Optimal control of behavior systems *

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Abstract

In this paper a general form of the infinite-horizon linear quadratic control problem is considered. We will discuss quadratic cost functionals which involve not only the state and input-variables but also derivatives of the state and input-variables of arbitrary order under constraints given by linear systems of higher order. We will examine two results that relate the linear quadratic control problem to an optimality system, which is given through a para-Hermitian matrix polynomial. The results can be applied to general rectangular descriptor systems (see Subsection 6.1) to obtain results which so far were only known for quadratic descriptor systems. Also we will see that the notion of dissipativity (when introduced in the proper way) is equivalent to the solvability of the linear quadratic control problem.

Key words. structured eigenvalue problem, generalized eigenvalue problem, polynomial matrix, rational matrix, optimal control, linear quadratic optimal control, Hamiltonian system, dissipativity, even polynomial, descriptor system, behavior approach, quadratic cost functional, para-Hermitian

1 Introduction

Let $\mathcal{C}^\infty(\mathbb{R},\mathbb{C}^n)$ denote the set of all infinitely often differentiable functions. Let $\pi \in \mathbb{N}_0$ and let $P_0,\ldots,P_\pi \in \mathbb{C}^{p,q}$ be matrices. Assume that a fixed function $\hat{z} \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^n)$ is given, which fulfills

$$P_\pi z^{(\pi)}(t) + \ldots + P_1 z^{(1)}(t) + P_0 z(t) = 0, \tag{1}$$

where $z^{(i)}$ denotes the i-th derivative of z. The identity (1) describes a linear system and all functions $z \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^n)$ which fulfill (1) are called trajectories. Assume that the function $\hat{z}$ encapsulates the history of events that occurred in the past, up to time point $t = 0$. We now ask the question how we can continue from $t = 0$ in such a way

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that a certain cost functional is minimized. More precisely, for $\ell \in \mathbb{N}_0$ and matrices $H_{i,j} \in \mathbb{C}^{q,q}$ with $H_{i,j} = H_{j,i}^*$ for $i,j = 0,\dots,\ell$ we want to compute

$$\inf_{z(t)=\hat{z}(t), t\leq 0} \int_0^\infty \begin{bmatrix} z(t) \\ z^{(1)}(t) \\ \vdots \\ z^{(\ell)}(t) \end{bmatrix}^* \begin{bmatrix} H_{0,0} & H_{0,1} & \dots & H_{0,\ell} \\ H_{1,0} & H_{1,1} & \dots & H_{1,\ell} \\ \vdots & \vdots & & \vdots \\ H_{\ell,0} & H_{\ell,1} & \dots & H_{\ell,\ell} \end{bmatrix} \begin{bmatrix} z(t) \\ z^{(1)}(t) \\ \vdots \\ z^{(\ell)}(t) \end{bmatrix} dt, \tag{2}$$

where the infimum is to be understood over all $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n)$ for which the expression under the integral is integrable and which fulfill (1). Also, we are looking for a trajectory of (1) which attains the infimum, if such a trajectory exists. We will derive an optimality condition for the solution of problem (2) which is well-known for the special case of quadratic descriptor systems. We can generalize these well-known results to rectangular descriptor systems with the results obtained in this paper. Of course, the problem may not be solvable in the sense that (2) becomes $-\infty$, i.e., that one can construct trajectories with arbitrary small (negative) cost. We will see that a proper notion of dissipativity is equivalent to the solvability of problem (2).

This paper is structured in the following way. We will first recall some known results about polynomial and rational matrices in Section 2 and about behavior systems and dissipativity in Section 3. In Section 4 we will see that linear differential equations with an exponentially decaying inhomogeneity always have (at least one) solution which itself is exponentially decaying. Section 5 contains the two main results and here the connection between the optimal control system and the optimality system is drawn. We finish the section with some remarks about how to interpret the results. Finally, in Section 6 the results are specialized to first order systems and then to state-space descriptor systems to obtain as Corollaries (a generalization of) some well-known results and a conclusion and outlook is presented in Section 7.

The notation used in this paper is summed up in Tables 1-3.

Table 1: Notation - 1/3

S^+	for an arbitrary set $S \subset \mathbb{C}$; denotes $\{z \in S : \mathrm{Re}\{z\} > 0\}$
S^-	for an arbitrary set $S \subset \mathbb{C}$; denotes $\{z \in S : \mathrm{Re}\{z\} < 0\}$
$\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n)$	$\{z : \mathbb{R} \to \mathbb{C}^n \mid z \text{ is infinitely often differentiable}\}$
$\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^n)$	$\{z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n) \mid z \text{ has compact support}\}$
$\mathcal{C}^\infty$	$\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$
$\mathcal{C}_c^\infty$	$\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$
$\mathbb{C}[\lambda]$	the ring of polynomials with coefficients in $\mathbb{C}$
$\mathbb{C}(\lambda)$	the field of rational functions with coefficients in $\mathbb{C}$
$\mathbb{C}[\lambda]^{p,q}$	a p-by-q matrix with polynomial entries
$\mathbb{C}(\lambda)^{p,q}$	a p-by-q matrix with entries from the field of the rational functions
polynomial matrix	an element of $\mathbb{C}[\lambda]^{p,q}$, i.e., a matrix with polynomial entries

Table 2: Notation - 2/3

matrix polynomial	the same as a polynomial matrix, since one can also look at a polynomial matrix as a polynomial that has matrices as coefficients
rational matrix	an element of $\mathbb{C}(\lambda)^{p,q}$, i.e., a matrix with entries from the field of rational functions
$\mathfrak{P}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$ is a rational matrix; denotes the set of poles of R, i.e., the set of all $\lambda \in \mathbb{C}$ where at least one entry of R has a pole
$\mathfrak{D}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$ is a rational matrix; denotes the set $\mathbb{C} \setminus \mathfrak{P}(R)$, i.e., the set of all λ such that $R(\lambda)$ is a well defined matrix in $\mathbb{C}^{p,q}$ and thus the domain of definition of R
$\operatorname{rank}_{\mathbb{C}(\lambda)}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the rank of R over the field $\mathbb{C}(\lambda)$
$\operatorname{kernel}_{\mathbb{C}(\lambda)}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the kernel of R over the field $\mathbb{C}(\lambda)$ which is a subset of $\mathbb{C}(\lambda)^q$
$\operatorname{range}_{\mathbb{C}(\lambda)}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the range of R over the field $\mathbb{C}(\lambda)$ with is a subset of $\mathbb{C}(\lambda)^p$
$\operatorname{rank}(R(\lambda))$	where $R \in \mathbb{C}(\lambda)^{p,q}$ and $\lambda \in \mathfrak{D}(R)$; denotes the rank of $R(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\operatorname{kernel}(R(\lambda))$	where $R \in \mathbb{C}(\lambda)^{p,q}$ and $\lambda \in \mathfrak{D}(R)$; denotes the kernel of $R(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\operatorname{range}(R(\lambda))$	where $R \in \mathbb{C}(\lambda)^{p,q}$ and $\lambda \in \mathfrak{D}(R)$; denotes the range of $R(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$z^{(i)}$	the i-th derivative of the function z
$P\left(\frac{d}{dt}\right) z$	where $P \in \mathbb{C}[\lambda]^{p,q}$ has the form $P(\lambda) = \sum_{i=0}^{d} \lambda^i P_i$ and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$; denotes the function $\sum_{i=0}^{d} P_i z^{(i)}$
Δ_ℓ^q	where $\ell, q \in \mathbb{N}$; denotes the polynomial given by $$\Delta_\ell^q(\lambda) := \begin{bmatrix} (\lambda)^0 I_q \\ (\lambda)^1 I_q \\ \vdots \\ (\lambda)^\ell I_q \end{bmatrix} \in \mathbb{C}[\lambda]^{q(l+1),q}$$
$\Delta_\ell z$	where $\ell \in \mathbb{N}$ and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$; denotes the function $$\Delta_\ell z := \begin{bmatrix} z \\ z^{(1)} \\ \vdots \\ z^{(\ell)} \end{bmatrix} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q(\ell+1)}),$$ and thus we have $\Delta_\ell z = \Delta_\ell^q\left(\frac{d}{dt}\right) z$

Table 3: Notation - 3/3

$\operatorname{diag}(\alpha_1, \ldots, \alpha_r)$	where $\alpha_1, \ldots, \alpha_r$ are scalars; denotes the r-by-r diagonal matrix which has the scalars $\alpha_1, \ldots, \alpha_r$ on the diagonal and is zero everywhere else
$\mathbf{0} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$	The trivial function which is zero everywhere on $\mathbb{R}$
$\mathcal{E}_+(\mathbb{R}, \mathbb{C}^q)$	denotes the set of all functions $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ for which z and all its derivatives are exponentially decaying, i.e., all $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ such that for every $i \in \mathbb{N}_0$ there exist $a_i, b_i > 0$ with $$\left\| z^{(i)}(t) \right\|_2 \leq a_i e^{-b_i t}$$ for all $t \geq 0$
$\mathcal{E}_+^q$	short for $\mathcal{E}_+(\mathbb{R}, \mathbb{C}^q)$
$\mathcal{E}_+$	short for $\mathcal{E}_+^1$
$\langle f, g \rangle_+$	where $f, g \in \mathcal{E}_+^n$; denotes the L_2 scalar product on the positive half axis given by $$\langle f, g \rangle_+ := \int_0^\infty g^*(t) f(t) dt$$
$\\|f\\|_+$	where $f \in \mathcal{E}_+^n$; denotes the L_2 norm on the positive half axis given by $$\\|f\\|_+ := \sqrt{\langle f, f \rangle_+} = \sqrt{\int_0^\infty \\|f(t)\\|_2^2 dt}$$

2 Preliminaries

In this section we will review some definitions and results which are needed in the following.

