A Note on Robust Representations of Law-Invariant Quasiconvex Functions

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We give robust representations of law-invariant monotone quasiconvex functions. The results are based on Jouini et al. [10] and Svindland [14], showing that law-invariant quasiconvex functions have the Fatou property.

Key Words: Fatou property, law-invariance, risk measure, robust representation

1 Introduction

The theory of *monetary risk measures* dates back to the end of the twentieth century where Artzner et al. in [1] introduced the *coherent cash additive risk measures* which were further extended to the *convex cash additive risk measures* by Föllmer and Schied in [7] and Frittelli and Gianin in [8]. Monetary risk measures aim at specifying the capital requirement that financial institutions have to reserve in order to cope with severe losses from their risky financial activities. Recently, motivated by the study of *risk orders* in a general framework, Drapeau and Kupper in [4] defined *risk measures* as quasiconvex monotone functions. Building upon the latter, the aim of this note is to specify the robust representation of risk measures in the law-invariant case.

Robust representation of law-invariant monetary risk measures for bou-nded random variables have first been studied by Kusuoka in [12], then Frittelli and Gianin in [9] and further Jouini et al. in [10]. In a recent paper [2], Cerreia-Voglio et al. provide a robust representation for law-invariant risk measures which are weakly[1] upper semicontinuous.

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[1]For the weak* topology $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$.

In this note, we provide a robust representation of law-invariant risk measures for bounded random variables which are norm lower semicontinuous. This is based on results by Jouini et al. in [10] and Svindland in [14] showing that law-invariant norm closed convex sets of bounded random variables are then Fatou closed. This robust representation takes the form

$$\rho(X) = \sup_{\psi} R\left(\psi, \int_0^1 q_{-X}(s)\,\psi(s)\,ds\right),$$

where R is a *maximal risk function* which is uniquely determined, ψ are some nondecreasing right-continuous functions whose integral is normalized to 1, and q_X is the quantile function of the random variable X. We further provide a representation in the special case of norm lower semicontinuous law-invariant *convex cash subadditive risk measures* introduced by El Karoui and Ravanelli in [13]. Finally, we give a representation of time-consistent law-invariant monotone quasiconcave functions in the spirit of Kupper and Schachermayer in [11]. We illustrate these results by a couple of explicit computations for examples of law-invariant risk measures given by certainty equivalents.

2 Notations, Definitions and the Fatou Property

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space. We identify random variables which are almost surely (a.s.) identical. All equalities and inequalities between random variables are understood in the a.s. sense. As usual, $\mathbb{L}^\infty := \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the space of bounded random variables with topological dual $(\mathbb{L}^\infty)^*$. Following [4], a risk measure is defined as follows.

Definition 2.1 *A* risk measure *on $\mathbb{L}^\infty$ is a function $\rho : \mathbb{L}^\infty \to [-\infty, +\infty]$ satisfying for any $X, Y \in \mathbb{L}^\infty$ the axioms of*

(i) Monotonicity:

$$\rho(X) \geq \rho(Y), \quad \textit{whenever } X \leq Y,$$

(ii) Quasiconvexity:

$$\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}, \quad \textit{for any } 0 \leq \lambda \leq 1.$$

Further particular risk measures used in this paper satisfy some of the following additional properties,

(i) cash additivity if $\rho(X + m) = \rho(X) - m$ for any $m \in \mathbb{R}$,

(ii) cash subadditivity if $\rho(X + m) \geq \rho(X) - m$ for any $m \geq 0$,

(iii) convexity if $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for any $0 \leq \lambda \leq 1$,

(iv) law-invariance if $\rho(X) = \rho(Y)$ whenever X and Y have the same law.

A risk measure satisfies the *Fatou property* if

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n) \quad \text{whenever} \quad \sup_n \|X_n\|_\infty < \infty \quad \text{and} \quad X_n \xrightarrow{\mathbb{P}} X, \tag{1}$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. A reformulation of the results in [10] in the context of quasiconvex law-invariant functions yields the following result.

