those of Z are the same, we obtain the desired formulas in (40). □

Remark 33 Theorem 32 has two computational consequences. By [20, Lemma 4.3], it follows that we can work with any minimal fundamental solution X. This is an advantage compared to the QR methods [37, 38] which require the use of a normal basis. In order to compute the Lyapunov exponents numerically, they must be stable. For the ODE case, the stability of distinct Lyapunov exponents and the integral separation of the ODE are equivalent. For DAEs, however, we need further boundedness conditions, see Example 12 and Theorem 14, which need to be checked in any computational method.

Theorem 34 *Suppose that (3) has distinct and stable Lyapunov exponents. Then, the Sacker-Sell spectrum of (3) is the same as that of the diagonal system*

$$\dot{\Sigma}(t) = \operatorname{diag}(C(t))\Sigma(t).$$

Furthermore, this Sacker-Sell spectrum is given by the union of the Bohl intervals associated with the scalar equations $\dot{\sigma}_i(t) = c_{i,i}(t)\sigma_i(t)$, $i = 1, 2, \ldots, d$.

Proof. The proof follows in the same way as the proof of Theorem 32 and [20, Theorem 4.6]. □

Similarly to the ODE case, the limit matrix $\bar{V}$ provides a normal basis, that is, using the columns of $X(0)\bar{V}$ for initial conditions, we obtain a fundamental matrix solution whose columns have Lyapunov spectral intervals $[\lambda_j^\ell, \lambda_j^u]$, $j = 1, 2, \ldots, d$.

Theorem 35 *Suppose that (3) has distinct and stable Lyapunov exponents. Let $X(t) = U(t)\Sigma(t)V(t)^T$ be a smooth SVD of an arbitrary fundamental solution. Let $\bar{V} = [\bar{v}_1, \ldots, \bar{v}_d]$ be the limit of the factor $V(t)$ as $t \to \infty$. Then*

$$\chi^u(X(\cdot)\bar{v}_j) = \lambda_j^u, \quad \chi^\ell(X(\cdot)\bar{v}_j) = \lambda_j^\ell, \quad j = 1, 2, \ldots, d.$$

Proof. We apply [20, Theorem 5.8] to the EUODE (5) with the corresponding fundamental solution Z. Then observing that $\chi^u(X(\cdot)\bar{v}_j) = \chi^u(Z(\cdot)\bar{v}_j)$ and $\chi^\ell(X(\cdot)\bar{v}_j) = \chi^\ell(Z(\cdot)\bar{v}_j)$, the assertion follows. □

Theorem 36 *Suppose that (3) has distinct and stable Lyapunov exponents and let $\Sigma_S = \bigcup_{j=1}^m [a_j, b_j]$. Then $\mathcal{N}_j = \hat{U}(0)\operatorname{span}\{\bar{v}_k, \ldots, \bar{v}_l\}$, where the integers k, l, $k < l$ are such that*

$$\lambda_{l+1}^u < a_j \le \lambda_l^\ell, \quad \lambda_k^u \le b_j < \lambda_{k-1}^\ell.$$

Proof. We apply [20, Theorem 5.12] to obtain the characterization of the subspaces $\mathcal{N}_j^d$ associated with (5). Then, the relationship between $\mathcal{N}_j$ and $\mathcal{N}_j^d$ expressed in (23) yields the assertion. □

Remark 37 In Theorem 36 we can also use $X(0)\operatorname{span}\{\bar{v}_k, \ldots, \bar{v}_l\}$ for an arbitrary fundamental solution matrix X and we also have $[a_j, b_j] = \bigcup_{i=k}^l [\kappa^\ell(X(\cdot)\bar{v}_i), \kappa^u(X(\cdot)\bar{v}_i)]$ for $j = 1, \ldots, m$.

In the following we show that the initial conditions given by $X(0)\bar{V}$ provide not only the directional information for a normal basis and for the subspaces associated with Sacker-Sell spectral intervals as stated in Theorem 35 and Theorem 36, but they also lead to an integrally separated fundamental solution. Since we do not need to assume that the DAE (3) has d disjoint Sacker-Sell spectral intervals, the following theorem significantly improves the result of [20, Theorem 5.14, Corollary 5.15].

Theorem 38 *Suppose that the DAE system (3) has distinct and stable Lyapunov exponents. Let $X(t) = U(t)\Sigma(t)V(t)^T$ be a smooth SVD of an arbitrary fundamental solution and let $\bar{V} = [\bar{v}_1, \dots \bar{v}_d]$ be the limit of $V(t)$ as $t \to \infty$. Then starting from $X(0)\bar{V}$ leads to an integral separated fundamental solution, i.e., $X(t)\bar{V}$ is integrally separated.*

Proof. Let $x_i(t)$, $i = 1, 2, \dots, d$ be the columns of $X(t)\bar{V}$ which is a fundamental solution for (3). By assumption, there exists an integrally separated fundamental solution which we denote by $\bar{X}(t) = [\bar{x}_1(t), \dots, \bar{x}_d(t)]$. Then there exist positive constants α_1 and α_2 such that

$$\frac{\|\bar{x}_i(t)\|}{\|\bar{x}_i(s)\|}\frac{\|\bar{x}_{i+1}(s)\|}{\|\bar{x}_{i+1}(t)\|} \geq \alpha_2 e^{\alpha_1(t-s)}, \quad t \geq s \geq 0,\ i = 1, \dots, d-1. \tag{41}$$