Lemma 1. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ a matrix polynomial. Then there exists a finite set $\sigma(P)$ such that*

1. $\operatorname{rank}_{\mathbb{C}(\lambda)}(P) = \operatorname{rank}(P(\lambda))$ *for all $\lambda \notin \sigma(P)$,*
2. $\operatorname{rank}_{\mathbb{C}(\lambda)}(P) > \operatorname{rank}(P(\lambda))$ *for all $\lambda \in \sigma(P)$.*

Proof. See [2, Lemma 2]. □

Definition 2. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and define the set $\sigma(P)$ as in Lemma 1. Then the elements of $\sigma(P)$ are called *eigenvalues of P*. For $\lambda \in \sigma(P)$ the quantity

$$g(\lambda) := \operatorname{rank}_{\mathbb{C}(\lambda)}(P) - \operatorname{rank}(P(\lambda))$$

is called *geometric multiplicity of λ*.

Remark 3. In the theory of matrix polynomials it is common practice to distinguish between *finite* and *infinite* eigenvalues, see [1, Section 3]. However, in this paper we are not concerned with infinite eigenvalues, as they do not seem to play a role for the results presented. Thus, if we speak of eigenvalues in the following we always refer to finite eigenvalues.

Definition 4. A quadratic rational matrix $R \in \mathbb{C}(\lambda)^{p,p}$ is called *unimodular* if its determinant is a non-zero constant, i.e., there exists a non-zero constant $c \in \mathbb{C} \setminus \{0\}$ such that

$$\det R(\lambda) = c,$$

for all $\lambda \in \mathfrak{D}(R)$.

Unimodular matrices have some convenient properties.

Lemma 5. *A polynomial, unimodular matrix $P \in \mathbb{C}[\lambda]^{p,p}$ is invertible over $\mathbb{C}(\lambda)$ and its inverse is again a polynomial, unimodular matrix. Also, a unimodular matrix has no eigenvalues.*

Proof. Invertibility follows since the determinant is non-zero. We see that the inverse (which we know exists in $\mathbb{C}(\lambda)^{p,p}$) is also a polynomial matrix use the adjoint formula for the inverse. To see that the inverse is unimodular, note that

$$1 = \det I = \det(PP^{-1}) = \det(P)\det(P^{-1}).$$

A unimodular matrix can have no eigenvalues, since at such an eigenvalue the determinant would vanish which contradicts the assumption that the determinant is a non-zero constant. □

Theorem 6. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Then there exist unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ such that*

$$P = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T, \tag{3}$$

where $d_1, \ldots, d_r \in \mathbb{C}[\lambda]$ with $d_i \neq 0$ for $i = 1, \ldots, r$ and d_{i+1} divides d_i for $i = 1, \ldots, r-1$.

Proof. The proof is quite simple and a completely self contained presentation can be found in [7, Chapter S1.1] and another in [5, p.141, Theorem 3]. □

The canonical form in (3) is called Smith form.

Theorem 7. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Then there exist polynomial matrices $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ with the properties*

1. $PU = 0$,
2. $\operatorname{rank}_{\mathbb{C}(\lambda)}(PV) = r$,

3. $\operatorname{rank}_{\mathbb{C}(\lambda)}(U) = \operatorname{rank}(U(\lambda)) = q - r$ *for all* $\lambda \in \mathbb{C}$,

4. $\operatorname{rank}_{\mathbb{C}(\lambda)}(V) = \operatorname{rank}(V(\lambda)) = r$ *for all* $\lambda \in \mathbb{C}$,

5. $\begin{bmatrix} U & V \end{bmatrix}$ *is unimodular.*

Proof. Let the Smith form (3) of P be given by

$$P = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

Partition the inverse of T (which is again a polynomial unimodular matrix due to Lemma 5) according to the block structure of the diagonal matrix in the Smith form as

$$T^{-1} =: \begin{bmatrix} V & U \end{bmatrix},$$

i.e., such that V has r columns and U has $q - r$ columns. Then clearly also $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular since it can be obtained from T^{-1} via a column permutation, which is itself a unimodular transformation, and then 5. is proved. Also we see that 1. holds, since

$$PU = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TU = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TT^{-1} \begin{bmatrix} 0 \\ I_{q-r} \end{bmatrix} = 0.$$

Property 2. holds since analogously

$$PV = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) \\ 0 \end{bmatrix},$$

and all $d_i \neq 0$ for $i = 1, \ldots, r$. Parts 3. and 4. follow since $\begin{bmatrix} U & V \end{bmatrix}$ has full rank (over $\mathbb{C}(\lambda)$) and due to the unimodularity also the matrix $\begin{bmatrix} U(\lambda) & V(\lambda) \end{bmatrix}$ has full rank (over $\mathbb{C}$) for every $\lambda \in \mathbb{C}$. □

Definition 8. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Then the rational matrix $U \in \mathbb{C}(\lambda)^{q,q-r}$ and the polynomial matrix $V \in \mathbb{C}[\lambda]^{q,r}$ are called *kernel matrix and co-kernel matrix* of P, resp., if they fulfill the following properties

1. $PU = 0$,

2. $\operatorname{rank}_{\mathbb{C}(\lambda)}(PV) = r$,

3. $\operatorname{rank}_{\mathbb{C}(\lambda)}(U) = \operatorname{rank}(U(\lambda)) = q - r$ for all $\lambda \in \mathfrak{D}(U)$,

4. $\operatorname{rank}_{\mathbb{C}(\lambda)}(V) = \operatorname{rank}(V(\lambda)) = r$ for all $\lambda \in \mathbb{C}$,

5. $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular.

Theorem 7 shows that for every matrix polynomial there also exists a kernel and co-kernel matrix, where the kernel matrix is even a polynomial. The kernel matrix is allowed to be a rational function, because for regular first order state-space systems one can give a kernel matrix in explicit form which happens to be a rational matrix.

The following Lemma 9 justifies that one may speak of a kernel matrix and a co-kernel matrix independently.

Lemma 9. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ together be a kernel and co-kernel matrix of P and independently of that, let also $\tilde{U} \in \mathbb{C}(\lambda)^{q,q-r}$ and $\tilde{V} \in \mathbb{C}[\lambda]^{q,r}$ together be a kernel and co-kernel matrix of P. Then also U and $\tilde{V}$ together are a kernel and co-kernel matrix of P as well as $\tilde{U}$ and V together are a kernel and co-kernel matrix of P.*

Proof. See [2, Lemma 13] □

Lemma 10. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P. Let a Smith form (3) of P be given by*

$$P = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T,$$

and partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q - r$ columns, partitioned analogously to the Smith form.
Then, there exists a unimodular rational matrix $U_2 \in \mathbb{C}(\lambda)^{q-r,q-r}$ with $\mathfrak{D}(U) = \mathfrak{D}(U_2)$ such that $U = T_2U_2$. If U is in addition a polynomial matrix, then the matrix U_2 is also polynomial.

Proof. See [2, Lemma 12]. □

Lemma 11. *Let $Q \in \mathbb{C}[\lambda]^{p,r}$ be a matrix polynomial with full column rank and no eigenvalues, i.e., let $\operatorname{rank}_{\mathbb{C}(\lambda)}(Q) = r = \operatorname{rank}(Q(\lambda))$ for all $\lambda \in \mathbb{C}$. Then there exists a polynomial left inverse $X \in \mathbb{C}[\lambda]^{r,q}$ of Q, i.e., a polynomial matrix X such that $XQ = I_r$.*

Proof. The Smith form can be used to prove the claim. See [2, Lemma 14] for a detailed presentation. □

3 The behavior approach and dissipativity

In the following we state a few results related to the behavior approach which will be needed subsequently. For a more detailed discussion on the behavior approach the reader is referred to [10], where a slightly different notation is used.

Definition 12. Let $P \in \mathbb{C}[\lambda]^{p,q}$ a polynomial matrix. Then we call

$$\mathfrak{B}(P) := \left\{ z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \middle| P\left(\frac{d}{dt}\right) z = 0 \right\} = \operatorname{kernel}_{\mathcal{C}^\infty}\left(P\left(\frac{d}{dt}\right)\right)$$

the *behavior of* P and

$$\mathfrak{B}_c(P) := \mathfrak{B}(P) \cap \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q) = \operatorname{kernel}_{\mathcal{C}_c^\infty}\left(P\left(\frac{d}{dt}\right)\right)$$

the *compact behavior of* P. The elements of $\mathfrak{B}(P)$ and $\mathfrak{B}_c(P)$ are also called *trajectories of* P. Furthermore, we are going to use the symbol

$$\mathfrak{B}_+(P) := \left\{ z \in \mathcal{E}_q^+ \middle| P\left(\frac{d}{dt}\right) z = 0 \right\},$$

to denote the set of exponentially decaying trajectories.

