Model Reduction for a Class of Nonlinear Electrical Circuits by Reduction of Linear Subcircuits

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Abstract

We analyze a model reduction approach for a class of electrical circuits containing nonlinear resistance. In our approach the linear subcircuits are extracted and replaced with linear passive reduced-order models. The resulting nonlinear reduced-order model preserves admissibility and passivity. Moreover, we derive a-priori bounds for the error between the input-output map of the original circuit equations and that of our reduced-order model. These bounds are valid for all inputs. Since only linear subcircuits are reduced, our approach is effective when the number of nonlinear resistances is relatively small. The performance of our approach is illustrated numerically.

Keywords: electrical circuits, passive systems, model reduction, modified nodal analysis

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1. Introduction

Model reduction is an attractive tool for the fast simulation of very large circuits. The goal of model reduction is to replace the very large system of differential algebraic equations (DAEs) that model the circuit by a small system of DAEs so that the small system approximates the input-output of the original system and preserves essential physical properties such as stability and passivity. While several model reduction approaches exists for linear time-invariant systems, only a few techniques have been proposed for

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nonlinear circuits and for none of the existing approaches global bounds for the error between full and reduced-order circuit equations exist. This paper presents and analyzes a model reduction approach for a class of circuits that contain few nonlinear resistors. Our approach extracts the linear subcircuit and applies standard passivity preserving model reduction to this subcircuit with auxiliary inputs and outputs given by the connectivity with the nonlinear subcircuit. We prove admissibility, stability and passivity of the resulting reduced-order model, and we prove global error bounds. The performance of our approach is illustrated numerically.

For linear and time-invariant circuits, a variety of passivity-preserving model reduction methods exist. These can be classified into interpolation-based, e.g., [1, 2, 3, 4, 5, 6] and methods based on balanced truncation [7, 8, 9, 10, 11]. An overview of several approaches and a comparison of their performance on linear circuit equations is given in [12]. Although they have been successfully applied in circuit simulation, interpolation based methods provide approximations which are only known to be good at certain local frequency ranges and, so far, there exist no global error bounds. In contrast, passivity-preserving balanced truncation provides computable error bounds that allow an adaptive choice of the order of the approximate model.

Model reduction approaches for nonlinear circuits are described in [13, 14, 15, 16, 17]. The technique in [16, 17] is based on piecewise linearization of the nonlinearities. The points around which the nonlinearity is linearized is detrmined by using 'training' inputs. A-posteriori error estimates are possible, but this approach does not allow for a-priori error estimates. The papers [13, 14, 15] use so-called Volterra series to expand the nonlinear systems into a system of linear time invariant systems and use moment matching to derive reduced order models for the latter. No bounds for the error in the input-output maps are given. The model reduction approach presented in this paper does not involve additional state variables, it preserves passivity, and we can prove global a-priori error bounds. However, our approach is only effective for circuits with relatively few nonlinear elements.

The idea behind our model reduction approach is simple. We extract the linear and the nonlinear subcircuit, and the interconnections between them. Then we apply a passivity preserving model reduction method to the linear subcircuit with auxiliary inputs and outputs to the linear subcircuit determined by the interconnections. The coupled reduced order model the consists of the reduced order linear subcircuit coupled to the original nonlinear subcircuit. We prove admissibility, stability and passivity of the coupled reduced order model. Reduction of the linear subcircuit is performed using the passivity-preserving balanced truncation method introduces in [11]. This method exploits the special structure of the circuit equations and also provides a bound for the error between the input-to-output map of the original linear subcircuit and the input-to-output map of the reduced order subcircuit. This error bound will be used to derive error bounds for our nonlinear reduced order model.

The development of reduced order model for nonlinear problems by extraction of large-scale linear time-invariant subproblems, and the model reduction of these linear subproblems is not new. For example, the papers [18, 19] present such an approach for systems of ordinary equations (ODEs) that arise from spatial discretizations of partial differential equations. The paper [18] presents an a-priori error estimate. The analysis of the reduced order model apprach presented in this paper is different from that in [18]. Our approach exploits the structure of the circuit equations. This enables us to prove admissibility and passivity of the reduced order models. Admissibility trivially satisfied for the ODE models in [18] and passivity plays no role in the problems studied in [18]. The techniques used to derive the a-priori error bound in this paper are tailored to the circuit equations and are different from those used in [18]. Some basic results for our approach are outlined in the proceedings paper [20]. This paper extends the error analysis given in [20] and presents numerical examples on a larger circuit provided by NEC Laboratories Europe, IT Research Division.

The paper is organized as follows. First, we briefly review the basic framework of linear and nonlinear circuit theory. In Section 3 we study properties of a class of descriptor systems which includes our circuit equations. In particular, we will introduce the concepts of incremental passivity and contractivity, we will provide sufficient conditions for these properties to hold, and we will verify incremental passivity of our circuit equations and contractivity of the corresponding scattering equations. The passivity-preserving balanced truncation model reduction method for circuit equations developed in [11] is reviewed in Section 4. We will present our model reduction approach in Section 5 and we will show that the resulting reduced order system remains incrementally passive. Global error bounds for our model reduction approach are derived in Section 6. Finally, the performance of our approach for the model reduction of a circuit provided by NEC Laboratories Europe, IT Research Division is illustrated in Section 7.

Throughout the paper $\mathbb{R}^{n,m}$ denotes the set of $n \times m$ real matrices. For symmetric matrices $P, Q \in \mathbb{R}^{n,n}$ we write $P \succ Q$ ($P \succeq Q$) if $P - Q$ is positive (semi)definite. The Euclidean vector norm and the spectral matrix norm are denoted by $\|\cdot\|$. The L_2-norm of a function $f : [t_0, t_f] \to \mathbb{R}^m$ is denoted by $\|f\|_{L_2([t_0,t_f])^m}$, or just $\|f\|_{L_2}$ if dimension and time horizon are clear from the context.

2. Modified Nodal Analysis

A general electrical circuit can be modeled as a directed graph whose nodes correspond to the nodes of the circuit and whose branches correspond to the circuit elements like capacitors, inductors, resistors, diodes and transistors. Using Kirchhoff's current and voltage laws as well as the branch constitutive relations, linear RLC circuits consisting only of linear resistors, inductors, capacitors and independent current and voltage sources can be described via Modified Nodal Analysis (MNA) by the following system of DAEs

$$\begin{aligned} A_{\mathcal{C}} \tfrac{d}{dt} q(A_{\mathcal{C}}^T \eta(t)) =& - A_{\mathcal{R}} g(A_{\mathcal{R}}^T \eta(t)) - A_{\mathcal{L}} i_{\mathcal{L}}(t) - A_{\mathcal{V}} i_{\mathcal{V}}(t) - A_{\mathcal{I}} i_{\mathcal{I}}(t), \\ \tfrac{d}{dt} \Psi(i_{\mathcal{L}}(t)) =& A_{\mathcal{L}}^T \eta(t), \\ 0 =& A_{\mathcal{V}}^T \eta(t) - v_{\mathcal{V}}(t) \end{aligned} \tag{1}$$

with initial conditions

$$\eta(t_0) = \eta_0, \qquad i_{\mathcal{L}}(t_0) = i_{\mathcal{L}0}, \qquad i_{\mathcal{V}}(t_0) = i_{\mathcal{V}0}. \tag{2}$$

Here η is a vector of node potentials, $i_{\mathcal{L}}$, $i_{\mathcal{I}}$ and $i_{\mathcal{V}}$ are vectors of currents through inductors, current and voltage sources, respectively. We denote by n_η the number of nodes excepting the grounding node, by $n_{\mathcal{R}}$, $n_{\mathcal{C}}$, $n_{\mathcal{L}}$, $n_{\mathcal{I}}$ and $n_{\mathcal{V}}$ the number of resistances, capacitances, inductors, current and voltage sources, respectively. Furthermore, $A_{\mathcal{R}}$, $A_{\mathcal{C}}$, $A_{\mathcal{L}}$, $A_{\mathcal{I}}$ and $A_{\mathcal{V}}$ are the element-specific incidence matrices describing the circuit topology, and the functions $g : \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}$, $q : \mathbb{R}^{n_{\mathcal{C}}} \to \mathbb{R}^{n_{\mathcal{C}}}$, $\Psi : \mathbb{R}^{n_{\mathcal{L}}} \to \mathbb{R}^{n_{\mathcal{L}}}$ are the describe the relations between resistive currents and voltages, capacitive voltages and charges and inductive currents and magnetic flux, respectively.

The input of the system is formed by voltages of voltage sources and currents of current sources, i.e., the number of inputs is $m = n_{\mathcal{I}} + n_{\mathcal{V}}$. The state consists of η, $i_{\mathcal{L}}$ and $i_{\mathcal{V}}$. The number of state variables, also referred to as the order of the system (1) is given by $n = n_\eta + n_{\mathcal{L}} + n_{\mathcal{V}}$. Finally, our system outputs are the negatives of the voltages of current sources and currents of voltages, i.e., our outputs are $-v_{\mathcal{I}}$ and $-i_{\mathcal{V}}$.

We make the following assumptions.

(A1) The matrix $A_{\mathcal{V}}$ has full column rank.

(A2) The matrix $[\,A_{\mathcal{R}},\, A_{\mathcal{C}},\, A_{\mathcal{L}},\, A_{\mathcal{V}}]$ has full row rank.

(A3) The function $g : \mathbb{R}^{n_{\mathcal{R}}} \to \mathbb{R}^{n_{\mathcal{R}}}$ has the form

$$g([\,u_1\,,\,\ldots\,,\,u_{n_{\mathcal{R}}}]^T) = [\,g_1(u_1)\,,\,\ldots\,,\,g_{n_{\mathcal{R}}}(u_{n_{\mathcal{R}}})\,]^T,$$

where for all $i = 1, \ldots, n_{\mathcal{R}}$, the scalar function g_i is strictly monotonically increasing with $g_i(0) = 0$.

(A4) There exists a diagonal matrix $\mathcal{C} \in \mathbb{R}^{n_{\mathcal{C}}, n_{\mathcal{C}}}$ with $\mathcal{C} \succ 0$ such that $q(u_{\mathcal{C}}) = \mathcal{C} u_{\mathcal{C}}$.

(A5) There exists a diagonal matrix $\mathcal{L} \in \mathbb{R}^{n_{\mathcal{L}}, n_{\mathcal{L}}}$ with $\mathcal{L} \succ 0$ such that $\Psi(i_{\mathcal{L}}) = \mathcal{L} i_{\mathcal{L}}$.

Assumption **(A1)** corresponds to the absence of loops of voltage sources, while **(A2)** forbids cutsets of current sources. Note that equations of circuits with loops of voltage sources or cutsets of current sources contain redundant and contradicting equations.
The property of diagonality of g, q and Ψ in **(A3)**–**(A5)** means that the circuit does not contain multiport elements. The monotonicity of g_i and the positive definiteness of C and L means that the elements of the circuit do not generate energy. These assumptions together guarantee that the DAE is neither under- nor overdetermined [21]. Additionally, **(A4)** and **(A5)** state that the capacitances and resistances are linear and time-invariant components.

3. Nonlinear Descriptor Systems and Incremental Passivity

Descriptor systems are input-output systems whose internal behavior is described by differential-algebraic equations (DAEs). This section presents results on incremental passivity and on Lipschitz-continuity (both will be fined below) for a class of descriptor systems that contains the circuit system (1). The properties of descriptor systems derived in this section will be used in the error analysis of our model reduction approach.