Proposition 2.2 *Let $f : \mathbb{L}^\infty \to [-\infty, +\infty]$ be a $||\cdot||_\infty$-lower semicontinuous, quasiconvex and law-invariant function. Then, f is $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-lower semicontinuous and has the Fatou property.*

Proof. Let $C \subset \mathbb{L}^\infty$ be a $||\cdot||_\infty$-closed, convex, law-invariant set with polar C° in $(\mathbb{L}^\infty)^*$. In view of Proposition 4.1 in [10], it follows that $C^\circ \cap \mathbb{L}^1$ is $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-dense in C. Hence, $C = (C^\circ \cap \mathbb{L}^1)^\circ$, showing that C is $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-closed.

Consider now a law-invariant, quasiconvex and $||\cdot||_\infty$-lower semicontinuous function $f : \mathbb{L}^\infty \to [-\infty, +\infty]$. By assumption, the level sets $\mathcal{A}_m := \{X \in \mathbb{L}^\infty \mid f(X) \leq m\}$, $m \in \mathbb{R}$, are $||\cdot||_\infty$-closed, convex and law-invariant. Hence, $\mathcal{A}_m$ are $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-closed, showing that f is $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-lower semicontinuus. Finally a similar argumentation as in [3] yields the Fatou property. $\square$

Remark 2.3 *Any law-invariant, proper convex function $f : \mathbb{L}^\infty \to [-\infty, \infty]$ is $\sigma(\mathbb{L}^\infty, \mathbb{L}^\infty)$-lower semicontinuous, see [5].*

Recently, a similar result is shown in [14] in the more general setting of non-atomic probability spaces rather than standard probability spaces.

3 Representation results for law-invariant risk measures

Throughout,

$$q_X(t) := \inf\{s \in \mathbb{R} | \mathbb{P}[X \leq s] \geq t\}, \qquad t \in (0,1)$$

denotes the quantile function of a random variable $X \in \mathbb{L}^\infty$. Further, let Ψ be the set of nondecreasing, right-continuous functions $\psi : (0,1) \to [0, +\infty)$ and define the subsets

$$\Psi_1 := \left\{ \psi \in \Psi \,\Big|\, \int_0^1 \psi(u)\, du = 1 \right\},$$
$$\Psi_{1,s} := \left\{ \psi \in \Psi \,\Big|\, \int_0^1 \psi(u)\, du \leq 1 \right\},$$

and denote by Ψ_1^∞ and $\Psi_{1,s}^\infty$ the set of all bounded functions in Ψ_1 and $\Psi_{1,s}$, respectively. It is shown in [6], Theorem 4.54, that any law-invariant cash additive risk measure ρ on $\mathbb{L}^\infty$ that satisfies the Fatou property has the robust representation

$$\rho(X) = \sup_{\psi \in \Psi_1} \left(\int_0^1 \psi(t) q_{-X}(t)\, dt - \alpha_{min}(\psi) \right), \quad X \in \mathbb{L}^\infty, \tag{2}$$

where $\alpha_{\min}(\psi) = \sup_{X \in \mathcal{A}_\rho} \int_0^1 \psi(t) q_{-X}(t)\, dt$ is the minimal penalty function for the acceptance set $\mathcal{A}_\rho := \{X \in \mathbb{L}^\infty \mid \rho(X) \leq 0\}$.

In a first step, we derive the following representation result for law-invariant cash sub-additive convex risk measures.

Proposition 3.1 *Let ρ be a law-invariant cash sub-additive convex risk measure on $\mathbb{L}^\infty$. Then ρ has the robust representation*

$$\rho(X) = \sup_{\psi \in \Psi_{1,s}^\infty} \left(\int_0^1 \psi(t) q_{-X}(t)\, dt - \alpha_{\min}(\psi) \right), \quad X \in \mathbb{L}^\infty,$$

for the minimal penalty function

$$\alpha_{\min}(\psi) = \sup_{X \in \mathbb{L}^\infty} \left(\int_0^1 \psi(t) q_{-X}(t)\, dt - \rho(X) \right), \quad \psi \in \Psi_{1,s}.$$