Lemma 13. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix according to Definition 8 and Theorem 7. Let $a \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ be arbitrary. Then*

$$z := U\left(\frac{d}{dt}\right) a \in \mathfrak{B}(P),$$

i.e., z is a trajectory of P.

Proof. See [2, Lemma 17]. □

The following Lemma 14 completely characterizes the compact behavior of a system with the help of a polynomial kernel matrix.

Lemma 14. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P. Then*

$$\mathfrak{B}_c(P) = \operatorname{range}_{\mathcal{C}_c^\infty}\left(U\left(\frac{d}{dt}\right)\right).$$

Proof. The inclusion "$\supset$" follows readily from Lemma 13. The inclusion "$\subset$" is more complicated, see [2, Lemma 18]. □

Lemma 15. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P. Let $z \in \mathfrak{B}(P)$ be such that for some fixed $t_0 \in \mathbb{R}$ we have*

$$z^{(k)}(t_0) = 0,$$

for all $k \in \mathbb{N}$. Then there exists an $\alpha \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ such that

$$z = U\left(\frac{d}{dt}\right) \alpha.$$

Proof. Let the Smith form (3) of P be given by

$$P = S \begin{bmatrix} \operatorname{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

Define $y := T\left(\frac{d}{dt}\right) z$. Since T has the representation

$$T(\lambda) = \sum_{i=0}^{\tau} \lambda^i T_i,$$

for some $\tau \in \mathbb{N}$ and $T_i \in \mathbb{C}^{q,q}$ we see that for all $k \in \mathbb{N}_0$ we have

$$y^{(k)}(t_0) = \sum_{i=0}^{\tau} T_i z^{(i+k)}(t_0) = 0,$$

due to the assumption. Denoting the elements of y by y_i and using that $P\left(\frac{d}{dt}\right) z = 0$, we find that $d_i\left(\frac{d}{dt}\right) y_i(t) = 0$ for $i = 1, \dots, r$. For each $i = 1, \dots, r$ we distinguish two cases. The first case is when $d_i(\lambda) \equiv \tilde{d}_i$ is a constant non-zero polynomial. Then $\tilde{d}_i y_i(t) = 0$ implies that $y_i(t) = 0$ for all $t \in \mathbb{R}$. The second case is when $d_i(\lambda)$ is a polynomial of degree higher than or equal to 1. In this case $d_i\left(\frac{d}{dt}\right) y_i(t) = 0$ constitutes a differential equation. Since we have already derived the initial conditions $y_i^{(k)}(t_0) = 0$ for all $k \in \mathbb{N}$ we see from the basic theory of homogeneous linear differential equations that in this case also $y_i(t) = 0$ for all $t \in \mathbb{R}$.

Thus, we have shown that y takes the form

$$y = \begin{bmatrix} 0 \\ \tilde{y} \end{bmatrix},$$

with $\tilde{y} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q-r})$. Partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q - r$ columns.
Applying Lemma 10 we obtain the existence of a unimodular $U_2 \in \mathbb{C}[\lambda]^{q-r,q-r}$ such that

$$U = T_2 U_2.$$

Setting $\alpha := U_2^{-1}\left(\frac{d}{dt}\right) \tilde{y}$ we find that

$$\begin{aligned} z &= T^{-1}\left(\frac{d}{dt}\right) y = \begin{bmatrix} T_1\left(\frac{d}{dt}\right) & T_2\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{y} \end{bmatrix} \\ &= T_2\left(\frac{d}{dt}\right) \tilde{y} = T_2\left(\frac{d}{dt}\right) U_2\left(\frac{d}{dt}\right) U_2^{-1}\left(\frac{d}{dt}\right) \tilde{y} = U\left(\frac{d}{dt}\right) \alpha, \end{aligned}$$

which proves the claim. □

In Definition 12 we have introduced a system (and its behavior) by means of a matrix polynomial $P \in \mathbb{C}[\lambda]^{p,q}$. Next, we introduce a notion of energy or cost for such systems. Therefore, let $\ell \in \mathbb{N}_0$ and let $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ be a Hermitian matrix. We then measure the cost that a given trajectory $z \in \mathfrak{B}(P)$ causes at the time point $t \in \mathbb{R}$ through the term $(\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))$. This term depends not only on z at time point t but also on derivatives of z up to order ℓ at time point t and might well become negative. One can also think of $(\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))$ as measuring the amount of energy that is supplied to the system P along the trajectory z, implying that every amount of energy we supply to the system costs energy. However, we are not interested in the cost that a trajectory causes at one specific time point but in the cost that a trajectory causes over a whole time span in a cumulative way. Since we consider the infinite horizon linear quadratic optimal control problem we measure the cumulative cost that a trajectory z causes by

$$\int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,$$

where we will assume that $z \in \mathfrak{B}_+(P)$ is exponentially decaying to ensure the existence of the integral.

Definition 16. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$. The P is called *dissipative* with respect to $\tilde{H}$ if the dissipation inequality

$$0 \leq \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \, \tilde{H} \, (\Delta_\ell z(t)) \, dt, \tag{4}$$

holds for all $z \in \mathfrak{B}_c(P)$.

Definition 16 plays an important role in control theory and there are a lot of other equivalent characterizations available see, e.g., [2, 16] which, however, will not be pursued in this paper.

To discuss the importance of Definition 16 for the optimal control problem, assume that there would exist a trajectory $\tilde{z} \in \mathfrak{B}_c(P)$ with compact support such that

$$0 > \int_{-\infty}^{\infty} (\Delta_\ell \tilde{z}(t))^* \, \tilde{H} \, (\Delta_\ell \tilde{z}(t)) \, dt.$$

Then we could concatenate the non-trivial part of $\tilde{z}$ over and over again (as depicted in Figure 1) to obtain a trajectory z_0 of arbitrary low cost.

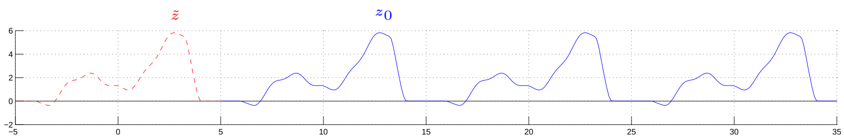

Figure 1: Concatenation of functions with compact support

Thus, in this case there could never exist trajectory of minimum cost. We need the following implication in the next section.

Lemma 17. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be dissipative with respect to $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$. Then also*

$$0 \leq \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \, \tilde{H} \, (\Delta_\ell z(t)) \, dt,$$

for all $z \in \mathfrak{B}_+(P)$ with $z(t) = 0$ for $t \leq 0$.

Proof. Assume to the contrary that there was a trajectory $z \in \mathfrak{B}_+(P)$ with $z(t) = 0$ for $t \leq 0$ and

$$0 > \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \, \tilde{H} \, (\Delta_\ell z(t)) \, dt = \int_{0}^{\infty} (\Delta_\ell z(t))^* \, \tilde{H} \, (\Delta_\ell z(t)) \, dt. \tag{5}$$

Let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P, where $r = \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$. According to Lemma 15 (with, e.g., $t_0 = 0$) this implies the existence of an $\alpha \in$

$\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ such that $z = U\left(\frac{d}{dt}\right)\alpha$. Using Lemma 11 we obtain the existence of a matrix $X \in \mathbb{C}[\lambda]^{q-r,q}$ such that $XU = I_{q-r}$. Since $z \in \mathcal{E}_q^+$ with $z(t) = 0$ for $t \leq 0$ so is

$$X\left(\frac{d}{dt}\right) z = X\left(\frac{d}{dt}\right) U\left(\frac{d}{dt}\right) \alpha = \alpha,$$

i.e., $\alpha \in \mathcal{E}_{q-r}^+$ and $\alpha(t) = 0$ for $t \leq 0$. Let U take the form

$$U(\lambda) = \sum_{i=0}^{\mu} U_i \lambda^i,$$

with $\mu \in \mathbb{N}$ and $U_i \in \mathbb{C}^{q,q-r}$ and let $\tilde{b}$ be a smooth transition from 1 to 0, i.e., let $\tilde{b} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with $\tilde{b}(t) = 1$ for $t \leq 0$ and $\tilde{b}(t) = 0$ for $t \geq q$. Define a family of functions $z_T \in \mathfrak{B}_c(P)$ through

$$z_T(t) := U\left(\frac{d}{dt}\right)\left[\alpha(t)\tilde{b}(t-T)\right].$$

With this we obtain

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} (\Delta_\ell z_T(t))^* \tilde{H} (\Delta_\ell z_T(t))\, dt \\ &= \int_0^T (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt + \int_T^\infty (\Delta_\ell z_T(t))^* \tilde{H} (\Delta_\ell z_T(t))\, dt \end{aligned}$$

Using (5) we see that there exists a $\epsilon > 0$ and a T_0 such that for all $T \geq T_0$ we have

$$\int_0^T (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt < -\epsilon,$$

e.g., one can choose

$$\epsilon = -\frac{1}{2}\int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt.$$