We consider descriptor systems of the form

$$E\dot{x}(t) = a(x(t)) + Bu(t), \qquad x(t_0) = x_0 \tag{3a}$$
$$y(t) = Cx(t) + Du(t), \tag{3b}$$

where $E \in \mathbb{R}^{n,n}$, $a : \mathbb{R}^n \to \mathbb{R}^n$, and $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$ and $D \in \mathbb{R}^{p,m}$. The function $u : [t_0, t_f) \to \mathbb{R}^m$ is called the input and $y : [t_0, t_f) \to \mathbb{R}^p$ is the output.

We say that an input $u : [t_0, t_f) \to \mathbb{R}^m$ is *admissible for the initial value* x_0 if the DAE (3a) has a unique solution $x : [t_0, t_f) \to \mathbb{R}^n$. The initial value x_0 *consistent with the input* u, if the DAE (3a) has a unique solution. A descriptor system is *admissible* if for all sufficiently smooth inputs $u : [t_0, t_f) \to \mathbb{R}^m$ there exists a consistent initial value x_0. Note that in the linear and time-invariant case, i.e., $a(x) = Ax$ with $A \in \mathbb{R}^{n,n}$, admissibility of the descriptor system (3a) is equivalent to the regularity of the matrix pencil $\lambda E - A$ ($\lambda \mapsto \det(\lambda E - A)$ does not vanish identically).

Next we define some special properties of descriptor systems.

Definition 1. *An admissible descriptor system (3) is called*

- incrementally passive*, if $p = m$ and for all $x_0 \in \mathbb{R}^n$ and for all inputs $u_1, u_2 : [t_0, t_f) \to \mathbb{R}^m$ admissible with x_0 the corresponding outputs $y_1(\cdot), y_2(\cdot) : [t_0, t_f) \to \mathbb{R}^m$ satisfy*

$$\int_{t_0}^{t_f} (u_1(\tau) - u_2(\tau))^T (y_1(\tau) - y_2(\tau)) d\tau \geq 0. \tag{4}$$

- L-Lipschitz*, if for all $x_0 \in \mathbb{R}^n$ and for all inputs $u_1, u_2 : [t_0, t_f) \to \mathbb{R}^m$ admissible with x_0 the corresponding outputs $y_1(\cdot), y_2(\cdot) : [t_0, t_f) \to \mathbb{R}^m$ satisfy*

$$\int_{t_0}^{t_f} \|y_1(\tau) - y_2(\tau)\|^2 d\tau \leq L^2 \int_{t_0}^{t_f} \|u_1(\tau) - u_2(\tau)\|^2 d\tau. \tag{5}$$

- (strictly) contractive *if it is L-Lipschitz with ($L < 1$) $L = 1$.*

The notion of incremental passivity is introduced for systems governed by ordinary differential equations in, e.g., [22, 23]. Our definition of incremental passivity specifies a property of the input-output behavior. In particular, for linear systems (3) (i.e., $a(x) = Ax$ with $A \in \mathbb{R}^{n,n}$) incremental passivity as defined above is equivalent to passivity.

The definition of incremental passivity given in [22] involves a so-called storage function. We give some sufficient criteria for incremental passivity (as defined above) and Lipschitz continuity of descriptor systems (3) via storage functions. The following result is a generalization of the *Positive Real Lemma* [24, 25, 26] for linear time-invariant systems governed by ordinary differential equations. In the case where the system is also controllable and observable, this criterion is also known to be necessary.

Theorem 2. *An admissible descriptor system (3) is incrementally passive if there exists a function $X : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *X is continuously differentiable with Jacobian $\frac{d}{de}X(\cdot)$,*

(ii) *$X(Ex) \geq 0$ for all $x \in \mathbb{R}^n$, and*

(iii) *for all vectors $x_1, x_2 \in \mathbb{R}^n$ and $u_1, u_2 \in \mathbb{R}^m$ with $a(x_1) + Bu_1 \in \operatorname{im} E$ and $a(x_2) + Bu_2 \in \operatorname{im} E$ the inequality*

$$\begin{aligned} &\frac{d}{de}X(E(x_1 - x_2))\,(a(x_1) - a(x_2) + B(u_1 - u_2)) \\ \leq &(u_1 - u_2)^T(C(x_1 - x_2) + D(u_1 - u_2)) \end{aligned} \tag{6}$$

holds.

An admissible descriptor system (3) is L-Lipschitz if there exists a function $X : \mathbb{R}^n \to \mathbb{R}$ with the properties (i), (ii), and such that

(iv) *for all vectors $x_1, x_2 \in \mathbb{R}^n$ and $u_1, u_2 \in \mathbb{R}^m$ with $a(x_1) + Bu_1 \in \operatorname{im} E$ and $a(x_2) + Bu_2 \in \operatorname{im} E$ the inequality*

$$\begin{aligned} &\frac{d}{de}X(E(x_1 - x_2))\,(a(x_1) - a(x_2) + B(u_1 - u_2)) \\ \leq &L^2(u_1 - u_2)^T(u_1 - u_2) \\ &- (C(x_1 - x_2) + D(u_1 - u_2))^T(C(x_1 - x_2) + D(u_1 - u_2)) \end{aligned} \tag{7}$$

holds.

If X satisfies (i)-(iii) [(i), (ii), (iv)], then the function $S : \mathbb{R}^n \to \mathbb{R}$ with $S(x) = X(Ex)$ is called a storage function for incremental passivity [*L*-Lipschitz continuity].

PROOF. Let $x_0 \in \mathbb{R}^n$ and $u_1, u_2 \in (L_2([t_0, t_f))^m$ be admissible with the initial value x_0. Let $x_1, x_2 : [t_0, t_f) \to \mathbb{R}^n$ and $y_1, y_2 : [t_0, t_f) \to \mathbb{R}^m$ be the corresponding states and outputs, respectively. Then we have for $t \in [t_0, t_f)$ that

$$\begin{aligned} &\int_{t_0}^{t} (u_1(\tau) - u_2(\tau))^T (y_1(\tau) - y_2(\tau)) d\tau \\ &= \int_{t_0}^{t} (u_1(\tau) - u_2(\tau))^T (C(x_1(\tau) - x_2(\tau)) + D(u_1(\tau) - u_2(\tau))) d\tau. \end{aligned}$$

Due to $E\dot{x}_i(\tau) = a(x_i(\tau)) + Bu_i(\tau)$ for all $\tau \in [t_0, t_f)$ and $i = 1, 2$, we have $a(x_i(\tau)) + Bu_i(\tau) \in \operatorname{im} E$. Hence, (6) implies that

$$\begin{aligned} &\int_{t_0}^{t} (u_1(\tau) - u_2(\tau))^T (y_1(\tau) - y_2(\tau)) d\tau \\ &\geq \int_{t_0}^{t} \frac{d}{de} X(E(x_1(\tau) - x_2(\tau))) \left(a(x_1(\tau)) - a(x_2(\tau)) + B(u_1(\tau) - u_2(\tau))\right) d\tau \\ &= \int_{t_0}^{t} \frac{d}{de} X(E(x_1(\tau) - x_2(\tau))) \cdot (E\dot{x}_1(\tau) - E\dot{x}_2(\tau)) d\tau \\ &= \int_{t_0}^{t} \frac{d}{d\tau} X(E(x_1(\tau) - x_2(\tau))) d\tau = X(E(x_1(t) - x_2(t))) \geq 0. \end{aligned}$$

The proof of the second part of the theorem concerning L-Lipschitz continuity is analogous. $\square$

For linear and time invariant systems $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$ the storage functions for incremental passivity or L-Lipschity continuity are quadratic. Let $H \in \mathbb{R}^{n,n}$ be a matrix such that $E^T H = H^T E \succeq 0$ and define the function $X(Ex) = x^T H^T E x$. For this function inequality (7) reads

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} -A^T H - H^T A - C^T C & -H^T B - C^T D \\ -B^T H - D^T C & L^2 I_m - D^T D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}^T \geq 0 \tag{8}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ with $Ax + Bu \in \operatorname{im} E$. Note that a similar criterion can be formulated for passivity of linear and time-invariant descriptor systems [27].

Next we give a connections between incrementally passive and contractive systems. From a descriptor system (3) with $D = 0$, we define the so-called *scattering system*

$$\begin{aligned} E\dot{\widehat{x}}(t) &= (a(\widehat{x}(t)) - BC\widehat{x}(t)) - \sqrt{2}B\widehat{u}(t), \\ \widehat{y}(t) &= \sqrt{2}C\widehat{x}(t) + \widehat{u}(t). \end{aligned} \tag{9}$$

The scattering system (9) can be obtained from the original system (3) using

$$\widehat{x}(t) = x(t), \quad \widehat{u}(t) = \tfrac{1}{\sqrt{2}}(u(t) + y(t)), \quad \widehat{y}(t) = \tfrac{1}{\sqrt{2}}(u(t) - y(t)). \tag{10}$$

Note that the scattering system could be also defined for general descriptor systems provided that $I - D$ is invertible [10].

Proposition 3. *Let an admissible system (3) with $D = 0$ be given and assume that the scattering system (9) is also admissible. The system (3) is incrementally passive if and only if (9) contractive.*

PROOF. Using the definitions (10) and elementary manipulations we obtain

$$\|\widehat{u}_1(\tau) - \widehat{u}_2(\tau)\|^2 - \|\widehat{y}_1(\tau) - \widehat{y}_2(\tau)\|^2 = 2(u_1(\tau) - u_2(\tau))^T(y_1(\tau) - y_2(\tau)),$$

which implies the desired result. □

Next we present some properties of the circuit descriptor system (1) which can be seen as a special case of (3) with

$$\begin{aligned} x(t) &= \begin{bmatrix} \eta(t) \\ i_{\mathcal{L}}(t) \\ i_{\mathcal{V}}(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} i_{\mathcal{I}}(t) \\ v_{\mathcal{V}}(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} -v_{\mathcal{I}}(t) \\ -i_{\mathcal{V}}(t) \end{bmatrix}, \\ E &= \begin{bmatrix} A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T, \\ a(\eta, i_{\mathcal{L}}, i_{\mathcal{V}}) &= \begin{bmatrix} -A_{\mathcal{R}}g(A_{\mathcal{R}}^T\eta) - A_{\mathcal{L}}i_{\mathcal{L}} - A_{\mathcal{V}}i_{\mathcal{V}} \\ A_{\mathcal{L}}^T\eta \\ A_{\mathcal{V}}^T\eta \end{bmatrix}, \quad \text{and } D = 0. \end{aligned} \tag{11}$$

Proposition 4. *If* **(A1)**–**(A5)** *hold, then the circuit descriptor system (3) with matrices and functions as in (11) is incrementally passive. Moreover, its scattering system is admissible.*

PROOF. Admissibility of the circuit descriptor system follows from the results in [21]. Incremental passivity follows by an application of Theorem 2 with $X(Ex) = x^T Ex$ and a projector K onto $\ker E$.

