

# Fredholmness and Smooth Dependence for Linear Hyperbolic Periodic-Dirichlet Problems

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## Abstract

This paper concerns  $n \times n$  linear one-dimensional hyperbolic systems of the type

$$\omega \partial_t u_j + a_j(x) \partial_x u_j + \sum_{k=1}^n b_{jk}(x) u_k = f_j(x, t), \quad j = 1, \dots, n,$$

with periodicity conditions in time and reflection boundary conditions in space. We state sufficient conditions on the data  $\omega, a_j$  and  $b_{jk}$  such that the system has a Fredholm like solvability behavior. Moreover, we state sufficient conditions on the data such that for any right hand side there exists exactly one solution, that the solution survives under small perturbations of the data, and that the corresponding data-to-solution-map is smooth with respect to appropriate function space norms. In particular, those sufficient conditions imply that no small denominator effects occur. Moreover, we show that perturbations of the coefficients  $\omega$  and  $a_j$  lead to essentially different results than perturbations of the coefficients  $b_{jk}$ , in general.

## 1 Introduction

### 1.1 Problem and main results

This paper concerns linear inhomogeneous hyperbolic systems of first order PDEs in one space dimension of the type

$$\omega \partial_t u_j + a_j(x) \partial_x u_j + \sum_{k=1}^n b_{jk}(x) u_k = f_j(x, t), \quad j = 1, \dots, n, \quad x \in (0, 1), \quad t \in \mathbb{R} \quad (1.1)$$

with time-periodicity conditions

$$u_j(x, t + 2\pi) = u_j(x, t), \quad j = 1, \dots, n, \quad x \in [0, 1], \quad t \in \mathbb{R} \quad (1.2)$$

and reflection boundary conditions

$$\left. \begin{aligned} u_j(0, t) &= \sum_{k=m+1}^n r_{jk}^0 u_k(0, t), \quad j = 1, \dots, m, \\ u_j(1, t) &= \sum_{k=1}^m r_{jk}^1 u_k(1, t), \quad j = m+1, \dots, n, \end{aligned} \right\} t \in \mathbb{R}. \quad (1.3)$$

Here  $1 \leq m < n$  are fixed natural numbers,  $\omega > 0$  and  $r_{jk}^0, r_{jk}^1$  are real numbers, and the right-hand sides  $f_j : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  are supposed to be  $2\pi$ -periodic with respect to  $t$ .

Roughly speaking, we will prove results of the following type:

First, we will state sufficient conditions on the data  $\omega, a_j, b_{jk}, r_{jk}^0$ , and  $r_{jk}^1$  such that the system (1.1)–(1.3) has a Fredholm type solution behavior, i.e. that it is solvable if and only if the right hand side is orthogonal to all solutions to the corresponding homogeneous adjoint system

$$\begin{aligned} -\omega \partial_t u_j - \partial_x (a_j(x) u_j) + \sum_{k=1}^n b_{kj}(x) u_k &= 0, \quad j = 1, \dots, n, \quad x \in [0, 1], \quad t \in \mathbb{R} \\ u_j(x, t + 2\pi) &= u_j(x, t), \quad j = 1, \dots, n, \quad x \in [0, 1], \quad t \in \mathbb{R}, \\ \left. \begin{aligned} a_j(0) u_j(0, t) &= - \sum_{k=1}^m r_{kj}^0 a_k(0) u_k(0, t), \quad j = m+1, \dots, n \\ a_j(1) u_j(1, t) &= - \sum_{k=m+1}^n r_{kj}^1 a_k(1) u_k(1, t), \quad j = 1, \dots, m, \end{aligned} \right\} t \in \mathbb{R}. \end{aligned} \quad (1.4)$$

And second, we will state sufficient conditions on the data  $\omega, a_j, b_{jk}, r_{jk}^0$ , and  $r_{jk}^1$  such that the system (1.1)–(1.3) is uniquely solvable for any right hand side, that this unique solvability property survives under small perturbations of the data, and that the corresponding data-to-solution-maps are smooth with respect to appropriate function space norms. For example, under those sufficient conditions the following is true:

(I) If  $\partial_t^j f \in L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  for  $j = 0, 1$ , then the map  $b \mapsto u$  is  $C^\infty$ -smooth from an open set in  $L^\infty((0, 1); \mathbb{M}_n)$  into  $L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$ .

(II) If  $\partial_t^j f \in L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  for  $j = 0, 1, 2$ , then the map  $b \mapsto u$  is  $C^\infty$ -smooth from an open set in  $L^\infty((0, 1); \mathbb{M}_n)$  into  $C([0, 1] \times [0, 2\pi]; \mathbb{R}^n)$ .

(III) If  $\partial_t^j f \in L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  for  $j = 0, 1, \dots, k$  with  $k \geq 2$ , then the map  $(\omega, a) \mapsto u$  is  $C^{k-1}$ -smooth (rsp.  $C^{k-2}$ -smooth) from an open subset of  $\mathbb{R} \times C^1([0, 1]; \mathbb{R}^n)$  into  $L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  (rsp.  $C([0, 1] \times [0, 2\pi]; \mathbb{R}^n)$ ).

Here and in what follows we denote by

$$a := (a_1, \dots, a_n), \quad f := (f_1, \dots, f_n), \quad \text{and} \quad b := [b_{jk}]_{j,k=1}^n$$

the vectors of the coefficient functions  $a_j$  and the right hand sides  $f_j$  and the matrix of the coefficient functions  $b_{jk}$ , respectively, and  $\mathbb{M}_n$  is the space of all real  $n \times n$  matrices.

In order to formulate our results more precisely, let us introduce the following function spaces: For  $\gamma \geq 0$  we denote by  $W^\gamma$  the vector space of all locally integrable functions  $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $u(x, t) = u(x, t + 2\pi)$  for almost all  $x \in (0, 1)$  and  $t \in \mathbb{R}$  and that

$$\|u\|_{W^\gamma}^2 := \sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 \left\| \int_0^{2\pi} u(x, t) e^{-ist} dt \right\|^2 dx < \infty. \quad (1.5)$$

Here and in what follows  $\|\cdot\|$  is the Hermitian norm in  $\mathbb{C}^n$ . It is well-known (see, e.g., [3], [19, Chapter 5.10] and [20, Chapter 2.4]) that  $W^\gamma$  is a Banach space with the norm (1.5). In fact, it is the anisotropic Sobolev space of all measurable functions  $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $u(x, t) = u(x, t + 2\pi)$  for almost all  $x \in (0, 1)$  and  $t \in \mathbb{R}$  and that the distributional partial derivatives of  $u$  with respect to  $t$  up to the order  $\gamma$  are locally quadratically integrable. Further, for  $\omega > 0$  and  $a \in L^\infty((0, 1); \mathbb{R}^n)$  with  $\text{ess inf } |a_j| > 0$  for all  $j = 1, \dots, n$  we will work with the function spaces

$$U^\gamma(\omega, a) := \left\{ u \in W^\gamma : [\omega \partial_t u_j + a_j \partial_x u_j]_{j=1}^n \in W^\gamma \right\}$$

endowed with the norms

$$\|u\|_{U^\gamma(\omega,a)}^2 := \|u\|_{W^\gamma}^2 + \left\| [\omega \partial_t u_j + a_j \partial_x u_j]_{j=1}^n \right\|_{W^\gamma}^2.$$

Remark that the space  $U(\omega, a)$  depends on  $\omega$  and  $a$ . In particular, it is larger than the space of all  $u \in W^\gamma$  such that  $\partial_t u \in W^\gamma$  and  $\partial_x u \in W^\gamma$  (which does not depend on  $\omega$  or  $a$ ).

For  $\gamma \geq 1$  and  $u \in U^\gamma(\omega, a)$  there exist traces  $u(0, \cdot), u(1, \cdot) \in L_{loc}^2(\mathbb{R}; \mathbb{R}^n)$  (see Section 2), and, hence, it makes sense to consider the closed subspaces in  $U^\gamma(\omega, a)$

$$\begin{aligned} V^\gamma(\omega, a, r) &:= \{u \in U^\gamma(\omega, a) : (1.3) \text{ is fulfilled}\}, \\ \tilde{V}^\gamma(\omega, a, r) &:= \{u \in U^\gamma(\omega, a) : (1.4) \text{ is fulfilled}\}. \end{aligned}$$

Here we use the notation

$$r := (r^0, r^1) \text{ with } r^0 := [r_{jk}^0]_{j=1, k=m+1}^m, r^1 := [r_{jk}^1]_{j=m+1, k=1}^m$$

for the matrices of the reflection coefficients  $r_{jk}^0$  and  $r_{jk}^1$ . Further, we denote by

$$b^0 := \text{diag}\{b_{11}, b_{22}, \dots, b_{nn}\} \text{ and } b^1 := b - b^0$$

the diagonal and the off-diagonal parts of the coefficient matrix  $b$ , respectively, and we introduce operators  $\mathcal{A}(\omega, a, b^0) \in \mathcal{L}(V^\gamma(\omega, a, r); W^\gamma)$ ,  $\tilde{\mathcal{A}}(\omega, a, b^0) \in \mathcal{L}(\tilde{V}^\gamma(\omega, a, r); W^\gamma)$  and  $\mathcal{B}(b^1), \tilde{\mathcal{B}}(b^1) \in \mathcal{L}(W^\gamma)$  by

$$\begin{aligned} \mathcal{A}(\omega, a, b^0)u &:= [\omega \partial_t u + a_j \partial_x u_j + b_{jj} u_j]_{j=1}^n, \\ \tilde{\mathcal{A}}(\omega, a, b^0)u &:= [-\omega \partial_t u - a_j \partial_x u_j + b_{jj} u_j]_{j=1}^n, \\ \mathcal{B}(b^1)u &:= \left[ \sum_{j \neq k} b_{jk} u_k \right]_{j=1}^n, \\ \tilde{\mathcal{B}}(b^1)u &:= \left[ \sum_{j \neq k} b_{kj} u_k \right]_{j=1}^n. \end{aligned}$$

Obviously, the operator equation

$$\mathcal{A}(\omega, a, b^0)u + \mathcal{B}(b^1)u = f \tag{1.6}$$

is an abstract representation of the periodic-Dirichlet problem (1.1)–(1.3). Finally, for  $s \in \mathbb{Z}$  we introduce the following complex  $(n - m) \times (n - m)$ -matrices

$$R_s(\omega, a, b^0, r) := \left[ \sum_{l=1}^m e^{is\omega(\alpha_j(1) - \alpha_l(1)) + \beta_j(1) - \beta_l(1)} r_{jl}^1 r_{lk}^0 \right]_{j, k=m+1}^n, \tag{1.7}$$

where

$$\alpha_j(x) := \int_0^x \frac{1}{a_j(y)} dy, \quad \beta_j(x) := \int_0^x \frac{b_{jj}(y)}{a_j(y)} dy. \tag{1.8}$$

Our first result concerns an isomorphism property of  $\mathcal{A}(\omega, a, b^0)$ :

**Theorem 1.1** *For all  $c \in (0, 1)$  there exists  $C > 0$  such that the following is true: If*

$$a_j, b_{jj} \in L^\infty(0, 1) \text{ and } \text{ess inf } |a_j| \geq c \text{ for all } j = 1, \dots, n, \tag{1.9}$$

$$|\omega| + \sum_{j=1}^n (\|a_j\|_\infty + \|b_{jj}\|_\infty) + \sum_{j=1}^m \sum_{k=m+1}^n |r_{jk}^0| + \sum_{j=m+1}^n \sum_{k=1}^m |r_{jk}^1| \leq \frac{1}{c}, \tag{1.10}$$

and

$$|\det(I - R_s(\omega, a, b^0, r))| \geq c \text{ for all } s \in \mathbb{Z}, \tag{1.11}$$

then for all  $\gamma \geq 1$  the operator  $\mathcal{A}(\omega, a, b^0)$  is an isomorphism from  $V^\gamma(\omega, a, r)$  onto  $W^\gamma$  and

$$\|\mathcal{A}(\omega, a, b^0)^{-1}\|_{\mathcal{L}(W^\gamma; V^\gamma(\omega, a, r))} \leq C.$$

Our second result concerns the following Fredholm type solvability behavior of (1.6):

**Theorem 1.2** *Suppose that*

$$a_j \in C^{0,1}[0, 1], \quad b_{jj} \in L^\infty(0, 1), \quad \min_{0 \leq x \leq 1} |a_j(x)| > 0 \text{ for all } j = 1, \dots, n \quad (1.12)$$

and

$$\left. \begin{array}{l} \text{for all } j \neq k \text{ there is } c_{jk} \in BV(0, 1) \text{ such that} \\ b_{jk}(x) = c_{jk}(x)(a_j(x) - a_k(x)) \text{ for a.a. } x \in [0, 1]. \end{array} \right\} \quad (1.13)$$

Further, suppose that there exists  $c > 0$  such that (1.11) is satisfied. Then the following is true:

(i) The operator  $\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)$  is a Fredholm operator with index zero from  $V^\gamma(\omega, a, r)$  into  $W^\gamma$  for all  $\gamma \geq 1$ .