We also observe that

$$\begin{aligned} &\left|\int_T^\infty (\Delta_\ell z_T(t))^* \tilde{H} (\Delta_\ell z_T(t))\, dt\right| \\ = &\left|\int_0^\infty (\Delta_\ell z_T(t+T))^* \tilde{H} (\Delta_\ell z_T(t+T))\, dt\right| \\ = &\left|\left\langle \tilde{H}\Delta_\ell z_T(\cdot + T), \Delta_\ell z_T(\cdot + T)\right\rangle_+\right| \\ \leq &\left\|\tilde{H}\right\|_+ \|\Delta_\ell z_T(\cdot + T)\|_+^2 \end{aligned}$$

$$
\begin{aligned}
&= \left\|\tilde{H}\right\|_+ \left\|\Delta_\ell U\left(\frac{d}{dt}\right)\left[\alpha(\cdot+T)\tilde{b}(\cdot)\right]\right\|_+^2 \\
&= \left\|\tilde{H}\right\|_+ \sum_{i=0}^{\ell} \left\|\left(\frac{d}{dt}\right)^i U\left(\frac{d}{dt}\right)\left[\alpha(\cdot+T)\tilde{b}(\cdot)\right]\right\|_+^2 \\
&= \left\|\tilde{H}\right\|_+ \sum_{i=0}^{\ell} \left\|\left(\frac{d}{dt}\right)^i \sum_{j=0}^{\mu}\left(\frac{d}{dt}\right)^j U_j\left[\alpha(\cdot+T)\tilde{b}(\cdot)\right]\right\|_+^2 \\
&\leq \left\|\tilde{H}\right\|_+ \sum_{i=0}^{\ell}\sum_{j=0}^{\mu} \left\|U_j\left(\frac{d}{dt}\right)^{i+j}\left[\alpha(\cdot+T)\tilde{b}(\cdot)\right]\right\|_+^2 \\
&\leq \left\|\tilde{H}\right\|_+ \sum_{i=0}^{\ell}\sum_{j=0}^{\mu} \|U_j\|_+^2 \left\|\left(\frac{d}{dt}\right)^{i+j}\left[\alpha(\cdot+T)\tilde{b}(\cdot)\right]\right\|_+^2 \\
&= \left\|\tilde{H}\right\|_+ \sum_{i=0}^{\ell}\sum_{j=0}^{\mu} \|U_j\|_+^2 \left\|\sum_{k=0}^{i+j}\binom{i+j}{k}\alpha^{(k)}(\cdot+T)\tilde{b}^{(i+j-k)}(\cdot)\right\|_+^2 \\
&\leq \left\|\tilde{H}\right\|_+ \sum_{i=0}^{\ell}\sum_{j=0}^{\mu} \|U_j\|_+^2 \sum_{k=0}^{i+j}\binom{i+j}{k}\left\|\alpha^{(k)}(\cdot+T)\tilde{b}^{(i+j-k)}(\cdot)\right\|_+^2 . \qquad (6)
\end{aligned}
$$

Defining

$$B := \max_{\substack{t\in[0,1] \\ k=0,\ldots,\mu+\ell}} \left|\tilde{b}^{(k)}(t)\right|_2$$

and choosing $C, D \in \mathbb{R}^+$ such that

$$\left\|\alpha^{(k)}(t)\right\| \leq Ce^{-Dt},$$

for all $t \in \mathbb{R}$ and all $k = 0, \ldots, \mu+\ell$ we obtain

$$
\begin{aligned}
&\left\|\alpha^{(k)}(\cdot+T)\tilde{b}^{(i+j-k)}(\cdot)\right\|_+^2 \\
&= \int_0^\infty \left\|\alpha^{(k)}(t+T)\tilde{b}^{(i+j-k)}(t)\right\|_2^2 dt \\
&\leq B^2 \int_0^\infty \left\|\alpha^{(k)}(t+T)\right\|_2^2 dt \\
&\leq B^2 \int_0^\infty C^2 e^{-2D(t+T)} dt \\
&\leq B^2 C^2 e^{-2DT} \int_0^\infty e^{-2Dt} dt \\
&\leq B^2 C^2 e^{-2DT} \frac{1}{-2D} e^{-2Dt}\Big|_0^\infty
\end{aligned}
$$

$$= \frac{B^2C^2}{2D} e^{-2DT}, \tag{7}$$

for all $k = 0, \dots, \mu + \ell$. Combining (6) and (7) yields

$$\begin{aligned}
& \left| \int_T^\infty (\Delta_\ell z_T(t))^* \tilde{H} (\Delta_\ell z_T(t))\, dt \right| \\
\leq \;& \left\| \tilde{H} \right\|_+ \sum_{i=0}^{\ell} \sum_{j=0}^{\mu} \|U_j\|_+^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \frac{B^2C^2}{2D} e^{-2DT} \\
= \;& e^{-2DT} A,
\end{aligned}$$

by defining

$$A := \left\| \tilde{H} \right\|_+ \sum_{i=0}^{\ell} \sum_{j=0}^{\mu} \|U_j\|_+^2 \sum_{k=0}^{i+j} \binom{i+j}{k} \frac{B^2C^2}{2D} \geq 0.$$

Now, choosing $T_1 \geq T_0$ such that

$$e^{-2DT_1} A \leq \frac{\epsilon}{2},$$

we finally see that

$$\begin{aligned}
0 \;\leq\; & \int_{-\infty}^{\infty} (\Delta_\ell z_{T_1}(t))^* \tilde{H} (\Delta_\ell z_{T_1}(t))\, dt \\
= \;& \int_0^{T_1} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt + \int_{T_1}^{\infty} (\Delta_\ell z_{T_1}(t))^* \tilde{H} (\Delta_\ell z_{T_1}(t))\, dt \\
< \;& -\epsilon + \left| \int_{T_1}^{\infty} (\Delta_\ell z_{T_1}(t))^* \tilde{H} (\Delta_\ell z_{T_1}(t))\, dt \right| \\
< \;& -\epsilon + e^{-2DT_1} A \leq -\epsilon + \frac{\epsilon}{2} = -\frac{\epsilon}{2} < 0,
\end{aligned}$$

which is a contradiction and thus the claim is proved. □

Assume that the dissipation inequality (4) holds for all $z \in \mathfrak{B}_+(P)$ with $z(t) = 0$ for $t \leq 0$. Then the dissipation inequality (4) is also fulfilled for all trajectories $z \in \mathfrak{B}_c(P)$ with compact support, i.e., P is dissipative. Lemma 17 shows that also the converse direction is true. In other words, by Lemma 17 we derived a condition which is equivalent to dissipativity which will be used in the following.

4 Differential equations with exponentially decaying inhomogeneity

In this section we will first see that ordinary differential equations of the form

$$\dot{x} = Ax + f,$$

with $A \in \mathbb{C}^{n,n}$ and $x, f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n)$ with exponentially decaying $f \in \mathcal{E}_+^n$ have at least one solution $x \in \mathcal{E}_+^n$ which is also exponentially decaying. Therefore, the Jordan form will be employed. We then generalize the results further to higher order systems with may also contain algebraic constraints by using the Smith form. Only the final results, Lemma 22, will be needed in the next section.

Lemma 18. *Let $f, g \in \mathcal{E}_+$ and $\alpha \in \mathbb{C}$. Then also $f + g \in \mathcal{E}_+$, $\alpha f \in \mathcal{E}_+$, and $\dot{f} \in \mathcal{E}_+$. This especially means that $\mathcal{E}_+$ is a linear subspace of $\mathcal{C}^\infty$.*

Proof. The proof is straightforward. □

Lemma 19. *Let $f \in \mathcal{E}_+$ and $a \in \mathbb{C}$ be arbitrary. Then there exists a $y \in \mathcal{E}_+$ such that*

$$\dot{y}(t) = ay(t) + f(t), \tag{8}$$

for all $t \in \mathbb{R}$, i.e., in the scalar first order case the linear differential equation with exponentially decaying inhomogeneity f has (at least one) exponentially decaying solution.