As a consequence, we obtain

$$\frac{\|\bar{x}_i(t)\|}{\|\bar{x}_{i+1}(t)\|} \geq \alpha_2 e^{\alpha_1 t}, \quad t \geq 0,\ i = 1, \dots, d-1. \tag{42}$$

To investigate the relation between $[x_1, \dots x_d]$ and $[\bar{x}_1, \dots, \bar{x}_d]$, observe that, since both form fundamental solutions, there exist coefficients $b_{i,j}$, $1 \leq i, j \leq d$ such that

$$x_i = b_{i,1}\bar{x}_1 + b_{i,2}\bar{x}_2 + \cdots + b_{i,d}\bar{x}_d, \quad i = 1, 2, \dots, d.$$

By Theorem 35 we have $\chi^u(x_i) = \chi^u(\bar{x}_i) = \lambda_i^u$, $i = 1, 2, \dots, d$ and the Lyapunov exponents are distinct. Then using (9) and (10) it follows that

$$b_{i,1} = b_{i,2} = \ldots = b_{i,i-1} = 0,\ b_{i,i} \neq 0, \quad i = 1, \dots, d,$$

and, thus, we can estimate x_i by $\bar{x}_i$. In fact, they are asymptotically equal for sufficiently large t and we have

$$\|x_i(t)\| = |b_{i,i}|\,\|\bar{x}_i(t)\| \left\| \frac{\bar{x}_i(t)}{\|\bar{x}_i\|} + \sum_{j=i+1}^{d} \frac{b_{i,j}\bar{x}_j(t)}{|b_{i,i}|\,\|\bar{x}_i(t)\|} \right\| \leq |b_{i,i}|\,\|\bar{x}_i(t)\| \left(1 + \sum_{j=i+1}^{d} \frac{|b_{i,j}|\,\|\bar{x}_j(t)\|}{|b_{i,i}|\,\|\bar{x}_i(t)\|} \right),$$

and simultaneously

$$\|x_i(t)\| \geq |b_{i,i}|\,\|\bar{x}_i(t)\| \left(1 - \sum_{j=i+1}^{d} \frac{|b_{i,j}|\,\|\bar{x}_j(t)\|}{|b_{i,i}|\,\|\bar{x}_i(t)\|} \right),$$

Due to (42), for an arbitrarily given constant $\varepsilon \in (0, 1)$, there exists a (sufficiently large) $\bar{t}_i$ such that for $t \geq \bar{t}_i$

$$|b_{i,i}|\,\|\bar{x}_i(t)\|\,(1-\varepsilon) \leq \|x_i(t)\| \leq |b_{i,i}|\,\|\bar{x}_i(t)\|\,(1+\varepsilon).$$

Choosing $\bar{t} = \max_{i=1,\dots,d}\{\bar{t}_i\}$, then this estimate holds for all $t \geq \bar{t}$ and for all $i = 1, \dots, d$. Hence, by invoking (41), for $t \geq s \geq \bar{t}$ and $i = 1, \dots, d-1$, we have the following uniform estimate

$$\frac{\|x_i(t)\|}{\|x_i(s)\|}\frac{\|x_{i+1}(s)\|}{\|x_{i+1}(t)\|} \geq \frac{\|\bar{x}_i(t)\|\,(1-\varepsilon)}{\|\bar{x}_i(s)\|\,(1+\varepsilon)}\frac{\|\bar{x}_{i+1}(s)\|\,(1-\varepsilon)}{\|\bar{x}_{i+1}(t)\|\,(1+\varepsilon)} \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \alpha_2 e^{\alpha_1(t-s)} = \bar{\alpha}_2 e^{\alpha_1(t-s)},$$

where $\bar{\alpha}_2 = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2 \alpha_1$. This implies that $[x_1, \dots, x_d]$ are integrally separated as well. ∎

For the implementation of the continuous SVD method several important issues have to be considered. First, we need the computation of $P(t)$ in (33) for every used time point t. This can be done via the pencil arithmetic of [5] as presented in [38]. Let us briefly recall this process here. One first performs a QR factorization

$$\begin{bmatrix} \bar{E} \\ U^T \end{bmatrix} = \begin{bmatrix} \tilde{T}_{1,1} & \tilde{T}_{1,2} \\ \tilde{T}_{2,1} & \tilde{T}_{2,2} \end{bmatrix} \begin{bmatrix} \tilde{M}_{1,1} \\ 0 \end{bmatrix},$$

which implies that $\tilde{T}_{1,2}^T \bar{E} = -\tilde{T}_{2,2}^T U^T$. In general this factorization does not guarantee that $\tilde{T}_{2,2}$ is invertible. To obtain this we compute the QR factorization of the augmented matrix

$$\begin{bmatrix} \bar{E} & 0 \\ U^T & I_d \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

where $T = [T_{i,j}]$ is orthogonal and $M = [M_{i,j}]$ is upper triangular. Then we have that $T_{2,2}^T = M_{2,2}$ is nonsingular and upper triangular. In order to get the desired matrices P and $\mathcal{E}$, we use an additional QR factorization

$$T_{1,2} = PL, \tag{43}$$

where P fulfills $P^T P = I_d$ and L is lower triangular (the fact that $T_{1,2}$ is full column-rank is implied directly by the nonsingularity of $T_{2,2}$). Finally, we set $\mathcal{E} = -L^{-T} T_{2,2}$.

