Perturbation analysis for complex symmetric, skew symmetric, even and odd matrix polynomials[†]

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Abstract

In this work we propose a general framework for the structured perturbation analysis of several classes of structured matrix polynomials in homogeneous form, including complex symmetric, skew-symmetric, even and odd matrix polynomials. We introduce structured backward errors for approximate eigenvalues and eigenvectors and we construct minimal structured perturbations such that an approximate eigenpair is an exact eigenpair of an appropriately perturbed matrix polynomial. This work extends previous work for the non-homogeneous case (we include infinite eigenvalues) and we show that the structured backward errors improve the known unstructured backward errors.

Keywords. Polynomial eigenvalue problem, even matrix polynomial, odd matrix polynomial, complex symmetric matrix polynomial, complex skew-symmetric matrix polynomial, perturbation theory, backward error.
AMS subject classification. 65F15, 15A18, 65F35, 15A12

1 Introduction

In this paper we study the perturbation analysis for eigenvalues and eigenvectors of matrix polynomials of degree m

$$\mathbf{L}(c,s) := \sum_{j=0}^{m} c^{m-j} s^j A_j, \tag{1}$$

with coefficient matrices, $A_j \in \mathbb{C}^{n\times n}$. In contrast to previous work on this topic [1, 3], we consider the homogeneous form where the eigenvalues are represented as pairs $(c,s) \in \mathbb{C}^2\backslash\{0\}$, which for $c \neq 0$ correspond to finite eigenvalues $\lambda = \dfrac{s}{c}$, while $(0,1)$ corresponds to the eigenvalue ∞.

The eigenvalue problem for matrix polynomials arises naturally in a large number of applications, e.g., see [17, 18, 24, 27, 29, 36, 37] and the references therein. In many applications, the coefficient matrices have further structure which reflects the properties of the underlying physical model, see [9, 11, 12, 32, 19, 28, 30, 37]. Since the polynomial eigenvalue problems

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typically arise from physical modelling, including numerical discretization methods such as finite element modelling [10, 31], and since the eigenvalue problem is usually solved with numerical methods that are subject to round-off as well as approximation errors it is very important to study the perturbation analysis of these problems. This analysis is necessary to study the sensitivity of the eigenvalue/eigenvectors under the modelling, discretization, approximation and roundoff errors, but also to judge whether the numerical methods that are used yield reliable results.

While the perturbation analysis for classical and generalized eigenvalue problems is well studied, see [21, 33, 38] for polynomial eigenvalue problems the situation is much less satisfactory and most research is very recent, see [22, 24, 23, 35, 36]. Here we are particularly interested in the behavior of the eigenvalues and eigenvectors under perturbations which preserve the structure of the matrix polynomial. This has recently been an important research topic and we will present a systematic extension of the work in [1, 3, 4, 6, 11, 12].

In this paper we will focus on complex matrix polynomials, where the coefficient matrices are complex symmetric or skew-symmetric, i.e., $\mathbf{L}(c,s) = \pm\mathbf{L}^T(c,s)$ or where the matrix polynomials are T-even or T-odd, i.e., $\mathbf{L}(c,s) = \pm\mathbf{L}^T(c,-s)$. Complex (skew)-symmetric problems arise in the finite element modelling of the acoustic field in car interiors and in the design of axisymmetric VCSEL devices, see e.g. [8, 34] while complex T-even/T-odd problems arise in the vibration analysis for high-speed trains, see e.g. [25, 26]. Many applications only need finite eigenvalues and associated eigenvectors, but the eigenvectors associated with the eigenvalue infinity play an important role as well, since often the infinite spectrum has to be deflated before classical methods can be employed, see [13, 14].

While the perturbation analysis and the construction of backward errors for finite eigenvalues has been studied in detail, there are only few results associated with the eigenvalue infinity. We will present a systematic general perturbation framework that covers finite and infinite eigenvalues and extends the structured theory of [1, 3, 2, 4, 11, 12] as well as the unstructured theory for the homogeneous case studied in [5, 7, 6, 16, 24, 33]. In particular, to present the backward error analysis, for a given approximation to an eigenvalue/eigenvector pair of a matrix polynomial $\mathbf{L}$, we will construct an appropriately structured minimal (in Frobenius and spectral norm) perturbation polynomial $\Delta\mathbf{L}$ such that the given eigenvalue/eigenvector pair is exact for $\mathbf{L}+\Delta\mathbf{L}$. It will turn out that the so constructed minimal perturbation is unique in the case of Frobenius norm and that there are infinitely many such minimal perturbations in the case of the spectral norm. We will compare the so constructed perturbations with those constructed for matrix pencils and matrix polynomials in [1, 2, 3] and show that our results generalize these results and provide the following further information on the eigenvalues 0 and ∞ of $\mathbf{L}+\Delta\mathbf{L}$.

- For the case of complex symmetric or skew-symmetric matrix polynomials we show that the nearest perturbed matrix polynomial can have all kinds of eigenvalues including 0 and ∞.
- When the degree is $m = 1$, we present the perturbation analysis for the case of T-even and T-odd matrix pencils and we show that the nearest perturbed pair can have 0 and ∞ as eigenvalues depending on the choice of (λ,μ) for which we want to compute the backward error. Furthermore, when $\lambda = 0$ or $\mu = 0$, then we show that the perturbed pair is the same for the spectral and the Frobenius norm.
- When the degree is $m > 1$ and even, then for the case of T-even matrix polynomials

we show that the nearest perturbed polynomial can have both 0 and ∞ eigenvalues depending on the choice of (λ, μ) for which we want to compute the backward error. Again, when $\lambda = 0,\ \mu \neq 0$ or $\lambda \neq 0,\ \mu = 0$, then the perturbed polynomial is the same for the spectral and the Frobenius norm.

- When $m > 1$ is odd, then for the case of T-even matrix polynomials we show that the nearest perturbed matrix polynomial can have all possible finite eigenvalues including 0 but not the eigenvalue ∞.
- When $m > 1$ is even, then for the case of T-odd matrix polynomials we show that the nearest perturbed polynomial can have non-zero finite eigenvalues but not the eigenvalue ∞.
- When $m > 1$ is odd, then for the case of T-odd matrix polynomials we show that the perturbed polynomial can have only ∞ and non-zero finite eigenvalues.

The paper is organized as follows: In Section 2, we review some known techniques that were developed in [5, 6, 7] for matrix pencils and identify the types of structured homogeneous matrix polynomials that we will analyze, as well as the eigenvalue symmetry that arises for these structured matrix polynomials. In Section 3 and in Section 4 we present the structured backward error analysis of an approximate eigenpair for complex symmetric, complex skew-symmetric, T-even, and T-odd matrix polynomials and compare these results to the corresponding unstructured backward errors. We also present a systematic general procedure for the construction of an appropriate structured minimal complex symmetric, complex skew-symmetric, T-even, and T-odd polynomial $\Delta\mathbf{L}$ such that the given approximate eigenvalue and eigenvector are exact for $\mathbf{L} + \Delta\mathbf{L}$. These results cover finite and infinite eigenvalues and generalize results of [1, 2, 3, 4, 11, 12] in a systematic way.

2 Notation and preliminaries

We denote by $\mathbb{R}^{n\times n}, \mathbb{C}^{n\times n}$ the sets of real and complex $n \times n$ matrices, respectively. For an elementwise nonnegative vector $w = [w_1, \ldots, w_n]^T \in \mathbb{R}^m$, a real or complex vector $x = [x_1, \ldots, x_n]^T$ and an integer p, $1 \leq p \leq \infty$, we define a weighted p-(semi)norm via

$$\|x\|_{w,p} := \|[w_1x_1, w_2x_2, \ldots, w_nx_n]^T\|_p.$$

If w is elementwise strictly positive, then this is a norm, and if w has zero components then it is a seminorm. We define the componentwise inverse of w via $w^{-1} := [w_1^{-1}, \ldots, w_m^{-1}]^T$, where we use the convention that $w_i^{-1} = 0$ if $w_i = 0$.

We will consider structured and unstructured backward errors both in spectral norm and Frobenius norm on $\mathbb{C}^{n\times n}$, which are given by

$$\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|,\ \|A\|_F := (\mathrm{trace} A^*A)^{1/2},$$

respectively.

By $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ we denote the largest and smallest singular values of a matrix A, respectively. The identity matrix is denoted by I and $\overline{A}$, A^T, and A^H stand for the conjugate, transpose, and conjugate transpose of a matrix A, respectively.

The set of all matrix polynomials of degree $m \geq 0$ with coefficients in $\mathbb{C}^{n\times n}$ is denoted by $\mathbf{L}_m(\mathbb{C}^{n\times n})$. This is a vector space which (for a given nonnegative weight vector $w := [w_0, w_1, \ldots, w_m]^T \in \mathbb{R}^{m+1} \setminus \{0\}$) we can equip with a weighted norm/seminorm $|\!|\!|.|\!|\!|_{w,2}$ defined as

$$|\!|\!|\mathbf{L}|\!|\!|_{w,F} := \|(A_0, \ldots, A_m)\|_{w,F} = (w_0^2\|A_0\|_F^2 + \ldots + w_m^2\|A_m\|_F^2)^{1/2},$$

for the Frobenius norm and

$$|\!|\!|\mathbf{L}|\!|\!|_{w,2} := \|(A_0, \ldots, A_m)\|_{w,2} = (w_0^2\|A_0\|_2^2 + \ldots + w_m^2\|A_m\|_2^2)^{1/2},$$

for the spectral norm. A matrix polynomial is called *regular* if $\det(\mathbf{L}(\lambda, \mu)) \neq 0$ for some $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, otherwise it is called *singular*. The *spectrum* of a homogeneous matrix polynomial $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ is defined as

$$\Lambda(\mathbf{L}) := \{(c, s) \in \mathbb{C}^2 \setminus \{(0, 0)\} : \text{ rank}(\mathbf{L}(c, s)) < n\}.$$

In the following we normalize the set of points $(c, s) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, such that $|c|^2 + |s|^2 = 1$ and that c is real. With this normalization, it follows that the spectrum $\Lambda(\mathbf{L})$ of a matrix polynomial $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ can be identified with a subset of the Riemann sphere, see e.g. [6].

In the following we will compute backward errors for structured matrix polynomials. These were introduced, e.g., in [20, 35], but here we follow [5, 6, 7] and define the backward error of an approximate eigenpair as follows. Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ be an approximate eigenvalue of $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ with corresponding normalized approximate right eigenvector $x \neq 0$ with $x^H x = 1$, i.e., $\mathbf{L}(\lambda, \mu)x = 0$. Then we consider the Frobenius and spectral norm backward errors associated with a given nonnegative weight vector $[w_0, w_1, \ldots, w_n]^T$,

$$\begin{aligned}
\eta_{w,F}(\lambda, \mu, x, \mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,F},\ \Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n}),\\
&\qquad (\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0\},\\
\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2},\ \Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n}),\\
&\qquad (\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0\},
\end{aligned}$$

respectively. When $w := [1, 1, \ldots, 1]^T$, then we just leave off the index w for convenience. For a given $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $x \in \mathbb{C}^n$ with $x^H x = 1$, we set $k := -\mathbf{L}(\lambda, \mu)x$ and, with a given nonnegative weight vector $[w_0, w_1, \ldots, w_m]^T$, we introduce $H_{w,2} := \|(\lambda^m\mu^0, \lambda^{m-1}\mu, \ldots, \lambda^0\mu^m)\|_{w,2}$, the map

$$\begin{aligned}
\mathbb{C} &\to \mathbb{R}\\
z_j &\mapsto H_{w,2}(z_0, z_1, \ldots, z_m),
\end{aligned}$$

and its gradient $\nabla_j H_{w,2}$ at $(\lambda^m\mu^0, \lambda^{m-1}\mu, \ldots, \lambda^0\mu^m)$.

Then we have

$$\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) = \frac{\|\mathbf{L}(\lambda, \mu)x\|}{H_{w^{-1},2}(\lambda, \mu)}. \tag{2}$$

Defining for each of the coefficients

$$z_{A_j} := \frac{\nabla_j H_{w^{-1},2}}{H_{w^{-1},2}} \tag{3}$$

and introducing the perturbations $\Delta A_j := \overline{z_{A_j}} k x^H$ for the coefficients, we form the matrix polynomial

$$\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j} s^j \Delta A_j,$$

with

$$|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2} = \frac{\|k\|}{H_{w^{-1},2}}.$$

The backward errors for structured matrix polynomials from a set $\mathbf{S} \subset \mathbf{L}_m(\mathbb{C}^{n\times n})$ are defined analogously as

$$\begin{aligned}
\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,F},\ \Delta\mathbf{L}\in\mathbf{S},\ (\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0\},\\
\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) &:= \inf\{|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2},\ \Delta\mathbf{L}\in\mathbf{S},\ (\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x=0\},
\end{aligned}$$

respectively.

In order to compute the backward errors, we will need the partial gradient $\nabla_i\|z\|_{w,2}$ of the map

$$\begin{aligned}
\mathbb{C}^{m+1} &\to \mathbb{R},\\
z &\mapsto \|z\|_{w,2},
\end{aligned}$$

which is just the gradient of the map

$$\begin{aligned}
\mathbb{C} &\to \mathbb{R},\\
z_i &\mapsto \|[z_0,z_1,\ldots,z_m]^T\|_{w,2},
\end{aligned}$$

obtained by fixing the variables $z_0, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$ as constants.

The gradient of the map

$$\begin{aligned}
\mathbb{C}^{m+1} &\to \mathbb{R},\\
z &\mapsto \|z\|_{w,2},
\end{aligned}$$

is then defined as

$$\nabla(\|z\|_{w,2}) = [\nabla_0\|z\|_{w,2}, \nabla_1\|z\|_{w,2}, \ldots, \nabla_m\|z\|_{w,2}]^T \in \mathbb{C}^{m+1}.$$

For $z\in\mathbb{C}$ we set $\operatorname{sign}(z) := \overline{z}/|z|$, when $z\neq 0$ and $\operatorname{sign}(z) := 0$ when $z=0$. With these definitions we have the following preliminary results which generalize the corresponding results of [5, 6] to matrix polynomials.