Proof. Since $f \in \mathcal{E}_+$ we know that for $i \in \mathbb{N}$ there exist constants $d_i, \delta_i > 0$ such that

$$\left| f^{(i)}(t) \right| \leq d_i e^{-\delta_i t},$$

for all $t \geq 0$. We distinguish two cases. First assume that $\operatorname{Re}\{a\} \geq 0$. Define

$$y_0 := -\int_0^\infty e^{-as} f(s) ds.$$

Note that y_0 is well-defined, since in this case e^{-as} is bounded for all $s \geq 0$ by 1 and f is exponentially decaying and infinitely often differentiable. With the variation-of-constants formula and y_0 as an initial condition we obtain that

$$\begin{aligned} y(t) &= e^{at} y_0 + e^{at} \int_0^t e^{-as} f(s) ds \\ &= e^{at} \left(-\int_0^\infty e^{-as} f(s) ds + \int_0^t e^{-as} f(s) ds \right) \\ &= -e^{at} \int_t^\infty e^{-as} f(s) ds \end{aligned}$$

is a solution of (8). We have

$$\begin{aligned} |y(t)| &= \left| e^{at} \right| \left| \int_t^\infty e^{-as} f(s) ds \right| \\ &\leq e^{\operatorname{Re}\{a\} t} \int_t^\infty \left| e^{-as} f(s) \right| ds \\ &\leq e^{\operatorname{Re}\{a\} t} \int_t^\infty e^{-\operatorname{Re}\{a\} s} d_0 e^{-\delta_0 t} ds \end{aligned}$$

$$\begin{aligned}
&= d_0 e^{\mathrm{Re}\{a\}t} \int_t^\infty e^{-(\mathrm{Re}\{a\}+\delta_0)s} ds \\
&= d_0 e^{\mathrm{Re}\{a\}t} \left(-\frac{1}{\mathrm{Re}\{a\}+\delta_0} e^{-(\mathrm{Re}\{a\}+\delta_0)s} \Big|_t^\infty \right) \\
&= \frac{d_0}{\mathrm{Re}\{a\}+\delta_0} e^{\mathrm{Re}\{a\}t} e^{-(\mathrm{Re}\{a\}+\delta_0)t} \\
&= \frac{d_0}{\mathrm{Re}\{a\}+\delta_0} e^{-\delta_0 t} =: c_0 e^{-\gamma_0 t},
\end{aligned}$$

by setting $c_0 := \frac{d_0}{\mathrm{Re}\{a\}+\delta_0}$ and $\gamma_0 := \delta_0$. Since y solves the differential equation (8), we see that

$$y^{(i)}(t) = a^i y(t) + \sum_{j=0}^{i-1} a^j f^{(i-1-j)}(t).$$

Using Lemma 18 shows that all derivatives of y are also exponentially decaying, which implies $y \in \mathcal{E}_+$.

For the second case assume that $\mathrm{Re}\{a\} < 0$. In this case there exist multiple solutions. We choose $y_0 := 0$ and observe that in this case

$$y(t) \quad = \quad \int_0^t e^{a(t-s)} f(s) ds,$$

is a solution of (8). W.l.o.g. we assume that $\delta_0 < -\mathrm{Re}\{a\}$ (otherwise chose δ_0 smaller, which is still appropriate). Then $\delta_0 + \mathrm{Re}\{a\} < 0$ and we have

$$\begin{aligned}
|y(t)| &\leq \int_0^t \left| e^{a(t-s)} f(s) \right| ds \\
&\leq \int_0^t e^{\mathrm{Re}\{a\}(t-s)} d_0 e^{-\delta_0 s} ds \\
&= d_0 e^{\mathrm{Re}\{a\}t} \int_0^t e^{-s(\mathrm{Re}\{a\}+\delta_0)} ds \\
&= -\frac{d_0}{\mathrm{Re}\{a\}+\delta_0} e^{\mathrm{Re}\{a\}t} \left(e^{-s(\mathrm{Re}\{a\}+\delta_0)} \Big|_0^t \right) \\
&= -\frac{d_0}{\mathrm{Re}\{a\}+\delta_0} e^{\mathrm{Re}\{a\}t} \left(e^{-t(\mathrm{Re}\{a\}+\delta_0)} - 1 \right) \\
&= -\frac{d_0}{\mathrm{Re}\{a\}+\delta_0} \left(e^{-\delta_0 t} - e^{\mathrm{Re}\{a\}t} \right) \\
&=: c_0 \left(e^{-\gamma_0 t} - e^{\mathrm{Re}\{a\}t} \right) \\
&\leq c_0 e^{-\gamma_0 t},
\end{aligned}$$

which shows that y is exponentially decaying. As above we deduce that then all derivatives of y are also exponentially decaying, since y solves the differential equation (8), and thus $y \in \mathcal{E}_+$. □

The proof of Lemma 19 shows that in the case $\mathrm{Re}\{a\} \geq 0$ there is one exponentially decaying solution. In the case $\mathrm{Re}\{a\} < 0$ we could have as well taken any other initial condition $y_0 \neq 0$ to obtain the result, i.e., in the cas $\mathrm{Re}\{a\} < 0$ there is more than one exonentially decaying solution.

Theorem 20. *Let $f \in \mathcal{E}_+^n$ and $A \in \mathbb{C}^{n,n}$ be arbitrary. Then there exists $y \in \mathcal{E}_+^n$ such that*

$$\dot{y}(t) = Ay(t) + f(t),$$

for all $t \in \mathbb{R}$, i.e., the linear differential equation with exponentially decaying inhomogeneity f has (at least one) exponentially decaying solution.

Proof. Using the Jordan canonical form of A the problem decomposes into a finite number of subproblems of which each has the form

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{n_i} \end{bmatrix} = \begin{bmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_i} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n_i} \end{bmatrix},$$

with $n_i \in \mathbb{N}$. Starting from the last variable and last equation one can use Lemma 19 to show that there exists a solution $y_{n_i} \in \mathcal{E}_+$. Using Lemma 18 we see that $y_{n_i} + f_{n_i-1} \in \mathcal{E}_+$. Thus, using Lemma 19 again, we find that there exists a solution $y_{n_i-1} \in \mathcal{E}_+$. Proceeding this way we obtain the claim for the subproblem and thus also for an arbitrary matrix A. $\square$

Lemma 21. *Let $d \in \mathbb{C}[\lambda] \setminus \{\mathbf{0}\}$ be a non-zero polynomial. Let $b \in \mathcal{E}_+$ be a function. Then there exists $x \in \mathcal{E}_+$ such that*

$$d\left(\frac{d}{dt}\right) x = b.$$

Proof. We distinguish two cases. For the first case assume that $d \equiv c \in \mathbb{C}$ is a non-zero constant. In this case we set $x := \frac{1}{c} b$ and obtain the assertion immediately. For the second case let

$$d = \sum_{i=0}^{\delta} d_i \lambda^i$$

with $\delta \in \mathbb{N} \setminus \{0\}$ and $d_\delta \neq 0$. This means that we are looking for a solution $x \in \mathcal{E}^+$ of the differential equation

$$\sum_{i=0}^{\delta} d_i x^{(i)}(t) = b(t).$$

Reducing this higher order, scalar differential equation to a first order, matrix differential equation with the help of the companion form of d, and applying Theorem 20 we immediately obtain the result. $\square$

Lemma 22. *Let $A \in \mathbb{C}[\lambda]^{p,q}$ be a polynomial matrix with full row rank* $\operatorname{rank}_{\mathbb{C}(\lambda)}(A) = p$. *Let $b \in \mathcal{E}_p^+$ be arbitrary. Then there exists $x \in \mathcal{E}_q^+$ such that*

$$A\left(\frac{d}{dt}\right)x(t) = b(t).$$

Proof. Let a Smith-form (3) of A be given by $A = UDV$ where $U \in \mathbb{C}[\lambda]^{p,p}$ and $V \in \mathbb{C}[\lambda]^{q,q}$ are unimodular and $D \in \mathbb{C}[\lambda]^{p,q}$ is diagonal of the form

$$D = \begin{bmatrix} d_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & d_p & 0 & \cdots & 0 \end{bmatrix},$$

with $d_1, \ldots, d_p \in \mathbb{C}[\lambda]$ and $d_i \neq 0$ for $i = 1, \ldots, p$. Set $\tilde{b} := U^{-1}\left(\frac{d}{dt}\right)b$. Then we see that $\tilde{b} \in \mathcal{E}_p^+$ is itself exponentially decaying. For $i = 1, \ldots, p$ define $x_i \in \mathcal{E}^+$ with the help of Lemma 21. Define $\tilde{x} \in \mathcal{E}_q^+$ through

$$\tilde{x} := \begin{bmatrix} x_1 & \cdots & x_p & 0 \cdots & 0 \end{bmatrix}^T$$

and notice that in this case

$$D\left(\frac{d}{dt}\right)\tilde{x} = \tilde{b}$$

Setting $x := V^{-1}\left(\frac{d}{dt}\right)\tilde{x} \in \mathcal{E}_q^+$ proves the claim. □

5 The main results

In this section we present our main results. Before this, we have to introduce some further notation.

Definition 23. Let $P \in \mathbb{C}[\lambda]^{p,q}$. Then we call $P^{\sim} \in \mathbb{C}[\lambda]^{q,p}$ defined through

$$P^{\sim}(\lambda) := P^*(-\overline{\lambda}),$$

for $\lambda \in \mathbb{C}$. The *para-Hermitian* of P. If $p = q$ and we have $P = P^{\sim}$ the matrix polynomial P is called *para-Hermitian*.

Remark 24. Para-Hermitian matrix polynomials are sometimes also called *even*, compare [8, 11, 12].

To see that $P^{\sim}$ in Definition 23 is indeed again a polynomial in λ let P take the form $P(\lambda) = \sum_{i=0}^{\pi} \lambda^i P_i$. Then we have

$$P^{\sim}(\lambda) \quad = \quad P^*(-\overline{\lambda}) = \left(\sum_{i=0}^{\pi} \left(-\overline{\lambda}\right)^i P_i\right)^* = \sum_{i=0}^{\pi} \lambda^i (-1)^i P_i^*, \tag{9}$$

i.e., a polynomial in λ. Equating coefficients and using the identity (9) we also see that for a matrix polynomial to be para-Hermitian its coefficients have to fulfill

$$P_i = (-1)^i P_i^*,$$

for all $i = 0, \ldots, \pi$, i.e., the coefficients of a para-Hermitian matrix polynomial have to be alternating Hermitian and skew-Hermitian. One can easily check that for polynomial matrices A and B of proper dimension we have that $(AB)^\sim = B^\sim A^\sim$ as well as $(A^\sim)^\sim = A$. Also, for a para-Hermitian matrix polynomial $D = D^\sim$ and a general matrix polynomial V of proper dimension one obtains that also $V^\sim DV$ is again a para-Hermitian matrix.