As an alternative approach, we may use the transformation as it is used to determine consistent initial conditions (7) as well as for the EUODEs in [37]. First, compute an orthogonal matrix $\tilde{Q}$ such that

$$\begin{bmatrix} E_{11} & E_{12} \\ A_{21} & A_{22} \end{bmatrix} \tilde{Q} = \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix},$$

where $\tilde{A}_{22}$ is upper triangular, for instance, by using Householder transformations. Let P^T and the transformed matrix function $U^T \tilde{Q}$ be decomposed accordingly into blocks as $\begin{bmatrix} P_1^T & P_2^T \end{bmatrix}$ and $\begin{bmatrix} \tilde{U}_1^T & \tilde{U}_2^T \end{bmatrix}$, respectively. Observe that then $\tilde{U}_2 = 0$, since the constraint $A_2 U = 0$ implies that $A_2 \tilde{Q} \tilde{Q}^T U = 0$ and hence $\tilde{U}_1$ is orthogonal. Multiplying (33) by $\tilde{Q}$ from right, one obtains

$$\begin{bmatrix} P_1^T & P_2^T \end{bmatrix} \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} = \mathcal{E} \begin{bmatrix} \tilde{U}_1^T & 0 \end{bmatrix},$$

or equivalently

$$P_1^T \tilde{E}_{11} = \mathcal{E} \tilde{U}_1^T, \quad P_1^T \tilde{E}_{12} + P_2^T \tilde{A}_{22} = 0.$$

To solve this system of matrix equations we calculate a QR factorization

$$\tilde{E}_{11} \tilde{U}_1 = \tilde{P}_1 \tilde{\mathcal{E}},$$

where $\tilde{P}_1$ is orthogonal and $\tilde{\mathcal{E}}$ is upper triangular. Then we solve the upper triangular matrix equation $\tilde{A}_{22}^T \tilde{P}_2 = -\tilde{E}_{12}^T \tilde{P}_1$ to obtain $\tilde{P}_2$. Computing another QR factorization

$$\begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \mathcal{R},$$

where $\mathcal{R}$ is lower triangular, and finally setting $\mathcal{E} = \mathcal{R}^{-T} \tilde{\mathcal{E}}$, we have finished the construction.

Remark 39 In practice, if the boundedness of $\tilde{E}_{12} \tilde{A}_{22}^{-1}$ is ensured, then we can accept the pair $\tilde{P}, \tilde{\mathcal{E}}$ as well, i.e., the last orthogonalization step (applied to $\tilde{P}$) can be omitted. In this case, if $\tilde{E}_{11}$ is boundedly invertible, then so is $\tilde{\mathcal{E}}$. See also Theorem 14.

Next, we discuss how to avoid the risk of overflow in calculating $\sigma_i(t)$, since the singular values may grow exponentially fast. For this we use the same approach as suggested for the ODE case in [21]. We introduce auxiliary functions

$$\nu_j(t) = \frac{\sigma_{j+1}(t)}{\sigma_j(t)} \text{ for } j = 1, \ldots, d-1; \quad \nu_d(t) = \ln \sigma_d(t), \quad t \geq 0. \tag{44}$$

Instead of integrating the diagonal elements $\sigma_i(t)$, we solve initial value problems for the ODEs

$$\frac{d}{dt}(\ln \nu_j(t)) = c_{j+1,j+1}(t) - c_{j,j}(t),\ j = 1, \ldots, d-1; \quad \dot{\nu}_d(t) = c_{d,d}(t). \tag{45}$$

Then, we define

$$\nu_{ij}(t) = \frac{\sigma_j(t)}{\sigma_i(t)} = \prod_{k=j-1}^{i} \nu_k(t) \text{ for } j = i+1, i+2, \ldots, d.$$

and rewrite the formulas for the entries of $H = [h_{i,j}]$ and $K = [k_{i,j}]$ as

$$h_{i,j} = \frac{c_{i,j}\nu_{i,j}^2 + c_{j,i}}{\nu_{i,j}^2 - 1},\ j > i, \quad h_{i,j} = -h_{j,i},\ j < i, \tag{46}$$

$$k_{i,j} = \frac{c_{i,j} + c_{j,i}}{\nu_{i,j}^2 - 1}\nu_{i,j},\ j > i, \quad k_{i,j} = -k_{j,i},\ j < i. \tag{47}$$

To compute the Lyapunov exponents, similarly to the discrete SVD method, we introduce

$$\lambda_j(t) = \frac{1}{t}\ln \sigma_j(t), \quad j = 1, 2, \ldots, d. \tag{48}$$

Then, we have

$$\lambda_d(t) = \frac{1}{t}\nu_d(t),\ \lambda_j(t) = \lambda_{j+1}(t) - \frac{1}{t}\ln \nu_j(t),\ j = 1, 2, \ldots, d-1. \tag{49}$$

In practice, choosing $\tau \geq 0$ large and $T \gg \tau$, we may use the approximation

$$\lambda_j^u \approx \max_{\tau \leq t \leq T} \lambda_j(t), \quad \lambda_j^\ell \approx \min_{\tau \leq t \leq T} \lambda_j(t),\ j = 1, 2, \ldots, d-1.$$