Proposition 2.1 *Consider the map $H_{w,2}:\mathbb{C}^{m+1}\to\mathbb{R}$ given by $H_{w,2}(z) := \|[z_0,z_1,\ldots,z_m]^T\|_{w,2}$. Then $H_{w,2}$ is differentiable on $\mathbb{C}^{m+1}$ and*

$$\nabla_i H_{w,2}(z) = \frac{w_i^2 z_i}{H_{w,2}(z)},\ i=1,2,\ldots,m.$$

Proof. The assertion follows from the fact that $\nabla(|z_i|^2) = 2z_i$. □

The proof of the following two propositions is analogous.

Proposition 2.2 *Let m be an integer and let $\tilde{m} = \dfrac{m}{2} + 1$, $\hat{m} = m$ if m is even and $\tilde{m} = \dfrac{m+1}{2}$, $\hat{m} = m - 1$ if m is odd. Consider the mapping*

$$\begin{aligned} K_{w,2} : \mathbb{C}^{\tilde{m}} &\to \mathbb{R} \\ z &\mapsto \|[z_0, z_2, z_4, \ldots, z_{\hat{m}}]^T\|_{w,2}. \end{aligned}$$

Then $K_{w,2}$ is differentiable and

$$\nabla_i K_{w,2}(z) = \frac{w_i^2 z_i}{K_{w,2}(z)},\ i = 0, 2, 4, \ldots, \hat{m}.$$

Proposition 2.3 *Let m be an integer and let $\tilde{m} = \dfrac{m}{2}$, $\hat{m} = m$ if m is even and $\tilde{m} = \dfrac{m+1}{2}$, $\hat{m} = m - 1$ if m is odd. Consider the mapping*

$$\begin{aligned} N_{w,2} : \mathbb{C}^{\tilde{m}} &\to \mathbb{R} \\ z &\mapsto \|[z_1, z_3, z_5, \ldots, z_{\hat{m}}]^T\|_{w,2}. \end{aligned}$$

Then $N_{w,2}$ is differentiable and

$$\nabla_i N_{w,2}(z) = \frac{w_i^2 z_i}{N_{w,2}(z)},\ i = 1, 3, 5, \ldots, \hat{m}.$$

Proposition 2.4 *Consider the functions*

$$\begin{aligned} H_{w,2}(c^m s^0, c^{m-1} s, \ldots, c^0 s^m) &= \|[c^m s^0, c^{m-1} s, \ldots, c^0 s^m]^T\|_{w,2}, \\ K_{w,2}(c^m s^0, c^{m-2} s^2, \ldots, c^0 s^m) &= \|[c^m s^0, c^{m-2} s^2, \ldots, c^0 s^m]^T\|_{w,2} \text{ if } m \text{ is even}, \\ K_{w,2}(c^m s^0, c^{m-2} s^2, \ldots, c^0 s^{m-1}) &= \|[c^m s^0, c^{m-2} s^2, \ldots, c s^{m-1}]^T\|_{w,2} \text{ if } m \text{ is odd}, \\ N_{w,2}(c^{m-1} s, c^{m-3} s^3, \ldots, c s^{m-1}) &= \|[c^{m-1} s, c^{m-3} s^3, \ldots, c s^{m-1}]^T\|_{w,2} \text{ if } m \text{ is even}, \\ N_{w,2}(c^{m-1} s, c^{m-3} s^3, \ldots, c^0 s^m) &= \|[c^{m-1} s, c^{m-3} s^3, \ldots, c^0 s^m]^T\|_{w,2} \text{ if } m \text{ is odd}. \end{aligned}$$

For even m, the following formulas hold:

$$\sum_{j=0, j \text{ even}}^{m} c^{m-j} s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{K_{w,2}^2}{H_{w,2}^2} \text{ and } \sum_{j=0}^{m} w_j^{-2} |\nabla_j K_{w,2}|^2 = 1,$$

$$\sum_{j=1, j \text{ odd}}^{m-1} c^{m-j} s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{N_{w,2}^2}{H_{w,2}^2} \text{ and } \sum_{j=1}^{m-1} w_j^{-2} |\nabla_j N_{w,2}|^2 = 1,$$

$$\sum_{j=0, j \text{ even}}^{m} c^{m-j} s^j \frac{\overline{\nabla_j K_{w,2}}}{K_{w,2}} = 1, \sum_{j=1, j \text{ odd}}^{m-1} c^{m-j} s^j \frac{\overline{\nabla_j N_{w,2}}}{N_{w,2}} = 1, \text{ and}$$

$$\sum_{j=0, j \text{ even}}^{m} c^{m-j} s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} + \sum_{j=odd}^{m-1} c^{m-j} s^j \frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1.$$

For odd m then the following formulas hold:

$$\sum_{j=0,j\text{ even}}^{m-1} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{K_{w,2}^2}{H_{w,2}^2},\ \sum_{j=1,j\text{ odd}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \frac{N_{w,2}^2}{H_{w,2}^2},$$
$$\sum_{j=0,j\text{ even}}^{m-1} c^{m-j}s^j\frac{\overline{\nabla_j K_{w,2}}}{K_{w,2}} = 1,\ \sum_{j=1,j\text{ odd}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j N_{w,2}}}{N_{w,2}} = 1,\ \textit{and}$$
$$\sum_{j=0,j\text{ even}}^{m-1} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} + \sum_{j=1,j\text{ odd}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1.$$

For all m the following formulas hold:

$$\sum_{j=0}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = 1,\ \sum_{j=0}^{m} w_j^{-2}|\nabla_j H_{w,2}|^2 = 1.$$

Proof. By Proposition 2.1, we have

$$\nabla_j H_{w,2}(c^m, c^{m-1}s, \ldots, s^m) = \frac{w_j^2 c^{m-j}s^j}{H_{w,2}(c^m, c^{m-1}s, \ldots, s^m)}.$$

Then

$$\sum_{j=0,\ j\text{ even}}^{m} c^{m-j}s^j\frac{\overline{\nabla_j H_{w,2}}}{H_{w,2}} = \sum_{j=0,\,j\text{ even}}^{m} w_j^2 c^{m-j}s^j\frac{\overline{c^{m-j}s^j}}{H_{w,2}^2} = \frac{K_{w,2}^2}{H_{w,2}}.$$

The other parts follow analogously, using Propositions 2.1–2.3. □

After establishing these formulas for general matrix polynomials, we now turn to the structured classes. These classes were discussed in detail in [28] but not in homogeneous form. So let us first introduce the homogeneous versions.

Definition 2.5 *Let $(c,s) \in \mathbb{C}^2 \setminus \{(0,0)\}$. A matrix polynomial $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ is called*

1. Symmetric/skew-symmetric *if* $\mathbf{L}(c,s) = \pm\mathbf{L}^T(c,s)$,
2. T-even/T-odd *if* $\mathbf{L}(c,s) = \pm\mathbf{L}^T(c,-s)$.

The spectra of these classes of structured matrices have a symmetry structure that is summarized in the following proposition which follows directly from the results for the non-homogeneous case in [28].

Proposition 2.6

1. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex symmetric or complex skew-symmetric matrix polynomial of the form (1). If $x \in \mathbb{C}^n$ is a right eigenvector of $\mathbf{L}$ corresponding to an eigenvalue $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, then $\overline{x}$ is a left eigenvector corresponding to the eigenvalue (λ,μ).*
2. *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex T-even or T-odd matrix polynomial of the form (1). If $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^n$ are right and left eigenvector associated to an eigenvalue $(\lambda,\mu) \in \mathbb{C}^2\setminus\{(0,0)\}$ of $\mathbf{L}$, then $\overline{y}$ and $\overline{x}$ are right and left eigenvectors associated to the eigenvalue $(\lambda,-\mu)$.*

S	Eigenvalues	Eigenpairs	$x^T A_j x$
symmetric	(λ, μ)	$((\lambda, \mu), x, \overline{x})$	
skew-symm.	(λ, μ)	$((\lambda, \mu), x, \overline{x})$	0
T-even	$((\lambda, \mu), (\lambda, -\mu))$	$((\lambda, \mu), x, \overline{y}), ((\lambda, -\mu), y, \overline{x})$	0 for all odd j
T-odd	$((\lambda, \mu), (-\lambda, \mu))$	$((\lambda, \mu), x, \overline{y}), ((-\lambda, \mu), y, \overline{x})$	0 for all even j

Table 1: Eigenvalues and eigenvectors of structured matrix polynomials.

Since T-odd and T-even matrix polynomials have coefficients that are alternating between symmetric and skew-symmetric matrices it is clear that if we form products $x^T(\mathbf{L}(\lambda, \mu))x$, then all those terms associated with skew-symmetric coefficients vanish; these are the coefficients with odd index for T-even matrix polynomials, and the ones with even index for T-odd matrix polynomials. We summarize the properties of these structured matrix polynomials in Table 1.

To derive the backward error formulas, in the following we will frequently need the following completion results in which for a symmetric matrix X, $X^{\frac{1}{2}}$ denotes the positive square root.

Theorem 2.7 *[15] Consider a block matrix* $T := \begin{bmatrix} A & C \\ B & X \end{bmatrix}$. *Then for any positive number*

$$\chi \geq \max\left\{ \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_2, \left\| \begin{bmatrix} A & C \end{bmatrix} \right\|_2 \right\},$$

the block X *can be chosen such that*

$$\left\| \begin{bmatrix} A & C \\ B & X \end{bmatrix} \right\|_2 \leq \chi,$$

where X *is of the form* $X = -KA^H L + \chi(I - KK^H)^{1/2} Z (I - L^H L)^{1/2}$, *where* $K := ((\chi^2 I - A^H A)^{-1/2} B^H)^H$, $L := (\chi^2 I - A^H A)^{-1/2} C$ *and* Z *is an arbitrary matrix such that* $\|Z\|_2 \leq 1$.

As a Corollary of Theorems 2.7 one has the following result for complex matrices.

Corollary 2.8 *Let* $A = \pm A^T$, $C = \pm B^T \in \mathbb{C}^{n \times n}$ *and* $\chi := \sigma_{\max}\left(\begin{bmatrix} A \\ B \end{bmatrix} \right)$. *Then there exists a symmetric/skew-symmetric matrix* $X \in \mathbb{C}^{n \times n}$ *such that* $\sigma_{max}\left(\begin{bmatrix} A & \pm B^T \\ B & X \end{bmatrix} \right) = \chi$, *and* X *has the form*

$$X := -K\overline{A}K^T + \chi(I - KK^H)^{1/2} Z (I - \overline{K}K^T)^{1/2},$$

$K := B(\chi^2 I - \overline{A}A)^{-1/2}$ *and where* $Z = \pm Z^T \in \mathbb{C}^{n \times n}$ *is an arbitrary matrix such that* $\|Z\|_2 \leq 1$.

In the results presented below, we always use $Z = 0$.

In the following section we derive backward errors for the different classes of structured matrix polynomials.

3 Backward errors for complex symmetric and skew-symmetric matrix polynomials

In this section we derive backward error formulas for homogeneous complex symmetric and skew-symmetric matrix polynomials. Throughout this section, we will make use of the partial gradients $\dfrac{\nabla_j H_{w^{-1},2}}{H_{w^{-1},2}}$ of $H_{w^{-1},2}$ and of z_{A_j} as defined in (3).

Theorem 3.1 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a regular, symmetric matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and let $k := -\mathbf{L}(\lambda,\mu)x$. Introduce the perturbation matrices*

$$\Delta A_j = -\overline{x}x^T A_j x x^H + \overline{z_{A_j}}\left[\overline{x}k^T + kx^H - 2(x^T k)\overline{x}x^H\right],\ j = 0,\dots,m$$

and form

$$\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j \in \mathbf{L}_m(\mathbb{C}^{n\times n}).$$

Then $\Delta\mathbf{L}$ is a symmetric matrix polynomial and $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

Proof. Since for all j we have $\Delta A_j = \Delta A_j^T$, it follows that $\Delta\mathbf{L}$ is symmetric and we have that

$$\begin{aligned}(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m}\lambda^{m-j}\mu^j(A_j + \Delta A_j)x\\ &= \sum_{j=0}^{m}\lambda^{m-j}\mu^j\left[A_j x - \overline{x}x^T A_j x + \overline{z_{A_j}}\left[\overline{x}k^T x + k - 2(x^T k)\overline{x}\right]\right]\\ &= -k(I - \overline{x}x^T) + \left[\overline{x}k^T x + k - 2(x^T k)\overline{x}\right]\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}}.\end{aligned}$$

By Proposition 2.4 we have that $\displaystyle\sum_{j=0}^{m}\lambda^{m-j}\mu^j\overline{z_{A_j}} = 1$. Then

$$\begin{aligned}&(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x\\ &\quad = -(I - \overline{x}x^T)k + \overline{x}k^T x + k - 2(x^T k)\overline{x} = 0,\end{aligned}$$

since $k^T x = x^T k$. ∎

As a corollary of Theorem 3.1 we obtain Theorem 4.2.1 of [1] for the case of non-homogeneous matrix polynomials that have only finite eigenvalues, i.e., for which $\det(A_m) \neq 0$.

Corollary 3.2 *Let $\mathbf{L}(s) := \sum_{j=0}^{m} s^j A_j \in \mathbb{C}^{n\times n}$ have only finite eigenvalues, let $\mu \in \mathbf{L}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and let $k := -\mathbf{L}(\mu)x$. Introduce the perturbation matrices*

$$\Delta A_j := -\overline{x}x^T A_j x x^H + \frac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[\overline{x}k^T + kx^H - 2(x^T k)\overline{x}x^H\right],\ j = 0,1,\dots,m,$$

with $\|\Lambda\|_2 = \|[1,\mu,\mu^2,\dots\mu^m]^T\|_2$. Then with $\Delta\mathbf{L}(s) := \sum_{j=0}^{m} s^j\Delta A_j$, $\Delta\mathbf{L}$ is symmetric and $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$.