Lemma 25. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial of the form $P(\lambda) = \sum_{i=0}^n \lambda^i P_i$, let $y \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^p)$, and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$. Then for all $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$ we have that*

$$\begin{aligned} \int_{t_0}^{t_1} z^*(t) P^\sim\left(\frac{d}{dt}\right) y(t) dt &= \int_{t_0}^{t_1} \left(P\left(\frac{d}{dt}\right) z(t)\right)^* y(t) dt \\ &+ \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} \left(P_i z^{(j)}(t)\right)^* y^{(i-1-j)}(t)\Big|_{t_0}^{t_1} \end{aligned} \tag{10}$$

Proof. See [2, Lemma 31]. □

Lemma 25 implies that $P\left(\frac{d}{dt}\right)$ and $P^\sim\left(\frac{d}{dt}\right)$ are adjoint operators when considered with respect to the scalar product $\langle f, g\rangle_{L_2} := \int_{\mathbb{R}} f^*(t) g(t) dt$ over the smooth functions with compact support, since in this case only the integrals in (10) are left and the sums (which stem from partial integration) vanish for functions with compact support.

Theorem 26. *Let $P \in \mathbb{C}[\lambda]^{p,q}$, $\ell \in \mathbb{N}$, and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1)}$ and define the para-Hermitian polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through*

$$H(\lambda) := \left(\Delta_\ell^q(\lambda)\right)^\sim \tilde{H} \left(\Delta_\ell^q(\lambda)\right).$$

Let P be dissipative with respect to $\tilde{H}$. Let $\hat{z} \in \mathcal{E}_q^+$ and $\hat{\mu} \in \mathcal{E}_p^+$ be such that

$$\begin{bmatrix} 0 & P\left(\frac{d}{dt}\right) \\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{z} \end{bmatrix} = 0.$$

Then, we have

$$\int_0^\infty \left(\Delta_\ell \hat{z}(t)\right)^* \tilde{H} \left(\Delta_\ell \hat{z}(t)\right) dt = \inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t) = \hat{z}(t), t \leq 0}} \int_0^\infty \left(\Delta_\ell z(t)\right)^* \tilde{H} \left(\Delta_\ell z(t)\right) dt.$$

Proof. Let $v \in \mathfrak{B}_+(P)$ be arbitrary such that $v(t) = \hat{z}(t)$ for $t \leq 0$. Then we see that for every $s \in \mathbb{R}$ the function $z = s\hat{z} + (1-s)v \in \mathfrak{B}_+(P)$ is also a trajectory of the system due to linearity. Also we have that $z(t) = \hat{z}(t)$ for all $t \leq 0$. Define the function $\Phi_v : \mathbb{R} \to \mathbb{R}$ through

$$\begin{aligned}
\Phi_v(s) \quad &:= \int_0^\infty (s\Delta_\ell \hat{z}(t) + (1-s)\Delta_\ell v(t))^* \tilde{H} (s\Delta_\ell \hat{z}(t) + (1-s)\Delta_\ell v(t))\, dt \\
&= \int_0^\infty s^2 (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) + 2s(1-s)\mathrm{Re}\left\{(\Delta_\ell v(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\right\} \\
&\quad +(1-s)^2 (\Delta_\ell v(t))^* \tilde{H} (\Delta_\ell v(t))\, dt.
\end{aligned}$$

In the following we are going to show that Φ_v has a minimum in $s = 1$. Since v is assumed to be arbitrary this then implies the optimality of $\hat{z}$. To show that Φ_v has a minimum in $s = 1$ we consider its first and second derivative. Differentiation of Φ_v yields

$$\begin{aligned}
\frac{d}{dt}\Phi_v(s) \quad &= \int_0^\infty 2s (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) + 2(1-2s)\mathrm{Re}\left\{(\Delta_\ell v(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\right\} \\
&\quad -2(1-s) (\Delta_\ell v(t))^* \tilde{H} (\Delta_\ell v(t))\, dt,
\end{aligned}$$

and evaluation at the point $s = 1$ implies

$$\begin{aligned}
\frac{d}{ds}\Phi_v(1) \quad &= \int_0^\infty 2 (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) - 2\mathrm{Re}\left\{(\Delta_\ell v(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\right\} dt \\
&= 2\int_0^\infty \mathrm{Re}\left\{(\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) - (\Delta_\ell v(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\right\} dt \\
&= 2\mathrm{Re}\left\{\int_0^\infty (\Delta_\ell \hat{z}(t) - \Delta_\ell v(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt\right\}.
\end{aligned}$$

Defining $y := \hat{z} - v$ we see that $y \in \mathfrak{B}_+(P)$ with

$$y(t) = 0 \tag{11}$$

for all $t \leq 0$ and we have that

$$\frac{d}{ds}\Phi_v(1) \quad = \quad 2\mathrm{Re}\left\{\int_0^\infty (\Delta_\ell y(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt\right\}.$$

Using Lemma 25 we find that for all $t_0 \leq t_1$ we have

$$\begin{aligned}
&\int_{t_0}^{t_1} y^*(t) H\left(\frac{d}{dt}\right) \hat{z}(t) dt \\
= &\int_{t_0}^{t_1} y^*(t) \begin{bmatrix} I_q & \left(-\frac{d}{dt}\right) I_q & \cdots & \left(-\frac{d}{dt}\right)^\ell I_q \end{bmatrix} \tilde{H} (\Delta_\ell \hat{z}(t))\, dt \\
= &\int_{t_0}^{t_1} (\Delta_\ell y(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt \\
&+ \sum_{i=0}^{\ell} \sum_{j=0}^{i-1} (-1)^{i+j} \left(y^{(j)}(t)\right)^* \begin{bmatrix} H_{i,0} & \cdots & H_{i,l} \end{bmatrix} (\Delta_\ell \hat{z}(t)) \Big|_{t_0}^{t_1},
\end{aligned}$$

and thus, since $\lim_{t\to\infty} \hat{z}^{(j)}(t) = 0$ for $j = 0, 1, \ldots$, also

$$
\begin{aligned}
& \frac{d}{ds}\Phi_v(1) \\
= \;& 2\mathrm{Re}\left\{\int_0^\infty y^*(t) H\left(\frac{d}{dt}\right)\hat{z}(t)dt\right\} \\
& -2\mathrm{Re}\left\{\sum_{i=0}^{\ell}\sum_{j=0}^{i-1}(-1)^{i+j}\underbrace{\left(y^{(j)}(0)\right)^*}_{=0}\begin{bmatrix} H_{i,0} & \cdots & H_{i,l}\end{bmatrix}(\Delta_\ell \hat{z}(0))\right\} \\
= \;& 2\mathrm{Re}\left\{-\int_0^\infty y^*(t) P^\sim\left(\frac{d}{dt}\right)\hat{\mu}(t)dt\right\} \\
= \;& 2\mathrm{Re}\left\{-\int_0^\infty \left(\underbrace{P\left(\frac{d}{dt}\right)y(t)}_{=0}\right)^*\hat{\mu}(t)dt\right\} \\
& +2\mathrm{Re}\left\{\sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}\left(P_i y^{(j)}(t)\right)^*\hat{\mu}^{(i-1-j)}(t)\Big|_0^\infty\right\} \\
= \;& 2\mathrm{Re}\left\{-\sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}\left(P_i y^{(j)}(0)\right)^*\hat{\mu}^{(i-1-j)}(0)\right\} = 0,
\end{aligned}
$$

where for the last identity we again used Lemma 25 and condition (11). The second derivative of Φ_v is given by

$$
\begin{aligned}
& \left(\frac{d}{dt}\right)^2\Phi_v(s) \\
= \;& 2\int_0^\infty (\Delta_\ell \hat{z}(t))^*\tilde{H}(\Delta_\ell\hat{z}(t)) - 2\mathrm{Re}\left\{(\Delta_\ell v(t))^*\tilde{H}(\Delta_\ell \hat{z}(t))\right\} \\
& + (\Delta_\ell v(t))^*\tilde{H}(\Delta_\ell v(t))\,dt \\
= \;& 2\int_0^\infty (\Delta_\ell\hat{z}(t) - \Delta_\ell v(t))^*\tilde{H}(\Delta_\ell\hat{z}(t) - \Delta_\ell v(t))\,dt \\
= \;& 2\int_0^\infty (\Delta_\ell y(t))^*\tilde{H}(\Delta_\ell y(t))\,dt \\
= \;& 2\int_{-\infty}^\infty (\Delta_\ell y(t))^*\tilde{H}(\Delta_\ell y(t))\,dt \geq 0,
\end{aligned}
$$

where the last inequality follows from the dissipativity of the system and Lemma 17 and the last identity follows from (11). □

Corollary 27. *Under the assumptions of Theorem 26 and with the representation of P given by*

$$P(\lambda) = \sum_{i=0}^{\pi} \lambda^i P_i$$

we also have that

$$\int_0^\infty (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt = \inf_{\substack{\Delta_{\ell-1} z(0) = \Delta_{\ell-1} \hat{z}(0) \\ P_i z^{(j)}(0) = P_i \hat{z}^{(j)}(0),\ 0 \le j < i \le \pi}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,$$

where the infimum has to be taken over all $z \in \mathfrak{B}_+(P)$.