The computation of Sacker-Sell intervals (in fact we compute the Bohl exponents of $\sigma_j(t)$, see Theorem 34) can be carried out using the same auxiliary functions. Similar to [37], for $\tilde{\tau} > 0$, we define the Steklov averages

$$\psi_{\tilde{\tau},j}(t) = \frac{1}{\tilde{\tau}}\left(\ln \sigma_j(t+\tilde{\tau}) - \ln \sigma_j(t)\right) = \frac{1}{\tilde{\tau}}\left((t+\tilde{\tau})\lambda_j(t+\tilde{\tau}) - t\lambda_j(t)\right),\ j = 1, 2, \ldots, d-1. \tag{50}$$

In practice, with $\tilde{\tau} > 0$ large and $T \gg \tilde{\tau}$, we approximate the desired Bohl exponents by

$$\kappa_j^u \approx \max_{0 \leq t \leq T-\tilde{\tau}} \psi_{\tilde{\tau},j}(t), \quad \kappa_j^\ell \approx \min_{0 \leq t \leq T-\tilde{\tau}} \psi_{\tilde{\tau},j}(t),\ j = 1, 2, \ldots, d-1. \tag{51}$$

Finally, we need to carefully integrate the the orthogonal factor U (and also V if we are interested in the growth directions). During the integration of the strangeness-free DAE (38), we have to guarantee that the computed factor U satisfies the algebraic constraint as well as the orthonormality at every mesh point t_i. This can be achieved by using a projected DAE solver such as the projected backward difference formula BDF, see [9]. To solve the nonlinear matrix-valued equation that is arising in every time-step, we suggest to use several simple fixpoint iterations instead of the faster converging but much more expensive Newton iteration. To illustrate this, applying the implicit Euler method to (38) at the time $t = t_n$ yields

$$E(t_n)\frac{U_n - U_{n-1}}{h} = E(t_n)U_n S(t_n, U_n) + A(t_n)U_n, \tag{52}$$

where U_n denotes the approximation of $U(t_n)$ and $S = H - C$ is the nonlinear function of t and U given in (5). Rearranging the terms, we obtain the fixpoint equation

$$\begin{bmatrix} E_1(t_n) \\ A_2(t_n) \end{bmatrix} U_n = \begin{bmatrix} E_1(t_n)U_{n-1} + h\left(E_1(t_n)U_n S(t_n, U_n) + A_1(t_n)U_n\right) \\ 0 \end{bmatrix}, \tag{53}$$

or alternatively

$$\left[\begin{array}{c} E_1(t_n) - hA_1(t_n) \\ A_2(t_n) \end{array}\right] U_n = \left[\begin{array}{c} E_1(t_n)U_{n-1} + hE_1(t_n)U_n S(t_n, U_n) \\ 0 \end{array}\right]. \tag{54}$$

To approximate U_n, we may use the simple fixpoint iteration

$$\left[\begin{array}{c} E_1(t_n) \\ A_2(t_n) \end{array}\right] U_n^{(k+1)} = \left[\begin{array}{c} E_1(t_n)U_{n-1} + h\left(E_1(t_n)U_n^{(k)} S(t_n, U_n^{(k)}) + A_1(t_n)U_n^{(k)}\right) \\ 0 \end{array}\right], \quad k = 0, 1, \ldots, \tag{55}$$

with starting value $U_n^{(0)} = U_{n-1}$. The iteration based on (54) is similar. Due to the assumption that the system is strangeness-free, the solution of the linear systems exists and if a direct solver is used then only one LU factorization is needed in each time-step. For sufficiently small time-size h, the iteration converges linearly and the approximate limit, denoted by $\bar{U}_n$, obtained in this way obviously satisfies the algebraic equation. The orthogonality can then be achieved by an additional QR factorization which yields the solution U_n with orthogonal columns.

To avoid having to solve a linear system and having to evaluate the nonlinear term repeatedly in each iteration step, one may exploit the special structure and the quasi-linearity of (38) and use instead so-called half-explicit methods (HEMs) [31]. That is, we apply an appropriate explicit discretization scheme to the differential part of (38) and simply write $A_2(t_n)U_n = 0$ for the algebraic part at the actual time $t = t_n$. As integrator for (38), we then may use e.g. an explicit Euler method. This then leads to a linear system that has to be solved in every time-step given by

$$\left[\begin{array}{c} E_1(t_{n-1}) \\ A_2(t_n) \end{array}\right] U_n = \left[\begin{array}{c} E_1(t_{n-1})U_{n-1} + h\left(E_1(t_{n-1})U_{n-1} S(t_{n-1}, U_{n-1}) + A_1(t_{n-1})U_{n-1}\right) \\ 0 \end{array}\right]. \tag{56}$$

If we assume, in addition, that the function $\dot{A}_2$ is bounded, which is a natural condition in the sensitivity analysis of the exponents, see Theorem 14, as well as in the convergence analysis of the Euler method, then for sufficiently small stepsize h, the coefficient matrix of the linear system for U_n is invertible and again only one linear system needs to be solved in each time step.