Proof. According to Theorem 4.3 in [13] it follows

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{M}_{1,s}(\mathbb{P})} \left(\mathbb{E}_{\mathbb{Q}}[-X] - \tilde{\alpha}_{\min}(\mathbb{Q}) \right),$$

where $\tilde{\alpha}_{\min}(\mathbb{Q}) = \sup_{X \in \mathbb{L}^\infty} (\mathbb{E}_{\mathbb{Q}}[-X] - \rho(X))$ and $\mathcal{M}_{1,s}(\mathbb{P})$ denotes the set of measures $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$, which density satisfy $\mathbb{E}[d\mathbb{Q}/d\mathbb{P}] \leq 1$. By Lemma 4.55 in [6] and the law-invariance of ρ we deduce

$$\begin{aligned} \tilde{\alpha}_{\min}(\mathbb{Q}) &= \sup_{X \in \mathbb{L}^\infty} \sup_{Y \sim X} \left(\mathbb{E}_{\mathbb{Q}}[-Y] - \rho(Y) \right) \\ &= \sup_{X \in \mathbb{L}^\infty} \left(\int_0^1 \psi(t) q_{-X}(t)\, dt - \rho(X) \right) = \alpha_{min}(\psi), \end{aligned}$$

for any $\psi \in \Psi_{1,s}$ and $\mathbb{Q} \in \mathcal{M}_{1,s}(\mathbb{P})$ with $\psi = q_{d\mathbb{Q}/d\mathbb{P}}$. Finally, under consideration of Remark 2.3 and Lemma 4.55 in [6], it follows

$$\begin{aligned} \rho(X) &= \sup_{\mathbb{Q} \in \mathcal{M}_{1,s}^\infty(\mathbb{P})} \left(\mathbb{E}_{\mathbb{Q}}[-X] - \tilde{\alpha}_{\min}(\mathbb{Q}) \right) \\ &= \sup_{\mathbb{Q} \in \mathcal{M}_{1,s}^\infty(\mathbb{P})} \sup_{\tilde{\mathbb{Q}} \sim \mathbb{Q}} \left(\mathbb{E}_{\tilde{\mathbb{Q}}}[-X] - \tilde{\alpha}_{\min}(\tilde{\mathbb{Q}}) \right) \\ &= \sup_{\psi \in \Psi_{1,s}^\infty} \left(\int_0^1 \psi(t) q_{-X}(t) dt - \alpha_{\min}(\psi) \right), \end{aligned}$$

where $\mathcal{M}_{1,s}^\infty(\mathbb{P})$ are those elements in $\mathcal{M}_{1,s}(\mathbb{P})$ with a bounded Radon-Nikodým derivative. $\square$

As a second step, we state our main result: a quantile representation for $||\cdot||_\infty$-lower semicontinuous law-invariant risk measures. Beforehand, as in [4], we define the class of *maximal risk functions* $\mathcal{R}^{\max}$ as the set of functions $R : \Psi_1 \times \mathbb{R} \to [-\infty, +\infty]$ which

(i) are nondecreasing and left-continuous in the second argument,

(ii) are jointly quasiconcave,

(iii) have a uniform asymptotic minimum, that is,

$$\lim_{s \to -\infty} R(\psi_1, s) = \lim_{s \to -\infty} R(\psi_2, s)$$

for any $\psi_1, \psi_2 \in \Psi_1$,

(iv) right-continuous version $R^+(\psi, s) := \inf_{s'>s} R(\psi, s')$, are $\sigma\left(\mathbb{L}^1, \mathbb{L}^\infty\right)$-upper semicontinous in the first argument.

Theorem 3.2 *Let $\rho : \mathbb{L}^\infty \to [-\infty, +\infty]$ be a risk measure which is $||\cdot||_\infty$-lower semicontinuous. Then, there exists a unique maximal risk function $R \in \mathcal{R}^{\max}$ such that*

$$\rho(X) = \sup_{\psi \in \Psi_1} R\left(\psi, \int_0^1 q_{-X}(t)\psi(t)\, dt\right), \quad X \in \mathbb{L}^\infty$$

where

$$R(\psi, x) = \sup_{m \in \mathbb{R}} \left\{ m \;\middle|\; \alpha_{\min}(\psi, m) < x \right\}, \quad \psi \in \Psi_1$$

for

$$\alpha_{\min}(\psi, m) = \sup_{X \in \mathcal{A}^m} \int_0^1 q_{-X}(t)\psi(t)\, dt$$

and $\mathcal{A}^m = \{X \in \mathbb{L}^\infty \mid \rho(X) \leq m\}$.

The proof of the previous theorem is based on the following proposition.