To show admissibility of the scattering system, we define

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A_{\mathcal{V}}^T & 0 & I \end{bmatrix}, \qquad \begin{bmatrix} \widehat{\eta}(t) \\ \widehat{i}_{\mathcal{L}}(t) \\ \widehat{i}_{\mathcal{V}}(t) \end{bmatrix} = \widehat{x}(t) = T^{-1}x(t)$$

and consider the state-space transformation

$$\begin{aligned} T^{-T}ET\dot{\widehat{x}}(t) =& T^{-T}\big(a(T\widehat{x}(t)) - BCT\widehat{x}(t)\big) - \sqrt{2}T^{-T}B\widehat{u}(t), \\ \widehat{y}(t) =& \sqrt{2}CT\widehat{x}(t) + \widehat{u}(t). \end{aligned} \tag{12}$$

In particular, we have $T^{-T}ET = E$,

$$T^{-T}B = \begin{bmatrix} -A_{\mathcal{I}}^T & A_{\mathcal{V}}^T \\ 0 & 0 \\ 0 & -I \end{bmatrix}, \quad (CT)^T = \begin{bmatrix} -A_{\mathcal{I}}^T & -A_{\mathcal{V}}^T \\ 0 & 0 \\ 0 & -I \end{bmatrix}, \text{ and}$$

$$\begin{aligned} & T^{-T}\left(a\left(T\begin{bmatrix} \widehat{\eta} \\ \widehat{i}_{\mathcal{L}} \\ \widehat{i}_{\mathcal{V}} \end{bmatrix}\right) - BCT\begin{bmatrix} \widehat{\eta} \\ \widehat{i}_{\mathcal{L}} \\ \widehat{i}_{\mathcal{V}} \end{bmatrix}\right) \\ & = \begin{bmatrix} -A_{\mathcal{R}}g(A_{\mathcal{R}}^T\widehat{\eta}) - (A_{\mathcal{V}}A_{\mathcal{V}}^T + A_{\mathcal{I}}A_{\mathcal{I}}^T)\widehat{\eta} - A_{\mathcal{L}}\widehat{i}_{\mathcal{L}} \\ A_{\mathcal{L}}^T\widehat{\eta} \\ -\widehat{i}_{\mathcal{V}} \end{bmatrix}. \end{aligned}$$

The first two rows of (12) can be seen as a circuit system satisfying **(A1)**–**(A5)**, and the last row consists of some explicit algebraic equation for $i_{\mathcal{V}}$. The results of [21] can be now again used to conclude admissibility. □

4. Model Reduction for Linear Time-Invariant Circuits

We will construct a reduced order model for the circuit system (1) by applying model reduction to the linear subsystem. In particular, we will apply the passivity-preserving balanced truncation method for electrical circuits (PABTEC) developed in [10, 11]. In this section we will review the main features of PABTEC.

PABTEC is applied to the MNA of linear time-invariant circuits

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{13}$$

with $u(t)$, $x(t)$, $y(t)$, E, B, C as in (11) and

$$A = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix},$$

where $\mathcal{G} \succ 0$ is the conductance matrix.

Let $\mathcal{P}_l$ and $\mathcal{P}_r$ be the so-called *spectral projectors* of the pencil $\lambda E - A + BC$, i.e.,

$$\mathcal{P}_l = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad \mathcal{P}_r = T_r^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_r,$$

where $T_l, T_r \in \mathbb{R}^{n,n}$ are invertible matrices bringing the matrix pencil $\lambda E - A + BC$ into Kronecker normal form, i.e.,

$$T_l(\lambda E - A + BC)T_r = \begin{bmatrix} \lambda I_{n_f} & 0 \\ 0 & \lambda N - I \end{bmatrix}$$

for some nilpotent N. Furthermore, define

$$\begin{aligned} \mathcal{M}_0 =& I - 2\lim_{s\to\infty} C(sE - A + BC)^{-1}B, \\ \mathcal{J}_c\mathcal{J}_c^T =& I - \mathcal{M}_0\mathcal{M}_0^T, \\ \mathcal{J}_o^T\mathcal{J}_c =& I - \mathcal{M}_0^T\mathcal{M}_0. \end{aligned}$$

PABTEC is based on the solution of the Lur'e equations

$$\begin{aligned} (A - BC)XE^T + EX(A - BC)^T + 2\mathcal{P}_l BB^T\mathcal{P}_l^T &= -2\,\mathcal{K}_c\mathcal{K}_c^T, \\ X = \mathcal{P}_r X \mathcal{P}_r^T &\succeq 0, \\ EXC^T - \mathcal{P}_l B\mathcal{M}_0^T &= -\mathcal{K}_c\mathcal{J}_c^T, \end{aligned} \tag{14}$$

and

$$\begin{aligned}(A-BC)^TYE+E^TY(A-BC)+2\mathcal{P}_r^TC^TC\mathcal{P}_r &= -2\,\mathcal{K}_o^T\mathcal{K}_o,\\ Y=\mathcal{P}_l^TY\mathcal{P}_l &\succeq 0,\\ -E^TYB+\mathcal{P}_r^TC^T\mathcal{M}_0 &= -\mathcal{K}_o^T\mathcal{J}_o\end{aligned} \tag{15}$$

for $X\in\mathbb{R}^{n,n}$, $\mathcal{K}_c\in\mathbb{R}^{n,m}$ and $Y\in\mathbb{R}^{n,n}$, $\mathcal{K}_o\in\mathbb{R}^{m,n}$, respectively. Furthermore, X and Y are required to be minimal solutions, that is, with respect to definiteness, they are below all other solutions of the Lur'e equations.

Based on X and Y a state-space transformations $W,T\in\mathbb{R}^{n,n}$ can be constructed such that

$$WET=\begin{bmatrix}I_q & 0\\ 0 & 0\end{bmatrix},\quad T^{-1}XT^{-T}=W^{-T}YW^{-1}=\begin{bmatrix}\Sigma_f & 0\\ 0 & 0\end{bmatrix}$$

for $\Sigma_f=\operatorname{diag}(\pi_1,\dots,\pi_q)$ with decreasing diagonal elements

$$\pi_1\geq\pi_2\geq\ldots\geq\pi_q\geq 0.$$

The values $\pi_1,\dots,\pi_q$ are called the *passivity characteristic values*.

The passive reduced order model of (13) is given by

$$\begin{aligned}\widetilde{E}\dot{\widetilde{x}}(t)&=\widetilde{A}\widetilde{x}(t)+\widetilde{B}u(t),\\ \widetilde{y}(t)&=\widetilde{C}\widetilde{x}(t).\end{aligned} \tag{16}$$

where $\widetilde{E}=Z^TWETZ$, $\widetilde{A}=Z^TWATZ$, $\widetilde{B}=Z^TWB$, $\widetilde{C}=CTZ$, and

$$Z=\begin{bmatrix}I_r & 0_{r,q-r} & 0_{r,n-q}\\ 0_{n-q,r} & 0_{n-q,q-r} & I_{n-q}\end{bmatrix}^T.$$

The structure of the circuit equations can be exploited to reduce the numerical effort of the model reduction method significantly. For instance, explicit expressions for $\mathcal{M}_0$ and the spectral projectors can be derived and it can be shown that the solutions of the Lur'e equations satisfy $X=SYS$ for $S=\operatorname{diag}(I_{n_\eta},-I_{n_{\mathcal{L}}+n_{\mathcal{V}}})$. See [10].

The passivity characteristic values $\pi_1,\dots,\pi_q$ can be used to derive a priori error bounds in the so-called $\mathbb{H}_\infty$-*norm* that is given by the maximal value $\|\boldsymbol{G}(s)\|$ for $s\in\mathbb{C}^+$, where $\boldsymbol{G}(s)=C(sE-A)^{-1}B$ is the *transfer function* of (13). Roughly speaking, the $\mathbb{H}_\infty$-norm a system (13) is the smallest constant L such that (13) is L-Lipschitz. The following error bounds are proven in [10]:

1. Let $\boldsymbol{G}(s) = C(sE - A)^{-1}B$ and $\widetilde{\boldsymbol{G}}(s) = \widetilde{C}(s\widetilde{E} - \widetilde{A})^{-1}\widetilde{B}$ be the transfer functions of the full order system (13) and of the reduced order system (16), respectively. If r is chosen such that $\|I + \boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{r+1} + \ldots + \pi_q) < 1$, then

$$\|\widetilde{\boldsymbol{G}} - \boldsymbol{G}\|_{\mathbb{H}_\infty} \leq \frac{\|I + \boldsymbol{G}\|^2_{\mathbb{H}_\infty}(\pi_{r+1} + \ldots + \pi_q)}{1 - \|I + \boldsymbol{G}\|_{\mathbb{H}_\infty}(\pi_{r+1} + \ldots + \pi_q)}. \tag{17}$$

2. The transfer function $\widehat{\boldsymbol{\mathcal{G}}}(s) = I - 2C(sE - A + BC)^{-1}B$ of the scattering system associated with (13) and the transfer function $\widehat{\widetilde{\boldsymbol{\mathcal{G}}}}(s) = I - 2\widetilde{C}(s\widetilde{E} - \widetilde{A} + \widetilde{B}\widetilde{C})^{-1}\widetilde{B}$ of the scattering system associated with the reduced problem (16) obey

$$\|\widehat{\widetilde{\boldsymbol{\mathcal{G}}}} - \widehat{\boldsymbol{\mathcal{G}}}\|_{\mathbb{H}_\infty} \leq 2(\pi_{r+1} + \ldots + \pi_q). \tag{18}$$

Additional error bounds are derived in [10].

5. Model Reduction of Nonlinear Circuits by Model Reduction of Linear Subcircuits

The idea behind our approach is simple. We identify the linear subcircuit corresponding to the linear resistances, determine its connectivity with the nonlinear subcircuit. The connectivities with the nonlinear subcircuit will be included as auxiliary inputs and outputs for the linear subcircuit when PABTEC is applied to compute a reduced order model for linear subcircuit. In this section we specify the linear and nonlinear subcircuits and their connectivity. Furthermore, we will establish admissibility and incremental passivity of the reduced order model that is obtained by applying PABTEC to the linear subcircuit. Error bounds will be discussed in Section 6.