To start the continuous SVD algorithm, we first integrate the DAE (3) with an appropriate initial condition $X(0) = X_0$, see (7) until $t = t_1 > 0$. Then, we compute the SVD of the matrix solution at $t = t_1$

$$X(t_1) = U_1 \Sigma_1 V_1 \tag{57}$$

and proceed with the continuous SVD method for computing Lyapunov and Sacker-Sell spectra from $t = t_1$.

Algorithm 2 (Continuous SVD algorithm for computing Lyapunov and Sacker-Sell spectra)

Input: *A pair of sufficiently matrix functions (E, A) in the form of the strangeness-free DAE (3) (if they are not available directly they must be obtained pointwise as output of a routine such as* `GELDA`*); the first derivative of A_2 (if it is not available directly, use a finite difference formula to approximate); the values $T, \tilde{\tau}, \tau$ such that $\tilde{\tau} \in (0, T)$ and $\tau \in (0, T)$;*

Output: *Endpoints for spectral intervals $\{\lambda_i^l, \lambda_i^u\}_{i=1}^p$.*

- **Initialization:**

  1. Set $j = 0, t_0 := 0$. Compute X_0 by (7).
  2. Integrate (3) with $X(t_0) = X_0$ on $[t_0, t_1]$, $t_1 \geq t_0$.
  3. Calculate the SVD (57). Set $U(t_1) = U_1$, $V(t_1) = V_1$ and $\sigma_i(t_1) = (\Sigma_1)_i$, $i = 1, 2, \ldots, d$. Evaluate also $\nu_i(t_1)$ via (44).
  4. Compute $P(t_1)$ as described in (33).
  5. Form $\lambda_i(t_1)$, $i = 1, ..., d$ by (48).

While $t_j < T$

1. $j := j + 1$.
2. Choose a stepsize h_j and set $t_j = t_{j-1} + h_j$.
3. Evaluate U (and V, if it is desired) and ν_k, $k = 1, \ldots, d$ by integrating (38), (37), and (45) using (34),(46), and (47).
4. Compute $P(t_j)$ as in (43).
5. Compute $\lambda_i(t_j)$ by (49) and $\psi_{\tilde{\tau},i}(t_j)$ by (50).
6. If desired, test integral separation via the Steklov difference.
7. Update $\min_{\tau \le t \le t_j} \lambda_i(t)$ and $\max_{\tau \le t \le t_j} \lambda_i(t)$.

The corresponding algorithm for computing Sacker-Sell spectra is almost the same (except for the last step) where, applying (51) $\min_{\tau \le t \le T-\tilde{\tau}} \psi_{\tilde{\tau},i}(t)$ and $\max_{\tau \le t \le T-\tilde{\tau}} \psi_{\tilde{\tau},i}(t)$ are computed. For the computation of the Sacker-Sell spectra via the continuous SVD algorithm, the memory requirement is increased, since the values of the values λ_i at the previous mesh points in $[t_j - \tilde{\tau}, t_j]$ must be stored and updated as j changes.

4.3 A comparison of the continuous and discrete SVD and QR methods

At first sight, the discrete version of the SVD method seems to be more simple and more easy to implement. However, there are some serious disadvantages, which make the use of this method become less preferable. The first problem is the exponential growth of the fundamental solution. Similarly to the discrete QR method, see [37, 38], if the system has one or some large positive exponents, then we must restrict the numerical integration to small stepsizes. Otherwise the columns of X become nearly linearly-dependent and then the QR factorization may produce inaccurate results. The second disadvantage is the need to use the product SVD. On the one hand both the computational cost and the memory requirement become extremely large as the problem dimensions d and ℓ and the number of steps N increase, on the other hand it is well-known that forming the product explicitly may lead to very inaccurate or even useless results if N is large. Hence the discrete SVD technique is not feasible for large problems and long products.

For this reason we prefer the continuous SVD algorithm. To compare this approach with the continuous QR algorithm proposed and investigated in [38], we should first note that if we do not need to integrate the V-component, then the complexity of the continuous SVD algorithm is only slightly higher than that of the continuous QR algorithm. However, in the SVD method, we do not need to work with a normal basis as it is required in the QR method. We can choose any fundamental solution and proceed with it. Furthermore, if we want to determine information on the leading directions, this is easily available by incorporating the evaluation of the V factor in the algorithm. Note that for integrally separated problems the factor V converges exponentially fast. A weak point of the continuous SVD method is that we have to assume the existence of a smooth SVD which can only be guaranteed if the coefficient functions are analytic or singular values are distinct for all time t, see (35). For integrally separated systems, this latter condition is ensured only from a sufficiently large time $\bar{t}$ on. In practice, the trick of integrating the system up to a (not necessarily large) time t_1 often helps. However, even if the singular values are different, but come very close to each other, then numerical instabilities may occur in the course of integration of U that need extra treatment as suggested in [11, 40].

Finally we comment on the extra difficulties that arise, when the continuous SVD method is applied to DAEs instead of ODEs. First of all we need the derivative of the block A_2. If it is not available explicitly, then a procedure based on automatic differentiation or finite differences can be used to evaluate $\dot{A}_2$. Second, the differential equation for the factor U is a strangeness-free DAE, as well. As we have already discussed, then for the numerical integration a DAE solver must be used, which is able to preserve both the (linear) algebraic constraint and the orthonormality of the solution at the mesh points. Finally, there are some extra numerical linear algebra tasks to be performed, such as the computation of the factor P in (33) via a QR factorization and the

calculation of C in (34) via the solution of linear systems, which however, are upper triangular and generally of smaller size than that of the original problem, in particular if only $\ell < d$ spectral intervals are needed. The conditioning of these two linear algebra problems is not worse than that of the original DAE (3) which is dominated by the condition number of $\bar{E}$.