Proposition 3.3 *Suppose that $\mathcal{A} \subset \mathbb{L}^\infty$ is law-invariant, $\|\cdot\|_\infty$-closed, convex and such that $\mathcal{A} + \mathbb{L}^\infty_+ \subset \mathcal{A}$. Then*

$$X \in \mathcal{A} \quad \Longleftrightarrow \quad \int_0^1 q_{-X}(t)\psi(t)\, dt \leq \alpha_{\min}(\psi) \quad \textit{for all } \psi \in \Psi_1, \tag{3}$$

where

$$\alpha_{\min}(\psi) := \sup_{X \in \mathcal{A}} \int_0^1 q_{-X}(t)\psi(t)\, dt, \quad \psi \in \Psi_1.$$

Proof. Associated to the set $\mathcal{A}$ we define

$$\rho_{\mathcal{A}}(X) := \inf \{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}, \quad X \in \mathbb{L}^\infty.$$

The function $\rho_{\mathcal{A}} : \mathbb{L}^\infty \to [-\infty, +\infty]$ is a law-invariant, convex risk measure. Since

$$\{X \in \mathbb{L}^\infty \,|\, \rho_{\mathcal{A}}(X) \leq m\} = \mathcal{A} - m, \tag{4}$$

which by Proposition 2.2 is $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-closed, it follows that $\rho_{\mathcal{A}}$ is $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-l.s.c.. Moreover, one of the following cases must be valid:

(i) $\mathcal{A} = \emptyset$, $\rho_{\mathcal{A}} \equiv +\infty$ and $\alpha_{\min} \equiv -\infty$;

(ii) $\mathcal{A} = \mathbb{L}^\infty$, $\rho_{\mathcal{A}} \equiv -\infty$ and $\alpha_{\min} \equiv +\infty$;

(iii) $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \neq \mathbb{L}^\infty$, in which case $\rho_{\mathcal{A}}$ is real valued. Indeed, if there is $X, Y \in \mathbb{L}^\infty$ such that $X \in \mathcal{A}$ and $Y \notin \mathcal{A}$, then there is $n \in \mathbb{R}$ such that $X + n \notin \mathcal{A}$ showing that $\rho_{\mathcal{A}}(X) \in \mathbb{R}$. By monotonicity and translation invariance of $\rho_{\mathcal{A}}$, it follows that $\rho_{\mathcal{A}}(Z) \in \mathbb{R}$ for all $Z \in \mathbb{L}^\infty$.

For the cases (i) and (ii), the equivalence (3) is obvious. As for the third case, it follows from (2) that

$$\rho_{\mathcal{A}}(X) = \sup_{\psi \in \Psi_1} \left(\int_0^1 q_{-X}(t)\psi(t)dt - \alpha_{\min}(\psi) \right),$$

which together with (4) implies (3). $\square$

We are now ready for the proof of Theorem 3.2.

Proof. The *risk acceptance family* $\mathcal{A} = (\mathcal{A}_m)_{m \in \mathbb{R}}$ defined as

$$\mathcal{A}^m := \{X \in \mathbb{L}^\infty \mid \rho(X) \leq m\},$$

are law-invariant, $||\cdot||_\infty$-closed, convex and such that $\mathcal{A} + \mathbb{L}^\infty_+ \subset \mathcal{A}$. Thus, Proposition 3.3 implies

$$X \in \mathcal{A}^m \quad \Longleftrightarrow \quad \int_0^1 q_{-X}(t)\psi(t)dt - \alpha_{\min}(\psi, m) \leq 0 \quad \text{for all } \psi \in \Psi_1, \tag{5}$$

for the family of penalty functions

$$\alpha_{\min}(\psi, m) = \sup_{X \in \mathcal{A}^m} \int_0^1 q_{-X}(t)\psi(t)\, dt, \quad \psi \in \Psi_1.$$

Since for all $X \in \mathbb{L}^\infty$

$$\rho(X) = \inf \left\{ m \in \mathbb{R} \;\middle|\; X \in \mathcal{A}^m \right\}, \tag{6}$$

it follows from (5) that

$$\rho(X) = \inf \left\{ m \in \mathbb{R} \;\middle|\; \int_0^1 q_{-X}(t)\psi(t)\, dt \leq \alpha_{\min}(\psi, m) \text{ for all } \psi \in \Psi_1 \right\}. \tag{7}$$