Let $n_{\mathcal{R}_l}$ and $n_{\mathcal{R}_n}$ be the number of linear and nonlinear resistances, respectively. To simplify the presentation, we assume that this first $n_{\mathcal{R}_l}$ resistances are linear. Consequently, let the matrix $\mathcal{G}_l \in \mathbb{R}^{n_{\mathcal{R}_l}, n_{\mathcal{R}_l}}$ and the function $g_n : \mathbb{R}^{n_{\mathcal{R}_n}} \to \mathbb{R}^{n_{\mathcal{R}_n}}$ such that for $u = [u_l^T, u_n^T]^T$ with $u_l \in \mathbb{R}^{n_{\mathcal{R}_l}}$, $u_n \in \mathbb{R}^{n_{\mathcal{R}_n}}$ the nonlinearity g in (11) is given by

$$g\left(\begin{bmatrix} u_l \\ u_n \end{bmatrix}\right) = \begin{bmatrix} \mathcal{G}_l u_l \\ g_n(u_n) \end{bmatrix}.$$

From **(A3)** we get that $\mathcal{G}_l$ is diagonal with $\mathcal{G}_l \succ 0$ and, furthermore, the function g_n also satisfies **(A3)**. We split the resistive incidence matrix $A_{\mathcal{R}} = [\, A_{\mathcal{R}_l}\,,\, A_{\mathcal{R}_n}\,]$ according to the linear and nonlinear parts. The circuit equations can now be rewritten as

$$\begin{aligned} E_l\dot{x}(t) &= A_l x(t) + B_l u(t) + B_n g_n(-C_n x(t)), \\ y(t) &= C_l x(t), \end{aligned} \tag{19}$$

with

$$E_l = E, \quad B_l = B, \quad C_l = C, \tag{20a}$$

where E, B, and C are given as in (11), and

$$A_l = \begin{bmatrix} -A_{\mathcal{R}_l}\mathcal{G}_l A_{\mathcal{R}_l}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \qquad B_n = C_n^T = \begin{bmatrix} -A_{\mathcal{R}_n} \\ 0 \\ 0 \end{bmatrix}. \tag{20b}$$

To facilitate model reduction we isolate the nonlinearity and write (19) in the form

$$\begin{aligned} E_l\dot{x}(t) &= A_l x(t) + B_l u(t) + B_n i_{\mathcal{R}_n}(t), \\ -v_{\mathcal{R}_n}(t) &= C_n x(t), \\ y(t) &= C_l x(t) \end{aligned} \tag{21}$$

with nonlinearity

$$i_{\mathcal{R}_n}(t) = g_n(v_{\mathcal{R}_n}(t)). \tag{22}$$

To guarantee that (21) is admissible, we employ the following assumption:

(A6) $\operatorname{rank}[\, A_{\mathcal{R}_l}\,, A_{\mathcal{C}}\,,\, A_{\mathcal{L}}\,,\, A_{\mathcal{V}}\,] = n_\eta$,

In terms of the circuit topology, this condition forbids cutsets of elements different from current sources and nonlinear resistances or, equivalently, the removal of current sources and nonlinear resistances still yields a connected circuit. Note that this may be rather restrictive. However, this property can be bypassed by a further subdivision of the linear part into connected subcircuits.

Remark 5. *The trajectories of linear subsystem (21) satisfy*

$$\int_{t_0}^{t_f} u^T(\tau)y(\tau) - i_{\mathcal{R}_n}^T(\tau)v_{\mathcal{R}_n}\, d\tau = \eta^T(t)A_{\mathcal{R}_l}\mathcal{G}_l A_{\mathcal{R}_l}^T\eta(t)\Big|_{t=t_0}^{t=t_f}.$$

Since $\mathcal{G}_l$ is positive semidefinite, (21) is passive.

To obtain a reduced order model for (19), we apply a passivity preserving reduced order model approach to the linear subsystem (21) and keep the nonlinearity (22). Let

$$
\begin{aligned}
\widetilde{E}_l\dot{\widetilde{x}}(t) &= \widetilde{A}_l\widetilde{x}(t) + \begin{bmatrix}\widetilde{B}_l & \widetilde{B}_n\end{bmatrix}\begin{bmatrix}u(t)\\ i_{\mathcal{R}_n}(t)\end{bmatrix},\\
\begin{bmatrix}\widetilde{y}(t)\\ -\widetilde{v}_{\mathcal{R}_n}(t)\end{bmatrix} &= \begin{bmatrix}\widetilde{C}_l\\ \widetilde{C}_n\end{bmatrix}\widetilde{x}(t)
\end{aligned}
\tag{23}
$$

be a passive reduced order model of (21). The reduced order model for (19) is given by

$$
\begin{aligned}
\widetilde{E}_l\dot{\widetilde{x}}(t) &= \widetilde{A}_l\widetilde{x}(t) + \widetilde{B}_n g(-\widetilde{C}_n\widetilde{x}(t)) + \widetilde{B}_l u(t)\\
\widetilde{y}(t) &= \widetilde{C}_l\widetilde{x}(t).
\end{aligned}
\tag{24}
$$

First we show that the global system with reduced order linear part is admissible and incrementally passive. Recall that for linear descriptor systems such as (23) incremental passivity is equivalent to passivity.

Theorem 6. *Let* **(A1)**–**(A6)** *be satisfied. If (23) is passive, then the system (24) is admissible and incrementally passive. Moreover, the reduced order scattering system*

$$
\begin{aligned}
\widetilde{E}_l\dot{\widehat{\widetilde{x}}}(t) &= (\widetilde{A}_l - \widetilde{B}_l\widetilde{C}_l)\widehat{\widetilde{x}}(t) + \widetilde{B}_n g(-\widetilde{C}_n\widehat{\widetilde{x}}(t)) - \sqrt{2}\widetilde{B}_l\widehat{u}(t)\\
\widehat{\widetilde{y}}(t) &= \sqrt{2}\widetilde{C}_l\widehat{\widetilde{x}}(t) + \widehat{u},
\end{aligned}
\tag{25}
$$

is admissible.

Proof. We only sketch the proofs for admissibility. By the results of [28] and references, a circuit realizing the system (23) can be synthesized, i.e., there exists a state space transform of (23) such that the resulting system has the form of equations of modified nodal analysis. The state space transformed system together with $i_{\mathcal{R}_n}(t) = -g_n(v_{\mathcal{R}_n}(t))$ then also yields circuit equations satisfying **(A1)**–**(A3)**. By the results of [21], (23) is then admissible. The proof for admissibility of (25) can be obtained by a combination of the previous arguments and those in the proof of Proposition 4.

Now we show incremental passivity of (24). Let $u_1, u_2 : [t_0, t_f] \to \mathbb{R}^m$ be admissible for (24) with the initial value $\widetilde{x}_0 \in \mathbb{R}^r$. Let $\widetilde{x}_1, \widetilde{x}_2 : [t_0, t_f] \to \mathbb{R}^r$ and $\widetilde{y}_1, \widetilde{y}_2 : [t_0, t_f] \to \mathbb{R}^m$ be the corresponding states and outputs for (24).

Furthermore, let $\widetilde{v}_{\mathcal{R}_n,j} = \widetilde{C}_n\widetilde{x}_j$ and $i_{\mathcal{R}_n,j} = -g(\widetilde{v}_{\mathcal{R}_n,j})$, $j = 1, 2$. Since the reduced linear subsystem (23) is passive we have

$$
\begin{aligned}
0 \leq & \int_{t_0}^{t_f} (u_1(\tau) - u_2(\tau))^T (\widetilde{y}_1(\tau) - \widetilde{y}_2(\tau))\, d\tau \\
& - \int_{t_0}^{t_f} (i_{\mathcal{R}_n,1}(\tau) - i_{\mathcal{R}_n,2}(\tau))^T (\widetilde{v}_{\mathcal{R}_n,1}(\tau) - \widetilde{v}_{\mathcal{R}_n,2}(\tau))\, d\tau \\
= & \int_{t_0}^{t_f} (u_1(\tau) - u_2(\tau))^T (\widetilde{y}_1(\tau) - \widetilde{y}_2(\tau))\, d\tau \\
& - \int_{t_0}^{t_f} \big(g_n(\widetilde{v}_{\mathcal{R},1}(\tau)) - g_n(\widetilde{v}_{\mathcal{R}_n,2}(\tau))\big)^T \big(\widetilde{v}_{\mathcal{R}_n,1}(\tau) - \widetilde{v}_{\mathcal{R}_n,2}(\tau)\big)\, d\tau.
\end{aligned}
$$

Note that since g_n is strictly monotonically increasing, we have

$$
\big(g_n(\widetilde{v}_{\mathcal{R},1}(\tau)) - g_n(\widetilde{v}_{\mathcal{R}_n,2}(\tau))\big)^T \big(\widetilde{v}_{\mathcal{R}_n,1}(\tau) - \widetilde{v}_{\mathcal{R}_n,2}(\tau)\big) \geq 0
$$

for all $\tau \in [t_0, t_f]$. Hence we have

$$
0 \leq \int_{t_0}^{t_f} (u_1(\tau) - u_2(\tau))(\widetilde{y}_1(\tau) - \widetilde{y}_2(\tau)) d\tau.
$$

□

6. Global Error Bounds for the System with Reduced-Order Linear Part

Throughout the remaining part we assume that the reduced-order system (23) is obtained from (21) via PABTEC. Let $\pi_{r+1}, \ldots, \pi_q$ be passivity characteristic values corresponding to the truncated states.

In the following, we formulate four possible special properties of the circuit. Note that we mention at every particular result which of these additional properties are required. The first three of them concern the topology of the circuit, whereas the latter one describes possible feature of the voltage-current characteristic of the nonlinear resistances.

(A7) $\operatorname{im}[\, A_{\mathcal{V}}\,,\ A_{\mathcal{I}}\,,\ A_{\mathcal{R}_n}\,] \subset \operatorname{im} A_{\mathcal{R}_l}$.

(A8) $\ker[\, A_{\mathcal{C}}\,,\ A_{\mathcal{L}}\,,\ A_{\mathcal{V}}\,,\ A_{\mathcal{I}}\,,\ A_{\mathcal{R}_n}\,] = \ker[\, A_{\mathcal{C}}\,,\ A_{\mathcal{L}}\,] \times \{0\}$.

(A9) $\ker[\, A_{\mathcal{C}}\,,\ A_{\mathcal{L}}\,,\ A_{\mathcal{R}_n}\,] = \ker[\, A_{\mathcal{C}}\,,\ A_{\mathcal{L}}\,] \times \{0\}$.

(A10) there exist positive constants c_{g1}, c_{g2} such that for all $u_1, u_2 \in \mathbb{R}^{n_{\mathcal{R}_n}}$ holds

$$(u_1 - u_2)^T(g_n(u_1) - g_n(u_2)) \geq c_{g1}\|u_1 - u_2\|^2,$$
$$(u_1 - u_2)^T(g_n(u_1) - g_n(u_2)) \geq c_{g2}\|g_n(u_1) - g_n(u_2)\|^2.$$

We now give some circuit topological interpretations of the above introduced matrix rank criteria by employing the argumentations in [21, 29]. The claim **(A7)** means that the incidence nodes of every source and every nonlinear resistance are connected by a path of linear resistances. The circuit topological interpretation of **(A8)** is that the circuit does not contain loops only consisting of elements different from linear resistances except for loops that only consist of inductances and capacitances. This criterion can be equivalently rewritten as

$$\ker Z_{\mathcal{LC}}^T[\, A_{\mathcal{I}}\,,\, A_{\mathcal{V}}\,,\, A_{\mathcal{R}_n}\,] = \{0\}, \tag{26}$$

where $Z_{\mathcal{LC}} \in \mathbb{R}^{n_\eta, k_{\mathcal{LC}}}$ is a matrix with $\operatorname{im} Z_{\mathcal{LC}} = \ker[\, A_{\mathcal{L}}\,,\, A_{\mathcal{C}}\,]^T$. Analogously, **(A9)** is equivalent to $\ker Z_{\mathcal{LC}}^T A_{\mathcal{R}_n} = \{0\}$. This means that there are no loops only consisting of elements different from resistances except for loops that only consist of inductances and capacitances. In **(A10)** it is essentially claimed that the voltage-current characteristic of the resistances is in some cone. This is, at least locally, fulfilled for any practically relevant resistance.

The first result provides sufficient conditions for the contractivity of scattering systems associated with the linear subsystem (21).