(ii) The subspaces  $\ker(\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1))$  and  $\ker(\tilde{\mathcal{A}}(\omega, a, b^0) + \tilde{\mathcal{B}}(b^1))$  do not depend on  $\gamma$ , and

$$\text{im}(\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)) = \left\{ f \in W^\gamma : \langle f, u \rangle_{L^2} = 0 \text{ for all } u \in \ker(\tilde{\mathcal{A}}(\omega, a, b^0) + \tilde{\mathcal{B}}(b^1)) \right\}.$$

Here we write

$$\langle f, u \rangle_{L^2} := \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \langle f(x, t), u(x, t) \rangle \, dx dt \quad (1.14)$$

for the usual scalar product in the Hilbert space  $L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$ , and  $\langle \cdot, \cdot \rangle$  denotes the Euclidean scalar product in  $\mathbb{R}^n$  (as well as the Hermitian scalar product in  $\mathbb{C}^n$ ).

The main tools of the proofs of Theorems 1.1 and 1.2 are separation of variables (cf. (3.2)–(3.3)), integral representation of the solutions of the corresponding boundary value problems of the ODE systems (cf. (3.10)), and an abstract criterion for Fredholmness (cf. Lemma 4.1). In the special case  $m = 1, n = 2, a_1(x) = 1, a_2(x) = -1$  Theorem 1.2 was proved in [9].

Our last results concern the solution behavior of (1.6) under small perturbations of the data  $\omega, a, b$ , and  $f$ . In order to describe this we use the following notation for the corresponding open balls (for  $\varepsilon > 0$ ):

$$\begin{aligned} A_\varepsilon(a) &:= \left\{ \tilde{a} \in C^{0,1}([0, 1]; \mathbb{R}^n) : \max_{j=1, \dots, n} \|\tilde{a}_j - a_j\|_\infty < \varepsilon \right\}, \\ B_\varepsilon^\infty(b) &:= \left\{ \tilde{b} \in L^\infty((0, 1); \mathbb{M}_n) : \max_{j, k=1, \dots, n} \|\tilde{b}_{jk} - b_{jk}\|_\infty < \varepsilon \right\}, \\ B_\varepsilon(b) &:= \left\{ \tilde{b} \in B_\varepsilon^\infty(b) : \tilde{b}_{jk} \in BV(0, 1) \text{ for all } 1 \leq j \neq k \leq n \right\}. \end{aligned}$$

Of course,  $B_\varepsilon^\infty(b)$  is an open set in the Banach space  $L^\infty((0, 1); \mathbb{M}_n)$ . The sets  $A_\varepsilon(a)$  and  $B_\varepsilon(b)$  will be considered as open sets in the (not complete) normed vector spaces  $C^{0,1}([0, 1]; \mathbb{R}^n)$  and  $\{\tilde{b} \in L^\infty((0, 1); \mathbb{M}_n) : \tilde{b}_{jk} \in BV(0, 1) \text{ for all } 1 \leq j \neq k \leq n\}$ , equipped with the corresponding  $L^\infty$ -norms.

The solution behavior of (1.6) under small perturbations of  $b$  and  $f$  follows directly from Theorem 1.2 and the Implicit Function Theorem, because the map

$$b \in L^\infty((0, 1); \mathbb{M}_n) \mapsto (\mathcal{A}(\omega, a, b^0), \mathcal{B}(b^1)) \in \mathcal{L}(V^\gamma(\omega, a, r); W^\gamma) \times \mathcal{L}(W^\gamma) \quad (1.15)$$

is affine and continuous:

**Corollary 1.3** *Suppose (1.11) for some  $c > 0$ , (1.12), (1.13), and*

$$\dim \ker(\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)) = 0. \quad (1.16)$$

Then there exists  $\varepsilon > 0$  such that for all  $\gamma \geq 1$ ,  $\tilde{b} \in B_\varepsilon^\infty(b)$ , and  $f \in W^\gamma$  there exists exactly one  $u \in V^\gamma(\omega, a, r)$  with  $\mathcal{A}(\omega, a, \tilde{b}^0)u + \mathcal{B}(\tilde{b}^1)u = f$ . Moreover, the map

$$(\tilde{b}, f) \in B_\varepsilon^\infty(b) \times W^\gamma \mapsto u \in V^\gamma(\omega, a, r)$$

is  $C^\infty$ -smooth.

In particular, Corollary 1.3 implies assertion (I) above, and, because of the continuous embedding  $V^\gamma(\omega, a, r) \hookrightarrow C([0, 1] \times [0, 2\pi]; \mathbb{R}^n)$  for  $\gamma > 3/2$  (see Lemma 2.2(iii)), also assertion (II).

The solution behavior of (1.6) under small perturbations of  $\omega$ ,  $a$ , and  $r$  seems to be more complicated. Under those perturbations the function spaces  $V^\gamma(\omega, a, r)$  change, in general. This makes them inappropriate. On the other hand, we don't know any Fredholmness results for the operator  $\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)$  besides that which is described in Theorem 1.2 and, hence, which is related to the choice of the function spaces  $V^\gamma(\omega, a, r)$  and  $W^\gamma$ .

**Theorem 1.4** *Suppose (1.12), (1.16), and*

$$\sum_{j,k=m+1}^n \sum_{l=1}^m e^{2(\beta_l(1) - \beta_j(1))} |r_{jl}^1 r_{lk}^0|^2 < 1. \quad (1.17)$$

*Further, assume that for all  $1 \leq j \neq k \leq n$  we have  $b_{jk} \in BV(0, 1)$  and*

$$a_j(x) \neq a_k(x) \text{ for all } x \in [0, 1]. \quad (1.18)$$

*Then there exists  $\varepsilon > 0$  such that for all  $\gamma \geq 1$ ,  $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$ ,  $\tilde{a} \in A_\varepsilon(a)$ ,  $\tilde{b} \in B_\varepsilon(b)$ , and  $f \in W^\gamma$  there exists exactly one  $u \in V^\gamma(\tilde{\omega}, a, r)$  with  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0)u + \mathcal{B}(\tilde{b}^1)u = f$ . Moreover, the map*

$$(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in (\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma \mapsto u \in W^{\gamma-k-1} \quad (1.19)$$

*is  $C^k$ -smooth for all nonnegative integers  $k \leq \gamma - 1$ .*

In particular, for  $k = \gamma - 1$  (rsp.  $k = \gamma - 2$ ) we get assertion (III) above.

The present paper has been motivated mainly by two reasons:

The first reason is that the Fredholm property of the linearization is a key for many local investigations for nonlinear equations, such as small periodic forcing of stationary solutions to nonlinear autonomous problems or Hopf bifurcation. In particular, those techniques are well established for nonlinear ODEs and nonlinear parabolic PDEs, but almost nothing is known about the question if those techniques work for nonlinear dissipative hyperbolic PDEs.

The second reason are applications to mathematical biology [4, 5, 6, 14], to kinetic gas dynamics [1, 7, 16], or to semiconductor laser dynamics [12, 17, 18]. Phenomena like Hopf bifurcation (describing the appearance of selfpulsations of lasers) and periodic forcing of stationary solutions (describing the modulation of stationary laser states by time periodic electric pumping) are essential for many applications of semiconductor laser devices in communication systems (see, e.g., [18]).

Remark that our smoothness assumptions concerning  $a_j$ ,  $b_{jk}$ , and  $f_j(\cdot, t)$  are quite weak. This is important for the applications to laser dynamics. But it turns out that any stronger smoothness assumption with respect to the space variable  $x$  would not essentially improve our results and would not simplify the proofs.

Our paper is organized as follows: In Subsection 1.2 we comment about sufficient conditions for the key assumptions (1.11), (1.13), (1.16), and (1.17) and about the question if those conditions as well as the assertions of Theorems 1.1 and 1.2 are stable under small perturbations of the data. In Section 2 we introduce the main properties of the function spaces, used in this paper. In Section 3 we prove Theorem 1.1, in Sections 4 and 5 we prove Theorem 1.2, and, finally, in Section 6 we prove Theorem 1.4.

## 1.2 Some Remarks

**Remark 1.5 about a sufficient condition for (1.11):** Let us formulate a sufficient condition for (1.11), in which the parameters  $\omega$  and  $s$  do not appear.

Condition (1.11) is satisfied iff for all  $s \in \mathbb{Z}$  the matrix  $I - R_s(\omega, a, b^0, r)$  is invertible and the Euclidean norm  $\|(I - R_s(\omega, a, b^0, r))^{-1}\|$  is bounded uniformly in  $s \in \mathbb{Z}$ . For the latter it is sufficient to have

$$\|R_s(\omega, a, b^0, r)\| \leq \text{const} < 1 \text{ for all } s \in \mathbb{Z}. \quad (1.20)$$

Condition (1.20) and, therefore, (1.11) are satisfied if, for example, condition (1.17) is satisfied.

**Remark 1.6 about small perturbations of the data in (1.11) and (1.17):** Let us comment about the behavior of the assumption (1.11) and its sufficient condition (1.17) under small perturbations of the data.

If condition (1.17) is satisfied for given data, then it remains to be satisfied under sufficiently small perturbations of the coefficients  $\omega, r_{jk}^0, r_{jk}^1$  and under sufficiently small (in  $L^\infty(0,1)$ ) perturbations of the coefficient functions  $a_j$  and  $b_{jk}$ .

If condition (1.11) is satisfied, then it remains to be satisfied under sufficiently small perturbations of  $r_{jk}^0, r_{jk}^1$ , and  $b_{jk}$ , but not under small perturbations of  $\omega$  and  $a_j$ , in general. In other words, (1.11) is not sufficient for (1.17). It may happen that there exist arbitrarily small perturbations of  $\omega$  and  $a_j$  that destroy the validity of (1.11):

For example, consider the case  $m = 1, n = 2, a_1(x) = 1, a_2(x) = -1, b_{jk}(x) = 0$  for  $j, k = 1, 2, r_{1,2}^0 = 1, r_{2,1}^1 = -1$ . Then (1.11) reads as

$$|1 + e^{2i\omega s}| \geq \text{const} > 0 \text{ for all } s \in \mathbb{Z}. \quad (1.21)$$

This is satisfied iff

$$\omega = \frac{k\pi}{2l+1} \text{ with } k \in \mathbb{Z} \text{ and } l \in \mathbb{N}. \quad (1.22)$$

The set of all  $\omega \in \mathbb{R}$  of the type (1.22) is dense in  $\mathbb{R}$ , but does not contain any inner point. In other words: In this case the set of all values  $\omega$  such that condition (1.11) is satisfied, is dense in  $\mathbb{R}$ , but the set of all values  $\omega$  such that (1.11) is not satisfied, is dense too.