Proof. The proof follows from Theorem 3.1 with $c = 1$ and $w =: [1, 1, \ldots, 1]^T$. ∎

As further corollary we obtain Theorem 2.2 of [3] for matrix pencils.

Corollary 3.3 *Let $\mathbf{L} = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a symmetric matrix pencil, let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda, \mu)x$. Introduce the perturbation matrices*

$$\begin{aligned} \Delta A_0 &:= -\overline{x}x^T A_0 x x^H + \overline{z_{A_0}}\left[\overline{x}k^T + kx^H - 2(x^T k)\overline{x}x^H\right], \\ \Delta A_1 &:= -\overline{x}x^T A_1 x x^H + \overline{z_{A_1}}\left[\overline{x}k^T + kx^H - 2(x^T k)\overline{x}x^H\right]. \end{aligned}$$

Then $\Delta\mathbf{L}(c, s) := c\Delta A_0 + s\Delta A_1$ is symmetric and $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$.

Using Theorem 3.1 we then obtain the following backward errors for complex symmetric matrix polynomials.

Theorem 3.4 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex symmetric matrix polynomial of the form (1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda, \mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L}) = \frac{\sqrt{2\|k\|_2^2 - |x^T k|^2}}{H_{w^{-1},2}}$$

and there exists a unique complex symmetric polynomial $\Delta\mathbf{L}(c, s) := \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$ with coefficients

$$\Delta A_j = \overline{z_{A_j}}\left[\overline{x}k^T + kx^H - (x^T k)\overline{x}x^H\right],\ j = 0, 1, \ldots, m$$

such that the structured backward error satisfies $\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_{w,2}$ and $\overline{x}$, x are left and right eigenvectors corresponding to the eigenvalue (λ, μ) of $\mathbf{L}+\Delta\mathbf{L}$, respectively.

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \frac{\|k\|_2}{H_{w^{-1},2}}$$

and there exist a complex symmetric polynomial $\Delta\mathbf{L}(c, s) := \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$ with coefficients

$$\Delta A_j := \overline{z_{A_j}}\left[\overline{x}k^T + kx^H - (k^T x)\overline{x}x^H - \frac{\overline{x^T k}(I - \overline{x}x^T)kk^T(I - xx^H)}{\|k\|_2^2 - |x^T k|^2}\right]$$

such that $\|\!|\Delta\mathbf{L}|\!\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L})$, and $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$.

Proof. By Theorem 3.1 we have that $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and hence $k = \Delta\mathbf{L}(\lambda, \mu)x$. Now we construct a unitary matrix U which has x as its first column, i.e., $U = [x, U_1] \in \mathbb{C}^{n\times n}$ and let $\widetilde{\Delta A_j} := U^T \Delta A_j U = \begin{bmatrix} d_{j,j} & d_j^T \\ d_j & D_{j,j} \end{bmatrix}$, where $D_{j,j} = D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}$. Then

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda, \mu)}U^H = \overline{U}U^T(\Delta\mathbf{L}(\lambda, \mu))U^H U = \Delta\mathbf{L}(\lambda, \mu),$$

and hence

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H x = \Delta\mathbf{L}(\lambda,\mu)x = k,$$

which implies that

$$\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H x = U^T k = \begin{bmatrix} x^T k \\ U_1^T k \end{bmatrix}.$$

Therefore, we get that

$$\begin{bmatrix} \sum_{j=0}^m \lambda^{m-j}\mu^j d_{j,j} \\ \sum_{j=0}^m \lambda^{m-j}\mu^j d_j \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^m w_j d_{j,j} \dfrac{\lambda^{m-j}\mu^j}{w_j} \\ \sum_{j=0}^m w_j \lambda^{m-j}\mu^j \dfrac{d_j}{w_j} \end{bmatrix} = \begin{bmatrix} x^T k \\ U_1^T k \end{bmatrix}.$$

To minimize the norm of the perturbation, we solve this system for the parameters $d_{j,j}, d_j$ in a least squares sense, and obtain

$$\begin{bmatrix} w_0 d_{0,0} \\ w_1 d_{1,1} \\ w_2 d_{2,2} \\ \vdots \\ w_m d_{m,m} \end{bmatrix} = \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \overline{z_{A_2}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} x^T k, \text{ and } \begin{bmatrix} w_0 d_0 \\ w_1 d_1 \\ \vdots \\ w_m d_m \end{bmatrix} = \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \overline{z_{A_2}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} U_1^T k,$$

Applying Proposition 2.1, we then get the following relations

$$d_{j,j} = \overline{z_{A_j}} x^T k, \; d_j = \overline{z_{A_j}} U_1^T k, \; j = 0, 1, \ldots, m.$$

From this we obtain

$$\begin{aligned} \Delta A_j &= \overline{U}\widetilde{\Delta A}U^H = \overline{x} d_{j,j} x^H + \overline{U_1} d_j x^H + \overline{x} d_j^T U_1^H + \overline{U_1} D_{j,j} U_1^H \\ &= \overline{z_{A_j}}[(\overline{x} x^T k x^H) + \overline{U_1} U_1^T k x^H + \overline{x} k^T U_1 U_1^H)] + \overline{U_1} D_{j,j} U_1^H \\ &= \overline{z_{A_j}}[(\overline{x} x^T k x^H) + (I - \overline{x} x^T) k x^H + \overline{x} k^T (I - x x^H))] + \overline{U_1} D_{j,j} U_1^H \\ &= \overline{z_{A_j}} \left[k x^H + \overline{x} k^T - (k^T x)\overline{x} x^H \right] + \overline{U_1} D_{j,j} U_1^H. \end{aligned} \tag{4}$$

In Frobenius norm, the minimal perturbation is obtained by taking $D_{j,j} = 0$ and hence we get

$$\begin{aligned} \|\Delta A_j\|_F^2 &= |d_{j,j}|^2 + 2\|d_j\|_2^2 = |z_{A_j}|^2 (|x^T k|^2 + 2\|U_1^T k\|_2^2) \\ &= |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2}, \end{aligned}$$

since $\|U^T k\|_2^2 = |x^T k|^2 + \|U_1^T k\|_2^2$. By Proposition 2.4, we have that $\displaystyle\sum_{j=0}^m w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$ and thus in the case of the Frobenius norm we have that

$$|||\Delta\mathbf{L}|||_{w,F}^2 = \sum_{j=0}^m w_j^2 |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2} = \frac{2\|k\|_2^2 - |x^T k|^2}{H_{w^{-1},2}^2},$$

and hence,

$$\|\!|\Delta\mathbf{L}|\!\|_{w,F} = \sqrt{\frac{2\|k\|_2^2 - |x^Tk|^2}{H_{w^{-1},2}^2}}.$$

As k^Tx is a scalar constant, it follows that all ΔA_j and thus also $\Delta\mathbf{L}$ are symmetric and

$$\begin{aligned}(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m} \lambda^{m-j}\mu^j(A_j + \Delta A_j)x \\ &= -k + (\sum_{j=0}^{m} \lambda^{m-j}\mu^j \Delta A_j)x \\ &= -k + \sum_{j=0}^{m} \lambda^{m-j}\mu^j\overline{z_{A_j}}[kx^H + \overline{x}k^T - \overline{x}k^Txx^H]x \\ &= -k + k + \overline{x}k^Tx - \overline{x}k^Tx = 0.\end{aligned}$$

Here we have used that by Proposition 2.4, we have that $\sum_{j=0}^{m} \lambda^{m-j}\mu^j\overline{z_{A_j}} = 1$, which implies that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$. Similarly, it follows that $\overline{x}^H(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu)) = 0$.

For the spectral norm we can apply Corollary 2.8 to (4) and get

$$\begin{aligned}D_{j,j} &= -\frac{\overline{z_{A_j}}}{P^2}\left[\overline{x^Tk}(U_1^Tk)(U_1^Tk)^T\right] \\ &+ \chi\left[I - \frac{(U_1^Tk)(U_1^Tk)^H}{P^2}\right]^{1/2} Z \left[I - \frac{\overline{U_1^Tk}(U_1^Tk)^T}{P^2}\right]^{1/2},\end{aligned}$$

where $Z = Z^T$ and $\|Z\|_2 \le 1$, $P^2 = \|k\|_2^2 - |x^Tk|^2$, $\chi := \sqrt{\|d_{j,j}\|^2 + \|d_j\|_2^2}$. With the special choice $Z = 0$ we get $D_{j,j} = -\frac{\overline{z_{A_j}}}{P^2}\left[\overline{x^Tk}(U_1^Tk)(U_1^Tk)^T\right]$ and

$$\overline{U_1}D_{j,j}U_1^H = -\frac{\overline{z_{A_j}}}{P^2}\overline{U_1}U_1^Tkk^TU_1U_1^H = -\frac{\overline{z_{A_j}}}{P^2}(I - \overline{x}x^T)kk^T(I - xx^H).$$

Hence,

$$\Delta A_j = \overline{z_{A_j}}\left[kx^H + \overline{x}k^T - \overline{x}(k^Tx)x^H\right] - \frac{\overline{z_{A_j}}}{P^2}(I - \overline{x}x^T)kk^T(I - xx^H),$$

$\Delta\mathbf{L}(c,s)$ is symmetric, and $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$. With

$$\chi := \sigma_{\max}\left(\begin{bmatrix}d_{j,j}\\ d_j\end{bmatrix}\right) = |z_{A_j}|\sqrt{|x^Tk|^2 + \|U_1^Tk\|^2} = \frac{|\nabla_j H_{w^{-1},2}|}{H_{w^{-1},2}}\|k\|_2,$$

then by Corollary 2.8 we have $\chi = \|\Delta A_j\|_2$, and since by Proposition 2.4, $\sum_{j=0}^{m} w_j^2|\nabla_j H_{w^{-1},2}|^2 = 1$, it follows that

$$\eta_{w,2}^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_{w,2} = \frac{\|k\|_2}{H_{w^{-1},2}}. \quad \square$$

Note that in the construction of a minimal spectral norm backward error we have infinitely many choices of an appropriate completion Z for which $\|Z\|_2 \le 1$, but here and in the following we always take $Z = 0$ to simplify the formulas.

Remark 3.5 If $w_j = 0$ for $j = 0, \ldots, m$, then $z_{A_j} = \dfrac{\nabla_j H_{w^{-1},2}(\lambda,\mu)}{H_{w^{-1},2}(\lambda,\mu)} = 0$ and hence by Theorem 3.4 we have that $\Delta A_j = 0$, $j = 0, \ldots, m$. This shows that $w_j = 0$ implies that A_j remains unperturbed.

We then have the following relations between structured and unstructured backward errors.

Corollary 3.6 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a regular, symmetric matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$. Then,*

$$\begin{aligned} \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) &\leq \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}) \\ \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) &= \eta_{w,2}(\lambda,\mu,x,\mathbf{L}). \end{aligned}$$

Proof. By Theorem 3.4 with $k := -\mathbf{L}(\lambda,\mu)x$, we have that

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \frac{\|k\|_2}{H_{w^{-1},2}}, \text{ and } \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \frac{\sqrt{2\|k\|_2^2 - |x^T k|^2}}{H_{w^{-1},2}}$$

and from (2) we have that $\eta_{w,2}(\lambda,\mu,x,\mathbf{L}) = \dfrac{\|k\|_2}{H_{w^{-1},2}}$. Thus the assertion follows. □

As a corollary we obtain the result for the case of non-homogeneous matrix polynomials that have no infinite eigenvalues of [1, 2].

Corollary 3.7 *Let $\mathbf{L}(s) = \sum_{j=0}^m s^j A_j \in \mathbb{C}^{n\times n}$ be complex symmetric and have only finite eigenvalues, let $\mu \in \mathbb{C}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and let $k := -\mathbf{L}(\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) = \frac{\sqrt{2\|k\|_2^2 - |x^T k|^2}}{\|\Lambda\|_2},$$

and there exists a unique complex symmetric polynomial $\Delta\mathbf{L}(s) := \sum_{j=0}^m s^j \Delta A_j$, with coefficients

$$\Delta A_j := \frac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[\overline{x}k^T + kx^H - (x^T k)\overline{x}x^H\right], \ j = 0, 1, \ldots, m$$

such that the structured backward error satisfies $\eta^{\mathbf{S}}_F(\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_F$ and $\overline{x}$, x are left and right eigenvectors corresponding to the eigenvalue μ of $\mathbf{L} + \Delta\mathbf{L}$, respectively.

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_2(\mu,x,\mathbf{L}) = \frac{\|k\|_2}{\|\Lambda\|_2} = \eta_2(\mu,x,\mathbf{L})$$

and there exist a complex symmetric polynomial $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta A_j \in \mathbb{C}^{n\times n}$ with coefficients

$$\Delta A_j := \frac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[\overline{x}k^T + kx^H - (k^T x)\overline{x}x^H - \frac{\overline{x^T k}(I - \overline{x}x^T)kk^T(I - xx^H)}{\|k\|_2^2 - |x^T k|^2}\right].$$

such that $\|\Delta\mathbf{L}\|_2 = \eta^{\mathbf{S}}_2(\mu,x,\mathbf{L})$, and $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$.

Proof. The proof follows from Theorem 3.4, by using $c = 1$, $w = [1, 1, \ldots, 1]^T$, $z_{A_j} = \dfrac{\nabla_j H_{w^{-1},2}}{H_{w^{-1},2}} = \dfrac{\mu^j}{H^2_{w^{-1},2}}$ and $H^2_{w^{-1},2} = \|\Lambda\|_2^2$. □

We also obtain as a corollary the result for homogeneous matrix pencils $\mathbf{L}(c, s) = cA + sB \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ and in the special case, i.e., for $c = 1$, we obtain results given in Theorems 3.1, and 3.2 of [3].