Proof. The proof is completely analogous to the proof of Theorem 26. □

Theorem 28. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ for some $\ell \in \mathbb{N}$ and define the para-Hermitian polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through*

$$H(\lambda) := (\Delta_\ell^q(\lambda))^\sim \tilde{H} (\Delta_\ell^q(\lambda)).$$

Let $\hat{z} \in \mathfrak{B}_+(P)$ be such that

$$\int_0^\infty (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt = \inf_{\substack{z \in \mathfrak{B}_+(P) \\ z(t) = \hat{z}(t), t \le 0}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt. \tag{12}$$

Then, P is dissipative with respect to $\tilde{H}$ and there exists a co-state function $\hat{\mu} \in \mathcal{E}_p^+$ such that we have

$$\begin{bmatrix} 0 & P\left(\frac{d}{dt}\right) \\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{z} \end{bmatrix} = 0. \tag{13}$$

Proof. Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ be a polynomial kernel and a co-kernel matrix, resp., which exist due to Theorem 7. Let $z_0 \in \mathfrak{B}_+(P)$ be an arbitrary function with $z_0(t) = 0$ for all $t \le 0$. Set

$$z_\epsilon(t) := (\hat{z} + \epsilon z_0)(t),$$

for $\epsilon \in \mathbb{R}$. Then we have

$$\begin{aligned}
& \int_0^\infty (\Delta_\ell z_\epsilon(t))^* \tilde{H} (\Delta_\ell z_\epsilon(t))\, dt \\
= \; & \int_0^\infty (\Delta_\ell \hat{z}(t) + \epsilon \Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t) + \epsilon \Delta_\ell z_0(t))\, dt \\
= \; & \int_0^\infty (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt + 2\epsilon \int_0^\infty \operatorname{Re}\left\{ (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) \right\} dt \\
& + \epsilon^2 \int_0^\infty (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell z_0(t))\, dt,
\end{aligned}$$

and thus, since $\hat{z}$ is an optimal trajectory in the sense of the assumption, we find that

$$\begin{aligned} 0 &\leq \int_0^\infty (\Delta_\ell z_\epsilon(t))^* \tilde{H} (\Delta_\ell z_\epsilon(t))\, dt - \int_0^\infty (\Delta_\ell \hat{z}(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt \\ &= 2\epsilon \int_0^\infty \operatorname{Re}\left\{ (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) \right\} dt \\ &\quad + \epsilon^2 \int_0^\infty (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell z_0(t))\, dt \end{aligned} \tag{14}$$

for all $\epsilon \in \mathbb{R}$. This implies that

$$0 \leq \int_0^\infty (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell z_0(t))\, dt = \int_{-\infty}^\infty (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell z_0(t))\, dt,$$

since $z_0(t) = 0$ for all $t \leq 0$. Because $\mathfrak{B}_c(P)$ is shift invariant and a subset of the functions which are in $\mathfrak{B}_+(P)$ with the additional property that they vanish on the negative time axis, this implies dissipativity of P with respect to $\tilde{H}$, since z_0 is allowed to be arbitrary. Also we see that (14) implies

$$0 = \int_0^\infty \operatorname{Re}\left\{ (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) \right\} dt.$$

If $z_0 \in \mathfrak{B}_+(P)$ is a trajectory of P with $z_0(t) = 0$ for $t \leq 0$, so is iz_0 with i begin the imaginary unit. Thus we obtain

$$\begin{aligned} 0 &= \int_0^\infty \operatorname{Re}\left\{ (\Delta_\ell i z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) \right\} dt = \int_0^\infty \operatorname{Re}\left\{ i\, (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) \right\} dt \\ &= -\int_0^\infty \operatorname{Im}\left\{ (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t)) \right\} dt. \end{aligned}$$

Using Lemma 25 this implies

$$\begin{aligned} 0 &= \int_0^\infty (\Delta_\ell z_0(t))^* \tilde{H} (\Delta_\ell \hat{z}(t))\, dt \\ &= \int_0^\infty \left(\Delta_\ell^q \left(\frac{d}{dt}\right) z_0(t) \right)^* \tilde{H} \left(\Delta_\ell^q \left(\frac{d}{dt}\right) \hat{z}(t) \right) dt \\ &= \int_0^\infty z_0^*(t) \Delta_\ell^{q\sim} \left(\frac{d}{dt}\right) \tilde{H} \Delta_\ell^q \left(\frac{d}{dt}\right) \hat{z}(t) dt \\ &= \int_0^\infty z_0^*(t) H \left(\frac{d}{dt}\right) \hat{z}(t) dt \end{aligned}$$

for all $z_0 \in \mathfrak{B}_+(P)$ with $z_0(t) = 0$ for $t \leq 0$. Using Lemma 13 we see that for every $\alpha \in \mathcal{E}_{q-r}^+$ with $\alpha(t) = 0$ for all $t \leq 0$, we have that the specific $z_0 := U\left(\frac{d}{dt}\right)\alpha \in \mathfrak{B}_+(P)$ is a trajectory of the system. In this case also $z_0(t) = U\left(\frac{d}{dt}\right)\alpha(t) = 0$ for all $t \leq 0$. Thus, with Lemma 25 we have

$$0 = \int_0^\infty \left(U \left(\frac{d}{dt}\right) \alpha(t) \right)^* H \left(\frac{d}{dt}\right) \hat{z}(t) dt$$

$$= \int_0^\infty \alpha^*(t) U^\sim \left(\frac{d}{dt}\right) H \left(\frac{d}{dt}\right) \hat{z}(t) dt,$$

for all $\alpha \in \mathcal{E}_{q-r}^+$ with $\alpha(t) = 0$ for $t \leq 0$. From this we deduce that

$$U^\sim \left(\frac{d}{dt}\right) H \left(\frac{d}{dt}\right) \hat{z}(t) = 0,$$

for all $t \geq 0$. Note, that we are looking for a $\hat{\mu}$ such that $P^\sim \left(\frac{d}{dt}\right) \hat{\mu} = -H \left(\frac{d}{dt}\right) \hat{z}$. Since the polynomial matrix $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular this problem is equivalent to the problem of finding a $\hat{\mu}$ such that

$$\begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^\sim P^\sim \left(\frac{d}{dt}\right) \hat{\mu} = - \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^\sim H \left(\frac{d}{dt}\right) \hat{z}.$$

Thus, let $\hat{\mu} \in \mathcal{E}_p^+$ be a solution of the problem

$$(PV)^\sim \left(\frac{d}{dt}\right) \hat{\mu}(t) = -V^\sim \left(\frac{d}{dt}\right) H \left(\frac{d}{dt}\right) \hat{z}(t),$$

which exists due to Lemma 22 and property 4. of Definition 8, i.e., since V is a co-kernel matrix. Using that U is a kernel matrix we find

$$\begin{aligned}
P^\sim \left(\frac{d}{dt}\right) \hat{\mu} &= \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^{-\sim} \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^\sim P^\sim \left(\frac{d}{dt}\right) \hat{\mu} \\
&= \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^{-\sim} \begin{bmatrix} (PU)^\sim \left(\frac{d}{dt}\right) \hat{\mu} \\ (PV)^\sim \left(\frac{d}{dt}\right) \hat{\mu} \end{bmatrix} \\
&= \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^{-\sim} \begin{bmatrix} 0 \\ (PV)^\sim \left(\frac{d}{dt}\right) \hat{\mu} \end{bmatrix} \\
&= \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^{-\sim} \begin{bmatrix} -U^\sim \left(\frac{d}{dt}\right) H \left(\frac{d}{dt}\right) \hat{z} \\ -V^\sim \left(\frac{d}{dt}\right) H \left(\frac{d}{dt}\right) \hat{z} \end{bmatrix} \\
&= \begin{bmatrix} U\left(\frac{d}{dt}\right) & V\left(\frac{d}{dt}\right) \end{bmatrix}^{-\sim} \begin{bmatrix} -U^\sim \left(\frac{d}{dt}\right) \\ -V^\sim \left(\frac{d}{dt}\right) \end{bmatrix} H \left(\frac{d}{dt}\right) \hat{z} \\
&= -H \left(\frac{d}{dt}\right) \hat{z},
\end{aligned}$$

which finishes the proof. □

Looking back at the optimality system (13) we see that such an optimality system is a boundary value problem for a homogeneous linear differential-algebraic equation. It is well-known that to determine the solution set of such a system we have to consider the finite eigenvalues and the right singular structure of the associated matrix polynomial

$$\begin{bmatrix} 0 & P(\lambda) \\ P^\sim(\lambda) & H(\lambda) \end{bmatrix}, \tag{15}$$

which is itself a para-Hermitian matrix polynomial, since H is para-Hermitian. The eigenvalues of a para-Hermitian matrix have an interesting property.