**Remark 1.7 about Fredholmness of  $\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)$  under small perturbations of the data:** Let us comment about the behavior of the conclusions of Theorem 1.2, mainly the Fredholmness of the operator  $\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)$ , under small perturbations of the data.

Suppose that for given data  $\omega, a$ , and  $b$  the assumptions of Theorem 1.2 are satisfied. Then, under sufficiently small perturbations of  $b_{jk}$  in  $L^\infty(0,1)$ , independently whether (1.13) remains to be true or not, the Fredholmness of  $\mathcal{A}(\omega, a, b^0)$  survives because the map (1.15) is continuous and because the set of index zero Fredholm operators between two fixed Banach spaces is open.

But if  $\omega, a_j, r_{jk}^0$ , or  $r_{jk}^1$  are perturbed, then the function space  $V^\gamma(\omega, a, r)$  is changed, in general, and it may happen that there exist arbitrarily small perturbations that destroy the Fredholmness:

For example, consider again the case  $m = 1, n = 2, a_1(x) = 1, a_2(x) = -1, f(x) = 0, b_{jk}(x) = 0$  for  $j, k = 1, 2, r_{1,2}^0 = 1, r_{2,1}^1 = -1$ . Then (1.11) reads as (1.21) which is equivalent to (1.22). Hence, by Theorem 1.2, if (1.22) is true, then  $\mathcal{A}(\omega, a, b^0)$  is Fredholm. Condition (1.22) and, hence, condition (1.21) is not satisfied, for example, if

$$\omega = \frac{2p+1}{2q}\pi \text{ with } p, q \in \mathbb{N}, \quad (1.23)$$

and in this case  $\mathcal{A}(\omega, a, b^0)$  is not Fredholm because  $\dim \ker \mathcal{A}(\omega, a, b^0) = \infty$ : Indeed, we have  $(u_1, u_2) \in \ker \mathcal{A}(\omega, a, b^0)$  iff

$$\omega \partial_t u_1 + \partial_x u_1 = \omega \partial_t u_2 - \partial_x u_2 = 0, \quad x \in [0, 1], \quad t \in \mathbb{R}, \quad (1.24)$$

$$u_j(x, t + 2\pi) = u_j(x, t), \quad j = 1, 2, \quad x \in [0, 1], \quad t \in \mathbb{R}, \quad (1.25)$$

$$u_1(0, t) = u_2(0, t), \quad u_2(1, t) = -u_1(1, t), \quad t \in \mathbb{R}. \quad (1.26)$$

The solutions of (1.24) are of the type  $u_1(x, t) = U_1(t - \omega x)$  and  $u_2(x, t) = U_2(t + \omega x)$ . They satisfy (1.25) iff the functions  $U_1$  and  $U_2$  are  $2\pi$ -periodic. From the boundary condition in  $x = 0$  follows  $U_1 = U_2$ , and, hence, the boundary condition in  $x = 1$  reads as

$$U_1(t - \omega) = -U_1(t + \omega). \quad (1.27)$$

Choosing  $U_1(y) = \sin(ry)$ ,  $r \in \mathbb{Z}$ , and using (1.23), condition (1.27) transforms into

$$\sin\left(r\left(t - \frac{2p+1}{2q}\pi\right)\right) = -\sin\left(r\left(t + \frac{2p+1}{2q}\pi\right)\right).$$

This is fulfilled, for example, for  $r = (2k+1)q$  and any choice of  $k \in \mathbb{Z}$ , i.e. we found infinitely many linearly independent solutions to (1.24)–(1.26).

The set of all values  $\omega$  of the type (1.23) is dense in  $[0, \infty)$ . Hence, we get: In this case the set of all  $\omega > 0$  such that  $\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)$  is Fredholm, is dense in  $[0, \infty)$ , but the set of all  $\omega > 0$  such that  $\mathcal{A}(\omega, a, b^0) + \mathcal{B}(b^1)$  is not Fredholm, is dense in  $[0, \infty)$  too.

**Remark 1.8 about assumptions (1.13) and (1.18):** Obviously, the condition (1.13) is not necessary for the conclusions of Theorem 1.2 because the conclusions of Theorem 1.2 survive under small (in  $L^\infty(0, 1)$ ) perturbations of the coefficients  $b_{jk}$ , but the assumption (1.13) does not, in general.

The following example shows that Theorem 1.2 is not true, in general, if all its assumptions are fulfilled with the exception of (1.13): Take  $m = 1, n = 2, \omega = a_1(x) = a_2(x) = 1, b_{11}(x) = b_{12}(x) = b_{22}(x) = f_1(x, t) = f_2(x, t) = 0, b_{21} = b = \text{const.}$  Then (1.1)–(1.3) looks like

$$\begin{aligned} \partial_t u_1 + \partial_x u_1 &= \partial_t u_2 + \partial_x u_2 + b u_1 &= 0, \\ u_1(x, t + 2\pi) - u_1(x, t) &= u_2(x, t + 2\pi) - u_2(x, t) &= 0, \\ u_1(0, t) - r_{12}^0 u_2(0, t) &= u_2(1, t) - r_{21}^1 u_1(1, t) &= 0. \end{aligned}$$

If  $r_{12}^0 r_{21}^1 < 1$  and  $b \neq 0$ , then all assumption of Theorem 1.2 are fulfilled with the exception of (1.13). If, moreover,

$$\frac{b r_{12}^0}{r_{12}^0 r_{21}^1 - 1} = 1,$$

then

$$u_1(x, t) = \sin l(t - x), \quad u_2(x, t) = b \left( \frac{1}{1 - r_{12}^0 r_{21}^1} - x \right) \sin l(t - x), \quad l \in \mathbb{N},$$

are infinitely many linearly independent solutions. Hence, the conclusion of Theorem 1.2 is not true.

On the other hand, the assumptions (1.13) and (1.18) of Theorems 1.2 and 1.4 can be weakened in order to handle cases with  $a_j(x) \equiv a_k(x)$  for certain  $1 \leq j \neq k \leq m$  or certain  $m + 1 \leq j \neq k \leq n$  (what is essential for applications to laser dynamics, cf. [12]). Remark that in the example above we have  $a_j(x) \equiv a_k(x)$  for  $1 = j = m < k = n = 2$ .

Let us describe this weakening of the assumptions (1.13) and (1.18): Consider a system of the type (1.1)–(1.3), but now with vector valued unknown functions:

$$u_j(x, t) \in \mathbb{R}^{p_j}, \quad p_j \in \mathbb{N}.$$

The coefficients  $\omega > 0$  and  $a_j(x)$  are still scalar valued, but the right hand sides  $f_j(x, t) \in \mathbb{R}^{p_j}$  are correspondingly vector valued, and the coefficients  $b_{jk}, r_{jk}^0$ , and  $r_{jk}^1$  are correspondingly matrix valued:

$$b_{jk}(x), r_{jk}^0, r_{jk}^1 \in \mathbb{M}(p_j \times p_k; \mathbb{R}).$$

Here we denote by  $\mathbb{M}(r \times s; \mathbb{K})$  the space of all real (if  $\mathbb{K} = \mathbb{R}$ ) or complex (if  $\mathbb{K} = \mathbb{C}$ )  $r \times s$ -matrices. Then our calculations and results can be translated to the new situation in a straightforward way. In particular,  $\alpha_j(x)$  has to be considered as the identity  $p_j \times p_j$ -matrix multiplied with  $\int_0^x 1/a_j(y) dy$ . Moreover,  $\beta_j(x) \in \mathbb{M}(p_j \times p_j; \mathbb{R})$ . Correspondingly,

$$R_s(\omega, a, b^0, r) \in \mathbb{M}((p_{m+1} + \dots + p_n) \times (p_{m+1} + \dots + p_n); \mathbb{C})$$

is a matrix with blocks

$$\sum_{l=1}^m e^{is\omega(\alpha_j(1) - \alpha_j(1)) + \beta_j(1) - \beta_l(1)} r_{jl}^0 r_{lk}^1 \in \mathbb{M}(p_j \times p_k; \mathbb{C}).$$

This matrix is the coefficient matrix for the system (3.8), again. Further, equation (4.3) holds again, but now as an equation in  $\mathbb{R}^{p_k}$ . Hence, the calculations to get (4.5) (which is the only place where assumptions (1.13) and (1.18) are used) can be repeated in the vectorial case.

Finally, let us remark that, surprisingly, the assumption (1.13) (as well as its weakening described above) is used also in quite another circumstances, for proving the spectrum-determined growth condition in  $L^p$ -spaces [2, 13, 15] and in  $C$ -spaces [11] for semiflows generated by hyperbolic systems of the type (1.1), (1.3).

**Remark 1.9 about weakening of assumptions (1.12) and (1.13):** The smoothness assumptions  $a_j \in C^{0,1}([0, 1])$  for all  $j = 1, \dots, n$  and  $b_{jk} \in BV(0, 1)$  for all  $1 \leq j \neq k \leq n$  of Theorem 1.2 (which follow from assumptions (1.12) and (1.13)) can be weakened to

$$a_j \in L^\infty(0, 1), \text{ ess inf } |a_j| > 0 \text{ for all } j = 1, \dots, n$$

and

$$\left. \begin{aligned} &\text{for all } j \neq k \text{ there is } c_{jk} \in BV(0, 1) \text{ such that} \\ &a_j(x)b_{jk}(x) = c_{jk}(x)(a_j(x) - a_k(x)) \text{ for a.a. } x \in [0, 1], \end{aligned} \right\} \quad (1.28)$$

(see (4.4), where (1.28) is used). In particular, if  $b_{jk} = 0$  for all  $1 \leq j \neq k \leq n$ , then (1.28) is satisfied. Hence, after this weakening Theorem 1.1 is a particular case or Theorem 1.2.

**Remark 1.10 about an open and dense condition:** Let  $\omega > 0$ ,  $r$ , and  $a \in C^{0,1}([0, 1]; \mathbb{R}^n)$  be fixed such that

$$a_j(x) \neq a_k(x) \text{ for all } 1 \leq j \neq k \leq n \text{ and } x \in [0, 1],$$

and denote

$$M := \left\{ b \in L^\infty((0, 1); \mathbb{M}_n) : \inf_{s \in \mathbb{Z}} \det |(I - R_s(\omega, a, b^0, r))| > 0 \right\}$$

and

$$M_{BV} := \{ b \in M : b_{jk} \in BV(0, 1) \text{ for all } 1 \leq j \neq k \leq n \}.$$

Then  $M$  is open in  $L^\infty((0, 1); \mathbb{M}_n)$ ,  $M_{BV}$  is dense in  $M$ , and Theorem 1.2 claims that  $A(\omega, a, b^0) + B(b^1)$  is Fredholm of index zero from  $V^\gamma(\omega, a, r)$  into  $W^\gamma$  for all  $\gamma \geq 1$  and  $b \in M_{BV}$ . But the map (1.15) is continuous, hence there is an open neighborhood  $M_0 \subseteq M$  of  $M_{BV}$  such that  $A(\omega, a, b^0) + B(b^1)$  is Fredholm of index zero from  $V^\gamma(\omega, a, r)$  into  $W^\gamma$  for all  $\gamma \geq 1$  and  $b \in M_0$ , and  $M_0$  is dense in  $M$ . Unfortunately we don't have examples showing that  $M_0 \neq M$ , in general.