5 Numerical results

We have implemented the continuous SVD method described in Section 4 in MATLAB. The following results are obtained on an IBM computer with Intel CPU T2300 1.66 GHz. For the orthogonal integration, we have used both the implicit Euler scheme (52) combined with the fixpoint iteration (54) and the half-explicit scheme (56) discussed in the previous section.

To illustrate the properties of the procedures, we consider two examples, which are slightly modified from examples in [37, 38]. One of the examples presents a Lyapunov-regular DAE system and the second system is not Lyapunov-regular. In the second case, we calculate not only the Lyapunov spectral intervals, but also the Sacker-Sell intervals.

Example 40 Our first example is a Lyapunov-regular DAE system which is constructed similar to ODE examples in [23]. It presents a DAE system of the form (3) which is constructed by beginning with an upper triangular implicit ODE system, $\bar{E}_{1,1}(t)\dot{\bar{x}}_1 = \bar{A}_{1,1}(t)\bar{x}_1$, where

$$\bar{E}_{1,1}(t) = \begin{bmatrix} 1+\frac{1}{(t+1)^2} & 1 \\ 0 & 1+\frac{1}{t+1} \end{bmatrix}, \quad \bar{A}_{1,1}(t) = \begin{bmatrix} \lambda_1 - \frac{1}{t+1} & \omega \sin t \\ 0 & \lambda_2 + \cos(t+1) \end{bmatrix}, \quad t \in \mathbb{I}, \quad (i = 1, 2),$$

and where $\lambda_i, i = 1, 2,\ (\lambda_1 > \lambda_2)$ are given real parameters.

By increasing the parameter ω we can make the problem of computing the spectral intervals more and more ill-conditioned.

We then performed a kinematic equivalence transformation to get the implicit ODE system $\tilde{E}_{1,1}(t)\dot{\tilde{x}}_1 = \tilde{A}_{1,1}(t)\tilde{x}_1$ with coefficients

$$\tilde{E}_{11} = \bar{U}_1 \bar{E}_{1,1} \bar{V}_1^T, \quad \tilde{A}_{11} = \bar{U}_1 \bar{A}_{1,1} \bar{V}_1^T + \bar{U}_1 \bar{E}_{1,1} \bar{V}_1^T \dot{\bar{V}}_1 \bar{V}_1^T,$$

where $\bar{U}_1(t) = G_{\gamma_1}(t),\ \bar{V}_1(t) = G_{\gamma_2}(t)$, and $G(\gamma_i)$ is a Givens rotation

$$G_\gamma(t) = \begin{bmatrix} \cos\gamma t & \sin\gamma t \\ -\sin\gamma t & \cos\gamma t \end{bmatrix}$$

with some real parameters γ_1, γ_2.

We then chose additional blocks $\tilde{E}_{12} = \bar{U}_1, \tilde{A}_{12} = \bar{V}_1, \tilde{A}_{22} = \bar{U}_1 \bar{V}_1$ and finally

$$\tilde{E} = \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}.$$

Using a 4×4 orthogonal matrix

$$G(t) = \begin{bmatrix} \cos\gamma_3 t & 0 & 0 & \sin\gamma_3 t \\ 0 & \cos\gamma_4 t & \sin\gamma_4 t & 0 \\ 0 & -\sin\gamma_4 t & \cos\gamma_4 t & 0 \\ -\sin\gamma_3 t & 0 & 0 & \cos\gamma_3 t \end{bmatrix},$$

with real values γ_3, γ_4, we obtain a strangeness-free DAE system of the form (3) with coefficients $E = \tilde{E}G^T,\ A = \tilde{A}G^T + \tilde{E}G^T\dot{G}G^T$. Furthermore, because Lyapunov-regularity as well Lyapunov exponents are invariant under orthogonal changes of variable, this system is Lyapunov-regular with the Lyapunov exponents λ_1, λ_2.

For our first numerical test we have used the values

$$\lambda_1 = 1, \lambda_2 = -1, \gamma_1 = \gamma_4 = 2, \gamma_2 = \gamma_3 = 1, \omega = 1. \tag{58}$$

Table 1: Lyapunov exponents computed via continuous SVD algorithm with half-explicit Euler integrator for Example 40.

T	h	λ_1	λ_2	CPU-time in s	CPU-time in s, $\ell = 1$
500	0.1	0.9539	-0.9579	3.0156	2.7344
500	0.05	0.9720	-0.9760	5.9375	5.4375
500	0.01	0.9850	-0.9890	29.5781	27.0625
1000	0.1	0.9591	-0.9592	5.9531	5.5000
1000	0.05	0.9772	-0.9773	11.7969	10.7969
1000	0.01	0.9902	-0.9903	58.7500	54.5000
2000	0.05	0.9801	-0.9805	23.4844	21.5938
2000	0.01	0.9932	-0.9936	117.4531	107.5156
5000	0.01	0.9952	-0.9955	294.1250	268.4531
10000	0.01	0.9960	-0.9962	586.9219	537.9375

Table 2: Lyapunov exponents computed via continuous SVD algorithm with implicit Euler integrator for Example 40.