The goal is to show that

$$\rho(X) = \sup_{\psi \in \Psi_1} \inf_{m \in \mathbb{R}} \left\{ m \;\middle|\; \int_0^1 q_{-X}(t)\psi(t)\, dt \leq \alpha_{\min}(\psi, m) \right\}. \tag{8}$$

To begin with, the equation (7) implies:

$$\rho(X) \geq \sup_{\psi \in \Psi_1} \inf_{m \in \mathbb{R}} \left\{ m \;\middle|\; \int_0^1 q_{-X}(t)\psi(t)\, dt \leq \alpha_{\min}(\psi, m) \right\}.$$

As for the reverse inequality, suppose that $\rho(X) > -\infty$, otherwise (8) is trivial, and fix $m_0 < \rho(X)$. Define $C = \{Y \in \mathbb{L}^\infty \mid \rho(Y) \leq m_0\}$, which is law-invariant, $\|\cdot\|_\infty$-closed, convex, such that $C + \mathbb{L}^\infty \subset C$. Thus, Proposition 3.3 yields

$$Y \in C \quad \Longleftrightarrow \quad \int_0^1 q_{-Y}(t)\psi(t)\, dt \leq \alpha_C(\psi) \quad \text{for all } \psi \in \Psi_1, \tag{9}$$

for the penalty function $\alpha_C(\psi) = \sup_{Y \in C} \int_0^1 q_{-Y}(t)\psi(t)\, dt$. Since $X \notin C$, it follows from (9) that there is $\psi^* \in \Psi_1$ such that

$$\int_0^1 q_{-X}(t)\psi^*(t)\, dt > \alpha_C(\psi^*) \geq \int_0^1 q_{-Y}(t)\psi^*(t)\, dt \quad \text{for all } Y \in C. \tag{10}$$

Since $\mathcal{A}^m \subset C$ for all $m \le m_0$ and therefore $\alpha_{\min}(\psi^*, m) \le \alpha_C(\psi^*)$, it follows

$$\int_0^1 q_{-X}(t)\psi^*(t)\,dt - \alpha_{\min}(\psi^*, m) \ge \int_0^1 q_{-X}(t)\psi^*(t)\,dt - \sup_{Y\in C}\int_0^1 q_{-Y}(t)\psi^*(t)\,dt > 0. \quad (11)$$

Hence,

$$m_0 \le \sup_{\psi\in\Psi_1} \inf_{m\in\mathbb{R}} \left\{ m \;\middle|\; \int_0^1 q_{-X}(t)\psi(t)\,dt \le \alpha_{\min}(\psi, m) \right\}. \quad (12)$$

Since (12) holds for all $m_0 < \rho(x)$ we deduce

$$\rho(X) \le \sup_{\psi\in\Psi_1} \inf_{m\in\mathbb{R}} \left\{ m \;\middle|\; \int_0^1 q_{-X}(t)\psi(t)\,dt \le \alpha_{\min}(\psi, m) \right\},$$

and (8) is established.

Let $R(\psi, x) := \sup_{m\in\mathbb{R}} \left\{ m \;\middle|\; \alpha_{\min}(\psi, m) < x \right\}$ be the left-inverse of $\alpha_{\min}$. Then

$$\rho(X) = \sup_{\psi\in\Psi_1} R\left(\psi, \int_0^1 q_{-X}(t)\psi(t)\,dt\right) \quad \text{for all } X \in \mathbb{L}^\infty. \quad (13)$$

The proof of the existence is completed. The uniqueness follows from a similar argumentation as in [4]. $\square$

4 Time-consistent law-invariant quasiconcave functions

As an application of Proposition 2.2 we discuss an extension of the representation results for time-consistent law-invariant strictly monotone functions given in [11]. In this subsection, we work on a standard filtered probability space[2] $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}_0}, \mathbb{P})$. We fix $-\infty \le a < b \le +\infty$ and denote by $\mathbb{L}_t^\infty(a,b)$ and $\mathbb{L}^\infty(a,b)$ the set of all random variables X such that $a < \operatorname{ess\,inf} X \le \operatorname{ess\,sup} X < b$ and which are $\mathcal{F}_t$-measurable and $\mathcal{F}$-measurable, respectively. A function $c_0 : \mathbb{L}^\infty(a,b) \to \mathbb{R}$ is

(i) *normalized on constants* if $c_0(m) = m$ for all $m \in (a,b)$;