Corollary 3.8 *Let $\mathbf{L}(c, s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a complex symmetric matrix pencil, let $(\lambda, \mu) \in \mathbb{C}^2\backslash\{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, let $k := -\mathbf{L}(\lambda, \mu)x$, and $w := [1, 1]^T$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \frac{\sqrt{2\|k\|_2^2 - |x^T k|^2}}{\|[\lambda, \mu]^T\|_2}.$$

and there exists a unique complex symmetric pencil $\Delta\mathbf{L}(c, s) := c\Delta A_0 + s\Delta A_1$, with coefficients

$$\begin{aligned} \Delta A_0 &:= \overline{z_{A_0}}\left[\overline{x}k^T + kx^H - (x^T k)\overline{x}x^H\right], \\ \Delta A_1 &:= \overline{z_{A_1}}\left[\overline{x}k^T + kx^H - (x^T k)\overline{x}x^H\right], \end{aligned}$$

such that $\eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \|\Delta\mathbf{L}\|_F$ and $\overline{x}, x$ are left and right eigenvectors corresponding to an eigenvalue (λ, μ) of $\mathbf{L} + \Delta\mathbf{L}$, respectively.

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \frac{\|k\|_2}{\|[\lambda, \mu]^T\|_2}$$

and there exists a complex symmetric pencil $\Delta\mathbf{L}(c, s) = c\Delta A + s\Delta B$, with coefficients

$$\begin{aligned} \Delta A_0 &:= \overline{z_{A_0}}\left[\overline{x}k^T + kx^H - (k^T x)\overline{x}x^H - \frac{\overline{x^T k}(I - \overline{x}x^T)kk^T(I - xx^H)}{\|k\|_2^2 - |x^T k|^2}\right], \\ \Delta A_1 &:= \overline{z_{A_1}}\left[\overline{x}k^T + kx^H - (k^T x)\overline{x}x^H - \frac{\overline{x^T k}(I - \overline{x}x^T)kk^T(I - xx^H)}{\|k\|_2^2 - |x^T k|^2}\right], \end{aligned}$$

such that $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_2 = \eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

In an analogous way we can derive the results for complex skew-symmetric matrix polynomials.

Theorem 3.9 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex skew-symmetric matrix polynomial of the form (1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ such that $x^H x = 1$ and let $k := -\mathbf{L}(\lambda, \mu)x$. Introduce the perturbation matrices*

$$\Delta A_j := -\overline{z_{A_j}}\left[\overline{x}k^T - kx^H\right], \ j = 0, 1, 2, \ldots, m.$$

Then the matrix polynomial $\Delta\mathbf{L}(c, s) = \displaystyle\sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$, is complex skew-symmetric and $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$.

Proof. By construction $\Delta\mathbf{L}$ is complex skew-symmetric and by Proposition 2.4, we have $\sum_{j=0}^{m} \lambda^{m-j}\mu^j\overline{z_{A_j}} = 1$. Thus, we have

$$\begin{aligned}
&(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x \\
&\quad = -k + \Delta\mathbf{L}(\lambda,\mu)x = -k + \sum_{j=0}^{m} \lambda^{m-j}\mu^j\overline{z_{A_j}}\left[\overline{x}k^T - kx^H\right]x \\
&\quad = -k + \overline{x}k^T x + k = 0,
\end{aligned}$$

as $\overline{x}k^T x = 0$, since the polynomial has skew-symmetric coefficients. □

Theorem 3.10 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex skew-symmetric matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and let $k := -\mathbf{L}(\lambda,\mu)x$. The structured backward errors with respect to the Frobenius norm and spectral norm are given by*

$$\begin{aligned}
\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) &= \frac{\sqrt{2\|k\|_2^2}}{H_{w^{-1},2}} \\
\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) &= \frac{\|k\|_2}{H_{w^{-1},2}},
\end{aligned}$$

respectively. Introducing the perturbation matrices

$$\Delta A_j = -\overline{z_{A_j}}\left[\overline{x}k^T - kx^H\right], j = 0,1,\ldots,m,$$

then $\Delta\mathbf{L}(c,s) := \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$ is skew-symmetric, $(\Delta\mathbf{L}(\lambda,\mu)+\mathbf{L}(\lambda,\mu))x = 0$, $\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_{w,F}$ and $\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_{w,2}$.

Proof. By Theorem 3.9 we have $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ and hence we have that $k = \Delta\mathbf{L}(\lambda,\mu)x$. Choose a unitary matrix $U = [x, U_1]$, $U_1 \in \mathbb{C}^{n\times n-1}$ and let $\widetilde{\Delta A_j} := U^T\Delta A_j U = \begin{bmatrix} 0 & d_j^T \\ -d_j & \Delta D_{j,j} \end{bmatrix}$, where

$$\Delta D_{j,j} = -\Delta D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}.$$

Then

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H = \overline{U}U^T(\Delta\mathbf{L}(\lambda,\mu))U^H U = \Delta\mathbf{L}(\lambda,\mu),$$

and hence

$$\overline{U}\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H x = \Delta\mathbf{L}(\lambda,\mu)x = k,$$

which implies that

$$\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^H x = U^T k = \begin{bmatrix} x^T k \\ U_1^T k \end{bmatrix}.$$

Since $U^H x = e_1$, it follows that $x^T k = 0$ and

$$U_1^T k = -\sum_{j=0}^{m} \lambda^{m-j}\mu^j d_j = \sum_{j=0}^{m} w_j\lambda^{m-j}\mu^j\frac{d_j}{w_j}.$$

To minimize the perturbation we solve the system for the parameters $d_{j,j}, d_j$ in a least square sense, and obtain $x^T k = 0$ and

$$\begin{bmatrix} w_0 d_0 \\ w_1 d_1 \\ \vdots \\ w_m d_m \end{bmatrix} = - \begin{bmatrix} \overline{z_{A_0}} \\ \overline{z_{A_1}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} U_1^T k,$$

where $H_{w,2} = \| \left[\lambda^m \mu^0, \lambda^{m-1}\mu, \ldots, \lambda^0 \mu^m\right]^T \|_{w,2}$. This yields $d_{j,j} = 0, d_j = -\overline{z_{A_j}} U_1^T k$ and then

$$\widetilde{\Delta A_j} = \begin{bmatrix} 0 & -\left(\overline{z_{A_j}} U_1^T k\right)^T \\ \overline{z_{A_j}} U_1^T k & \Delta D_{j,j} \end{bmatrix}.$$

The Frobenius norm can be minimized by taking $\Delta D_{j,j} = 0$ and then we have

$$\|\Delta A_j\|_F^2 = 2\|d_j\|_2^2 = 2|z_{A_j}|^2 \|U_1^T k\|^2 = |\nabla_j H_{w^{-1},2}|^2 \frac{2\|k\|_2^2}{H_{w^{-1},2}^2},$$

since $\|k\|_2^2 = \|U^T k\|_2^2 = |x^T k|^2 + \|U_1^T k\|_2^2 = \|U_1^T k\|_2^2$. Also by Proposition 2.4, we have that $\sum_{j=0}^m w_j^2 |\nabla_j H_{w^{-1},2}|^2 = 1$. Thus we have $\|\!|\Delta \mathbf{L}|\!\|_{w,F} = \dfrac{\sqrt{2\|k\|_2^2}}{H_{w^{-1},2}}$, and

$$\begin{aligned} \Delta A_j &= \overline{U}\widetilde{\Delta A} U^H = \begin{bmatrix} \overline{x} & \overline{U_1} \end{bmatrix} \begin{bmatrix} 0 & d_j^T \\ -d_j & \Delta D_{j,j} \end{bmatrix} \begin{bmatrix} x^H \\ U_1^H \end{bmatrix} \\ &= -\overline{U_1} d_j x^H + \overline{x} d_j^T U_1^H + \overline{U_1} D_{j,j} U_1^H \\ &= \overline{U_1}\overline{z_{A_j}} U_1^T k x^H - \overline{x}(\overline{z_{A_j}} U_1^T k)^T U_1^H + \overline{U_1}\Delta D_{j,j} U_1^H \\ &= \overline{z_{A_j}} \left[\overline{U_1} U_1^T k x^H - \overline{x} k^T U_1 U_1^H)\right] + \overline{U_1}\Delta D_{j,j} U_1^H \\ &= \overline{z_{A_j}}[(I - \overline{x}x^T) k x^H - \overline{x} k^T (I - x x^H))]. \end{aligned} \tag{5}$$

Therefore

$$\Delta A_j = \overline{z_{A_j}}[k x^H - \overline{x} k^T]$$

is complex skew-symmetric and we have that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

To minimize the spectral norm we make use of Corollary 2.8 and obtain

$$\begin{aligned} \Delta D_{j,j} &= -\frac{\overline{z_{A_j}}}{P^2}[\overline{x^T k}(U_1^T k)(U_1^T k)^T] \\ &+ \left[I - \frac{(U_1^T k)(U_1^T k)^H}{P^2}\right] Z \left[I - \frac{\overline{U_1^T k}(U_1^T k)^T}{P^2}\right], \end{aligned}$$

where $Z = -Z^T$ with $\|Z\|_2 \le 1$, and $P^2 = \|k\|_2^2 - |x^T k|^2$. Choosing $Z = 0$, we get

$$\Delta D_{j,j} = -\frac{\overline{z_{A_j}}}{P^2}[\overline{x^T k}(U_1^T k)(U_1^T k)^T]$$

and using (5), we get

$$\overline{U_1}\Delta D_{j,j} U_1^H = -\frac{\overline{z_{A_j}}}{P^2}\overline{x^T k}\,\overline{U_1} U_1^T k k^T U_1 U_1^H = -\frac{\overline{z_{A_j}}}{P^2}\overline{x^T k}(I - \overline{x}x^T) k k^T (I - x x^H),$$

and hence

$$\Delta A_j = \overline{z_{A_j}}\left[-kx^H + \overline{x}k^T - 2\overline{x}(k^Tx)x^H\right] - \frac{\overline{z_{A_j}}}{P^2}\overline{x^Tk}(I-\overline{x}x^T)kk^T(I-xx^H).$$

The skew-symmetry of A_j implies that $x^Tk = 0$ and thus $\Delta A_j = \overline{z_{A_j}}\left[kx^H - \overline{x}k^T\right]$ is complex skew-symmetric. Then $\Delta\mathbf{L}(c,s)$ is complex skew-symmetric and $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$ with $\chi_{\Delta A_j} = |\overline{z_{A_j}}|\|U_1^Tk\|_2$.

Then by Corollary 2.8 we obtain

$$\|\Delta A_j\|_2 = |z_{A_j}|\|U_1^Tk\|_2 = |z_{A_j}|\sqrt{\|k\|_2^2 - |x^Tk|^2} = |z_{A_j}|\|k\|_2$$

and hence $\eta_{w,2}^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_{w,2}$. □

As a direct Corollary of Theorem 3.10 we have the following relation between structured and unstructured backward errors of an approximate eigenpair.

Corollary 3.11 *Let $\mathbf{L}\in\mathbf{L}_m(\mathbb{C}^{n\times n})$ be a skew-symmetric matrix polynomial of the form (1), let $(\lambda,\mu)\in\mathbb{C}^2\setminus\{(0,0)\}$, let $x\in\mathbb{C}^n$ be such that $x^Hx = 1$, and set $k := -\mathbf{L}(\lambda,\mu)x$. Then the structured and unstructured backward errors are related via*

$$\begin{aligned}\eta_{w,F}^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) &= \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}),\\ \eta_{w,2}^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) &= \eta_{w,2}(\lambda,\mu,x,\mathbf{L}).\end{aligned}$$

As further corollary we obtain Theorem 4.3.4 of [1], see also [2] for non-homogeneous matrix polynomials with no infinite eigenvalues.

Corollary 3.12 *Let $\mathbf{L}(s) = \sum_{j=0}^m s^jA_j\in\mathbb{C}^{n\times n}$ be skew-symmetric, let $\mu\in\mathbb{C}$, let $x\in\mathbb{C}^n$ be such that $x^Hx = 1$ and let $k := -\mathbf{L}(\mu)x$. Then the structured backward errors with respect to the Frobenius and spectral norm are given by*

$$\begin{aligned}\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L}) &= \sqrt{2}\eta_2(\mu,x,\mathbf{L}),\\ \eta_2^{\mathbf{S}}(\mu,x,\mathbf{L}) &= \eta_2(\mu,x,\mathbf{L}),\end{aligned}$$

respectively.

Introducing the perturbation matrices

$$\Delta A_j = -\frac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[\overline{x}k^T - kx^H\right],\ j = 0,1,\ldots,m,$$

then $\Delta\mathbf{L}(\lambda) := \sum_{j=0}^m\lambda^j\Delta A_j$ is complex skew-symmetric, $(\Delta\mathbf{L}(\mu)+\mathbf{L}(\mu))x = 0$, $\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_F$, and $\eta_2^{\mathbf{S}}(\mu,x,\mathbf{L}) = \|\!|\Delta\mathbf{L}|\!\|_2$.

Proof. With $w := [1,\ldots,1]^T$, $c = 1$, and $H_{w^{-1},2}^2 = \|\Lambda\|_2^2$ the results follows directly from Theorem 3.10. □

For matrix pencils $\mathbf{L}(c,s) = cA_0 + sA_1\in\mathbf{L}_1(\mathbb{C}^{n\times n})$, Theorem 3.10 in the special case $c = 1$, also implies the results given in Theorem 3.3 and Theorem 4.2 of [3].

Example	**S**	$\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_2(\lambda,\mu,x,\mathbf{L})$
1	symmetric	0.7071	0.8660	0.7071
2	skew-symmetric	2	2.8284	2

Table 2: Structured and unstructured backward errors for Examples 3.14 and 3.15.