Proposition 7. *Let a circuit satisfying* **(A1)**–**(A6)** *be given. Let E_l, A_l, B_l, B_n, C_l, C_n be given as in (20).*

– *If* **(A7)** *and* **(A8)** *are satisfied, then the scattering system of (21), i.e.,*

$$\begin{aligned} E_l\dot{\widehat{x}}(t) &= (A_l - B_lB_l^T - B_nB_n^T)\widehat{x}(t) - \sqrt{2}B_l\widehat{u}(t) - \sqrt{2}B_n\widehat{i}_{\mathcal{R}_n}(t), \\ \widehat{v}_{\mathcal{R}_n}(t) &= \sqrt{2}C_n\widehat{x}(t) + \widehat{u}(t), \\ \widehat{y}(t) &= \sqrt{2}C_l\widehat{x}(t) + \widehat{i}_{\mathcal{R}_n}(t) \end{aligned} \tag{27}$$

is strictly contractive.

– *If* **(A7)** *and* **(A9)** *hold true, then the system*

$$\begin{aligned} E_l\dot{\widehat{x}}(t) &= (A_l - B_lB_l^T - B_nB_n^T)\widehat{x}(t) - \sqrt{2}B_n\widehat{i}_{\mathcal{R}_n}(t), \\ \widehat{v}_{\mathcal{R}_n}(t) &= \sqrt{2}C_n\widehat{x}(t) + \widehat{i}_{\mathcal{R}_n}(t) \end{aligned} \tag{28}$$

is strictly contractive.

PROOF. We only show the first statement since the second one can be proven using analogous arguments.

Let $\widehat{E} = E_l$, $\widehat{A} = A_l - B_l B_l^T - B_n B_n^T$, $\widehat{B} = -\widehat{C}^T = -\sqrt{2}[\, B_l\, ,\, B_n\,]$ and $\widehat{D} = I$. We will show strict contractivity by employing (8). Define $H = I_n$ and a matrix $Z_{\mathcal{C}} \in \mathbb{R}^{n_\eta, k_{\mathcal{C}}}$ with $\operatorname{im} Z_{\mathcal{C}} = \ker A_{\mathcal{C}}^T$. Moreover, let

$$K = \begin{bmatrix} Z_{\mathcal{C}}^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

By construction, we have $H^T \widehat{E} = \widehat{E}^T H \succeq 0$ and $K\widehat{E} = 0$. The latter implies that

$$\left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \widehat{A}x + \widehat{B}u \in \operatorname{im} \widehat{E} \right\} = \ker \begin{bmatrix} K\widehat{A} & K\widehat{B} \end{bmatrix}.$$

Let $x = [\, x_1^T\, ,\, x_2^T\, ,\, x_3^T\,]^T \in \mathbb{R}^{n_\eta + n_{\mathcal{L}} + n_{\mathcal{V}}}$ and $u = [\, u_1^T\, ,\, u_2^T\, ,\, i_{\mathcal{R}}^T\,]^T \in \mathbb{R}^{n_{\mathcal{I}} + n_{\mathcal{V}} + n_{\mathcal{R}_n}}$ be partitioned according to the block structure of A and B. Our aim is to show that there exists $L < 1$ such that for all $x \in \mathbb{R}^{n_\eta + n_{\mathcal{L}} + n_{\mathcal{V}}}$ and $u \in \mathbb{R}^{n_{\mathcal{I}} + n_{\mathcal{V}}}$ with $K\widehat{A}x + K\widehat{B}u = 0$ the inequality

$$\begin{aligned} 0 \leq& \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} -\widehat{A}^T H - H^T \widehat{A} - \widehat{C}^T \widehat{C} & -H^T \widehat{B} - \widehat{C}^T \widehat{D} \\ -\widehat{B}^T H - \widehat{D}^T \widehat{C} & L^2 I_m - \widehat{D}^T \widehat{D} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ =& x_1^T A_{\mathcal{R}_l} \mathcal{G}_l A_{\mathcal{R}_l}^T x_1 + (L^2 - 1)(u_1^T u_1 + u_2^T u_2 + i_{\mathcal{R}}^T i_{\mathcal{R}}) \end{aligned} \tag{29}$$

holds. From $K\widehat{A}x + K\widehat{B}u = 0$, we obtain

$$\begin{aligned} -Z_{\mathcal{C}}^T(A_{\mathcal{R}_l} \mathcal{G}_l A_{\mathcal{R}_l}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{R}_n} A_{\mathcal{R}_n}^T) x_1 = Z_{\mathcal{C}}^T A_{\mathcal{L}} x_2 + Z_{\mathcal{C}}^T A_{\mathcal{V}} x_3 \\ - \sqrt{2} Z_{\mathcal{C}}^T A_{\mathcal{I}} u_1 - \sqrt{2} Z_{\mathcal{C}}^T A_{\mathcal{R}_n} i_{\mathcal{R}}, \end{aligned} \tag{30}$$

$$A_{\mathcal{V}}^T x_1 - x_3 = -\sqrt{2} u_2. \tag{31}$$

Since $\operatorname{im} Z_{\mathcal{LC}} = \ker[\, A_{\mathcal{L}}\, ,\, A_{\mathcal{C}}\,]^T \subset \ker A_{\mathcal{C}}^T = \operatorname{im} Z_{\mathcal{C}}$, there exists a matrix $Z_{\mathcal{L}-\mathcal{C}} \in \mathbb{R}^{k_{\mathcal{L}\mathcal{V}}, k_{\mathcal{C}}}$ with $Z_{\mathcal{L}-\mathcal{C}} Z_{\mathcal{C}} = Z_{\mathcal{LC}}$. Then a multiplication of (30) from the left with $Z_{\mathcal{L}-\mathcal{C}}^T$ yields

$$\begin{aligned} & - Z_{\mathcal{LC}}^T (A_{\mathcal{R}_l} \mathcal{G}_l A_{\mathcal{R}_l}^T + A_{\mathcal{I}} A_{\mathcal{I}}^T + A_{\mathcal{R}_n} A_{\mathcal{R}_n}^T) x_1 \\ &= Z_{\mathcal{LC}}^T A_{\mathcal{V}} x_3 - \sqrt{2} Z_{\mathcal{LC}}^T A_{\mathcal{I}} u_1 - \sqrt{2} Z_{\mathcal{LC}}^T A_{\mathcal{R}_n} i_{\mathcal{R}}. \end{aligned} \tag{32}$$

By **(A7)** holds that $\operatorname{im}[\,A_{\mathcal{I}}\,,\,A_{\mathcal{R}_n}\,] \subset \operatorname{im} A_{\mathcal{R}_l}$, i.e., $[\,A_{\mathcal{I}}\,,\,A_{\mathcal{R}_n}\,] = A_{\mathcal{R}_l}F$ for some $F \in \mathbb{R}^{n_{\mathcal{R}_l}, n_{\mathcal{I}}+n_{\mathcal{R}_n}}$. Hence, equation (32) reads

$$\begin{aligned} &- Z_{\mathcal{LC}}^T A_{\mathcal{R}_l}(\mathcal{G}_l + FF^T)A_{\mathcal{R}_l}^T x_1 \\ &= Z_{\mathcal{LC}}^T A_{\mathcal{V}} x_3 - \sqrt{2} Z_{\mathcal{LC}}^T A_{\mathcal{I}} u_1 - \sqrt{2} Z_{\mathcal{LC}}^T A_{\mathcal{R}_n} i_{\mathcal{R}}. \end{aligned} \tag{33}$$

Since $Z_{\mathcal{LC}}^T[\,A_{\mathcal{V}}\,,\,A_{\mathcal{I}}\,,\,A_{\mathcal{R}_n}\,]$ is assumed to have full column rank, we get from (32) that the vectors x_3, u_1 and $i_{\mathcal{R}}$ are uniquely determined by $A_{\mathcal{R}_l}^T x_1$. This means that there exist matrices $G \in \mathbb{R}^{n_{\mathcal{V}}, n_\eta}$, $H \in \mathbb{R}^{n_{\mathcal{I}}, n_\eta}$ and $J \in \mathbb{R}^{n_{\mathcal{R}_n}, n_\eta}$ such that $x_3 = GA_{\mathcal{R}_l}^T x_1$ $u_1 = HA_{\mathcal{R}_l}^T x_1$ and $i_{\mathcal{R}} = JA_{\mathcal{R}_l}^T x_1$. Employing these relations and (31), the expression on the right hand side of (29) is given by

$$\begin{aligned} & x_1^T A_{\mathcal{R}_l} \mathcal{G}_l A_{\mathcal{R}_l}^T x_1 \\ & + (L^2-1)x_1^T(A_{\mathcal{R}_l}(H^TH + J^TJ)A_{\mathcal{R}_l}^T + \frac{1}{2}(A_{\mathcal{V}}^T - GA_{\mathcal{R}_l})^T(A_{\mathcal{V}}^T - GA_{\mathcal{R}_l}^T))x_1. \end{aligned}$$

Since $\operatorname{im} A_{\mathcal{V}} \subset \operatorname{im} A_{\mathcal{R}_l}$ and $\mathcal{G} > 0$, there exists some $L < 1$ such that the matrix

$$\begin{aligned} & A_{\mathcal{R}_l}\mathcal{G}A_{\mathcal{R}_l}^T + (L^2-1)(A_{\mathcal{R}_l}(H^TH + J^TJ)A_{\mathcal{R}_l} \\ & + \frac{1}{2}(A_{\mathcal{V}}^T - GA_{\mathcal{R}_l})^T(A_{\mathcal{V}}^T - GA_{\mathcal{R}_l}^T) \end{aligned}$$

is positive semidefinite. Thus, the system is strictly contractive. □

6.1. Error Bounds for the Scattering System

Now we present error bounds for the reduced-order system (24). First we give an estimate for the difference of the outputs of the scattering systems (9) and (25) corresponding to the full order system (11) and the reduced-order system (24), respectively. We will give an interpretation of the scattering error bound in terms of error of the behavioral system at the end of this section.

Theorem 8. *Let a circuit satisfying* **(A1)**–**(A6)**, **(A9)** *be given and let the reduced-order system (24) be obtained by applying PABTEC to (21). Furthermore, let the linear system (28) be strictly contractive with Lipschitz constant $L < 1$. If the size r of the reduced linear subsystem (23) is chosen such that*

$$\varepsilon \equiv 2 \sum_{k=r+1}^{q} \pi_k < 1 - L, \tag{34}$$

then the following holds true. If $\widehat{u} : [t_0, t_f] \to \mathbb{R}^m$ is admissible for the scattering systems (9) and (25) corresponding to (11) and (24), respectively, with initial conditions $x(t_0) = \widehat{x}(t_0) = 0$, then the corresponding outputs satisfy

$$\|\widehat{y} - \widetilde{\widehat{y}}\|_{L_2([t_0,t_f])} \leq \varepsilon \cdot \frac{(2-L)(2-L-\varepsilon)}{(1-L)(1-L-\varepsilon)} \|\widehat{u}\|_{L_2([t_0,t_f])}. \tag{35}$$

Before Theorem 8 is proven we remark that strict contractivity of the linear system (28) can be guaranteed if **(A7)**, **(A9)** hold (see Proposition 7). The passivity of the reduced-order linear part (23) together with Proposition 3 and Theorem 6 guarantees the contractivity of the reduced-order scattering system (25) and the reduced-order scattering linear subsystem (27).