(ii) *strictly monotone* if $X \ge Y$ and $\mathbb{P}[X > Y] > 0$ imply $c_0(X) > c_0(Y)$;

(iii) *time-consistent* if for any $t \in \mathbb{N}$ there exists a mapping $c_t : \mathbb{L}^\infty(a,b) \to \mathbb{L}_t^\infty(a,b)$ which satisfies the *local property*, that is, for any $X, Y \in \mathbb{L}^\infty(a,b)$

$$1_A X = 1_A Y \quad \text{implies} \quad 1_A c_t(X) = 1_A c_t(Y) \quad \text{for all } A \in \mathcal{F}_t, \quad (14)$$

and

$$c_0(X) = c_0(c_t(X)) \quad \text{for all } X \in \mathbb{L}^\infty(a,b). \quad (15)$$

[2]Recall that a standard filtered probability space is isomorphic to $\left([0,1]^{\mathbb{N}_0}, \mathcal{B}([0,1]^{\mathbb{N}_0}), (\mathcal{F}_t)_{t\in\mathbb{N}_0}, \lambda^{\mathbb{N}_0}\right)$ where $\mathcal{B}([0,1]^{\mathbb{N}_0})$ is the Borel sigma-algebra, $\lambda^{\mathbb{N}_0}$ is the product of Borel measures, and $(\mathcal{F}_t)_{t\in\mathbb{N}_0}$ is the filtration generated by the coordinate functions.

Under the additional assumtion of quasiconcavity we deduce as a corallary of Theorem 1.4 in [11]:

Theorem 4.1 *A function $c_0 : \mathbb{L}^\infty(a,b) \to \mathbb{R}$ is normalized on constants, strictly monotone, $||\cdot||_\infty$-continuous, law-invariant, time-consistent and quasiconcave if and only if*

$$c_0(X) = u^{-1} \circ \mathbb{E}\left[u(X)\right], \tag{16}$$

for an increasing, concave function $u : (a,b) \to \mathbb{R}$. In this case, the function u is uniquely defined up to positive affine transformations, and

$$c_t(X) = u^{-1} \circ \mathbb{E}\left[u(X) \mid \mathcal{F}_t\right] \quad \text{for all } t \in \mathbb{N}. \tag{17}$$

Proof. Fix a compact interval $[A,B] \subset (a,b)$. Since $\mathbb{L}^\infty[A,B] := \{X \in \mathbb{L}^\infty \mid A \le X \le B\}$ is $||\cdot||_\infty$-closed in $\mathbb{L}^\infty$, it follows that

$$c_0^{A,B}(X) := \begin{cases} c_0(X) & \text{if } X \in \mathbb{L}^\infty[A,B] \\ -\infty & \text{else} \end{cases}, \quad X \in \mathbb{L}^\infty,$$

is law-invariant, quasiconcave and $||\cdot||_\infty$-upper semicontinuous. Due to Proposition 2.2, the function $c_0^{A,B}$ has the Fatou property and consequently the condition (C) in [11] is satisfied. Hence, by Theorem 1.4 in [11] there is $u_{A,B} : (A,B) \to \mathbb{R}$ such that

$$c_0^{A,B}(X) = u_{A,B}^{-1} \circ \mathbb{E}\left[u_{A,B}(X)\right], \quad X \in \mathbb{L}^\infty(A,B).$$

Exhausting (a,b) by increasing compact intervals, as in the proof of the "only if"-part of Theorem 1.4 in [11], yields $u : (a,b) \to \mathbb{R}$ such that $c_0(X) = u^{-1} \circ \mathbb{E}[u(X)]$ for all $X \in \mathbb{L}^\infty(a,b)$. Finally, it is shown in Lemma 2 in [2] that c_0 is quasiconcave if and only if u is concave. □

5 Examples

The certainty equivalent of a random variable provides a typical example of a law-invariant risk measure which is not necessarily convex nor cash additive. Let $l : \mathbb{R} \to]-\infty, +\infty]$ be a loss function, that is, a lower semicontinuous proper convex nondecreasing function. By $l^{-1} : \mathbb{R} \to [-\infty, +\infty[$ we denote the left inverse of l given by

$$l^{-1}(s) = \inf\left\{x \in \mathbb{R} \,\middle|\, l(x) \ge s\right\}, \quad s \in \mathbb{R}.$$