Corollary 3.13 *Let $\mathbf{L}(c,s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a skew-symmetric matrix pencil, let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and let $k := -\mathbf{L}(\lambda,\mu)x$. Then the structured and unstructured backward errors are related via*

$$\begin{aligned} \eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) &= \sqrt{2}\eta_2(\lambda,\mu,x,\mathbf{L}), \\ \eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) &= \eta_2(\lambda,\mu,x,\mathbf{L}), \end{aligned}$$

respectively.

Introducing the perturbation matrices

$$\Delta A_0 = -\overline{z_{A_0}}\left[\overline{x}k^T - kx^H\right] \text{ and} \Delta A_1 = -\overline{z_{A_1}}\left[\overline{x}k^T - kx^H\right],$$

then $\Delta\mathbf{L}(c,s) := c\Delta A_0 + s\Delta A_1$ is skew-symmetric, $(\Delta\mathbf{L}(\lambda,\mu)+\mathbf{L}(\lambda,\mu))x = 0$, $\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_F$, and $\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_2$.

To illustrate our results, in the following we present some examples.

Example 3.14 Consider the complex symmetric pencil $\mathbf{L} \in \mathbf{L}_1(\mathbb{C}^{2\times 2})$ with coefficients $A_0 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and take $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$, $(\lambda,\mu) = (0,1)$.

For the Frobenius norm we obtain the coefficients of the perturbation pencil $\Delta\mathbf{L}$ as $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\Delta A_1 := \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$. Then $(0,1)$ is an eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$ and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = 0.8660$.

For the spectral norm we obtain $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $\Delta A_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$. Again $(0,1)$ is an eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$ and $\|\Delta\mathbf{L}\|_2 = \eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = 0.7071$, see also Table 4.

Example 3.15 Consider the complex skew-symmetric pencil $\mathbf{L} \in \mathbf{L}_1(\mathbb{C}^{2\times 2})$ with coefficients $A_0 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, A_1 := \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ and take $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$, $(\lambda,\mu) = (0,1)$.

For the Frobenius norm and spectral norm the coefficients or the perturbation pencil are $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Delta A_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $(0,1)$ is an eigenvalue of $\mathbf{L} + \Delta\mathbf{L}$, and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = 2.8284$, while for the spectral norm we obtain $\|\Delta\mathbf{L}\|_2 = \eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = 2$, see also Table 4.

4 Backward errors for complex T-odd and T-even matrix polynomials

In this section we derive backward error formulas for homogeneous T-odd and T-even matrix polynomials. Throughout this section we assume that the coefficient matrix A_0 is in the even

position, i.e., it is symmetric for a T-even and skew-symmetric for a T-odd matrix polynomial, the other case can be treated analogously via a multiplication with the imaginary unit $\imath$.

For a given nonnegative vector w, an eigenvalue (λ, μ) and the partial gradients as introduced in Propositions 2.1–2.4, we use the following abbreviations.

$$z_{A_j} := \frac{\nabla_j H_{w^{-1},2}(\lambda,\mu)}{H_{w^{-1},2}(\lambda,\mu)},\ n_{A_j} := \frac{\nabla_j N_{w^{-1},2}(\lambda,\mu)}{N_{w^{-1},2}(\lambda,\mu)}, k_{A_j} := \frac{\nabla_j K_{w^{-1},2}(\lambda,\mu)}{K_{w^{-1},2}(\lambda,\mu)}.$$

We then have the following backward errors.

Theorem 4.1 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a complex T-even or T-odd matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\lambda,\mu)x$. For $j = 0, 1, 2, \ldots, m$, and different cases, we introduce the following perturbation matrices.*

- *In the case that m is even and $\lambda \neq 0$, or when $m > 1$ is odd then let for T-even matrix polynomials*

$$\Delta A_j := \begin{cases} \overline{k_{A_j}}(x^T k)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j, \end{cases};$$

  so that the perturbation preserves the structure,

- *in the case that $m > 1$ is even and both $\lambda \neq 0, \mu \neq 0$, or in the case that m is odd and $\mu \neq 0$, let for T-odd matrix polynomials*

$$\Delta A_j := \begin{cases} \overline{n_{A_j}}(x^T k)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j, \end{cases};$$

  so that the perturbation again preserves the structure,

- *in the case that $\lambda \neq 0, \mu \neq 0$ consider perturbation matrices*

$$\Delta A_j := \begin{cases} -\overline{x}x^T A_j x x^H + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for symm. coeff.}, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for skew-symm. coeff.}, \end{cases}$$

  then there exists a matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j \in \mathbb{C}^{n\times n}$ that is appropriately structure preserving T-odd or T-even and $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$.

Proof. Let $\Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be of the form $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$. Then by the construction it is easy to see that $\Delta\mathbf{L}$ is either T-even or T-odd and it remains to show that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$. We begin with a T-odd polynomial $\mathbf{L}$. In both cases that m is even or odd, we have

$$\begin{aligned}(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x &= \sum_{j=0}^{m} \lambda^{m-j}\mu^j(A_j + \Delta A_j)x \\ &= \left(\sum_{j=0,j \text{ even}}^{m} \lambda^{m-j}\mu^j A_j\right)x - \left[-k + \overline{x}x^T k\right] \sum_{j=0,\, j \text{ even}}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}} \\ &+ \sum_{j=1,\, j \text{ odd}}^{m-1} \lambda^{m-j}\mu^j A_j x + \left[(x^T k)\overline{x} + \sum_{j=1,\, j \text{ odd}}^{m-1} \lambda^{m-j}\mu^j \overline{z_{A_j}}(I - \overline{x}x^T)k\right]\end{aligned}$$

$$= -k + \big[\sum_{j=0, j \text{ even}}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}} + \sum_{j=1,\, j \text{ odd}}^{m-1} \lambda^{m-j}\mu^j \overline{z_{A_j}} \big](I - \overline{x}x^T)k + x^T k\overline{x}$$
$$= -k + k - \overline{x}x^T k + x^T k\overline{x} = 0,$$

since by Proposition 2.4 we have that

$$\sum_{j=0, j \text{ even}}^{m} \lambda^{m-j}\mu^j \overline{z_{A_j}} + \sum_{j=1,\, j \text{ odd}}^{m-1} \lambda^{m-j}\mu^j \overline{z_{A_j}} = 1.$$

The proof for T-even polynomials is analogous. □

In the special case of linear matrix polynomials, i.e., for $m = 1$, we have the following expressions. For even pencils we have

$$\begin{aligned} \Delta A_0 &:= -|\text{sign}(\mu)|^2 \overline{x}x^T A_0 xx^H + \overline{z_{A_0}} \left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] \\ \Delta A_1 &:= -\overline{z_{A_1}} \left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \end{aligned}$$

and for odd pencils we have

$$\begin{aligned} \Delta A_1 &:= -|\text{sign}(\lambda)|^2 \overline{x}x^T A_1 xx^H + \overline{z_{A_1}} \left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \\ \Delta A_0 &:= -\overline{z_{A_0}} \left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \end{aligned}$$

where $|\text{sign}(z)| = 1$, if $z \neq 0$ and $|\text{sign}(z)| = 0$, for $z = 0$.

As a corollary we obtain the results for the case of non-homogeneous matrix polynomials with no infinite eigenvalues of Theorem 4.2.1 in [1], see also [2]. This case follows by setting $c = 1, \mathbf{L}(s) := \mathbf{L}(1, s), \Lambda := [1, \mu, \ldots, \mu^m]^T$ and $w := [1, 1, \ldots, 1]^T$.

The minimal backward errors for complex T-even polynomials and $m > 1$ are as follows.

Theorem 4.2 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form (1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\lambda, \mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^T k|^2}{K^2_{w^{-1},2}} + 2\dfrac{\|k\|_2^2 - |x^T k|^2}{H^2_{w^{-1},2}}} & \text{if } m \text{ is even or if} \\ & \mu \neq 0 \text{ and } m \text{ is odd,} \\ \sqrt{\dfrac{2\|k\|_2^2 - |x^T k|^2}{H^2_{w^{-1},2}}} & \text{if } \lambda = 0 \text{ and } m \text{ is even,} \\ \sqrt{\dfrac{2\|k\|_2^2 - |x^T k|^2}{H^2_{w^{-1},2}}} & \text{if } \mu = 0 \text{ and } m \text{ is odd.} \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^T k|^2}{K^2_{w^{-1},2}} + \dfrac{\|k\|_2^2 - |x^T k|^2}{H^2_{w^{-1},2}}} & \text{if } m \text{ is even or} \\ & \mu \neq 0 \text{ and } m \text{ is odd,} \\ \dfrac{\|k\|_2}{H_{w^{-1},2}} & \text{if } \lambda = 0, \text{ and } m \text{ is even,} \\ \dfrac{\|k\|_2}{H_{w^{-1},2}} & \text{if } \mu = 0 \text{ and } m \text{ is odd.} \end{cases}$$

When m is even, or when m is odd and $\lambda \neq 0$, introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \overline{k_{A_j}}(x^T k)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for even } j, \\ -\overline{z_{A_j}}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for odd } j, \end{cases}.$$

Then $\Delta\mathbf{L}(c,s) = \sum_{j=0}^{m} c^{m-j}s^j \Delta A_j$ is the unique T-even matrix polynomial satisfying $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$, and $\|\!|\Delta\mathbf{L}|\!\|_{w,F} = \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L})$. Similarly, for the spectral norm, when m is even or when m is odd and $\lambda \neq 0$, introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \Delta A_j - \dfrac{\overline{k_{A_j}}\,\overline{x^T k}(I - xx^H)kk^T(I - \overline{x}x^T)}{\|k\|^2 - |x^T k|^2} & \text{for even } j, \\ \Delta A_j & \text{for odd } j, \end{cases}.$$

Then the matrix polynomial $\Delta\mathbf{L}(c,s) = \displaystyle\sum_{j=0}^{m} c^{m-j}s^j\Delta A_j$ is T-even and satisfies $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ as well as $\|\!|\Delta\mathbf{L}|\!\|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$.

Proof. Theorem 4.1 implies that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ and hence $k = \Delta\mathbf{L}(\lambda,\mu)x$. Now choose a unitary matrix $U = [x, U_1]$, $U_1 \in \mathbb{C}^{n\times n-1}$ and let $\widetilde{\Delta A_j} := U^T\Delta A_j U = \begin{bmatrix} d_{j,j} & d_j^T \\ d_j & \Delta D_{j,j} \end{bmatrix}$, where $\Delta D_{j,j} = \Delta D_{j,j}^T \in \mathbb{C}^{(n-1)\times(n-1)}$, when j is even and $\Delta A_j = \overline{U}\begin{bmatrix} 0 & b_j^T \\ -b_j & \Delta B_{j,j} \end{bmatrix} U^H$, $\Delta B_{j,j}^T = -\Delta B_{j,j}$, when j is odd. Then, since $U\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^T = UU^T(\Delta\mathbf{L}(\lambda,\mu))U^TU = \Delta\mathbf{L}(\lambda,\mu)$, it follows that $U\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Tx = \Delta\mathbf{L}(\lambda,\mu)x = k$, and hence $\widetilde{\Delta\mathbf{L}(\lambda,\mu)}U^Tx = U^Tk = \begin{bmatrix} x^Tk \\ U_1^Tk \end{bmatrix}$. Using

$$\begin{bmatrix} \displaystyle\sum_{j=0}^{m} w_j d_{j,j}\frac{\lambda^{m-j}\mu^j}{w_j} \\ \displaystyle\sum_{j=0,\ j \text{ even}}^{m} w_j\lambda^{m-j}\mu^j\frac{d_j}{w_j} - \sum_{j=1,j \text{ odd}}^{m} w_j\lambda^{m-j}\mu^j\frac{b_j}{w_j} \end{bmatrix} = \begin{bmatrix} x^Tk \\ U_1^Tk \end{bmatrix}, \tag{6}$$

to minimize the perturbation, we solve this system for the parameters $d_{j,j}, d_j$ in a least square sense, and we obtain

$$\begin{bmatrix} w_0a_{0,0} \\ w_2a_{2,2} \\ \vdots \\ w_ma_{m,m} \end{bmatrix} = \begin{bmatrix} \overline{z_{A_m}} \\ \overline{z_{A_2}} \\ \vdots \\ \overline{z_{A_m}} \end{bmatrix} x^Tk.$$

Then $d_{j,j} = \overline{k_{A_j}}x^Tk$, $d_j = \overline{z_{A_j}}U_1^Tk$ for even j and $b_j = \overline{z_{A_j}}U_1^Tk$ for odd j and we obtain

$$\Delta A_j := \begin{cases} \overline{U}\begin{bmatrix} \overline{k_{A_j}}x^Tk & \left(\overline{z_{A_j}}U_1^Tk\right)^T \\ \overline{z_{A_j}}U_1^Tk & \Delta D_{j,j} \end{bmatrix} U^H & \text{for even } j, \\ \overline{U}\begin{bmatrix} 0 & -\left(\overline{z_{A_j}}U_1^Tk\right)^T \\ \overline{z_{A_j}}U_1^Tk & \Delta B_{j,j} \end{bmatrix} U^H & \text{for odd } j, \end{cases}.$$

This implies that

$$\Delta A_j = -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] + \overline{U_1}\Delta D_{j,j}U_1^H, \tag{7}$$

when j is odd. For even j, we get

$$\begin{aligned}\Delta A_j &= \begin{bmatrix}\overline{x} & \overline{U_1}\end{bmatrix}\begin{bmatrix}\overline{k_{A_j}}x^Tk & \left(\overline{z_{A_j}}U_1^Tk\right)^T \\ \overline{z_{A_j}}U_1^Tk & \Delta D_{j,j}\end{bmatrix}\begin{bmatrix}x^H \\ U_1^H\end{bmatrix} \\ &= \overline{k_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[\overline{U_1}(U_1^T)kx^H + \overline{x}k^TU_1U_1^H\right] + \overline{U_1}\Delta D_{j,j}U_1^H.\end{aligned}$$

Thus

$$\Delta A_j = \overline{k_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] + \overline{U_1}\Delta D_{j,j}U_1^H. \tag{8}$$