Proof. Consider the scattering systems corresponding to the linear subsystems (21) and (23), respectively. Taking into account that the output of (21) and (23) is composed of the output of the overall system and the negative of the voltages of nonlinear resistances, the inputs $(\widehat{u}(t), \widehat{i}_{\mathcal{R}_n}(t))$, $(\widehat{u}(t), \widetilde{\widehat{i}}_{\mathcal{R}_n}(t))$ and the outputs $(\widehat{y}(t), \widehat{v}_{\mathcal{R}_n}(t))$, $(\widetilde{\widehat{y}}(t), \widetilde{\widehat{v}}_{\mathcal{R}_n}(t))$ of the corresponding scattering systems satisfy

$$\begin{aligned} \widehat{i}_{\mathcal{R}_n}(t) &= \tfrac{1}{\sqrt{2}}(i_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)), \quad \widehat{v}_{\mathcal{R}_n}(t) = \tfrac{1}{\sqrt{2}}(i_{\mathcal{R}_n}(t) + v_{\mathcal{R}_n}(t)), \\ \widetilde{\widehat{i}}_{\mathcal{R}_n}(t) &= \tfrac{1}{\sqrt{2}}(\widetilde{i}_{\mathcal{R}_n}(t) - \widetilde{v}_{\mathcal{R}_n}(t)), \quad \widetilde{\widehat{v}}_{\mathcal{R}_n}(t) = \tfrac{1}{\sqrt{2}}(\widetilde{i}_{\mathcal{R}_n}(t) + \widetilde{v}_{\mathcal{R}_n}(t)). \end{aligned} \tag{36}$$

Furthermore, recall that

$$i_{\mathcal{R}_n} = g_n(v_{\mathcal{R}_n}), \quad \widetilde{i}_{\mathcal{R}_n} = g_n(\widetilde{v}_{\mathcal{R}_n}). \tag{37}$$

Since the scattering systems corresponding to (21) and (23) are linear, their input-to-output maps define linear operators $\mathcal{F} : (L_2([t_0, t_f])^{m+n_{\mathcal{R}_n}} \to (L_2([t_0, t_f]))^{m+n_{\mathcal{R}_n}}$ and $\widetilde{\mathcal{F}} : (L_2([t_0, t_f])^{m+n_{\mathcal{R}_n}} \to (L_2([t_0, t_f]))^{m+n_{\mathcal{R}_n}}$. Partitioning these mappings according to the block structure of the input and outputs, we have the structure

$$\begin{bmatrix} \widehat{y} \\ \widehat{v}_{\mathcal{R}_n} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{i}_{\mathcal{R}_n} \end{bmatrix}, \quad \begin{bmatrix} \widetilde{\widehat{y}} \\ \widetilde{\widehat{v}}_{\mathcal{R}_n} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathcal{F}}_{11} & \widetilde{\mathcal{F}}_{12} \\ \widetilde{\mathcal{F}}_{21} & \widetilde{\mathcal{F}}_{22} \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widetilde{\widehat{i}}_{\mathcal{R}_n} \end{bmatrix}. \tag{38}$$

The error bound (18) for PABTEC stated in the time domain implies the bound $\|\mathcal{F} - \widetilde{\mathcal{F}}\| \leq \varepsilon$ for the operator norms. This implies $\|\mathcal{F}_{ij} - \widetilde{\mathcal{F}}_{ij}\| \leq \varepsilon$,

$i, j = 1, 2$. Contractivity of the scattering linear subsystems imply $\|\mathcal{F}_{ij}\| \leq 1$, $\|\widetilde{\mathcal{F}}_{ij}\| \leq 1$, $i, j = 1, 2$. In particular, since $\mathcal{F}_{22}$ is the input-output-mapping of the system (28), we have $\|\mathcal{F}_{22}\| \leq L < 1$. Using (36) and (37) we can estimate

$$\begin{aligned}&\|\widehat{\widetilde{v}}_{\mathcal{R}_n}(t) - \widehat{v}_{\mathcal{R}_n}(t)\|^2 - \|\widehat{\widetilde{i}}_{\mathcal{R}_n}(t) - \widehat{i}_{\mathcal{R}_n}(t)\|^2 \\ =&2(g_n(\widetilde{v}_{\mathcal{R}_n}(t)) - g_n(v_{\mathcal{R}_n}(t)))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)).\end{aligned}$$

Since g_n is strictly monotonically increasing, the last expression is positive, and thus

$$\|\widehat{\widetilde{i}}_{\mathcal{R}_n}(t) - \widehat{i}_{\mathcal{R}_n}(t)\| \leq \|\widehat{\widetilde{v}}_{\mathcal{R}_n}(t) - \widehat{v}_{\mathcal{R}_n}(t)\|. \tag{39}$$

For convenience, we use $\|\cdot\|$ instead of $\|\cdot\|_{L_2}$ and L_2-induced operator norms. We have

$$\begin{aligned}\|\widehat{y} - \widehat{\widetilde{y}}\| \leq&\|\mathcal{F}_{11} - \widetilde{\mathcal{F}}_{11}\|\|\widehat{u}\| + \|\mathcal{F}_{12}\widehat{i}_{\mathcal{R}_n} - \widetilde{\mathcal{F}}_{12}\widehat{\widetilde{i}}_{\mathcal{R}_n}\| \\ \leq&\varepsilon\|\widehat{u}\| + \|(\mathcal{F}_{12} - \widetilde{\mathcal{F}}_{12})\widehat{\widetilde{i}}_{\mathcal{R}_n}\| + \|\mathcal{F}_{12}(\widehat{\widetilde{i}}_{\mathcal{R}_n} - \widehat{i}_{\mathcal{R}_n})\| \\ \leq&\varepsilon\|\widehat{u}\| + \varepsilon\|\widehat{\widetilde{i}}_{\mathcal{R}_n}\| + \|\widehat{i}_{\mathcal{R}_n} - \widehat{\widetilde{i}}_{\mathcal{R}_n}\| \\ \leq&\varepsilon\|\widehat{u}\| + \varepsilon\|\widehat{\widetilde{v}}_{\mathcal{R}_n}\| + \|\widehat{v}_{\mathcal{R}_n} - \widehat{\widetilde{v}}_{\mathcal{R}_n}\|.\end{aligned} \tag{40}$$

Analogously, by also incorporating $\|\mathcal{F}_{22}\| < L$, we obtain

$$\|\widehat{v}_{\mathcal{R}_n} - \widehat{\widetilde{v}}_{\mathcal{R}_n}\| \leq \varepsilon\|\widehat{u}\| + \varepsilon\|\widehat{\widetilde{v}}_{\mathcal{R}_n}\| + L\|\widehat{v}_{\mathcal{R}_n} - \widehat{\widetilde{v}}_{\mathcal{R}_n}\|,$$

which implies

$$\|\widehat{v}_{\mathcal{R}_n} - \widehat{\widetilde{v}}_{\mathcal{R}_n}\| \leq \frac{\varepsilon}{1-L}\left(\|\widehat{u}\| + \|\widehat{\widetilde{v}}_{\mathcal{R}_n}\|\right). \tag{41}$$

Moreover, we can derive

$$\begin{aligned}\|\widehat{\widetilde{v}}_{\mathcal{R}_n}\| \leq&\|\widetilde{\mathcal{F}}_{21}\|\|\widehat{u}\| + \|\widetilde{\mathcal{F}}_{22}\|\|\widehat{\widetilde{i}}_{\mathcal{R}_n}\| \leq \|\widehat{u}\| + (L+\varepsilon)\|\widehat{\widetilde{i}}_{\mathcal{R}_n}\| \\ \leq&\|\widehat{u}\| + (L+\varepsilon)\|\widehat{\widetilde{v}}_{\mathcal{R}_n}\|.\end{aligned}$$

and thus

$$\|\|\widehat{\widetilde{v}}_{\mathcal{R}_n}\| \leq \frac{1}{1-L-\varepsilon}\|\widehat{u}\|. \tag{42}$$

Now plugging the inequalities (41) and (42) into (40), we obtain (35). $\square$

Remark 9. *Since Assumption* **(A8)** *implies* **(A9)**, *Theorem 8 remain of course valid if* **(A9)** *is replaced by* **(A8)**.

In the following we provide an alternative error bound for the difference between the scattering systems corresponding to the original and reduced-order systems. Instead of assuming strict contractivity of the linear system (28), which can be guaranteed when certain conditions on the topology of the linear subsystem are satisfied (see **(A7)** and **(A8)** or **(A9)** in Proposition 7), we now prove an error estimate under assumption **(A10)** on the nonlinearities of the resistances.

Theorem 10. *Let a circuit satisfying* **(A1)**–**(A6)**, **(A10)** *be given and let (24) be a reduced-order system obtained by applied PABTEC to the linear part. If $\widehat{u} : [t_0, t_f] \to \mathbb{R}^m$ is admissible for both the scattering systems of (24) and (11) with initial conditions $x(t_0) = 0$ and $\widehat{x}(t_0) = 0$, respectively, then the corresponding outputs satisfy*

$$\|\widehat{y} - \widetilde{\widehat{y}}\|_{L_2([t_0,t_f])} \leq \frac{\varepsilon}{(1-c_g)^2} \cdot \|\widehat{u}\|_{L_2([t_0,t_f])}, \tag{43}$$

where $\varepsilon \equiv 2\sum_{k=r+1}^{q} \pi_k$ and $c_g = (1 + \min\{c_{g1}, c_{g2}\})^{-1/2}$ with the constants c_{g1}, c_{g2} in **(A10)**.

Proof. We define $\widehat{v}_{\mathcal{R}_n}$, $\widehat{i}_{\mathcal{R}_n}$, $\widetilde{\widehat{v}}_{\mathcal{R}_n}$ and $\widetilde{\widehat{i}}_{\mathcal{R}_n}$ as in (36) and we use the the block operator notation(38).

The passivity of the reduced-order linear part (23) together with Proposition 3 and Theorem 6 guarantees the contractivity of the reduced-order scattering system (25) and the reduced-order scattering linear subsystem (27). Together with the PABTEC error bound, this implies $\|\mathcal{F}_{ij}\| \leq 1$, $\|\widetilde{\mathcal{F}}_{ij}\| \leq 1$, $\|\mathcal{F}_{ij} - \widetilde{\mathcal{F}}_{ij}\| \leq \varepsilon$.