T	h	λ_1	λ_2	CPU-time in s	CPU-time in s, $\ell = 1$
500	0.1	1.0169	-1.0209	5.0781	3.6406
500	0.05	1.0028	-1.0069	9.0469	6.8594
500	0.01	0.9915	-0.9955	37.8438	32.5156
1000	0.1	1.0221	-1.0222	10.0781	7.2031
1000	0.05	1.0080	-1.0082	18.0625	13.5625
1000	0.01	0.9967	-0.9968	75.7188	63.9531
2000	0.05	1.0110	-1.0115	36.2344	26.8125
2000	0.01	0.9997	-1.0001	151.1875	127.6719
5000	0.01	1.0017	-1.0020	377.7813	319.5938
10000	0.01	1.0025	-1.0027	754.9688	638.2813

The results by the half-explicit Euler and the implicit Euler schemes are given in Tables 1 and 2. Time-savings for the reduced case $\ell = 1$ are noticeable. By comparing the two integrators, it is clearly seen that half-explicit methods promise to be competitive alternatives to fully implicit methods when solving the special class of matrix DAEs of the form (38).

Next, we investigate the dependence of the numerical results on the rotation parameters γ in this example. We set $\gamma_i = 10$ for $i = 1, 2, 3, 4$ and recalculated the Lyapunov exponents. The results by the half-explicit Euler and the implicit Euler schemes are displayed in Table 3. Clearly, smaller stepsizes are necessary. The $*$ indicates that with some larger stepsizes, the implicit Euler method even failed because the simple fixpoint iteration does not converge. Furthermore, the CPU-time of the implicit Euler method is significantly increased, since more iterations are needed.

The dependence on ω, i.e., the magnitude of the upper triangular part in $\bar{A}_{1,1}$ is presented in Tables 4 and 5, which show the numerically computed Lyapunov exponents in the case when $\omega = 10$ and $\omega = 100$, respectively. The other parameters are as in (58). We see that for larger parameters ω the computation of the Lyapunov exponents is much harder.

We have also tested the (exponential) convergence of the V-factor for different values of λ_is. In Figure 1, we plot the components V_{11} and V_{21} for $\lambda_1 = -\lambda_2 = 1$ and for $\lambda_1 = -\lambda_2 = 0.3$, respectively. Due to the larger difference between the exponents, the V-components of the first case (the highest and the lowest curves) converge very quickly to their constant limits, while those of the second case (the intermediate curves) oscillate at the beginning and only slowly converge. This illustrated the comments in Remark 31.

Table 3: Lyapunov exponents computed via half-explicit Euler and implicit Euler method for Example 40, and rotation parameter $\gamma_i = 10$.

T	h	λ_1	λ_2	CPU-time in s	λ_1	λ_2	CPU-time in s
500	0.1	0.3850	-0.3890	3.0469	*	*	*
500	0.05	0.6536	-0.6577	5.8906	*	*	*
500	0.01	0.9901	-0.9941	29.5313	0.9901	-0.9941	51.6094
1000	0.1	0.3893	-0.3894	5.9375	*	*	*
1000	0.05	0.6552	-0.6553	11.7344	*	*	*
1000	0.01	0.9952	-0.9953	58.7656	0.9951	-0.9952	103.2344
2000	0.05	0.6594	-0.6599	23.5938	*	*	*
2000	0.01	0.9981	-0.9985	117.5469	0.9979	-0.9983	205.9531
5000	0.01	1.0001	-1.0004	293.5156	0.9998	-1.0001	516.0469

Table 4: Lyapunov exponents computed via half-explicit Euler and implicit Euler method for Example 40, $\omega = 10$.

T	h	λ_1	λ_2	CPU-time in s	λ_1	λ_2	CPU-time in s
500	0.1	0.8755	-0.8796	3.0313	*	*	*
500	0.05	0.9340	-0.9380	5.9375	0.1887	-0.1927	5.9063
500	0.01	0.9777	-0.9817	29.2969	0.8901	-0.8941	29.3125
1000	0.1	0.7721	-0.7722	5.9375	1.1845	-1.1846	12.4688
1000	0.05	0.8884	-0.8885	11.7500	1.0948	-1.0949	19.3438
1000	0.01	0.9734	-0.9735	58.5781	1.0143	-1.0144	75.6250
5000	0.01	0.9780	-0.9783	293.2656	1.0190	-1.0193	377.3750

Table 5: Lyapunov exponents computed via half-explicit Euler and implicit Euler method for Example 40, $\omega = 100$.

T	h	λ_1	λ_2	CPU-time in s	λ_1	λ_2	CPU-time in s
1000	0.05	*	*	*	*	*	*
1000	0.01	0.7780	-0.7781	60.2188	1.1700	-1.1701	98.9375
1000	0.005	0.8941	-0.8942	119.0469	1.0866	-1.0868	176.4688
1000	0.001	0.9764	-0.9765	593.1719	1.0147	-1.0148	769.4375

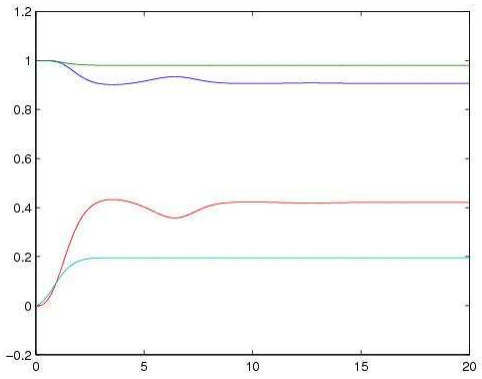

Figure 1: Graph of $V_{11}(t)$ and $V_{21}(t)$ for different λ_is in Example 40.