**Remark 1.11 about sufficient conditions for (1.16):** Similarly to [8], one can provide a wide range of  $\omega$ -independent sufficient conditions for (1.16). We here concentrate on the physically relevant case

$$\sum_{j=1}^m \sum_{k=m+1}^n |r_{jk}^0|^2 \leq 1 \text{ and } \sum_{j=m+1}^n \sum_{k=1}^m |r_{jk}^1|^2 \leq 1. \quad (1.29)$$

If (1.1)–(1.3) with  $f = 0$  is satisfied, then

$$0 = \int_0^{2\pi} (u_j^2(1, t) - u_j^2(0, t)) dt + 2 \int_0^{2\pi} \int_0^1 a_j^{-1}(x) \left( b_{jj}(x)u_j^2 + \sum_{k \neq j} b_{jk}(x)u_j u_k \right) dx dt. \quad (1.30)$$

Using the reflection boundary conditions, summing up separately the first  $m$  equations of (1.30) and the rest  $n - m$  equations of (1.30), and subtracting the second resulting equality from the first one, we get

$$\begin{aligned} &\int_0^{2\pi} \left( \sum_{j=1}^m u_j^2(1, t) - \sum_{j=m+1}^n \left( \sum_{k=1}^m r_{jk}^1 u_k(1, t) \right)^2 + \sum_{j=m+1}^n u_j^2(0, t) - \sum_{j=1}^m \left( \sum_{k=m+1}^n r_{jk}^0 u_k(0, t) \right)^2 \right) dt \\ &+ 2 \sum_{j=1}^m \int_0^{2\pi} \int_0^1 \frac{1}{a_j(x)} \left( b_{jj}(x)u_j^2 + \sum_{k \neq j} b_{jk}(x)u_j u_k \right) dx dt \\ &- 2 \sum_{j=m+1}^n \int_0^{2\pi} \int_0^1 \frac{1}{a_j(x)} \left( b_{jj}(x)u_j^2 + \sum_{k \neq j} b_{jk}(x)u_j u_k \right) dx dt = 0. \end{aligned} \quad (1.31)$$

Applying Hölder's inequality and assumption (1.29), we derive that

$$\int_0^{2\pi} \left( \sum_{j=1}^m u_j^2(1, t) - \sum_{j=m+1}^n \left( \sum_{k=1}^m r_{jk}^1 u_k(1, t) \right)^2 \right) dt \geq \left( 1 - \sum_{j=m+1}^n \sum_{k=1}^m |r_{jk}^1|^2 \right) \int_0^{2\pi} \sum_{j=1}^m u_j(1, t)^2 dt \geq 0.$$



A similar estimate is true for the second boundary summand in (1.31) as well. Set

$$c_{jk}(x) := \frac{b_{jk}(x)}{a_j(x)} \text{ for } 1 \leq j \leq m \quad \text{and} \quad c_{jk}(x) := -\frac{b_{jk}(x)}{a_j(x)} \text{ for } m+1 \leq j \leq n.$$

Then (1.29) together with

$$\sum_{j,k=1}^n c_{jk}(x) \xi_j \xi_k \geq C \sum_{j=1}^n |\xi_j|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and a.a. } x \in (0, 1), \quad (1.32)$$

where the constant  $C > 0$  does not depend on  $\xi$  and  $x$ , is sufficient for (1.16). It is easily seen that estimate (1.32) is true if, for instance,

$$\begin{aligned} \operatorname{ess\,inf} \left\{ \frac{b_{jj}}{a_j} - \sum_{k \neq j} \left( \left| \frac{b_{jk}}{a_j} \right| + \left| \frac{b_{jk}}{a_k} \right| \right) \right\} &> 0 \quad \text{for all } j = 1, \dots, m, \\ \operatorname{ess\,inf} \left\{ -\frac{b_{jj}}{a_j} - \sum_{k \neq j} \left( \left| \frac{b_{jk}}{a_j} \right| + \left| \frac{b_{jk}}{a_k} \right| \right) \right\} &> 0 \quad \text{for all } j = m+1, \dots, n. \end{aligned}$$

Summarizing, we get: In order the main conditions (1.16) and (1.17) to be satisfied, it is sufficient that (1.29) is fulfilled as well as

$$\begin{aligned} \operatorname{ess\,inf} a_j &> 0 \quad \text{for } j = 1, \dots, m, \\ \operatorname{ess\,sup} a_j &< 0 \quad \text{for } j = m+1, \dots, n, \\ \operatorname{ess\,inf} b_{jj} &> 0 \quad \text{for } j = 1, \dots, n, \\ \operatorname{ess\,sup} |b_{jk}| &\approx 0 \quad \text{for } 1 \leq j \neq k \leq n. \end{aligned}$$

**Remark 1.12 about very weak solutions:** Let the assumptions of Theorem 1.4 be satisfied. Denote by  $M$  the set of all  $(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in \mathbb{R} \times C([0, 1]; \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n) \times W^1$  such that

$$|\tilde{\omega} - \omega| < \varepsilon, \quad \max_{j=1, \dots, n} \|\tilde{a}_j - a_j\|_\infty < \varepsilon, \quad \max_{j,k=1, \dots, n} \|\tilde{b}_{j,k} - b_{jk}\|_\infty < \varepsilon,$$

and by  $M_0$  the set of all  $(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in M$  such that  $\tilde{a}_j \in C^{0,1}([0, 1])$  for all  $j = 1, \dots, n$  and  $\tilde{b}_{jk} \in BV(0, 1)$  for all  $1 \leq j \neq k \leq n$ . Then  $M_0$  is dense in  $M$  (with respect to the usual norm in  $\mathbb{R} \times C([0, 1]; \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n) \times W^1$ ). Moreover, the data-to-solution map (1.19) is defined on  $M_0$ , and it is uniformly continuous (with respect to the usual norm in  $\mathbb{R} \times C([0, 1]; \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n) \times W^1$ ) from  $M_0$  into  $W^0$  (cf. Lemma 6.3). Hence, there exists a unique continuous continuation of the data-to-solution map (1.19) on the set  $M$ . But the images  $u \in W^0$  of this continuation, corresponding to arguments  $(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in M \setminus M_0$ , are not strong solutions to (1.1)–(1.3), in general, they are a kind of very weak solutions to (1.1)–(1.3).

## 2 Some properties of the used function spaces

In this section we formulate some properties of the function spaces  $W^\gamma$ ,  $V^\gamma(\omega, a, r)$ , and  $U^\gamma(\omega, a)$  introduced in Section 1. For each  $u \in W^\gamma$  we have

$$u(x, t) = \sum_{s \in \mathbb{Z}} u^s(x) e^{ist} \quad \text{with} \quad u^s(x) := \frac{1}{2\pi} \int_0^{2\pi} u(x, t) e^{-ist} dt, \quad (2.1)$$

where  $u^s \in L^2((0, 1); \mathbb{C}^n)$ , and the series in (2.1) converges to  $u$  in the complexification of  $W^\gamma$ . And vice versa: For any sequence  $(u^s)_{s \in \mathbb{Z}}$  with

$$u^s \in L^2((0, 1); \mathbb{C}^n), \quad u^{-s} = \overline{u^s}, \quad \sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \|u^s\|_{H^k((0, 1); \mathbb{C}^n)}^2 < \infty \quad (2.2)$$

there exists exactly one  $u \in W^\gamma$  with (2.1). In what follows, we will identify functions  $u \in W^\gamma$  and sequences  $(u^s)_{s \in \mathbb{Z}}$  with (2.2) by means of (2.1), and we will keep for the functions and the sequences the notations  $u$  and  $(u^s)_{s \in \mathbb{Z}}$ , respectively.

The following lemma gives a compactness criterion in  $W^\gamma$  (see [9, Lemma 6]):

**Lemma 2.1** *A set  $M \subset W^\gamma$  is precompact in  $W^\gamma$  if and only if the following two conditions are satisfied:*

(i) *There exists  $C > 0$  such that for all  $u \in M$  it holds*

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 \|u^s(x)\|^2 dx \leq C.$$

(ii) *For all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $h \in (-\delta, \delta)$  and all  $u \in M$  it holds*

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 \|u^s(x+h)e^{ish} - u^s(x)\|^2 dx < \varepsilon,$$

where  $u^s(x+h) := 0$  for  $x+h \notin [0, 1]$ .

Concerning the spaces  $U^\gamma(\omega, a)$  we have the following result (see [9]):

**Lemma 2.2** (i) *The space  $U^\gamma(\omega, a)$  is complete.*

(ii) *If  $\gamma \geq 1$ , then for any  $x \in [0, 1]$  there exists a continuous trace map  $u \in U^\gamma(\omega, a) \mapsto u(x, \cdot) \in L^2((0, 2\pi); \mathbb{R}^n)$ .*

(iii) *If  $\gamma > 3/2$ , then  $U^\gamma(\omega, a)$  is continuously embedded into  $C([0, 1] \times [0, 2\pi]; \mathbb{R}^n)$ .*

Now, let us consider the dual spaces  $(W^\gamma)^*$ .

Obviously, for any  $\gamma \geq 0$  the spaces  $W^\gamma$  are densely and continuously embedded into the Hilbert space  $L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$ . Hence, there is a canonical dense continuous embedding

$$L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n) \hookrightarrow (W^\gamma)^* : [u, v]_{W^\gamma} = \langle u, v \rangle_{L^2} \quad (2.3)$$

for all  $u \in L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  and  $v \in W^\gamma$ . Here  $[\cdot, \cdot]_{W^\gamma} : (W^\gamma)^* \times W^\gamma \rightarrow \mathbb{R}$  is the dual pairing, and  $\langle \cdot, \cdot \rangle_{L^2}$  is the scalar product introduced in (1.14).

Let us denote

$$e_s(t) := e^{ist} \text{ for } s \in \mathbb{Z} \text{ and } t \in \mathbb{R}. \quad (2.4)$$

If a sequence  $(\varphi^s)_{s \in \mathbb{Z}}$  with  $\varphi^s \in L^2((0, 1); \mathbb{C}^n)$  is given, then the pointwise products  $\varphi^s e_s$  belong to  $L^2((0, 1) \times (0, 2\pi); \mathbb{C}^n)$ . Hence, they belong to the complexification of  $(W^\gamma)^*$  (by means of the complexified version of (2.3)), and it makes sense to ask if the series

$$\sum_{s \in \mathbb{Z}} \varphi^s e_s \quad (2.5)$$

converges in the complexification of  $(W^\gamma)^*$ . Moreover, we have (see [9, Lemma 10])

**Lemma 2.3** (i) *For any  $\varphi \in (W^\gamma)^*$  there exists a sequence  $(\varphi^s)_{s \in \mathbb{Z}}$  with*

$$\varphi^s \in L^2((0, 1); \mathbb{C}^n), \quad \varphi^{-s} = \overline{\varphi^s}, \quad \sum_{s \in \mathbb{Z}} (1 + s^2)^{-\gamma} \|\varphi^s(x)\|_{L^2((0, 1); \mathbb{C}^n)}^2 < \infty, \quad (2.6)$$

and the series (2.5) converges to  $\varphi$  in the complexification of  $(W^\gamma)^*$ . Moreover, it holds

$$\int_0^1 \langle \varphi^s(x), u(x) \rangle dx = [\varphi, u e_{-s}]_{W^\gamma} \text{ for all } u \in L^2(0, 1). \quad (2.7)$$

(ii) *For any sequence  $(\varphi^s)_{s \in \mathbb{Z}}$  with (2.6) the series (2.5) converges in the complexification of  $(W^\gamma)^*$  to some  $\varphi \in (W^\gamma)^*$ , and (2.7) is satisfied.*

### 3 Isomorphism property (proof of Theorem 1.1)

Let  $\gamma \geq 1$  and  $f \in W^\gamma$  be arbitrarily fixed. We have  $f(x, t) = \sum_{s \in \mathbb{Z}} f^s(x) e^{ist}$  with

$$f^s \in L^2((0, 1); \mathbb{C}^n), \quad \sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 \|f^s(x)\|^2 dx < \infty. \quad (3.1)$$

We have to show that, if (1.9), (1.10) and (1.11) hold, then there exists exactly one  $u \in V^\gamma(\omega, a, r)$  with (1.6) and

$$\|u\|_{V^\gamma(\omega, a, r)} \leq C \|f\|_{W^\gamma}.$$

Writing  $u$  as series according to (2.1) and (2.2), we have to show that there exists exactly one tuple of sequences  $(u_j^s)_{s \in \mathbb{Z}}, j = 1, \dots, n$ , with  $u_j^s \in H^1((0, 1); \mathbb{C})$  satisfying

$$a_j(x) \frac{d}{dx} u_j^s(x) + (is\omega + b_{jj}(x)) u_j^s(x) = f_j^s(x), \quad j = 1, \dots, n, \quad (3.2)$$

$$\left. \begin{aligned} u_j^s(0) &= \sum_{k=m+1}^n r_{jk}^0 u_k^s(0), \quad j = 1, \dots, m, \\ u_j^s(1) &= \sum_{k=1}^m r_{jk}^1 u_k^s(1), \quad j = m+1, \dots, n, \end{aligned} \right\} \quad (3.3)$$

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 |u_j^s(x)|^2 dx \leq C \|f\|_{W^\gamma}, \quad j = 1, \dots, n, \quad (3.4)$$

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 \left| a_j(x) \frac{d}{dx} u_j^s(x) + is\omega u_j^s(x) \right|^2 dx \leq C \|f\|_{W^\gamma}, \quad j = 1, \dots, n. \quad (3.5)$$

The estimate (3.5) follows from (3.2) and (3.4). Hence, we have to show that there exists exactly one tuple of sequences  $(u_j^s)_{s \in \mathbb{Z}}, j = 1, \dots, n$ , with  $u_j^s \in H^1((0, 1); \mathbb{C})$  satisfying (3.2)–(3.4).