We further denote by $l(x+) := \lim_{t \searrow x} l(t)$ for $x \in \mathbb{R}$ the right-continuous version of l. By Proposition B.2 in [4], we have

$$l^{-1}(s) \le x \quad \Longleftrightarrow \quad s \le l(x+). \tag{18}$$

We now define the risk measure

$$\rho(X) := l^{-1}\mathbb{E}\left[l(-X)\right], \qquad X \in \mathbb{L}^\infty, \tag{19}$$

with convention that $l^{-1}(+\infty) = \lim_{s\to+\infty} l^{-1}(s)$. In [2, 4] it is shown that ρ is a risk measure. Note that in [2], it is assumed that $l : \mathbb{R} \to \mathbb{R}$ and increasing, and therefore does not include some of the examples below. In [4], a constructive method is given to compute the robust representation. To be self contained, we present this method in the law-invariant context hereafter. For simplicity we suppose that l is differentiable on the interior of its domain and first compute the minimal penalty function at any risk level m. From relation (18) follows

$$\begin{aligned}
\alpha_{\min}(\psi, m) &= \sup_{X\in\mathcal{A}^m} \int_0^1 q_{-X}(s)\psi(s)\,ds \\
&= \sup_{\left\{X \mid \int_0^1 l(q_{-X}(s))ds \leq l(m+)\right\}} \int_0^1 q_{-X}(s)\psi(s)\,ds \\
&= \sup_{X\in\mathbb{L}^\infty} \int_0^1 \left[q_{-X}(s)\psi(s) - \frac{1}{\beta}\Big(l(q_{-X}(s)) - l(m+)\Big)\right] ds.
\end{aligned} \tag{20}$$

for some Lagrange multiplier $\beta := \beta(\psi, m) > 0$. The first order conditions implies

$$\psi - \frac{1}{\beta} l'\left(q_{-\hat{X}}\right) = 0 \qquad \Longleftrightarrow \qquad q_{-\hat{X}} = h(\beta\psi),$$

for the optimizer $q_{-\hat{X}}$ of (20) where h denotes the left inverse of l' as it is nondecreasing. Therefore, β is determined through the equation

$$\int_0^1 l\Big(h\big(\beta\psi(s)\big)\Big)ds = l(m+). \tag{21}$$

Plugging the optimizer $q_{-\hat{X}}$ in (20) yields

$$\alpha(\psi, m) = \int_0^1 h(\beta(\psi, m)\psi(s))\psi(s)\,ds. \tag{22}$$

We subsequently list closed form solutions for some specific l whereby we explicit the computations only for the first one.

- **Quadratic Functional:** Suppose that $l(x) = x^2/2 + x$ for $x \geq -1$ and $l(x) = -1/2$ elsewhere. In this case, $E[l(-X)]$ corresponds to a monotone version of the mean-variance risk measure of MARKOWITZ. Here, $l^{-1}(s) = \sqrt{2s+1} - 1$ if $s \geq -1/2$ and $-\infty$ elsewhere, therefore

$$\rho(X) := \begin{cases} \sqrt{2\mathbb{E}\left[-X + \dfrac{X^2}{2}\right] + 1} - 1 & \text{if } \mathbb{E}[-X] + \dfrac{1}{2}\mathbb{E}\left[X^2\right] \geq -\dfrac{1}{2} \\ -\infty & \text{else} \end{cases}. \tag{23}$$

  Direct computations yields $l'(x) = x + 1$ for $x \geq -1$ and $l'(x) = 0$ elsewhere, $h(s) = s - 1$ for $s \geq 0$ and $h(s) = +\infty$ elsewhere implying $l(h(s)) = 1/2\left(s^2 - 1\right)$ for $s \geq 0$ and $l(h(s)) = -1/2$ elsewhere. From relation (21) follows $\beta(\psi, m) = (m+1)/(\int_0^1 \psi(s)^2\,ds)^{1/2}$. Plugging into relation (22) yields

$$\alpha(\psi, m) = \int_0^1 \beta(\psi, m)\psi(s)^2\,ds - 1 = (1+m)\left(\int_0^1 \psi(s)^2 ds\right)^{1/2} - 1$$

for $m \geq -1$ and $\alpha(\psi, m) = -1$ otherwise. Inverting yields

$$R(\psi, s) = (s+1) / (\int_0^1 \psi(s)^2 ds)^{1/2} - 1$$

if $s > 1$, and $R(\psi, s) = -\infty$ elsewhere, and therefore

$$\rho(X) = \sup_{\psi \in \Psi_1} \left\{ \frac{\int_0^1 q_{-X}(s) \psi(s) ds + 1}{\left(\int_0^1 \psi(s)^2 ds\right)^{1/2}} \,\middle|\, \int_0^1 q_{-X}(s) \psi(s) ds > 1 \right\}. \tag{24}$$