Recall (37). Since g is monotonically increasing, we have

$$\begin{aligned}
&(\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \\
&= (g_n(\widetilde{v}_{\mathcal{R}_n}(t)) - g_n(v_{\mathcal{R}_n}(t)))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \geq 0.
\end{aligned}$$

Using this inequality and (36), we obtain

$$\begin{aligned}
2\|\widetilde{\widehat{i}}_{\mathcal{R}_n}(t) - \widehat{i}_{\mathcal{R}_n}(t)\|^2 =& \|\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t)\|^2 + \|\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)\|^2 \\
&- 2(\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \\
\leq& \|\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t)\|^2 + \|\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)\|^2.
\end{aligned}$$

The conditions **(A10)** imply

$$
\begin{aligned}
&(\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \\
&= (g_n(\widetilde{v}_{\mathcal{R}_n}(t)) - g_n(v_{\mathcal{R}_n}(t)))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \geq c_{g1}\|\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&(\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \\
&\geq c_{g2}\|g_n(\widetilde{v}_{\mathcal{R}_n}(t)) - g_n(v_{\mathcal{R}_n}(t))\|^2 = c_{g2}\|\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t)\|^2
\end{aligned}
$$

Using this inequality and (36), we obtain

$$
\begin{aligned}
&2\|\widetilde{\widehat{v}}_{\mathcal{R}_n}(t) - \widehat{v}_{\mathcal{R}_n}(t)\|^2 \\
&= \|\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t)\|^2 + \|\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)\|^2 \\
&\qquad + 2(\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t))^T(\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)) \\
&\geq (1 + c_{g1})\|\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t)\|^2 + (1 + c_{g2})\|\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)\|^2 \\
&\geq (1 + \min\{c_{g1}, c_{g2}\})(\|\widetilde{i}_{\mathcal{R}_n}(t) - i_{\mathcal{R}_n}(t)\|^2 + \|\widetilde{v}_{\mathcal{R}_n}(t) - v_{\mathcal{R}_n}(t)\|^2).
\end{aligned}
$$

This yields

$$
\|\widetilde{\widehat{i}}_{\mathcal{R}_n}(t) - \widehat{i}_{\mathcal{R}_n}(t)\| \leq c_g\|\widetilde{\widehat{v}}_{\mathcal{R}_n}(t) - \widehat{v}_{\mathcal{R}_n}(t)\|. \tag{44}
$$

In the following we derive the L_2 error bound. Note that we omit the subscripts $L_2([t_0, t_f])$ in the norms in the subsequent calculations. We use the block operator notation (38), the bounds $\|\mathcal{F}_{ij}\| \leq 1$, $\|\widetilde{\mathcal{F}}_{ij}\| \leq 1$, $\|\mathcal{F}_{ij} - \widetilde{\mathcal{F}}_{ij}\| \leq \varepsilon$ and the (44) (once directly and one time with $\widetilde{\widehat{v}}_{\mathcal{R}_n} = 0$ and $\widetilde{\widehat{i}}_{\mathcal{R}_n} = g(\widetilde{\widehat{v}}_{\mathcal{R}_n}) = g_n(0) = 0$) to estimate

$$
\begin{aligned}
\|\widehat{y} - \widetilde{\widehat{y}}\| \leq& \|\mathcal{F}_{11} - \widetilde{\mathcal{F}}_{11}\|\|\widehat{u}\| + \|\mathcal{F}_{12}\widehat{i}_{\mathcal{R}} - \widetilde{\mathcal{F}}_{12}\widehat{\widetilde{i}}_{\mathcal{R}}\| \\
\leq& \varepsilon\|\widehat{u}\| + \|(\mathcal{F}_{12} - \widetilde{\mathcal{F}}_{12})\widetilde{\widehat{i}}_{\mathcal{R}}\| + \|\widetilde{\mathcal{F}}_{12}(\widetilde{\widehat{i}}_{\mathcal{R}} - \widehat{i}_{\mathcal{R}})\| \\
\leq& \varepsilon\|\widehat{u}\| + \varepsilon\|\widehat{i}_{\mathcal{R}}\| + \|\widehat{i}_{\mathcal{R}} - \widetilde{\widehat{i}}_{\mathcal{R}}\| \\
\leq& \varepsilon\|\widehat{u}\| + \varepsilon c_g\|\widehat{v}_{\mathcal{R}_n}\| + c_g\|\widehat{v}_{\mathcal{R}_n} - \widetilde{\widehat{v}}_{\mathcal{R}_n}\|.
\end{aligned} \tag{45}
$$

Analogously, we obtain

$$
\|\widehat{v}_{\mathcal{R}_n} - \widetilde{\widehat{v}}_{\mathcal{R}_n}\| \leq \varepsilon\|\widehat{u}\| + \varepsilon c_g\|\widehat{v}_{\mathcal{R}_n}\| + c_g\|\widehat{v}_{\mathcal{R}_n} - \widetilde{\widehat{v}}_{\mathcal{R}_n}\|
$$

which implies that

$$\|\widehat{v}_{\mathcal{R}_n} - \widetilde{\widehat{v}}_{\mathcal{R}_n}\| \le \frac{\varepsilon}{1-c_g}\|\widehat{u}\| + \frac{\varepsilon c_g}{1-c_g}\|\widehat{v}_{\mathcal{R}_n}\| \tag{46}$$

Moreover, we can derive $\|\widehat{v}_{\mathcal{R}_n}\| \le \|\widehat{u}\| + c_g\|\widehat{v}_{\mathcal{R}_n}\|$, which yields

$$\|\widehat{v}_{\mathcal{R}_n}\| \le \frac{1}{1-c_g}\|\widehat{u}\|. \tag{47}$$

Now plugging the inequalities (46) and (47) into (45), we obtain (43). $\square$

The previous two theorems derived error bounds to the scattering system. In the next subsection we will use these error bounds for the scattering system to derive bounds for the errors between the input-output maps of the full order model (19) and the reduced order model (24). Next we state a result that links error bounds for scattering systems to error bounds for the graph of the system, i.e., the set of input-output trajectories. Model reduction with error bounds of this type is considered, e.g., in [30, 31, 32].

Theorem 11. *Let an admissible descriptor system (3) with initial value $x_0 = 0$ and a reduced-order system*

$$\begin{aligned} \widetilde{E}\dot{\widetilde{x}}(t) &= \widetilde{a}(\widetilde{x}(t)) + \widetilde{B}u(t), \qquad \widetilde{x}(t_0) = 0, \\ \widetilde{y}(t) &= C\widetilde{x}(t) + Du(t) \end{aligned} \tag{48}$$

be given. Moreover, assume that the scattering systems of (3) and (48) satisfy an error bound

$$\|\widehat{y} - \widetilde{\widehat{y}}\|_{L_2([t_0,t_f])} \le e \cdot \|\widehat{u}\|_{L_2([t_0,t_f])},$$

Then, for all inputs $u \in L_2([t_0,t_f])$ of the original system admissible with the initial value $x_0 = 0$, there exists an input $\widetilde{u} \in L_2([t_0,t_f])$ admissible for the reduced order system with $\widetilde{x}_0 = 0$ such that

$$\left\| \begin{bmatrix} u - \widetilde{u} \\ y - \widetilde{y} \end{bmatrix} \right\|_{L_2([t_0,t_f])} \le e \left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{L_2([t_0,t_f])}. \tag{49}$$

Proof. Let $u \in L_2([t_0,t_f])$ be an input admissible with the initial value $x_0 = 0$ and let $y \in L_2([t_0,t_f])$ be the corresponding output of the original system. Now define $\widehat{u} = \frac{1}{\sqrt{2}}(u+y)$. Then the output of the scattering system satisfies $\widehat{y} = \frac{1}{\sqrt{2}}(u - y)$. Further, let $\widetilde{\widehat{y}}$ be the output of the reduced-order scattering system. If we define $\widetilde{u} = \frac{1}{\sqrt{2}}(\widehat{u} + \widetilde{\widehat{y}})$, then the output of the reduced-order system is given by $\widetilde{y} = \frac{1}{\sqrt{2}}(\widehat{u} - \widetilde{y})$. Moreover, we have

$$\left\| \begin{bmatrix} u - \widetilde{u} \\ y - \widetilde{y} \end{bmatrix} \right\|_{L_2([t_0,t_f])} = \left\| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \widehat{u} - \widehat{u} \\ \widehat{y} - \widetilde{\widehat{y}} \end{bmatrix} \right\|_{L_2([t_0,t_f])}$$

Since the 2×2 matrix in front of the vectorial function is orthogonal, we have

$$\left\| \begin{bmatrix} u - \widetilde{u} \\ y - \widetilde{y} \end{bmatrix} \right\|_{L_2} = \|\widehat{y} - \widetilde{\widehat{y}}\|_{L_2} \leq e\|\widehat{u}\|_{L_2} \ \leq e \left\| \begin{bmatrix} \widehat{u} \\ \widehat{y} \end{bmatrix} \right\|_{L_2}. \tag{50}$$

Using the same arguments as above, we get

$$\left\| \begin{bmatrix} \widehat{u} \\ \widehat{y} \end{bmatrix} \right\|_{L_2([t_0,t_f])} = \left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{L_2([t_0,t_f])},$$

which together with (50) gives the desired estimate. □

6.2. Error Bounds for the Output of the Circuit System

To provide a bound for the error between the original system (19) and the reduced-order model (23) we need the following auxiliary result.

Lemma 12. *Let* **(A1)**–**(A6)** *be satisfied and assume that the scattering system corresponding to (19) is strictly contractive with Lipschitz constant $L < 1$. If the scattering systems of (19) and (24) satisfy an error bound*

$$\|\widehat{y} - \widetilde{\widehat{y}}\|_{L_2([t_0,t_f])} \leq e \cdot \|\widehat{u}\|_{L_2([t_0,t_f])},$$

for all admissible inputs $\widehat{u} \in L_2([t_0, t_f])$, then the following statements are true.

(i) The scattering system of (19) is strictly contractive with Lipschitz constant L.

(ii) The system (19) is K-Lipschitz with $K = (1 - L)^{-1}(1 + L)$.

(iii) If $e < 1 - L$, then the system (24) is $\widetilde{K}$-Lipschitz with

$$\widetilde{K} = (1 - L - e)^{-1}(1 + L + e).$$

(iv) For all inputs $u \in L_2([t_0, t_f])$ admissible for (19) and (24) with $x_0 = 0$ and $\widetilde{x}_0 = 0$, the corresponding outputs fulfill

$$\|y - \widetilde{y}\|_{L_2([t_0,t_f])} \leq \frac{e}{1-L} \left(\|u\|_{L_2([t_0,t_f])} + \|\widetilde{y}\|_{L_2([t_0,t_f])}\right) \tag{51}$$

(v) If $e < 1 - L$, then for all inputs $u \in L_2([t_0, t_f])$ admissible for (19) and (24) with $x_0 = 0$ and $\widetilde{x}_0 = 0$, the corresponding outputs fulfill

$$\|y - \widetilde{y}\|_{L_2([t_0,t_f])} \leq \frac{2e}{(1-L)(1-L-e)} \|u\|_{L_2([t_0,t_f])} \tag{52}$$

PROOF. (i) We apply the representation of the scattering system as in the proof of Theorem 8. Using the strict contractivity of the scattering system corresponding to (21) and inequality (39), we find that

$$\begin{aligned} &\|\widehat{y}_1 - \widehat{y}_2\|^2 + \|\widehat{v}_{\mathcal{R}_n 1} - \widehat{v}_{\mathcal{R}_n 2}\|^2 \\ \leq &L^2(\|\widehat{u}_1 - \widehat{u}_2\|^2 + \|\widehat{i}_{\mathcal{R}_n 1} - \widehat{i}_{\mathcal{R}_n 2}\|^2) \\ \leq &L^2(\|\widehat{u}_1 - \widehat{u}_2\|^2 + \|\widehat{v}_{\mathcal{R}_n 1} - \widehat{v}_{\mathcal{R}_n 2}\|^2). \end{aligned}$$

Subtracting $L^2\|\widehat{v}_{\mathcal{R}_n 1} - \widehat{v}_{\mathcal{R}_n 2}\|^2$ on both sides of the inequality, we obtain the desired result.

(ii) Let u_1, u_2 and y_1, y_2 be inputs and corresponding outputs of the system (19). Then, for $i = 1, 2$, the functions $\widehat{u}_i = \frac{1}{\sqrt{2}}(u_i + y_i)$ and $\widehat{y}_i = \frac{1}{\sqrt{2}}(u_i - y_i)$ are inputs and outputs of the corresponding scattering system. Strict contractivity of the scattering system corresponding to (19). implies that

$$\|u_1 - u_2 - (y_1 - y_2)\| \leq L\|u_1 - u_2 + y_1 - y_2\|$$

and thus

$$\|y_1 - y_2\| - \|u_1 - u_2\| \leq L\|u_1 - u_2\| + L\|y_1 - y_2\|$$

Rearranging terms, we obtain that $(1+L)/(1-L)$ is a Lipschitz constant for (19)).