By means of the variation of constants formula, (3.2) is fulfilled if and only if

$$u_j^s(x) = e^{-is\omega\alpha_j(x) - \beta_j(x)} \left( u_j^s(0) + \int_0^x e^{is\omega\alpha_j(y) + \beta_j(y)} \frac{f_j^s(y)}{a_j(y)} dy \right), \quad (3.6)$$

where the functions  $\alpha_j$  and  $\beta_j$  are defined in (1.8). The boundary conditions (3.3) are satisfied if and only if

$$u_j^s(0) = \sum_{k=m+1}^n r_{jk}^0 u_k^s(0), \quad j = 1, \dots, m, \quad (3.7)$$

and

$$\begin{aligned} & e^{-is\omega\alpha_j(1) - \beta_j(1)} \left( u_j^s(0) + \int_0^1 e^{is\omega\alpha_j(y) + \beta_j(y)} \frac{f_j^s(y)}{a_j(y)} dy \right) \\ &= \sum_{k=1}^m r_{jk}^1 e^{-is\omega\alpha_k(1) - \beta_k(1)} \left( u_k^s(0) + \int_0^1 e^{is\omega\alpha_k(y) + \beta_k(y)} \frac{f_k^s(y)}{a_k(y)} dy \right), \quad j = m+1, \dots, n. \end{aligned}$$

This is equivalent to (3.7),

$$e^{-is\omega\alpha_j(1) - \beta_j(1)} u_j^s(0) - \sum_{k=1}^m \sum_{p=m+1}^n e^{-is\omega\alpha_k(1) - \beta_k(1)} r_{jk}^1 r_{kp}^0 u_p^s(0)$$

$$\begin{aligned}
&= -e^{-is\omega\alpha_j(1)-\beta_j(1)} \int_0^1 e^{is\omega\alpha_j(y)+\beta_j(y)} \frac{f_j^s(y)}{a_j(y)} dy \\
&+ \sum_{k=1}^m e^{-is\omega\alpha_k(1)-\beta_k(1)} r_{jk}^1 \int_0^1 e^{is\omega\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)} dy, \quad j = m+1, \dots, n.
\end{aligned} \tag{3.8}$$

The linear inhomogeneous system (3.8) has a unique solution  $(u_{m+1}^s(0), \dots, u_n^s(0))$  if and only if its coefficient matrix  $I - R_s(\omega, a, b^0, r)$  (where  $R_s(\omega, a, b^0, r)$  is introduced in (1.7)) is regular. If, moreover, assumptions (1.10)–(1.11) are satisfied, then there exist coefficients  $c_{jk}^s$  and a constant  $C$  such that

$$u_j^s(0) = \sum_{k=1}^n c_{jk}^s e^{-is\omega\alpha_k(1)-\beta_k(1)} \int_0^1 e^{is\omega\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)} dy, \quad j = m+1, \dots, n, \tag{3.9}$$

and  $|c_{jk}^s| \leq C$  uniformly with respect to  $\omega, a, b^0, r$ , and  $s \in \mathbb{Z}$  with (1.10)–(1.11). Hence, for each  $s \in \mathbb{Z}$  the boundary value problem (3.2)–(3.3) is uniquely solvable, and we have the integral representation (3.6) of the solution, where  $u_j^s(0)$  for  $1 \leq j \leq m$  is given by (3.7) and for  $m+1 \leq j \leq n$  by (3.9). Putting this together, we get

$$\begin{aligned}
u_j^s(x) &= e^{-is\omega\alpha_j(x)-\beta_j(x)} \left( \int_0^x e^{is\omega\alpha_j(y)+\beta_j(y)} \frac{f_j^s(y)}{a_j(y)} dy \right. \\
&\left. + \sum_{k=1}^n d_{jk}^s e^{-is\omega\alpha_k(1)-\beta_k(1)} \int_0^1 e^{is\omega\alpha_k(y)+\beta_k(y)} \frac{f_k^s(y)}{a_k(y)} dy \right), \quad j = 1, \dots, n,
\end{aligned} \tag{3.10}$$

with certain coefficients  $d_{jk}^s$  such that there exists a constant  $C$  (not depending on  $\omega, a, b^0, r$ , and  $s \in \mathbb{Z}$  with (1.10)–(1.11) and on  $x$  and  $f$ ) with

$$|d_{jk}^s| \leq C. \tag{3.11}$$

In addition, (3.10) and (3.11) imply that there exists a constant  $C$  (not depending on  $\omega, a, b^0, r$ , and  $s \in \mathbb{Z}$  with (1.10)–(1.11) and on  $x$  and  $f$ ) such that

$$|u_j^s(x)| \leq C \int_0^1 \|f^s(x)\| dx. \tag{3.12}$$

The estimate (3.4) now follows from (3.1).

## 4 Fredholmness property (proof of Theorem 1.2)

In Sections 4 and 5 we suppose the data  $\omega, a, b$ , and  $r$  to be fixed and to satisfy (1.11)–(1.13). Hence we will omit the arguments in the operators and the spaces:

$$\mathcal{A} := \mathcal{A}(\omega, a, b^0, r), \quad \mathcal{B} := \mathcal{B}(b^1), \quad \tilde{\mathcal{A}} := \tilde{\mathcal{A}}(\omega, a, b^0, r), \quad \tilde{\mathcal{B}} := \tilde{\mathcal{B}}(b^1),$$

$$V^\gamma := V^\gamma(\omega, a, r), \quad \tilde{V}^\gamma := \tilde{V}^\gamma(\omega, a, r).$$

In this section we prove that  $\mathcal{A} + \mathcal{B}$  is Fredholm from  $V^\gamma$  into  $W^\gamma$ , which is part of the assertions of Theorem 1.2.

Obviously,  $\mathcal{A} + \mathcal{B}$  is Fredholm from  $V^\gamma$  into  $W^\gamma$  if and only if  $I + \mathcal{B}\mathcal{A}^{-1}$  is Fredholm from  $W^\gamma$  into  $W^\gamma$ . Here  $I$  is the identity in  $W^\gamma$ .

We will prove that  $I + \mathcal{B}\mathcal{A}^{-1}$  is Fredholm from  $W^\gamma$  into  $W^\gamma$  using the following abstract criterion for Fredholmness (see, e.g., [9, Lemma 11] and [21, Proposition 5.7.1]):

**Lemma 4.1** *Let  $W$  be a Banach space,  $I$  the identity in  $W$ , and  $\mathcal{C} \in \mathcal{L}(W)$  such that  $\mathcal{C}^2$  is compact. Then  $I + \mathcal{C}$  is Fredholm.*

In order to use Lemma 4.1 with  $W := W^\gamma$  and  $\mathcal{C} := \mathcal{BA}^{-1}$  we have to show that  $(\mathcal{BA}^{-1})^2$  is compact from  $W^\gamma$  into  $W^\gamma$ . For this purpose we will use Lemma 2.1.

Condition (i) of Lemma 2.1 is satisfied because  $\mathcal{BA}^{-1}$  is a bounded operator from  $W^\gamma$  into  $W^\gamma$ .

It remains to check condition (ii) of Lemma 2.1. For this purpose we will use the integral representation (3.10) of  $\mathcal{A}^{-1}$ :

Take a bounded set  $N \subset W^\gamma$  and  $f \in N$ . Denote  $u := \mathcal{A}^{-1}f$  and  $\tilde{u} := (\mathcal{BA}^{-1})^2 f$ . Then

$$\begin{aligned} \tilde{u}_j^s(x) &= \sum_{\substack{k=1 \\ k \neq j}}^n b_{jk}(x) e^{-is\omega\alpha_k(x) - \beta_k(x)} \left( \int_0^x e^{is\omega\alpha_k(y) + \beta_k(y)} a_k^{-1}(y) \sum_{\substack{l=1 \\ l \neq k}}^n b_{kl}(y) u_l^s(y) dy \right. \\ &\quad \left. + \sum_{l=1}^n d_{kl}^s e^{-is\omega\alpha_l(1) - \beta_l(1)} \int_0^1 e^{is\omega\alpha_l(y) + \beta_l(y)} a_l^{-1}(y) \sum_{\substack{r=1 \\ r \neq l}}^n b_{lr}(y) u_r^s(y) dy \right). \end{aligned}$$

Therefore  $\tilde{u}_j^s(x+h)e^{is\omega h} - \tilde{u}_j^s(x) = P_j^s(x, h) + Q_j^s(x, h) + R_j^s(x, h)$  with

$$P_j^s(x, h) := \sum_{\substack{k, l=1 \\ j \neq k \neq l}}^n \int_x^{x+h} e^{is\omega(-\alpha_k(x+h) + h + \alpha_k(y) - \beta_k(x+h) + \beta_k(y))} a_k^{-1}(y) b_{jk}(x+h) b_{kl}(y) u_l^s(y) dy,$$

$$Q_j^s(x, h) := \sum_{\substack{k=1 \\ k \neq j}}^n b_{jk}(x+h) e^{-\beta_k(x+h)} \left( e^{is\omega(-\alpha_k(x+h) + h)} - e^{-is\omega\alpha_k(x)} \right) S_k^s(x),$$

$$R_j^s(x, h) := \sum_{\substack{k=1 \\ k \neq j}}^n \left( b_{jk}(x+h) e^{-\beta_k(x+h)} - b_{jk}(x) e^{-\beta_k(x)} \right) S_k^s(x)$$

and

$$\begin{aligned} S_k^s(x) &:= \int_0^x e^{is\omega\alpha_k(y) + \beta_k(y)} a_k^{-1}(y) \sum_{\substack{l=1 \\ l \neq k}}^n b_{kl}(y) u_l^s(y) dy \\ &\quad + \sum_{l=1}^n d_{kl}^s e^{-is\omega\alpha_l(1) - \beta_l(1)} \int_0^1 e^{is\omega\alpha_l(y) + \beta_l(y)} a_l^{-1}(y) \sum_{\substack{r=1 \\ r \neq l}}^n b_{lr}(y) u_r^s(y) dy. \end{aligned} \quad (4.1)$$

We have to show that

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 (|P_j^s(x, h)|^2 + |Q_j^s(x, h)|^2 + |R_j^s(x, h)|^2) dx \rightarrow 0 \text{ for } h \rightarrow 0$$

uniformly with respect to  $f \in N$ .

Because of  $\mathcal{A}u = f$  we have (3.12). This implies

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 |P_j^s(x, h)|^2 dx \leq Ch^2 \|f\|_{W^\gamma}^2, \quad (4.2)$$

where the constant  $C$  does not depend on  $j, h$ , and  $f$ . Hence, the left hand side of (4.2) tends to zero for  $h \rightarrow 0$  uniformly with respect to  $f \in N$ .