Table 6: Lyapunov spectral intervals computed via continuous SVD algorithm with half-explicit Euler integrator for Example 41.

T	$\tilde{\tau}$	h	$[\lambda_1^l, \lambda_1^u]$	$[\lambda_2^l, \lambda_2^u]$	CPU-time in s
1000	100	0.10	[-1.0332 0.5704]	[-5.9311 -4.6909]	6.2500
5000	100	0.10	[-1.0332 0.9851]	[-5.9311 -4.3592]	31.5469
10000	100	0.10	[-1.0332 0.9851]	[-5.9311 -3.9980]	61.8906
10000	100	0.05	[-1.0183 0.9946]	[-5.9421 -4.0107]	123.2500
20000	100	0.10	[-1.0332 0.9851]	[-5.9311 -3.9746]	123.6563
20000	100	0.05	[-1.0183 0.9946]	[-5.9421 -3.9882]	248.7969
50000	100	0.05	[-1.0183 0.9946]	[-5.9421 -3.9882]	619.2344
50000	500	0.05	[-0.9935 0.9946]	[-5.9421 -3.9882]	627.0000
100000	100	0.05	[-1.0183 0.9946]	[-5.9421 -3.9882]	1283.3
100000	500	0.05	[-1.0087 0.9946]	[-5.9421 -3.9882]	1243.4

Table 7: Sacker-Sell spectral intervals computed by continuous SVD algorithm with half-explicit Euler integrator for Example 41.

T	$\tilde{\tau}$	h	$[\kappa_1^l, \kappa_1^u]$	$[\kappa_2^l, \kappa_2^u]$	CPU-time in s
5000	100	0.10	[-0.9723 1.4003]	[-6.3636 -3.5761]	39.5469
10000	100	0.10	[-0.9723 1.4003]	[-6.3636 -3.5626]	79.4531
10000	100	0.05	[-0.9617 1.4088]	[-6.3734 -3.5764]	186.1719
20000	100	0.10	[-1.3757 1.4003]	[-6.3636 -3.5626]	158.5313
20000	500	0.10	[-1.3708 1.3898]	[-6.4497 -3.5633]	277.9063
20000	100	0.05	[-1.3577 1.4088]	[-6.3734 -3.5764]	380.7656
50000	100	0.10	[-1.4412 1.4003]	[-6.3636 -3.5626]	395.0000
50000	500	0.10	[-1.4407 1.3898]	[-6.4497 -3.5633]	705.7031
50000	100	0.05	[-1.4241 1.4088]	[-6.3734 -3.5764]	971.3750
100000	100	0.10	[-1.4412 1.4003]	[-6.3636 -3.5626]	799.8281
100000	500	0.10	[-1.4407 1.3898]	[-6.4497 -3.5633]	1370.1
100000	100	0.05	[-1.4241 1.4088]	[-6.3734 -3.5764]	1897.2

Example 41 (A DAE system which is not Lyapunov-regular) With the same transformations as in Example 40 we also constructed a DAE that is not Lyapunov-regular by changing the matrix $\bar{A}(t)$ in Example 40 to

$$\bar{A}(t) = \begin{bmatrix} \sin(\ln(t+1)) + \cos(\ln(t+1)) + \lambda_1 & \omega \sin t \\ 0 & \sin(\ln(t+1)) - \cos(\ln(t+1)) + \lambda_2 \end{bmatrix}, \quad t \in \mathbb{I}.$$

Here we have chosen $\lambda_1 = 0, \lambda_2 = -5$. The other parameters are again as in (58). Since Lyapunov and Sacker-Sell spectra are invariant with respect to global kinematical equivalence transformation, it is easy to compute the Lyapunov spectral intervals as $[-1, 1]$ and $[-6, -4]$ and the Sacker-Sell spectral intervals as $[-\sqrt{2}, \sqrt{2}]$ and $[-5-\sqrt{2}, -5+\sqrt{2}]$. The calculated Lyapunov spectral intervals are displayed in Table 6 and the calculated Sacker-Sell intervals are given in Table 7.

6 Conclusion

In this paper we have improved the spectral analysis for linear DAEs introduced in [37]. Based on the construction of an essentially underlying implicit ordinary differential equation (EUODE) which has the same spectral properties as the original DAE, we have presented new methods that are based on smooth singular value decompositions (SVD). This approach provides a unified

insight into different kinds of computational techniques for approximating spectral intervals for DAEs. A characterization of the leading directions as well as the stable and unstable solution subspaces has been given. We have also developed SVD-based methods for just few spectral intervals and their associated leading directions. Unlike the QR-based methods proposed in [37], the new SVD-methods are applied directly to DAEs of the form (3). It has been shown that, under the integral separation and some other boundedness assumptions, not only the spectral intervals, but also their associated growth directions can be approximated efficiently by the continuous SVD method.

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