- **Exponential Function:** If $l(x) = e^x - 1$, then

$$\rho(X) := \ln\left(\mathbb{E}\left[e^{-X}\right]\right) = \sup_{\psi \in \Psi_1} \left\{ \int_0^1 \left(q_{-X}(s) \psi(s) - \psi(s) \log \psi(s) \right) ds \right\}. \tag{25}$$

- **Logarithm Function:** If $l(x) = -\ln(-x)$ for $x < 0$ and $l = -\infty$ elsewhere, then

$$\begin{aligned} \rho(X) &:= -\exp(\mathbb{E}[\ln(X)]) \\ &= \sup_{\psi \in \Psi_1} \left\{ \frac{1}{\exp\left(\int_0^1 \ln \psi(s)\, ds\right)} \int_0^1 q_{-X}(s) \psi(s) ds \right\}. \end{aligned} \tag{26}$$

- **Power Functions:** If $l(x) = -(-x)^{1-\gamma} / (1-\gamma)$ for $x \leq 0$ and $l = -\infty$ elsewhere whereby $0 < \gamma < 1$, we obtain

$$\rho(X) = \sup_{\psi \in \Psi_1} \left\{ \left(\int_0^1 \psi(s)^{\frac{\gamma - 1}{\gamma}} ds \right)^{\frac{\gamma}{1-\gamma}} \int_0^1 q_{-X}(s) \psi(s) ds \right\}. \tag{27}$$

References

[1] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D. (1999). Coherent Risk Measures. *Mathematical Finance,* **9(3)**, 203–228.

[2] Cerreia-Vioglio, S.,Maccheroni, F., Marinacci, M., Montrucchio, L. (2010). Risk Measures: Rationality and Diversification. Forthcoming in *Mathematical Finance*.

[3] Delbaen, F. (2002). Coherent measure of risk on general probability spaces. In: *Advances in Finance and Stochastics, Essays in Honor of Dieter Sondermann* (K. Sandmann and P.J. Schonbucher eds.) 1-37, Springer, Berlin.

[4] Drapeau, S., Kupper, M. (2010). Risk Preferences and their Robust Representation, Preprint (SSRN).

[5] Filipović, D., Svindland, G. (2008). The Canonical Model Space for Law-invariant Convex Risk Measures is L^1. Forthcoming in *Mathematical Finance*

[6] Föllmer, H., Schied, A., (2002). Stochastic Finance, An Introduction in Discrete Time. *de Gruyter Studies in Mathematics 27.*

[7] Föllmer, H., Schied, A., (2002) Convex measure of risk and trading constraints. *Finance Stoch.* 6, 429-447

[8] Frittelli, M., Rosazza Gianin, E (2002). Putting order in risk measures. *Journal of Banking and Finance,* **26(7)**, 1473-1486.

[9] Frittelli, M., Rosazza Gianin, E. (2005). Law-invariant convex risk measures. *Advances in Mathematical Economics* **7**, 33–46.

[10] Jouini, E., Schachermayer, W., Touzi, N. (2006). Law invariant risk measures have the Fatou property. *Advances in Mathematical Economics,* **9**, 49–71, Springer, Tokyo.

[11] Kupper, M., Schachermayer, W. (2009). Representation Results for Law Invariant Time Consistent Functions. *Mathematics and Financial Economics*, **2(3)**.

[12] Kusuoka S. (2001). On law-invariant coherent risk measures. *Advances in Mathematical Economics*, **3**, 83–95.

[13] El Karoui, N., Ravanelli, C. (2008) Cash Sub-additive Risk Measures and Interest Rate Ambiguity. Forthcoming in *Mathematical Finance*

[14] Svindland, G. (2010). Continuity Properties of Law-Invariant (Quasi-)Convex Risk Functions, Forthcoming in *Mathematics and Financial Economics.*