In order to estimate  $Q_j^s(x, h)$  and  $R_j^s(x, h)$ , let us first estimate  $S_j^s(x)$ . Again we use  $\mathcal{A}u = f$ . From (3.2) it follows

$$\frac{d}{dy} \left( e^{is\omega\alpha_l(y)} u_l^s(y) \right) = e^{is\omega\alpha_l(y)} \frac{f_l^s(y) - b_{ll}(y)u_l^s(y)}{a_l(y)}.$$

Using this, we get

$$\begin{aligned} & i s \omega \frac{a_l(y) - a_k(y)}{a_k(y)a_l(y)} e^{is\omega\alpha_k(y)} u_l^s(y) \\ &= e^{is\omega(\alpha_k(y) - \alpha_l(y))} \frac{d}{dy} \left( e^{is\omega\alpha_l(y)} u_l^s(y) \right) - \frac{d}{dy} \left( e^{is\omega\alpha_k(y)} u_l^s(y) \right) \\ &= e^{il\omega\alpha_l(y)} \frac{f_l^s(y) - b_{ll}(y)u_l^s(y)}{a_l(y)} - \frac{d}{dy} \left( e^{is\omega\alpha_k(y)} u_l^s(y) \right). \end{aligned}$$

Therefore

$$\begin{aligned} & e^{is\omega\alpha_k(y) + \beta_k(y)} \frac{b_{kl}(y)}{a_k(y)} u_l^s(y) \\ &= \frac{e^{\beta_k(y)}}{is\omega} \frac{a_l(y)b_{kl}(y)}{a_j(y) - a_k(y)} \left( e^{il\omega\alpha_l(y)} \frac{f_l^s(y) - b_{ll}(y)u_l^s(y)}{a_l(y)} - \frac{d}{dy} \left( e^{is\omega\alpha_k(y)} u_l^s(y) \right) \right). \end{aligned} \quad (4.3)$$

Moreover, because of assumptions (1.13) and  $a_l \in C^{0,1}([0, 1])$ , for all  $k \neq l$  the function

$$y \in [0, 1] \mapsto e^{\beta_k(y)} \frac{a_l(y)b_{kl}(y)}{a_l(y) - a_k(y)}$$

is in  $BV(0, 1)$ . Hence,

$$\left| \int_0^x e^{\beta_k(y)} \frac{a_l(y)b_{kl}(y)}{a_l(y) - a_k(y)} \frac{d}{dy} \left( e^{is\omega\alpha_k(y)} u_l^s(y) \right) dy \right| \leq C \|u_l^s\|_\infty, \quad (4.4)$$

the constant  $C$  being independent of  $x, k, l, s$ , and  $u$ . Here we used assumption (1.13). Therefore, (3.12) and (4.3) imply

$$\left| \int_0^x e^{is\omega\alpha_k(y) + \beta_k(y)} \frac{b_{kl}(y)}{a_k(y)} u_l^s(y) dy \right| \leq \frac{C}{1 + |s|} \int_0^1 \|f^s(y)\| dy \quad (4.5)$$

for some constant  $C$  being independent of  $x, k, l, s$ , and  $f$ . Similar estimates are true for all other integrals in (4.1). As a consequence,

$$|S_j^s(x)| \leq \frac{C}{1 + |s|} \int_0^1 \|f^s(y)\| dy, \quad (4.6)$$

where  $C$  does not depend on  $x, j, s$ , and  $f$ . This gives

$$\int_0^1 |Q_j^s(x, h)|^2 dx \leq \frac{C}{1 + |s|} \max_{k=1, \dots, n} |e^{is\omega(-\alpha_k(x+h)+h)} - e^{-is\omega\alpha_k(x)}|^2 \int_0^1 \|f^s(y)\|^2 dy,$$

where  $C$  does not depend on  $x, h, j, s$ , and  $f$ . But assumption (1.12) and notation (1.8) imply that

$$|e^{is\omega(-\alpha_k(x+h)+h)} - e^{-is\omega\alpha_k(x)}| \leq Csh,$$

hence

$$\sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 |Q_j^s(x, h)|^2 dx \leq Ch^2 \|f\|_{W^\gamma}^2, \quad (4.7)$$

where the constants, again, do not depend on  $j, k, h$ , and  $f$ . Hence, the left hand side of (4.7) tends to zero for  $h \rightarrow 0$  uniformly with respect to  $f \in N$ .

Finally, (4.6) gives

$$\begin{aligned} & \sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \int_0^1 |R_j^s(x, h)|^2 dx \\ & \leq C \max_{k=1, \dots, n} \int_0^1 |b_{jk}(x+h)e^{-\beta_k(x+h)} - b_{jk}(x)e^{-\beta_k(x)}|^2 dx \|f\|_{W^\gamma}^2, \end{aligned} \quad (4.8)$$

where the constant  $C$  does not depend on  $j, h$ , and  $f$ . Hence, the left hand side of (4.8) tends to zero for  $h \rightarrow 0$  uniformly with respect to  $f \in N$  because of the continuity in the mean of the functions  $x \mapsto b_{jk}(x)e^{-\beta_k(x)}$ .

In order to finish the proof of assertion (i) of Theorem 1.2, it remains to show that the index of  $A + B$  is zero. This is the topic of the next section.

## 5 Index zero property (still proof of Theorem 1.2)

Directly from the definitions of the operators  $\mathcal{A}$ ,  $\tilde{\mathcal{A}}$ ,  $\mathcal{B}$ , and  $\tilde{\mathcal{B}}$  it follows

$$\langle (\mathcal{A} + \mathcal{B})u, \tilde{u} \rangle_{L^2} = \langle u, (\tilde{\mathcal{A}} + \tilde{\mathcal{B}})\tilde{u} \rangle_{L^2} \text{ for all } u \in V^\gamma \text{ and } \tilde{u} \in \tilde{V}^\gamma. \quad (5.1)$$

Here  $\langle \cdot, \cdot \rangle_{L^2}$  is the scalar product in  $H^{0,0} = L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$  introduced in (1.14).

In order to prove that the Fredholm operator  $\mathcal{A} + \mathcal{B}$  has index zero, it suffices to show that

$$\dim \ker(\mathcal{A} + \mathcal{B}) = \dim \ker(\mathcal{A} + \mathcal{B})^*. \quad (5.2)$$

Here  $(\mathcal{A} + \mathcal{B})^*$  is the dual operator to  $\mathcal{A} + \mathcal{B}$ , i.e. a linear bounded operator from  $(W^\gamma)^*$  into  $(V^\gamma)^*$ . Using the continuous dense embedding

$$\tilde{V}^\gamma \hookrightarrow W^\gamma \hookrightarrow (H^{0,0})^2 \hookrightarrow (W^\gamma)^*,$$

it makes sense to compare the subspaces  $\ker(\mathcal{A} + \mathcal{B})^*$  of  $(W^\gamma)^*$  and  $\ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})$  of  $\tilde{V}^\gamma$ :

**Lemma 5.1**  $\ker(\mathcal{A} + \mathcal{B})^* = \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})$ .

*Proof.* For all  $u \in V^\gamma$  and  $\tilde{u} \in \tilde{V}^\gamma$  we have

$$\langle (\tilde{\mathcal{A}} + \tilde{\mathcal{B}})\tilde{u}, u \rangle_{L^2} = \langle \tilde{u}, (\mathcal{A} + \mathcal{B})u \rangle_{L^2} = [\tilde{u}, (\mathcal{A} + \mathcal{B})u]_{W^\gamma} = [(\mathcal{A} + \mathcal{B})^* \tilde{u}, u]_{W^\gamma}. \quad (5.3)$$

Here we used (2.3) and (5.1). Obviously, (5.3) implies  $\ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) \subseteq \ker(\mathcal{A} + \mathcal{B})^*$ .

Now, take an arbitrary  $\varphi \in \ker(\mathcal{A} + \mathcal{B})^*$  and show that  $\varphi \in \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})$ . By Lemma 2.3, we have (using notation (2.4))  $\varphi = \sum_{s \in \mathbb{Z}} \varphi^s e_s$  with

$$\varphi^s \in L^2((0, 1); \mathbb{C}^n), \quad \sum_{s \in \mathbb{Z}} (1 + s^2)^\gamma \|\varphi^s(x)\|_{L^2((0,1); \mathbb{C}^n)}^2 < \infty.$$

It follows that for all  $u \in V^\gamma$

$$0 = [(\mathcal{A} + \mathcal{B})^* \varphi, u]_{W^\gamma} = [\varphi, (\mathcal{A} + \mathcal{B})u]_{W^\gamma} = \sum_{s \in \mathbb{Z}} \int_0^1 \left\langle \varphi^s, a(x) \frac{d}{dx} u^{-s} - i s \omega u^{-s} + b(x)^T u^{-s} \right\rangle_{L^2}.$$

Therefore

$$\left\langle \varphi^s, a(x) \frac{d}{dx} u^{-s} - i s \omega u^{-s} + b(x)^T u^{-s} \right\rangle_{L^2} = 0$$

for all  $u^s \in H^1((0, 1); \mathbb{C}^n)$  with (3.3). By a standard argument, we conclude that  $\varphi^s \in H^1((0, 1); \mathbb{C}^n)$  and that it satisfies the differential equation

$$-\frac{d}{dx}(a(x)\varphi^s) - i s \omega \varphi^s + b(x)^T \varphi^s = 0 \quad (5.4)$$

and the boundary conditions

$$\left. \begin{aligned} a_j(0)\varphi_j^s(0) &= -\sum_{k=1}^m r_{kj}^0 a_k(0)\varphi_k^s(0), & m+1 \leq j \leq n, \\ a_j(1)\varphi_j^s(1) &= -\sum_{k=m+1}^n r_{kj}^1 a_k(1)\varphi_k^s(1), & 1 \leq j \leq m. \end{aligned} \right\} \quad (5.5)$$

This yields, as in Section 3, that  $\varphi \in \tilde{V}^\gamma$  and  $(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})\varphi = 0$ .  $\square$

To finish with the proof of assertion (ii) of Theorem 1.2, it remains to prove (5.2), i.e.

$$\dim \ker(\mathcal{A} + \mathcal{B}) = \dim \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}). \quad (5.6)$$

**Lemma 5.2** *There exists  $s_0 \in \mathbb{N}$  such that for all  $u \in \ker(\mathcal{A} + \mathcal{B})$ ,  $\tilde{u} \in \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})$ , and  $s \in \mathbb{Z}$  with  $|s| > s_0$  we have  $u^s = \tilde{u}^s = 0$ .*

*Proof.* Suppose, contrary to our claim, that there exists a sequence  $(u_l)_{l \in \mathbb{N}} \in \ker(\mathcal{A} + \mathcal{B})$  such that for all  $l \in \mathbb{N}$  there exists  $s_l \in \mathbb{Z}$  with  $|s_l| \geq l$  and  $(u_l)^{s_l} \neq 0$ . Without loss of generality we can assume that  $|s_k| < |s_l|$  for  $k < l$ . This means that the functions  $(u_l)^{s_l} e_{s_l}$  belong to  $\ker(\mathcal{A} + \mathcal{B})$  and are linearly independent, a contradiction with the fact that  $\ker(\mathcal{A} + \mathcal{B}) < \infty$ .  $\square$

Lemma 5.2 implies that

$$\begin{aligned} \ker(\mathcal{A} + \mathcal{B}) &= \left\{ \sum_{|s| \leq s_0} u^s e_s \mid u^s \text{ solves (5.7), (3.3)} \right\}, \\ \ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}}) &= \left\{ \sum_{|s| \leq s_0} \varphi^s e_s \mid \varphi^s \text{ solves (5.4), (5.5)} \right\}, \end{aligned}$$

where

$$a(x) \frac{d}{dx} u^s + i s \omega u^s + b(x) u^s = 0. \quad (5.7)$$

It is known that, given  $s \in \mathbb{Z}$ , the number of linearly independent solutions to (5.7), (3.3) equals to the number of linearly independent solutions to (5.4), (5.5). Hence, (5.6) is proved. In particular, we have proved that  $\ker(\mathcal{A} + \mathcal{B})$  and  $\ker(\tilde{\mathcal{A}} + \tilde{\mathcal{B}})$  do not depend on  $\gamma$ , i.e. that claim (ii) of Theorem 1.2 is true.