The Frobenius norm can be minimized by taking $\Delta A_{j,j} = 0$, so we obtain

$$\Delta A_j := \begin{cases} \overline{k_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for even } j, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j. \end{cases}$$

Since the Frobenius norm is unitarily invariant, it follows that for even j we have

$$\begin{aligned}\|\Delta A_j\|_F &= \sqrt{|a_{j,j}|^2 + 2\|a_j\|_2^2} = \sqrt{|k_{A_j}|^2|x^Tk|^2 + 2|z_{A_j}|^2\|U_1^Tk\|_2^2} \\ &= \sqrt{\frac{|\nabla_j K_{w^{-1},2}|^2|x^Tk|^2}{K_{w^{-1},2}^2} + 2\frac{|\nabla_j H_{w^{-1},2}|^2\|U_1^Tk\|_2^2}{H_{w^{-1},2}^2}}.\end{aligned}$$

Similarly for odd j, we have $\|\Delta A_j\|_F = \sqrt{2}|z_{A_j}|\|U_1^Tk\|_2$. Furthermore, by Proposition 2.4, we have $\sum_{j=\text{even}}^{m} w_j^2|\nabla_j K_{w^{-1},2}|^2 = 1$ and $\sum_{j=0}^{m} w_j^2|\nabla_j H_{w^{-1},2}|^2 = 1$ when m is even. Then it follows that

$$\begin{aligned}|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,F} &= \sqrt{\sum_{j=0}^{m} w_j^2\|\Delta A_j\|_F^2} = \sqrt{\frac{|x^Tk|^2}{K_{w^{-1},2}^2} + \frac{2\|U_1^Tk\|_2^2}{H_{w^{-1},2}^2}} \\ &= \sqrt{\frac{|x^Tk|^2}{K_{w^{-1},2}^2} + \frac{2(\|k\|_2^2 - |x^Tk|^2)}{H_{w^{-1},2}^2}}.\end{aligned}$$

For the spectral norm, we have from (7) and (8) that

$$\Delta A_j := \begin{cases} \overline{k_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] + S_j & \text{for even } j, \\ -\overline{z_{A_j}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right] & \text{for odd } j, \end{cases};$$

where $S_j := \overline{U_1}\Delta D_{j,j}U_1^H = \dfrac{\overline{z_{A_j}}}{P^2}\overline{x^Tk}(I-\overline{x}x^T)kk^T(I-xx^H)$, and $P^2 = \|k\|_2^2 - |x^Tk|^2$. Now let

$$\chi_{\Delta A_j} := \begin{cases} \sqrt{|k_{A_j}|^2|x^Tk|^2 + |z_{A_j}|^2(\|k\|^2 - |x^Tk|^2)} & \text{for even } j, \\ \sqrt{|z_{A_j}|^2(\|k\|_2^2 - |x^Tk|^2)} & \text{for odd } j, \end{cases}.$$

Hence by Corollary 2.8 it follows that $\|A\|_2 = \chi_{\Delta A_j}$. Then

$$|||\Delta\mathbf{L}|||_{w,2} = \sqrt{\sum_{j=0}^{m} w_j^2 \|\Delta A\|_2^2} = \sqrt{\frac{|x^T k|^2}{K_{w^{-1},2}^2} + \frac{\|k\|^2 - |x^T k|^2}{H_{w^{-1},2}^2}}$$

and

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{\frac{|x^T k|^2}{K_{w^{-1},2}^2} + \frac{\|k\|^2 - |x^T k|^2}{H_{w^{-1},2}^2}}. \quad \square$$

We obtain the following relations between the structured and unstructured backward errors.

Corollary 4.3 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form (1), let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda, \mu)x$.*

1. *If $w := [1, 1, \ldots, 1]^T$, $|\lambda| = |\mu| = 1$ and if m is odd, then $H_{w^{-1},2}^2 = 2K_{w^{-1},2}^2$ and for the Frobenius norm we get*

$$\eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}).$$

   Similarly, for the spectral norm we have

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{H_{w^{-1},2}}.$$

2. *If m is even or if m is odd and $\lambda \neq 0$, then for the Frobenius and the spectral norm we have*

$$\begin{aligned} \eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) &\leq \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \\ \eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) &= \eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \end{aligned}$$

   respectively.

Proof. Consider the case that $|\lambda| = |\mu| = 1$, $w = [1, 1, \ldots, 1]^T$ and that m is odd. Then $H_{w^{-1},2}^2 = 2K_{w^{-1},2}^2$. Substituting these in Theorem 4.2 and then applying (2), we get for the Frobenius norm that

$$\eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L})$$

and for the spectral norm that

$$\eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \sqrt{\frac{\|k\|_2^2 + |x^T k|^2}{H_{w^{-1},2}^2}}.$$

If m is even and $\lambda = 0$, then we have $K_{w^{-1},2} = w_m^{-1}|\mu|^m$ and $H_{w^{-1},2} = w_m^{-1}|\mu|^m$ and hence

$$\begin{aligned} \eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) &\leq \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \\ \eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) &= \eta_{w,2}(\lambda, \mu, x, \mathbf{L}). \end{aligned}$$

Similarly, for $\mu = 0$ we have $K_{w^{-1},2} = w_0^{-1}|\lambda|^m$ and $H_{w^{-1},2} = w_0^{-1}|\lambda|^m$, and hence

$$\begin{aligned} \eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) &\leq \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}), \\ \eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) &= \eta_{w,2}(\lambda, \mu, x, \mathbf{L}). \end{aligned}$$

The assertion for the case that $\lambda \neq 0$ and m is odd follows analogously. ∎

As a corollary we obtain the results for non-homogeneous matrix polynomials with no infinite eigenvalues of [1, 2] using the notation $\Lambda_e := [1, \mu^2, \ldots, \mu^m]^T$ if m is even and $\Lambda_e := [1, \mu^2, \ldots, \mu^{m-1}]^T$ if m is odd.

Corollary 4.4 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form $\mathbf{L}(s) = \sum_{j=0}^m s^j A_j \in \mathbb{C}^{n\times n}$ that has only finite eigenvalues. Let $\mu \in \mathbb{C}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\mu, x, \mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^T k|_2^2}{\|\Lambda_e\|_2^2} + 2\dfrac{\|k\|_2^2 - |x^T k|^2}{\|\Lambda\|_2^2}} & \text{if } \mu \in \mathbb{C} \setminus \{0\}, \\ \sqrt{2\|k\|_2^2 - |x^T k|^2} & \text{if } \mu = 0. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^T k|^2}{\|\Lambda_e\|_2^2} + \dfrac{\|k\|_2^2 - |x^T k|^2}{\|\Lambda\|_2^2}} & \text{if } \mu \in \mathbb{C} \setminus \{0\}, \\ \eta_2(\mu, x, \mathbf{L}) & \text{if } \mu = 0. \end{cases}$$

In particular, if $|\mu| = 1$ and m is odd, then we have $\|\Lambda\|_2^2 = 2\|\Lambda_e\|_2^2$. Moreover, for the Frobenius norm $\eta_F^{\mathbf{S}}(\mu, x, \mathbf{L}) = \sqrt{2}\eta_2(\mu, x, \mathbf{L})$ and for the 2-norm we have $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \dfrac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{\|\Lambda\|_2}$.

If we introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \dfrac{\overline{\mu^j}(x^T k)(\overline{x}x^H)}{\|\Lambda_e\|_2^2} + \dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for even } j, \\ -\dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for odd } j, \end{cases}.$$

Then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta A_j$ is the unique T-even matrix polynomial that satisfies $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$, and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\mu, x, \mathbf{L})$ for the Frobenius norm.

For the spectral norm, we introduce

$$\Delta A_j := \begin{cases} \Delta A_j - \dfrac{\overline{\mu^j x^T k}(I - xx^H)kk^T(I - \overline{x}x^T)}{\|\Lambda_e\|_2^2(\|k\|^2 - |x^T k|^2)} & \text{for even } j, \\ \Delta A_j & \text{for odd } j, \end{cases}.$$

Then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta A_j$ a T-even matrix polynomial such that $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$ and $\|\Delta\mathbf{L}\|_2 = \eta_2^{\mathbf{S}}(\mu, x, \mathbf{L})$.

Proof. The proof follows from Theorem 4.2 using $w = [1, 1, \ldots, 1]^T$, $c = 1$ and using that $H_{w^{-1},2} := \|\Lambda\|_2, K_{w^{-1},2} := \|\Lambda_e\|_2$. ∎

Remark 4.5 Corollary 4.3 implies that for $|\mu| = 1$, and for the spectral norm we have that

$$\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{\|\Lambda\|_2},$$

while in Theorem 4.3.6 in [1] and Theorem 3.7 in [2] it is shown that $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \eta_2(\mu, x, \mathbf{L})$ when $w = [1, 1, \ldots, 1]^T$ and m is odd.

For complex T-even pencils we obtain the following result.

Corollary 4.6 *Let $\mathbf{L}(c, s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a T-even matrix pencil. Let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$, and set $k := -\mathbf{L}(\lambda, \mu)x$, $w := [1, 1]^T$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_0 x|^2 + 2\dfrac{\|k\|_2^2 - |\lambda|^2 |x^T A_0 x|^2}{\|[\lambda, \mu]^T\|_2^2}} & \\ = \sqrt{\dfrac{\left(\frac{|\mu|^2}{|\lambda|^2} - 1\right)|x^T k|^2 + 2\|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \lambda \neq 0, \\ \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) & \text{if } \mu = 0, \\ \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \\ \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L}) & \text{if } |\lambda| = 1, |\mu| = 1. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_0 x|^2 + \dfrac{\|k\|_2^2 - |\lambda|^2 |x^T A_0 x|^2}{\|[\lambda, \mu]^T\|_2^2}} & \\ = \sqrt{\dfrac{|\mu|^2 |x^T A_0 x|^2 + \|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \lambda \neq 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \mu = 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \\ \sqrt{\dfrac{|x^T A_0 x|^2 + \|k\|_2^2}{2}} & \text{if } |\lambda| = |\mu| = 1. \end{cases}$$

Defining the perturbation matrices

$$\begin{aligned} \Delta A_0 &:= -|\mathrm{sign}(\lambda)|^2 \overline{x} x^T A_0 x x^H + \overline{z_{A_0}} \left[(I - \overline{x}x^T)k x^H + \overline{x} k^T (I - x x^H)\right], \\ \Delta A_1 &:= -\overline{z_{A_1}} \left[-(I - \overline{x}x^T)k x^H + \overline{x} k^T (I - x x^H)\right], \end{aligned}$$

we have for the Frobenius norm that $\Delta\mathbf{L}(c, s) = c\Delta A_0 + s\Delta A_1$ is the unique T-even matrix polynomial that satisfies $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_{w,F} = \eta_{w,F}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

For the spectral norm we introduce the perturbation matrices

$$\begin{aligned} \Delta A_0 &:= \Delta A_0 - \frac{sign(\lambda^2)\overline{x^T A_0 x}(I - \overline{x}x^T)k k^T (I - x x^H)}{(\|k\|^2 - |x^T A_0 x|^2)}, \\ \Delta A_1 &:= -\overline{z_{A_1}} \left[-(I - \overline{x}x^T)k x^H + \overline{x} k^T (I - x x^H)\right], \end{aligned}$$

then $\Delta\mathbf{L}(c, s) = c\Delta A_0 + s\Delta A_1$ is T-even and satisfies $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_{w,2} = \eta_{w,2}^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

(λ,μ)	$\mathbf{S}$	$\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_2(\lambda,\mu,x,\mathbf{L})$
$(1,0)$	T-even	1.2247	1.6583	1.2247
$(0,1)$	T-even	1	1.414	1
$(2,1)$	T-even	1.0247	1.3601	1
$(4,3)$	T-even	0.9644	1.2689	0.9165
$(2i,i)$	T-even	1.0247	1.3601	1
$(2+3i,1+i)$	T-even	1.1255	1.5111	1.1106
$(1,2)$	T-even	0.9487	1.2450	0.8365
$(1,1)$	T-even	0.9354	1.2247	0.8660

Table 3: Computed structured and unstructured backward errors for Example 4.7

Proof. The proof follows as in Theorem 4.1, using $m = 1$ and $w := [1,1]^T$. ∎

It follows that for $\lambda = 0$ in the T-even case we have $\Delta A_0 = 0$ and

$$\Delta A_1 := -\overline{z_{A_1}}\left[-(I-\overline{x}x^T)kx^H + \overline{x}k^T(I-xx^H)\right].$$

These perturbations are the same for the spectral and the Frobenius norm. Furthermore, Corollary 4.6 shows that

$$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) \le \begin{cases} \sqrt{2}\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } |\mu|<|\lambda|, \\ \\ \|[\lambda,\mu]^T\|_2\,\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } |\mu|>|\lambda|, \end{cases}.$$

For a non-homogeneous pencil $\mathbf{L}(s) = A_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ we then have

$$\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L}) \le \begin{cases} \sqrt{2}\eta_2(\mu,x,\mathbf{L}) & \text{if } |\mu|<1, \\ \\ \|[1,\mu]^T\|_2\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } |\mu|>1, \end{cases};$$

which has been shown in Theorem 3.4 of [3] for the case that $\mu \neq 0$.