(iii) Taking into account that the scattering system of (24) is strictly contractive with Lipschitz constant $L + e$, the result can be obtained by the same arguments used in (ii).

(iv) Define $\widehat{u}(t) = \frac{1}{\sqrt{2}}(u+y)$, $\widetilde{\widehat{u}}(t) = \frac{1}{\sqrt{2}}(u+\widetilde{y})$ and the corresponding outputs $\widehat{y}(t) = \frac{1}{\sqrt{2}}(u-y)$ and $\widetilde{\widehat{u}}(t) = \frac{1}{\sqrt{2}}(u-\widetilde{y})$. Moreover, denote by $\widehat{\mathcal{F}}$ and $\widetilde{\widehat{\mathcal{F}}}$ the (nonlinear) input-output mapping of the scattering systems of the original and reduced-order system, respectively. Then we have

$$\begin{aligned}\|\tfrac{1}{\sqrt{2}}(y-\widetilde{y})\| =& \|\widehat{y}-\widetilde{\widehat{y}}\| = \|\widehat{\mathcal{F}}\widehat{u}-\widetilde{\widehat{\mathcal{F}}}\widetilde{\widehat{u}}\| \le \|\widehat{\mathcal{F}}\widehat{u}-\widehat{\mathcal{F}}\widetilde{\widehat{u}}\| + \|\widehat{\mathcal{F}}\widetilde{\widehat{u}}-\widetilde{\widehat{\mathcal{F}}}\widetilde{\widehat{u}}\| \\ \le& L\|\widehat{u}-\widetilde{\widehat{u}}\| + e\|\widetilde{\widehat{u}}\| \le L\|\tfrac{1}{\sqrt{2}}(\widetilde{y}-y)\| + e\|\tfrac{1}{\sqrt{2}}(u-\widetilde{y})\|.\end{aligned} \tag{53}$$

This directly implies (51).

(v) This follows by a combination of (iii) and (iv).

□

We now apply Lemma 12 for deriving estimates for the output error of the reduced-order circuit. Note that (51) gives an a posteriori error bound whereas (52) is of a priori nature.

Theorem 13. *Let a circuit satisfying* **(A1)**–**(A8)** *be given and let the reduced-order system (24) be obtained by applying PABTEC to (21). Furthermore, let the linear system (28) be strictly contractive with Lipschitz constant $L < 1$. If the size r of the reduced linear subsystem (23) is chosen such that*

$$\varepsilon \equiv 2\sum_{k=r+1}^{q} \pi_k < 1 - L,$$

then the following holds true. If $\widehat{u} : [t_0, t_f] \to \mathbb{R}^m$ is admissible for both (19) and (24) with initial conditions $x(t_0) = \widetilde{x}(t_0) = 0$, then the following holds true for the outputs sof the full and reduced-order system corresponding to the input u and initial values $x_0 = 0$, $\widetilde{x}_0 = 0$:

(i)

$$\|y-\widetilde{y}\|_{L_2([t_0,t_f])} \le \varepsilon\frac{(2-L)(2-L-\varepsilon)}{(1-L)^2(1-L-\varepsilon)}(\|u\|_{L_2([t_0,t_f])} + \|\widetilde{y}\|_{L_2([t_0,t_f])}). \tag{54}$$

(ii) If $\varepsilon < \frac{1}{5}(1-L)^3$, then

$$\|y-\widetilde{y}\|_{L_2([t_0,t_f])} \leq 2\varepsilon \frac{(2-L)(2-L-\varepsilon)}{(1-L)((1-L)^2(1-L-\varepsilon)-\varepsilon(2-L)(2-L-\varepsilon))}\|u\|_{L_2([t_0,t_f])}. \tag{55}$$

(iii) If, additionally, **(A10)** *holds true, then with $c_g = (1+\min\{c_{g1}, c_{g2}\})^{-1/2}$ for the constants c_{g1}, c_{g2} in* **(A10)**, *the corresponding outputs satisfy*

$$\|y-\widetilde{y}\|_{L_2([t_0,t_f])} \leq \varepsilon \frac{1}{(1-L)(1-c_g)^2}(\|u\|_{L_2([t_0,t_f])} + \|\widetilde{y}\|_{L_2([t_0,t_f])}). \tag{56}$$

(iv) If, additionally, **(A10)** *and $\varepsilon < (1-c_g)^2(1-L)$ hold true, then*

$$\|y-\widetilde{y}\|_{L_2([t_0,t_f])} \leq \varepsilon \frac{1}{(1-L)(1-L)(1-c_g)^2-\varepsilon}\|u\|_{L_2([t_0,t_f])}. \tag{57}$$

Proof. The statement (i) follows from Theorem 8 and (iv) of Lemma 12. Taking into account that $\varepsilon < \frac{1}{5}(1-L)^3$ implies $\varepsilon\frac{(2-L)(2-L-\varepsilon)}{(1-L)(1-L-\varepsilon)} \leq 1-L$, (ii) is a consequence of Theorem 8 and (v) of Lemma 12. (iii) and (iv) are also direct implications of Theorem 8 and Lemma 12. □

7. Numerical Example

We apply our model reduction approach to a circuit provided by NEC Laboratories Europe, IT Research Division. This circuit consists of 2002 capacitances, 2004 linear resistances, no inductance, one voltage source and two nonlinear resistances with voltage current characteristic

$$g_n(u) = \exp(1000 \cdot u) - 1.$$

The linear part is described by the passive system of order $n = 2007$. We apply trivial initial values $x_0 = 0$ and the input

$$u(t) = t \cdot (\exp(-100 \cdot (t-0.2)^2) - 5 \cdot \exp(-100 \cdot ((t-0.5)^2))), \qquad t \in [0,1],$$

which is shown in Figure 1.

Computations were done on IBM RS 6000 44P Modell 270 with machine precision $\varepsilon = 2.22 \times 10^{-16}$ using MATLAB 2008a.

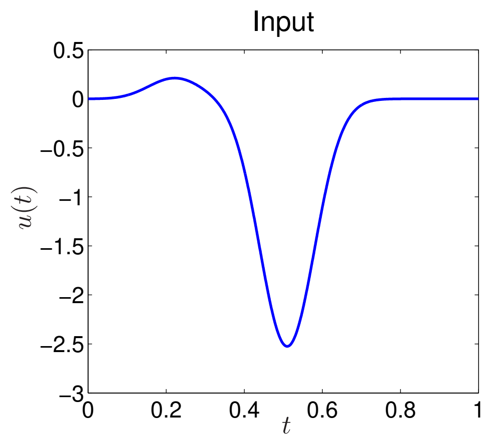

Figure 1: Input into the circuit equations

The minimal solution of the projected Lur'e equation (14) is approximated by a low-rank matrix $X \approx L^T L$ with $L \in \mathbb{R}^{92,2007}$ using Newton's method as presented in [10]. Figure 2 shows the largest 90 passivity characteristic values. One can see that the characteristic values decay rapidly. The original system was approximated by a model of order $r = 59$.

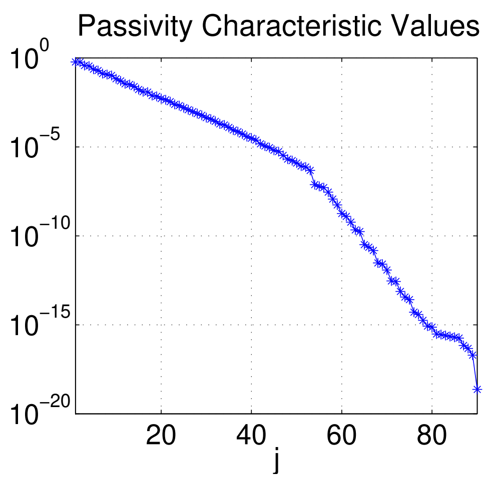

Figure 2: The passivity characteristic values decrease rapidly. The reduced-order model is obtained by truncating the passivity characteristic values with index greater than $r = 59$.

The numerical solution of the full order and reduced-order DAEs are computed using the MATLAB built in DAE solver `ode23t`. Figure 3 displays the output of the reduced-order model and the difference between outputs of original and reduced system. Our reduced-order model is in excellent agreement with the full order model output, as guaranteed by the error bound. Numerical computation yields that the actual model reduction error is around $\|y - \tilde{y}\| \approx 5.8 \cdot 10^{-4}$. The a posteriori and a priori error bounds according to Theorem 13 (i) and (ii) are given by 0.0716 and 0.1665, respectively.

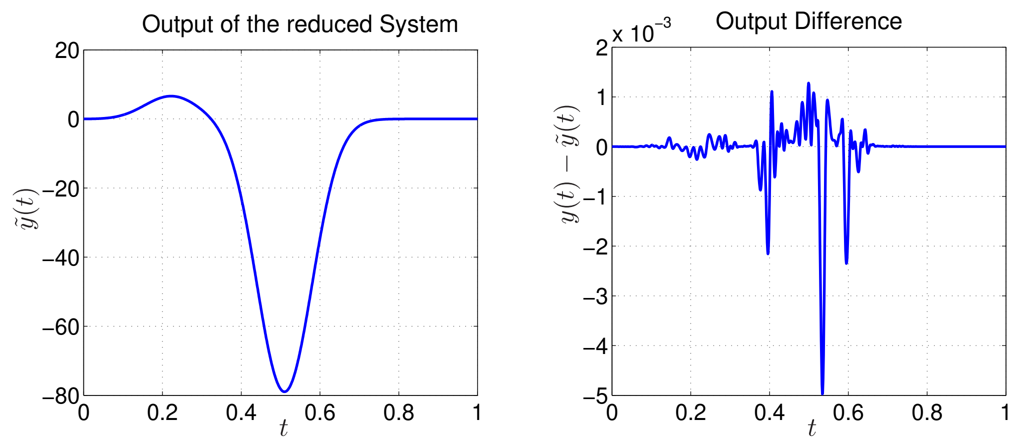

Figure 3: Output of the reduced system (left plot) and error between the outputs of the full and the reduced-order system (right plot). The full and the reduced-order system outputs are in excellent agreement.

The simulation of the full order system took 107.00 seconds. The computation of the reduced-order model using PABTEC required 9.04 seconds and the simulation of the reduced-order system required 0.31 seconds. Thus model reduction already amortizes after only one single output computation.

8. Conclusion

We have introduced a model reduction technique for electrical circuits with nonlinear resistances. Our approach extract linear time invariant subcircuits, model reduced these subcircuits by the passivity preserving method PABTEC, and then coupled the reduced order linear subcircuit with the nonlinear subcircuit of the original model. Since the nonlinear subcircuit is not reduced, our approach is only efficient if there are relatively few nonlinear resistors. We have used the properties of the circuit to show that our reduced order models maintain admissibility and passivity, and we have proven a-priori bounds for the errors between the input-output maps of the original nonlinear system and the reduced order nonlinear system. These a-priori bounds are valid for all inputs. The development of error bounds for larger classes of circuits is subject of further research.

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