## 6 $\mathbb{C}^k$ -smoothness of the data-to-solution map (proof of Theorem 1.4)

In this section we prove Theorem 1.4. Hence, we suppose the assumptions of Theorem 1.4 to be satisfied, i.e. the data  $\omega, a, b$ , and  $r$ , which satisfy (1.12) and (1.16)–(1.17), are given and fixed.

By Theorem 1.1, for all  $(\tilde{\omega}, \tilde{a}, \tilde{b}) \approx (\omega, a, b)$  in  $\mathbb{R} \times L^\infty((0, 1); \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n)$  and  $\gamma \geq 1$  the operator  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0)$  is an isomorphism from  $V^\gamma(\tilde{\omega}, \tilde{a}, r)$  onto  $W^\gamma$  and

$$\|\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0)^{-1}\|_{\mathcal{L}(W^\gamma; V^\gamma(\tilde{\omega}, \tilde{a}, r))} \leq \text{const}. \quad (6.8)$$

**Lemma 6.1** *For all  $\gamma \geq 2$  the map  $(\tilde{\omega}, \tilde{a}, \tilde{b}) \mapsto \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0)^{-1}$  is uniformly continuous as a map from a subset of  $\mathbb{R} \times L^\infty([0, 1]; \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n)$  into  $\mathcal{L}(W^\gamma; W^{\gamma-1})$ .*



*Proof.* Take  $(\omega', a', b'), (\omega'', a'', b'') \approx (\omega, a, b)$  in  $\mathbb{R} \times L^\infty((0, 1); \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n)$ ,  $\gamma \geq 2$ ,  $f \in W^\gamma$ ,  $u' \in V^\gamma(\omega', a', r)$ , and  $u'' \in V^\gamma(\omega'', a'', r)$  such that

$$\mathcal{A}(\omega', a', b')u' = \mathcal{A}(\omega'', a'', b'')u'' = f.$$

Then (6.8) yields

$$\|\partial_t u'\|_{W^{\gamma-1}} \leq \|u'\|_{W^\gamma} \leq \|u'\|_{V^\gamma(\omega', a', r)} \leq C\|f\|_{W^\gamma} \quad (6.9)$$

and

$$\begin{aligned} & \min_{j=1, \dots, n} \min_{0 \leq x \leq 1} |a'_j(x)| \|\partial_x u'\|_{W^{\gamma-1}} \leq \left\| [a'_j \partial_x u'_j]_{j=1}^n \right\|_{W^{\gamma-1}} \\ & \leq \left\| [\omega' \partial_t u'_j + a'_j \partial_x u'_j]_{j=1}^n \right\|_{W^{\gamma-1}} + \omega' \|\partial_t u'\|_{W^{\gamma-1}} \\ & \leq \|u'\|_{V^\gamma(\omega', a', r)} + C\omega' \|f\|_{W^\gamma} \leq C(1 + \omega') \|f\|_{W^\gamma}, \end{aligned} \quad (6.10)$$

where the constant  $C > 0$  is the same as in (6.8) and is independent of  $\gamma, \omega', \omega'', a', a'', b',$  and  $b''$ . Moreover, we have

$$\begin{aligned} & \mathcal{A}(\omega'', a'', b'')(u'' - u') = (\mathcal{A}(\omega', a', b') - \mathcal{A}(\omega'', a'', b''))u' \\ & + (\mathcal{A}(\omega', a', b') - \mathcal{A}(\omega', a'', b''))u' + (\mathcal{A}(\omega', a', b') - \mathcal{A}(\omega', a', b''))u' \\ & = [(\omega' - \omega'')\partial_t u'_j + (a'_j - a''_j)\partial_x u'_j + (b'_{jj} - b''_{jj})u'_j]_{j=1}^n. \end{aligned}$$

This is a well-defined equation in  $W^{\gamma-1}$ , and (6.8), (6.9), and (6.10) yield

$$\|u'' - u'\|_{W^{\gamma-1}} \leq C \left( |\omega'' - \omega'| + \max_{1 \leq j \leq n} \|a''_j - a'_j\|_\infty + \max_{1 \leq j, k \leq n} \|b''_{jk} - b'_{jk}\|_\infty \right) \|f\|_{W^\gamma},$$

with a new constant  $C$  being independent of  $\gamma, \omega', \omega'', a', a'', b', b''$ , again.  $\square$

**Lemma 6.2** *There exist  $\varepsilon > 0$  and  $C > 0$  such that for all  $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$ ,  $\tilde{a} \in A_\varepsilon(a)$ ,  $\tilde{b} \in B_\varepsilon(b)$  and  $\gamma \geq 2$  the operator  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$  is an isomorphism from  $V^\gamma(\tilde{\omega}, \tilde{a}, r)$  onto  $W^\gamma$ , and*

$$\left\| \left( \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right)^{-1} \right\|_{\mathcal{L}(W^\gamma; V^\gamma(\tilde{\omega}, \tilde{a}, r))} \leq C. \quad (6.11)$$

*Proof.* From Theorem 1.2 follows that for all  $\gamma \geq 1$  there exists  $\varepsilon > 0$  such that for all  $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$ ,  $\tilde{a} \in A_\varepsilon(a)$ , and  $\tilde{b} \in B_\varepsilon(b)$  the operator  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$  is Fredholm of index zero from  $V^\gamma(\tilde{\omega}, \tilde{a}, r)$  into  $W^\gamma$ . Moreover,  $\varepsilon$  can be chosen independently on  $\gamma$ , because, as we showed in Section 5, if  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$  is Fredholm of index zero from  $V^\gamma(\tilde{\omega}, \tilde{a}, r)$  onto  $W^\gamma$  for certain  $\gamma \geq 1$ , then  $\dim \ker(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1))$  and  $\text{codim im}(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1))$  do not depend on  $\gamma$ , and, hence,  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$  is Fredholm of index zero from  $V^\gamma(\tilde{\omega}, \tilde{a}, r)$  into  $W^\gamma$  for any  $\gamma \geq 1$ .

Let us show that  $\varepsilon$  can be chosen so small that for all  $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$ ,  $\tilde{a} \in A_\varepsilon(a)$ , and  $\tilde{b} \in B_\varepsilon(b)$  the operator  $\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)$  is injective (and, hence, bijective).

Suppose the contrary. Then there exist sequences  $\omega_k \rightarrow \omega$  in  $\mathbb{R}$ ,  $\alpha_k \rightarrow a$  in  $L^\infty((0, 1); \mathbb{R}^n)$ ,  $\beta_k \rightarrow b$  in  $L^\infty((0, 1); \mathbb{M}_n)$ , and  $v_k \in V^\gamma(\omega_k, \alpha_k, r)$  such that

$$(\mathcal{A}(\omega_k, \alpha_k, \beta_k^0) + \mathcal{B}(\beta_k^1))v_k = 0 \text{ and } v_k \neq 0. \quad (6.12)$$

Set  $w_k := \mathcal{A}(\omega_k, \alpha_k, \beta_k^0)v_k$ . Then there is  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$

$$(I + \mathcal{B}(\beta_k^1)\mathcal{A}(\omega_k, \alpha_k, \beta_k^0)^{-1})w_k = 0. \quad (6.13)$$

Hence

$$w_k = (\mathcal{B}(\beta_k^1)\mathcal{A}(\omega_k, \alpha_k, \beta_k^0)^{-1})^2 w_k,$$

and, consequently,

$$\begin{aligned} z_k & := \frac{w_k}{\|w_k\|_{W^\gamma}} = (\mathcal{B}(\beta_k^1)\mathcal{A}(\omega_k, \alpha_k, \beta_k^0)^{-1})^2 z_k \\ & + \left( (\mathcal{B}(\beta_k^1)\mathcal{A}(\omega_k, \alpha_k, \beta_k^0)^{-1})^2 - (\mathcal{B}(\beta_k^1)\mathcal{A}(\omega_k, \alpha_k, \beta_k^0)^{-1})^2 \right) z_k. \end{aligned} \quad (6.14)$$

Because the operator  $(\mathcal{B}(b^1)\mathcal{A}(\omega, a, b^0)^{-1})^2$  is compact from  $W^\gamma$  into  $W^\gamma$ , it is also compact from  $W^\gamma$  into  $W^{\gamma-1}$ . Hence there exist  $z \in W^{\gamma-1}$  and a subsequence  $z_{k_l}$  such that  $(\mathcal{B}(b^1)\mathcal{A}(\omega, a, b^0)^{-1})^2 z_{k_l} \rightarrow z$  in  $W^{\gamma-1}$ . Therefore Lemma 6.1 and (6.14) yield that  $z_{k_l} \rightarrow z$  in  $W^{\gamma-1}$ , and Lemma 6.1 and (6.13) yield that

$$(I + \mathcal{B}(b^1)\mathcal{A}(\omega, a, b^0)^{-1})z = 0 \text{ in } W^{\gamma-1}.$$

Hence  $z$  belongs to  $\ker(I + \mathcal{B}(b^1)\mathcal{A}(\omega, a, b^0)^{-1})$ . Since  $\|z\|_{W^\gamma} = 1$  and  $\ker(I + \mathcal{B}(b^1)\mathcal{A}(\omega, a, b^0)^{-1})$  does not depend on  $\gamma$ , we get a contradiction to assumption (1.16).

It remains to prove (6.11). Suppose the contrary, i.e. that for any choice of  $\varepsilon$  (6.11) is not true. Then there exist sequences  $\omega_k \rightarrow \omega$  in  $\mathbb{R}$ ,  $\alpha_k \rightarrow a$  in  $L^\infty((0, 1); \mathbb{R}^n)$ ,  $\beta_k \rightarrow b$  in  $L^\infty((0, 1); \mathbb{M}_n)$ , and  $v_k \in V^\gamma(\omega_k, \alpha_k, r)$  such that

$$(\mathcal{A}(\omega_k, \alpha_k, \beta_k^0) + \mathcal{B}(\beta_k^1))v_k \rightarrow 0 \text{ and } \|v_k\|_{V^\gamma(\omega_k, \alpha_k, r)} = 1. \quad (6.15)$$

Now we proceed as above, replacing (6.12) by (6.15), to get a contradiction.  $\square$

Similarly to Lemma 6.1 one can prove

**Lemma 6.3** *For all  $\gamma \geq 2$  the map*

$$(\tilde{\omega}, \tilde{a}, \tilde{b}) \in (\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \mapsto \left(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)\right)^{-1} \in \mathcal{L}(W^\gamma; W^{\gamma-1}) \quad (6.16)$$

*is uniformly continuous as a map from a subset of  $\mathbb{R} \times C([0, 1]; \mathbb{R}^n) \times L^\infty((0, 1); \mathbb{M}_n)$  into  $\mathcal{L}(W^\gamma; W^{\gamma-1})$ .*

Let us introduce the data-to-solution map

$$\begin{aligned} (\tilde{\omega}, \tilde{a}, \tilde{b}, f) &\in (\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma \\ &\mapsto \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) := \left(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)\right)^{-1} f \in W^\gamma. \end{aligned} \quad (6.17)$$

**Lemma 6.4** *The map  $\hat{u}$  is  $C^1$ -smooth as a map into  $W^{\gamma-2}$  for all  $\gamma \geq 2$ .*

*Proof.* We have to show that all partial derivatives  $\partial_\omega \hat{u}$ ,  $\partial_a \hat{u}$ ,  $\partial_b \hat{u}$ , and  $\partial_f \hat{u}$  exist and are continuous. First, consider  $\partial_f \hat{u}$ . From the definition (6.17) follows that it exists and that

$$\partial_f \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{f} = \left(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)\right)^{-1} \bar{f}.$$

The continuity of the map

$$(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in (\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma \mapsto \partial_f \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in \mathcal{L}(W^\gamma; W^{\gamma-1})$$

follows from Lemma 6.3.