Example 4.7 Consider a T-even matrix pencil with coefficients $A_0 := \begin{bmatrix} 2 & 1 \\ 1 & \imath \end{bmatrix}$, $A_1 := \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}$, let $x = \begin{bmatrix} -\imath/\sqrt{2} \\ \imath/\sqrt{2} \end{bmatrix}$ and $(\lambda,\mu) = (1,0)$. Then we obtain the following perturbation matrices. For the Frobenius norm we have $\Delta A_0 = \begin{bmatrix} -1+0.25\imath & 0+0.25\imath \\ 0+0.25\imath & 1-0.75\imath \end{bmatrix}$, $\Delta A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_0+\Delta A_0 = \begin{bmatrix} 1+0.25\imath & 1+0.25\imath \\ 1+0.25\imath & 1+0.25\imath \end{bmatrix}$, $A_1+\Delta A_1 = \begin{bmatrix} 0 & -\imath \\ +\imath & 0 \end{bmatrix}$, and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$. For the spectral norm we obtain $\Delta A_0 = \begin{bmatrix} -1.2+0.10\imath & -0.20+0.10\imath \\ -0.20+0.10\imath & 0.80-0.90\imath \end{bmatrix}$, $\Delta A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $A_0+\Delta A_0 = \begin{bmatrix} 0.80+0.10\imath & 0.80+0.10\imath \\ 0.80+0.10\imath & 0.80+0.10\imath \end{bmatrix}$, $A_1+\Delta A_1 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}$, and $\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) = \|\Delta\mathbf{L}\|_2 = 1.2247$, see also Table 4.

In a similar way we can derive the results for T-odd matrix polynomials.

Theorem 4.8 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-odd matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\lambda,\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{N^2_{w^{-1},2}} + 2\dfrac{\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \mu \neq 0 \text{ and } m \text{ odd} \\ & \text{or if } \lambda \neq 0, \mu \neq 0 \text{ and } m \text{ even,} \\ \sqrt{\dfrac{2\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \lambda = 0 \text{ and } m \text{ odd.} \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \begin{cases} \sqrt{\dfrac{|x^Tk|^2}{N^2_{w^{-1},2}} + \dfrac{\|k\|_2^2 - |x^Tk|^2}{H^2_{w^{-1},2}}} & \text{if } \mu \neq 0 \text{ and } m \text{ odd,} \\ & \text{or } \lambda \neq 0, \mu \neq 0, \text{ and } m \text{ even,} \\ \dfrac{\|k\|_2}{H_{w^{-1},2}} & \text{if } \lambda = 0 \text{ and } m \text{ odd.} \end{cases}$$

For $\mu \neq 0$ and odd m or for $\lambda \neq 0$, $\mu \neq 0$ and m even, introduce the perturbation matrices

$$\Delta A_j := \begin{cases} \overline{n_{A_j}}(x^Tk)(\overline{x}x^H) + \overline{z_{A_j}}\left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for odd } j, \\ -\overline{z_{A_j}}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for even } j. \end{cases}$$

Then for the Frobenius norm we obtain the unique T-odd matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^m c^{m-j}s^j\Delta A_j$ such that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ and $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,F} = \eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L})$.

For $\mu \neq 0$ and odd m or for $\lambda \neq 0$, $\mu \neq 0$ and even m and the spectral norm consider the perturbation matrices

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\overline{n_{A_j}}\overline{x^Tk}(I - xx^H)kk^T(I - \overline{x}x^T)}{(\|k\|^2 - |x^Tk|^2)} & \text{for odd } j, \\ \Delta A_j & \text{for even } j, \end{cases}.$$

Then the T-odd matrix polynomial $\Delta\mathbf{L}(c,s) = \sum_{j=0}^m c^{m-j}s^j\Delta E_j$ satisfies $(\mathbf{L}(\lambda,\mu)+\Delta\mathbf{L}(\lambda,\mu))x = 0$ and $|\!|\!|\Delta\mathbf{L}|\!|\!|_{w,2} = \eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L})$.

Proof. The proof is analogous to that for T-even matrix polynomials. □

We then obtain the following relations between structured and unstructured backward errors of an approximate eigenpair.

Corollary 4.9 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-even matrix polynomial of the form (1), let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, let $x \in \mathbb{C}^n$ be such that $x^Hx = 1$, and set $k := -\mathbf{L}(\lambda,\mu)x$.*

1. *If $\lambda = 0$ and m is odd, then for the Frobenius norm we have*

$$\eta^{\mathbf{S}}_{w,F}(\lambda,\mu,x,\mathbf{L}) \leq \sqrt{2}\eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

2. *If $\lambda = 0$ and m is odd, then for the spectral norm we have*

$$\eta^{\mathbf{S}}_{w,2}(\lambda,\mu,x,\mathbf{L}) = \eta_{w,2}(\lambda,\mu,x,\mathbf{L}).$$

3. Let $w := [1, 1, \ldots, 1]^T$ and $|\lambda| = |\mu| = 1$ for odd m. Then for the Frobenius-norm

$$\eta^{\mathbf{S}}_{w,F}(\lambda, \mu, x, \mathbf{L}) = \sqrt{2}\eta_{w,2}(\lambda, \mu, x, \mathbf{L})$$

and for the spectral-norm we have

$$\eta^{\mathbf{S}}_{w,2}(\lambda, \mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{H_{w^{-1};2}}.$$

Proof. The proof follows from the fact that if $w := [1, 1, \ldots, 1]^T$ and $|\lambda| = |\mu| = 1$ and m is odd, then we have $H^2_{w^{-1},2} = 2N^2_{w^{-1},2}$ and then applying (2) the results follow. □

As a corollary we also obtain the results for the case of non-homogeneous matrix polynomials with no infinite eigenvalues of [1, 2]. By introducing the notation $\Lambda_o := [\mu, \mu^3, \ldots, \mu^{m-1}]^T$ when m is even and $\Lambda_o := [\mu, \mu^3, \ldots, \mu^m]^T$ when m is odd and by choosing the weight vector $w := [1, 1, \ldots, 1]^T$. we have the following result similar to Theorem 4.3.8, [1].

Corollary 4.10 *Let $\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ be a T-odd matrix polynomial of the form $\mathbf{L}(s) = \sum_{j=0}^m s^j A_j \in \mathbb{C}^{n\times n}$ with $\det(A_m) \neq 0$, let $\mu \in \mathbb{C} \setminus \{0\}$ and let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta^{\mathbf{S}}_F(\mu, x, \mathbf{L}) = \left\{ \sqrt{\frac{|x^T k|^2}{\|\Lambda_o\|_2^2} + 2\frac{\|k\|_2^2 - |x^T k|^2}{\|\Lambda\|_2^2}}. \right.$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta^{\mathbf{S}}_2(\mu, x, \mathbf{L}) = \left\{ \sqrt{\frac{|x^T k|^2}{\|\Lambda_o\|_2^2} + \frac{\|k\|_2^2 - |x^T k|^2}{\|\Lambda\|_2^2}}. \right.$$

In particular, if m is odd and $|\mu| = 1$, then for the Frobenius norm we have $\|\Lambda\|_2^2 = 2\|\Lambda_o\|_2^2$ and $\eta^{\mathbf{S}}_F(\mu, x, \mathbf{L}) = \sqrt{2}\eta_2(\mu, x, \mathbf{L})$ and for the spectral norm we have $\eta^{\mathbf{S}}_2(\mu, x, \mathbf{L}) = \dfrac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{\|\Lambda\|_2}$.

Defining the perturbation matrices

$$\Delta A_j := \begin{cases} \dfrac{\overline{\mu^j}(x^T k)(\overline{x}x^H)}{\|\Lambda_o\|_2^2} + \dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for odd } j, \\ -\dfrac{\overline{\mu^j}}{\|\Lambda\|_2^2}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right] & \text{for even } j, \end{cases}$$

then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta A_j$ is the unique T-odd matrix polynomial such that $(\mathbf{L}(\mu)+\Delta\mathbf{L}(\mu))x = 0$ and $\|\!|\Delta\mathbf{L}|\!\|_F = \eta^{\mathbf{S}}_F(\mu, x, \mathbf{L})$ in Frobenius norm.

For the spectral-norm, we introduce the perturbation matrices

$$\Delta E_j := \begin{cases} \Delta A_j - \dfrac{\overline{\mu^j x^T k}(I - xx^H)kk^T(I - \overline{x}x^T)}{\|\Lambda_o\|_2^2(\|k\|^2 - |x^T k|^2)} & \text{for odd } j, \\ \Delta A_j & \text{for even } j. \end{cases}$$

Then $\Delta\mathbf{L}(s) = \sum_{j=0}^m s^j \Delta E_j$ is a T-odd matrix polynomial such that $(\mathbf{L}(\mu) + \Delta\mathbf{L}(\mu))x = 0$ and $\|\!|\Delta\mathbf{L}|\!\|_2 = \eta^{\mathbf{S}}_2(\mu, x, \mathbf{L})$.

Proof. The proof follows from Theorem 4.8 using the fact that $H_{w^{-1},2} := \|\Lambda\|_2, K_{w^{-1},2} := \|\Lambda_o\|_2$ when $w = [1, 1, \ldots, 1]^T$ and $c = 1$. □

Remark 4.11 The case that $\mu = 0$ is not covered by the formulas in Corollary 4.10 for the case $m > 1$. But it has been shown in Theorem 4.3.8 of [1]) that for $\mu = 0, \eta_F^{\mathbf{S}}(\mu, x, \mathbf{L})$ $= \sqrt{2}\eta_2(\mu, x, \mathbf{L})$ and $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \eta_2(\mu, x, \mathbf{L})$, respectively, for Frobenius norm and spectral norm. For $|\mu| = 1$ and spectral norm we have

$$\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \frac{\sqrt{\|k\|_2^2 + |x^T k|^2}}{\|\Lambda\|_2},$$

while again it has been shown in Theorem 4.3.8 of [1] that $\eta_2^{\mathbf{S}}(\mu, x, \mathbf{L}) = \eta_2(\mu, x, \mathbf{L})$.

For the pencil case we have the following Corollary.

Corollary 4.12 *Let $\mathbf{L}(c, s) = cA_0 + sA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a T-odd matrix pencil, let $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, let $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and set $k := -\mathbf{L}(\lambda, \mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_1 x|^2 + 2\dfrac{\|k\|_2^2 - |\mu|^2 |x^T A_1 x|^2}{\|[\lambda, \mu]^T\|_2^2}} & \\ = \sqrt{\dfrac{\left(\frac{|\lambda|^2}{|\mu|^2} - 1\right)|x^T k|^2 + 2\|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \mu \neq 0, \\ \sqrt{2}\eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \\ \sqrt{2}\eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \mu = 0, \\ \sqrt{2}\eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } |\lambda| = 1, |\mu| = 1. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = \begin{cases} \sqrt{|x^T A_1 x|^2 + \dfrac{\|k\|_2^2 - |\mu|^2 |x^T A_1 x|^2}{\|[\lambda, \mu]^T\|_2^2}} & \\ = \sqrt{\dfrac{|\lambda|^2 |x^T A_1 x|^2 + \|k\|_2^2}{\|[\lambda, \mu]^T\|_2^2}} & \text{if } \mu \neq 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda = 0, \mu \neq 0, \\ \eta_2(\lambda, \mu, x, \mathbf{L}) & \text{if } \lambda \neq 0, \mu = 0, \\ \sqrt{\dfrac{|x^T A_1 x|^2 + \|k\|_2^2}{2}} & \text{if } |\lambda| = 1, |\mu| = 1. \end{cases}$$

iii) Introduce the perturbation matrices

$$\begin{aligned} \Delta A_0 &:= -\overline{z_{A_0}}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right], \\ \Delta A_1 &:= -|sign(\mu)|^2 \overline{x}x^T A_1 xx^H + \overline{z_{A_1}}\left[(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right]. \end{aligned}$$

Then for the Frobenius norm we obtain the unique T-odd pencil $\Delta\mathbf{L}(c, s) = c\Delta A_0 + s\Delta A_1$ such that $(\mathbf{L}(\lambda, \mu) + \Delta\mathbf{L}(\lambda, \mu))x = 0$ and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L})$.

For the spectral norm, defining

$$\Delta E_1 := \Delta A_1 - \frac{sign(\mu^2)\overline{x^T A_1 x}(I - \overline{x}x^T)kk^T(I - xx^H)}{(\|k\|^2 - |x^T A_1 x|^2)} \text{ and } \Delta E_0 := \Delta A_0,$$

then we obtain a T-odd pencil $\Delta\mathbf{L}(c,s) = c\Delta E_0 + s\Delta E_1$ such that $(\mathbf{L}(\lambda,\mu) + \Delta\mathbf{L}(\lambda,\mu))x = 0$ and $\|\Delta\mathbf{L}\|_2 = \eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$.

Proof. The proof is analogous to that of Theorem 4.2 using $m = 1$ and $w := [1,1]^T$. □

By the above results it is clear that if $\mu = 0$, then for the T-odd case we have $\Delta A_1 = 0$ and $\Delta A_0 = -\overline{z_{A_0}}\left[-(I - \overline{x}x^T)kx^H + \overline{x}k^T(I - xx^H)\right]$. These perturbations are the same for the spectral and Frobenius norm.

Furthermore, Corollary 4.12 shows that

$$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}) \leq \begin{cases} \sqrt{2}\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{when } |\mu| > |\lambda|, \\ \\ \|[\lambda,\mu]^T\|_2\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{when } |\mu| < |\lambda|. \end{cases}$$

Now consider a pencil $\mathbf{L}(z) = A_0 + zA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$. Then for given $\mu \in \mathbb{C}$ and for let $x \in \mathbb{C}^n$ such that $x^H x = 1$, we have

$$\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L}) \leq \begin{cases} \sqrt{2}\eta_2(\mu,x,\mathbf{L}) & \text{when } |\mu| > 1, \\ \\ \|[1,\mu^{-1}]^T\|_2\eta_2(\lambda,\mu,x,\mathbf{L}) & \text{when } |\mu| < 1, \end{cases}$$

which has been shown in [3].

We also have the following Corollary for T-odd matrix pencils of the form $\mathbf{L}(z) := A_0 + zA_1$ which is immediate from Corollary 4.12 and is presented in [1].