Now, consider  $\partial_b \hat{u}$ . From Corollary 1.3 follows that it exists, and (6.17) yields

$$\partial_b \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{b} = - \left(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)\right)^{-1} \left(\partial_b \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) \bar{b} + \mathcal{B}'(\tilde{b}^1) \bar{b}\right) \left(\mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1)\right)^{-1} f.$$

Moreover, we have

$$\left(\partial_b \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) \bar{b}\right) u = [\bar{b}_{jj} u_j]_{j=1}^n \text{ and } \left(\mathcal{B}'(\tilde{b}^1) \bar{b}\right) u = \left[ \sum_{k \neq j} \bar{b}_{jk} u_k \right]_{j=1}^n. \quad (6.18)$$

Hence,  $\partial_b \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0)$  and  $\mathcal{B}'(\tilde{b}^1)$  do not depend on  $\tilde{\omega}$ ,  $\tilde{a}$ , and  $\tilde{b}^0$ . Therefore, again Lemma 6.3 yields the continuity of the map

$$(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in (\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma \mapsto \partial_b \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \in \mathcal{L}(L^\infty((0, 1), \mathbb{M}_n); W^{\gamma-1}).$$

Further, consider  $\partial_a \hat{u}$ . If  $\partial_a \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)$  exists as an element of  $\mathcal{L}(L^\infty((0, 1), \mathbb{R}^n); W^{\gamma-2})$ , then for any  $\bar{a} \in L^\infty((0, 1), \mathbb{R}^n)$  we have

$$\begin{aligned} & \left( \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) \partial_a \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{a} = - \left( \partial_a \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) \bar{a} \right) \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \\ & = - \left[ \bar{a}_j \partial_x \hat{u}_j(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \right]_{j=1}^n \\ & = \left[ \frac{\bar{a}_j}{\tilde{a}_j} \left( \tilde{\omega} \partial_t \hat{u}_j(\tilde{\omega}, \tilde{a}, \tilde{b}, f) + \sum_{k \neq j} \tilde{b}_{jk} \hat{u}_k(\tilde{\omega}, \tilde{a}, \tilde{b}, f) - f_j \right) \right]_{j=1}^n. \end{aligned} \quad (6.19)$$

The right hand side belongs to  $W^{\gamma-1}$ , hence this equation determines uniquely the candidate

$$\hat{v}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{a} := - \left( \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right)^{-1} \left( \partial_a \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) \bar{a} \right) \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)$$

for  $\partial_a \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)$  in  $\mathcal{L}(L^\infty((0, 1), \mathbb{R}^n); W^{\gamma-1})$ . Moreover, because of Lemma 6.3 the candidate  $\hat{v}$  for  $\partial_a \hat{u}$  is continuous as a map from  $(\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma$  into  $\mathcal{L}(L^\infty((0, 1), \mathbb{R}^n); W^{\gamma-2})$ .

It remains to prove that  $\hat{v}$  is really  $\partial_a \hat{u}$ . In order to show this, take  $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$ ,  $a_0, a_1 \in A_\varepsilon(a)$ ,  $\tilde{b} \in B_\varepsilon(b)$ ,  $\gamma \geq 2$ ,  $f \in W^\gamma$ ,  $u_0 \in V^\gamma(\omega_0, a, r)$ , and  $u_1 \in V^\gamma(\omega_1, a, r)$  such that

$$\left( \mathcal{A}(\tilde{\omega}, a_0, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) u_0 = \left( \mathcal{A}(\tilde{\omega}, a_1, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) u_1 = f.$$

Then

$$\begin{aligned} & \left( \mathcal{A}(\tilde{\omega}, a_1, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) \left( u_1 - u_0 - \hat{v}(\tilde{\omega}, a_0, \tilde{b}^0)(a_1 - a_0) \right) \\ & = \left( \mathcal{A}(\tilde{\omega}, a_0, \tilde{b}^0) - \mathcal{A}(\tilde{\omega}, a_1, \tilde{b}^0) \right) \hat{v}(\tilde{\omega}, a_0, \tilde{b}^0)(a_1 - a_0) \\ & + \left( \mathcal{A}(\tilde{\omega}, a_0, \tilde{b}^0) + \partial_a \mathcal{A}(\omega_0, a_0, b_0^0)(a_1 - a_0) - \mathcal{A}(\tilde{\omega}, a_1, \tilde{b}^0) \right) u_0. \end{aligned}$$

But  $\mathcal{A}(\tilde{\omega}, a_0, \tilde{b}^0) + \partial_a \mathcal{A}(\omega_0, a_0, b_0^0)(a_1 - a_0) - \mathcal{A}(\tilde{\omega}, a_1, \tilde{b}^0) = 0$  because  $\mathcal{A}(\tilde{\omega}, \cdot, \tilde{b}^0)$  is affine. Hence

$$\left\| \left( \mathcal{A}(\tilde{\omega}, a_1, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) \left( u_1 - u_0 - \hat{v}(\tilde{\omega}, a_0, \tilde{b}^0)(a_1 - a_0) \right) \right\|_{W^{\gamma-2}} = o \left( \|a_1 - a_0\|_{L^\infty((0,1), \mathbb{R}^n)} \right),$$

and Lemma 6.2 yields

$$\left\| u_1 - u_0 - \hat{v}(\tilde{\omega}, a_0, \tilde{b}^0)(a_1 - a_0) \right\|_{W^{\tilde{\gamma}-2}} = o \left( \|a_1 - a_0\|_{L^\infty((0,1), \mathbb{R}^n)} \right),$$

Finally, we consider  $\partial_\omega \hat{u}$ . If  $\partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)$  exists as an element of  $W^{\gamma-2}$ , then we have

$$\left( \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) \partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) = - \left( \partial_\omega \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) \right) \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) = - \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f).$$

This equation determines uniquely the candidate

$$\hat{w}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) := - \left( \mathcal{A}(\tilde{\omega}, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right)^{-1} \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f).$$

for  $\partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)$  in  $W^{\gamma-2}$ . Moreover, because of Lemma 6.3 the candidate  $\hat{w}$  for  $\partial_\omega \hat{u}$  is continuous as a map from  $(\omega - \varepsilon, \omega + \varepsilon) \times A_\varepsilon(a) \times B_\varepsilon(b) \times W^\gamma$  into  $W^{\gamma-2}$ .

Again, it remains to prove that  $\hat{w}$  is really  $\partial_\omega \hat{u}$ . In order to show this, take  $\omega_0, \omega_1 \in (\omega - \varepsilon, \omega + \varepsilon)$ ,  $\tilde{a} \in A_\varepsilon(a)$ ,  $\tilde{b} \in B_\varepsilon(b)$ ,  $\gamma \geq 2$ ,  $f \in W^\gamma$ ,  $u_0 \in V^\gamma(\omega_0, a, r)$  and  $u_1 \in V^\gamma(\omega_1, a, r)$  such that

$$\left( \mathcal{A}(\omega_0, \tilde{a}, \tilde{b}) + \mathcal{B}(\tilde{b}^1) \right) u_0 = \left( \mathcal{A}(\omega_1, \tilde{a}, \tilde{b}) + \mathcal{B}(\tilde{b}^1) \right) u_1 = f.$$

Then

$$\begin{aligned} & \left\| \left( \mathcal{A}(\omega_1, \tilde{a}, \tilde{b}^0) + \mathcal{B}(\tilde{b}^1) \right) \left( u_1 - u_0 - (\omega_1 - \omega_0) \hat{w}(\tilde{\omega}, a_0, \tilde{b}^0) \right) \right\|_{W^{\gamma-2}} \\ &= \left\| (\omega_1 - \omega_0) \left( \mathcal{A}(\omega_1, \tilde{a}, \tilde{b}^0) - \mathcal{A}(\omega_0, \tilde{a}, \tilde{b}^0) \right) \hat{w}(\tilde{\omega}, a_0, \tilde{b}^0) \right\|_{W^{\gamma-2}} = o(|\omega_1 - \omega_0|). \end{aligned}$$

Hence, Lemma 6.2 yields

$$\left\| u_1 - u_0 - (\omega_1 - \omega_0) \hat{w}(\tilde{\omega}, a_0, \tilde{b}^0) \right\|_{W^{\gamma-2}} = o(|\omega_1 - \omega_0|).$$

□

**Lemma 6.5** *The map  $\hat{u}$  is  $C^k$ -smooth as a map into  $W^{\gamma-k-1}$  for all  $1 \leq k \leq \gamma - 1$ .*

*Proof.* For  $k = 1$  the lemma is true, and the first partial derivatives satisfy

$$\partial_f \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{f} = \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}), \quad (6.20)$$

$$\partial_b \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{b} = -\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{\mathcal{B}}(\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f), \bar{b})), \quad (6.21)$$

$$\partial_a \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{a} = -\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{\mathcal{A}}(\tilde{\omega}, \tilde{a}, \tilde{b}, f, \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)) \bar{a}), \quad (6.22)$$

$$\partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) = -\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)). \quad (6.23)$$

Here we denoted by  $\bar{\mathcal{A}}(\tilde{\omega}, \tilde{a}, \tilde{b}, f, u) : L^\infty((0, 1); \mathbb{R}^n) \rightarrow W^{\gamma-1}$  the linear bounded operator which is defined by (cf.(6.19))

$$\bar{\mathcal{A}}(\tilde{\omega}, \tilde{a}, \tilde{b}, f, u) a := \left[ \frac{\bar{a}_j}{\bar{a}_j} \left( \tilde{\omega} \partial_t u_j + \sum_{k \neq j} \tilde{b}_{jk} u_k - f_j \right) \right]_{j=1}^n,$$

and  $\bar{\mathcal{B}} : W^\gamma \times L^\infty((0, 1); \mathbb{M}_n) \rightarrow W^\gamma$  is the bilinear bounded operator which is defined by (cf. (6.18))

$$\bar{\mathcal{B}}(u, b) := \left[ \sum_{k=1}^n b_{jk} u_k \right]_{j=1}^n.$$

Obviously, the map

$$\left( \tilde{\omega}, \tilde{a}, \tilde{b}, f, u \right) \in \mathbb{R} \times A \times L^\infty((0, 1); \mathbb{M}_n) \times W^\gamma \times W^\gamma \mapsto \bar{\mathcal{A}}(\tilde{\omega}, \tilde{a}, \tilde{b}, f, u) \in \mathcal{L}(L^\infty((0, 1); \mathbb{R}^n); W^{\gamma-1})$$

is  $C^\infty$ -smooth. Hence, (6.20)–(6.23), Lemma 6.4 and the chain rule imply that the data-to-solution map  $\hat{u}$  is  $C^2$ -smooth, and one gets corresponding formulae for the second partial derivatives by differentiating the identities (6.20)–(6.23). For example, it holds

$$\begin{aligned} \partial_f^2 \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) &= 0, \\ \partial_f \partial_b \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) (\bar{f}, \bar{b}) &= \partial_b \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}) \bar{b} = -\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{\mathcal{B}}(\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}), \bar{b})), \\ \partial_f \partial_a \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) (\bar{f}, \bar{a}) &= \partial_a \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}) \bar{a} = -\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{\mathcal{A}}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}, \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}))) \bar{a}), \\ \partial_f \partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) \bar{f} &= \partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f}) = -\hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \bar{f})) \end{aligned}$$

and

$$\begin{aligned} \partial_\omega^2 \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f) &= -\partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)) \\ -\partial_f \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)) \partial_t \partial_\omega \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \partial_t \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)) &= \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, \partial_t^2 \hat{u}(\tilde{\omega}, \tilde{a}, \tilde{b}, f)). \end{aligned}$$

If  $\gamma \geq 4$ , then those formulae and the chain rule imply that all second partial derivatives of  $\hat{u}$  are  $C^1$ -smooth etc. □

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