Corollary 4.13 *Let $\mathbf{L}(z) = A_0 + zA_1 \in \mathbf{L}_1(\mathbb{C}^{n\times n})$ be a T-odd matrix pencil, let $\mu \in \mathbb{C}$, $x \in \mathbb{C}^n$ be such that $x^H x = 1$ and let $k := -\mathbf{L}(\mu)x$.*

i) The structured backward error with respect to the Frobenius norm is given by

$$\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L}) = \begin{cases} \sqrt{|x^T A_1 x|^2 + 2\dfrac{\|k\|_2^2 - |x^T k|^2}{\|[1,\mu]^T\|_2^2}}, \\ = \sqrt{\dfrac{2\|k\|_2^2 + (|\mu|^{-2} - 1)|x^T k|^2}{\|[1,\mu]^T\|_2^2}} & \text{if } \mu \neq 0, \\ \sqrt{2}\eta_F(\lambda,\mu,x,\mathbf{L}) & \text{if } \mu = 0. \end{cases}$$

ii) The structured backward error with respect to the spectral norm is given by

$$\eta_2^{\mathbf{S}}(\mu,x,\mathbf{L}) = \begin{cases} \sqrt{|x^T A_1 x|^2 + \dfrac{\|k\|_2^2 - |x^T k|^2}{\|[1,\mu]^T\|_2^2}} \\ \\ = \sqrt{\dfrac{\|k\|_2^2 + |\mu|^{-2}|x^T k|^2}{\|[1,\mu]^T\|_2^2}} & \text{if } \mu \neq 0, \\ \eta_2(\lambda,\mu,x,\mathbf{L}) & \text{if } \mu = 0. \end{cases}$$

(λ,μ)	$\mathbf{S}$	$\eta_2^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L})$	$\eta_2(\lambda,\mu,x,\mathbf{L})$
$(0,1)$	T-odd	1	1.2247	1
$(1,0)$	T-odd	2.2361	3.1623	2.2361
$(2,1)$	T-odd	2.2361	3.0822	2.1448
$(4,3)$	T-odd	2.0881	2.8671	2.0100
$(2i,i)$	T-odd	2.2361	3.0822	2.1448
$(2+3i,1+i)$	T-odd	2.3310	3.2197	2.2361
$(1,2)$	T-odd	1.5166	2.0248	1.4832
$(1,1)$	T-odd	1.9365	2.6458	1.8708

Table 4: Computed structured and unstructured backward errors for Example 4.14

In particular, if $|\mu|=1$, *then we have* $\|[1,\mu]^T\|_2^2=2$, *and for the spectral norm we have*

$$\eta_2^{\mathbf{S}}(\mu,x,\mathbf{L})=\frac{\sqrt{\|k\|_2^2+|x^Tk|^2}}{H_{w^{-1},2}},$$

and for the Frobenius norm we have $\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L})=\sqrt{2}\eta_2(\mu,x,\mathbf{L})$.

Introduce the perturbation matrices

$$\begin{aligned}\Delta A_1 &:= -|sign(\mu)|^2\overline{x}x^TA_1xx^H+\overline{z_{A_1}}\left[(I-\overline{x}x^T)kx^H+\overline{x}k^T(I-xx^H)\right],\\ \Delta A_0 &:= -\overline{z_{A_0}}\left[-(I-\overline{x}x^T)kx^H+\overline{x}k^T(I-xx^H)\right].\end{aligned}$$

Then for the Frobenius norm we obtain the unique T*-odd pencil* $\Delta\mathbf{L}(z)=\Delta A_0+z\Delta A_1$ *such that* $(\mathbf{L}(\mu)+\Delta\mathbf{L}(\mu))x=0$ *and* $\|\Delta\mathbf{L}\|_F=\eta_F^{\mathbf{S}}(\mu,x,\mathbf{L})$.

For the spectral-norm, we define

$$\Delta E_1:=\Delta A_1-\frac{sign(\mu^2)\overline{x^TA_1x}(I-\overline{x}x^T)kk^T(I-xx^H)}{(\|k\|^2-|x^Tk|^2)},\ and\,\Delta E_0=\Delta A_0.$$

Then we obtain a T*-odd pencil* $\Delta\mathbf{L}(z)=\Delta E_0+z\Delta E_1$ *such that* $(\mathbf{L}(\mu)+\Delta\mathbf{L}(\mu))x=0$ *and* $\|\Delta\mathbf{L}\|_2=\eta_2^{\mathbf{S}}(\mu,x,\mathbf{L})$.

Let us illustrate these perturbation results with a few examples.

Example 4.14 Consider a T-odd matrix pencil with coefficients $A_0:=\begin{bmatrix}0 & -2+\imath\\ 2-\imath & 0\end{bmatrix}$, $A_1:=\begin{bmatrix}1+\imath & 0\\ 0 & 0\end{bmatrix}$. Let $x=\begin{bmatrix}-\imath/\sqrt{2}\\ \imath/\sqrt{2}\end{bmatrix}$ and $(\lambda,\mu)=(0,1)$.

i) For the Frobenius norm we obtain the minimal perturbation coefficients

$$\Delta A_0=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ \Delta A_1=\begin{bmatrix}-0.75-0.75\imath & 0.25+0.25\imath\\ 0.25+0.25\imath & 0.25+0.25\imath\end{bmatrix}\text{ and }A_0+\Delta A_0=\begin{bmatrix}0 & -2+\imath\\ 2-\imath & 0\end{bmatrix},$$
$$A_1+\Delta A_1=\begin{bmatrix}0.25+0.25\imath & 0.25+0.25\imath\\ 0.25+0.25\imath & 0.25+0.25\imath\end{bmatrix},\text{ and }\|\Delta\mathbf{L}\|_F=\eta_F^{\mathbf{S}}(\lambda,\mu,x,\mathbf{L}).$$

ii) For the spectral norm we obtain $\Delta A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\Delta A_1 = \begin{bmatrix} -0.5 - 0.5\imath & 0.5 + 0.5\imath \\ 0.5 + 0.5\imath & 0.5 + 0.5\imath \end{bmatrix}$ and $A_0 + \Delta A_0 = \begin{bmatrix} 0 & -2 + \imath \\ 2 - \imath & 0 \end{bmatrix}$, $A_1 + \Delta A_1 = \begin{bmatrix} 0.5 + 0.5\imath & 0.5 + 0.5\imath \\ 0.5 + 0.5\imath & 0.5 + 0.5\imath \end{bmatrix}$, and $\|\Delta\mathbf{L}\|_F = \eta_F^{\mathbf{S}}(\lambda, \mu, x, \mathbf{L}) = 1$,

see also Table 4.14

5 Conclusion

The structured backward errors for an approximate eigenpair and the construction of minimal structured matrix polynomials have been introduced in [1, 2, 3] such that an approximate eigenpair of $\mathbf{L}$ becomes exact for $\mathbf{L} + \Delta\mathbf{L}$ in Frobenius and spectral norm has been derived. However, this theory has been based on the condition that the polynomial eigenvalue problem has no eigenvalue at ∞. Also for T-odd matrix pencil case there are no informations of the backward error for the 0 eigenvalue. In this paper we have extended these results in the homogeneous setup of matrix polynomials which is a more convenient way to do the general perturbation analysis for matrix polynomials in that it equally treats all eigenvalues of a regular matrix polynomial. We have presented a systematic general procedure for the construction of appropriately structured minimal norm polynomials $\Delta\mathbf{L} \in \mathbf{L}_m(\mathbb{C}^{n\times n})$ such that approximate eigenvector and eigenvalue become exact of $\mathbf{L} + \Delta\mathbf{L}$. The resulting minimal perturbation is unique in the case of the Frobenius norm and there are infinitely many solutions for the case of the spectral norm. Furthermore, we derived as corollaries the known results for matrix pencils and polynomials of [1, 2, 3] and we have illustrated the results with several examples.

References

[1] B. Adhikari. *Backward perturbation and sensitivity analysis of structured polynomial eigenvalue problem.* PhD thesis, IIT Guwahati, Dept. of Mathematics, 2008.

[2] B. Adhikari and R. Alam. On backward errors of structured polynomial eigenproblems solved by structure preserving linearizations. Technical report, arxiv.org/pdf/0907.2545.

[3] B. Adhikari and R. Alam. Structured mapping problems for a class of linearly structured matrices. Preprint, IIT Guwahati, Dept. of Mathematics, 2009.

[4] B. Adhikari and D. Kressner. Backward errors and linearizations for palindromic matrix polynomials. Preprint, IIT Guwahati, Dept. of Mathematics, 2009.

[5] S. S. Ahmad. *Pseudospectra of matrix pencils and their applications in the perturbation analysis of eigenvalues and eigendecompositions.* PhD thesis, IIT Guwahati, Dept. of Mathematics, 2007.

[6] S. S. Ahmad and R. Alam. Pseudospectra, critical points and multiple eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 430:1171–1195, 2009.

[7] S. S. Ahmad and R. Alam. On pseudospectra, critical points and multiple eigenvalues of matrix pencils. *SIAM J. Matrix Anal. Appl.*, 2010, to appear.

[8] P. Arbenz and O. Chinellato. On solving complex-symmetric eigenvalue problems arising in the design of axisymmetric VCSEL devices. *Appl. Numer. Math.*, 58:381–394, 2008.

[9] T. Betcke, N. J. Higham, V. Mehrmann, C. Schröder, and F. Tisseur. NLEV: A collection of nonlinear eigenvalue problems. MIMS Eprint, The University of Manchester, UK, 2008.

[10] F. Blömeling and H. Voss. Model reduction methods for solving symmetric rational eigenvalue problems. *Proc. Appl. Math. Mech.*, 4:660–661, 2004.

[11] S. Bora. Structured eigenvalue condition number and backward error of a class of polynomial eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 31:900–917, 2009.

[12] S. Bora and V. Mehrmann. Perturbation theory for structured matrix pencils arising in control theory. *SIAM J. Matrix Anal. Appl.*, 28:148–169, 2006.

[13] T. Brüll and V. Mehrmann. STCSSP: A FORTRAN 77 routine to compute a structured staircase form for a (skew-)symmetric/(skew-)symmetric pencil. Preprint 2007-31, Institut für Mathematik, TU Berlin, D-10623 Berlin, FRG, 2007. `url: http://www.math.tu-berlin.de/preprints/`.

[14] R. Byers, V. Mehrmann, and H. Xu. A structured staircase algorithm for skew-symmetric/symmetric pencils. *Electr. Trans. Num. Anal.*, 26:1–33, 2007.

[15] C. Davis, W. Kahan, and H. Weinberger. Norm-preserving dialations and their applications to optimal error bounds. *SIAM J. Math. Anal.*, 19:445–469, 1982.

[16] J. P. Dedieu and F. Tisseur. Perturbation theory for homogeneous polynomial eigenvalue problems. *Linear Algebra Appl.*, 358:71–94, 2003.

[17] I. Gohberg, P. Lancaster, and L. Rodman. Spectral analysis of self adjoint matrix polynomials. *Ann. of Math.(2)*, 112:33–71, 1980.

[18] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.

[19] C. H. Guo. Numerical solution of a quadratic eigenvalue problem. *Linear Algebra Appl.*, 385:391–406, 2004.

[20] D. J. Higham and N. J. Higham. Structured backward error and condition of generalized eigenvalue problems. *SIAM J. Matrix Anal. Appl.*, 20:493–512, 1998.

[21] N. J. Higham. *The Accuracy and Stability of Numerical Algorithims*. SIAM, Philadelphia, USA, 1996.

[22] N. J. Higham, R. C. Li, and F. Tisseur. Backward error of polynomial eigenproblems solved by linearization. *SIAM J. Matrix Anal. Appl.*, 29:1218–1241, 2007.

[23] N. J. Higham, D. S. Mackey, and F. Tisseur. Conditioning of linearizations of matrix polynomials.

[24] N. J. Higham and F. Tisseur. More on pseudospectra for polynomial eigenvalue problems and applications in control theory. *Linear Algebra Appl.*, 351–352:435–453, 2002.

[25] A. Hilliges. Numerische Lösung von quadratischen Eigenwertproblemen mit Anwendungen in der Schienendynamik. Diploma Thesis, TU Berlin, Berlin, FRG.

[26] A. Hilliges, C. Mehl, and V. Mehrmann. On the solution of palindromic eigenvalue problems. In *Proc. of 4th European Congress on Computational Methods in Applied Science and Engineering (ECCOMAS)*, Jyväskylä, Finland, 2004. CD-ROM.

[27] P. Lancaster. *Lambda-matrices and Vibrating Systems*. Pergamon Press, Oxford, 1966.

[28] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28:1029–1051, 2006.

[29] V. Mehrmann and H. Voss. Nonlinear eigenvalue problems: a chalenge for modern eigenvalue methods. 27:121–152, 2004.

[30] J. Qian and W. W. Lin. A numerical method for quadratic eigenvalue problems of gyroscopic system. *J. Sound Vibration*, 306:284–296, 2007.

[31] J. Roomes. *Methods for eigenvalue problems with applications in model order reduction*. PhD thesis, Utrecht University, Netherland, 2008.

[32] C. Schröder. *Palindromic and even eigenvalue problems-analysis and numerical methods*. PhD thesis, Technical University Berlin, Germany, 2008.

[33] G.W. Stewart and J.-G. Sun. *Matrix Perturbation Theory*. Academic Press, New York, 1990.

[34] E. Teidelt. Numerical Solution of Eigenvalue Problems Arising in Acoustics and Structural mechanics. Diploma thesis, TU Berlin, Fakutät II, Institut für Mathematik, 2009.

[35] F. Tisseur. Backward error and condition of polynomial eigenvalue problem. *Linear Algebra Appl.*, 309:339–361, 2000.

[36] F. Tisseur and N. J. Higham. Structured pseudospectra for polynomial eigenvalue problems, with applications. *SIAM J. Matrix Anal. Appl.*, 23:187–208, 2001.

[37] F. Tisseur and K. Meerbergen. A survey of the quadratic eigenvalue problem. *SIAM Rev.*, 43:234–286, 2001.

[38] J.H. Wilkinson. *The Algebraic Eigenvalue Problem*. Oxford University Press, Oxford, 1965.