Lemma 29. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial. Then we have*

$$\sigma(P^{\sim}) = -\overline{\sigma(P)}.$$

Especially, if $p = q$ and $P = P^{\sim}$ is para-Hermitian, we have that

$$\sigma(P) = -\overline{\sigma(P)},$$

i.e., the spectrum of an para-Hermitian matrix is symmetric to the imaginary axis.

Proof. First note that for every unimodular matrix $T \in \mathbb{C}[\lambda]^{q,q}$ also its para-Hermitian $T^{\sim}$ is unimodular. Let a Smith form (3) of P be given by

$$P = S \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

Then, we have that

$$\begin{aligned} P^{\sim} &= T^{\sim} \begin{bmatrix} \operatorname{diag}(d_1, \ldots, d_r)^{\sim} & 0 \\ 0 & 0 \end{bmatrix} S^{\sim} \\ &= T^{\sim} \begin{bmatrix} \operatorname{diag}(d_1^{\sim}, \ldots, d_r^{\sim}) & 0 \\ 0 & 0 \end{bmatrix} S^{\sim}, \end{aligned}$$

and since every d_i can be factored into a product of linear polynomials, it is sufficient to show that for $p(\lambda) := \lambda - a$ with $a \in \mathbb{C}$ we have $\sigma(p^{\sim}) = -\overline{\sigma(p)}$. Since, however,

$$(\lambda - a)^{\sim} = (-\overline{\lambda} - a)^* = -\lambda - \overline{a},$$

this is clearly the case. □

Lemma 29 shows that for a para-Hermitian matrix polynomial there are exactly as many eigenvalues in the strict left half plane $\mathbb{C}^-$ as there are in the strict right half plane $\mathbb{C}^+$. Since we are looking for solutions $z \in \mathcal{E}_+^q$ and $\mu \in \mathcal{E}_+^p$ we are especially interested in the eigenvalues in the strict left half plane. To be more specific, assume that $\lambda \in \mathbb{C}^-$ is an eigenvalue in the left half plane associated with the eigenvector $\begin{bmatrix} \hat{z}^T & \hat{\mu}^T \end{bmatrix}^T \in \mathbb{C}^{p+q}$ of the matrix polynomial (15). Then we can use Theorem 26 to obtain that

$$\hat{z}(t) = z(t) := e^{\lambda t} \hat{z},$$

is an optimal trajectory of the system. Theorem 28, on the other hand, states that every optimal trajectory $\hat{z}$ can be extended by a co-state function $\hat{\mu}$ to a solution of the system (13).

Using the Smith form of (15) this shows that for a dissipative system we obtain all optimal solutions of the optimal control problem by considering all eigenvalues of (15) which are in the strict left half plane, the complete right singular structure of (15), and the eigenvalues of (15) which are on the imaginary axis.

Finally, note that Theorem 26 and Theorem 28 justify to say that dissipativity is equivalent to the solvability of the optimal control problem.

6 Specialization to first order systems

In the last section we have derived optimality conditions for problems where the infimum is taken over all trajectories z which match a given prescribed trajectory $\hat{z}$ (which has been assumed to be the optimal trajectory) along the negative time axis, i.e., over all trajectories z with $z(t) = \hat{z}(t)$ for $t \le 0$. Nevertheless, it seems quite intuitive that the optimal trajectory will not depend on the complete history of $\hat{z}$, i.e., on $\hat{z}(t)$ with $t \le 0$ but only on the value of $\hat{z}$ (and maybe its derivatives) at the time point $t = 0$. Corollary 27 gives further evidence that this is the case.

In this section we will first show that for systems given by first order matrix polynomials, the optimal trajectory indeed only depends on the value of $\hat{z}$ at 0 by using the Kronecker canonical form and conjecture that it may also hold true for higher order systems. In the subsequent subsection we will use this to derive well-known results for descriptor systems and standard systems.

Theorem 30. *Let $P_1, P_0 \in \mathbb{C}^{p,q}$. Then there exist nonsingular matrices $X \in \mathbb{C}^{p,p}$ and $Y \in \mathbb{C}^{q,q}$ such that*

$$X(\lambda P_1 + P_0)Y = \operatorname{diag}\left(\mathcal{L}_{\zeta_1}, \ldots, \mathcal{L}_{\zeta_\mu}, \mathcal{J}_{\rho_1}, \ldots, \mathcal{J}_{\rho_r}, \mathcal{N}_{\sigma_1}, \ldots, \mathcal{N}_{\sigma_s}, \mathcal{M}_{\eta_1}, \ldots, \mathcal{M}_{\eta_v}\right), \tag{16}$$

where the block entries have the following properties:

1. *Every entry $\mathcal{L}_{\zeta_j}$ has the size $\zeta_j \times (\zeta_j + 1)$, $\zeta_j \in \mathbb{N}_0$ and the form*

$$\mathcal{L}_{\zeta_j}(\lambda) := \lambda \begin{bmatrix} -1 & 0 & & \\ & \ddots & \ddots & \\ & & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}. \tag{17}$$

2. *Every entry $\mathcal{J}_{\rho_j}$ has the size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$ and the form*

$$\mathcal{J}_{\rho_j}(\lambda) := \lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \tag{18}$$

   where $\lambda_j \in \mathbb{C}$.

3. *Every entry $\mathcal{N}_{\sigma_j}$ has the size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$ and the form*

$$\mathcal{N}_{\sigma_j}(\lambda) := \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} + \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{19}$$

4. *Every entry $\mathcal{M}_{\eta_j}$ has the size $(\eta_j + 1) \times \eta_j$, $\eta_j \in \mathbb{N}_0$ and the form*

$$\mathcal{M}_{\eta_j}(\lambda) := \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}. \tag{20}$$

Proof. See, e.g., [6, Theorem 3, §2, p.28]. □

The canonical form (16) is called Kronecker canonical form.

Lemma 31. *Let $\theta \in \mathbb{N}_0$ and $\alpha \in \mathcal{C}_c^\infty$ be such that*

$$\Delta_{\theta-1}\alpha(0) = 0.$$

Let $\tilde{b} : \mathbb{R} \to \mathbb{R}$ be a smooth transition from 1 to 0, i.e., a function such that

$$\tilde{b}(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 1 \end{cases},$$

while $\tilde{b} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is infinitely often differentiable. Then, there exists a constant $C \in \mathbb{R}^+$ such that we have

$$\left\| \left(\frac{d}{dt}\right)^i \left[\tilde{b}(k\cdot)\alpha(\cdot)\right] \right\|_+ \leq \frac{C}{k^{\theta-i}} \frac{1}{\sqrt{k}},$$

for $i = 0, \ldots, \theta$.

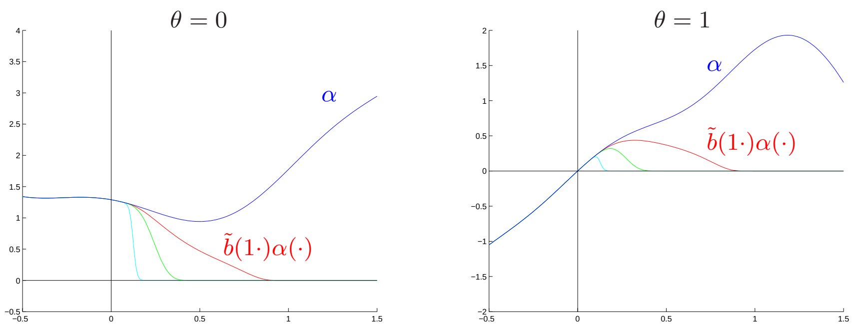

Figure 2: Two different cases of θ

Proof. Define the constant $B \in \mathbb{R}^+$ through

$$B := \max_{j=0,\dots,\theta} \max_{t\in[0,1]} \left|\tilde{b}^{(j)}(t)\right|,$$

and the constant $A \in \mathbb{R}^+$ through

$$A := \max_{t\in[0,1]} \left|\alpha^{(\theta)}(t)\right|.$$

Using Taylor expansion we find that for $\theta \geq i \geq j \geq 0$ and for $t \in [0,1]$ there exists a $\xi \in [0,1]$ such that we have

$$\begin{aligned} \left|\alpha^{(i-j)}(t)\right| &= \left|\left(\sum_{k=i-j}^{\theta-1} \frac{t^{k-i+j}}{(k-i+j)!}\alpha^{(k)}(0)\right) + \frac{t^{\theta-i+j}}{(\theta-i+j)!}\alpha^{(\theta)}(\xi)\right| \\ &= \left|\frac{t^{\theta-i+j}}{(\theta-i+j)!}\alpha^{(\theta)}(\xi)\right| \leq \frac{t^{\theta-i+j}}{(\theta-i+j)!}A. \end{aligned}$$

With this and using the Leibniz rule for differentiation we see that for $i = 0,\dots,\theta$ we have

$$\begin{aligned} &\left\|\left(\frac{d}{dt}\right)^i \left[\tilde{b}(k\cdot)\alpha(\cdot)\right]\right\|_+ \\ =\ &\left\|\sum_{j=0}^{i} \binom{i}{j} k^j \tilde{b}^{(j)}(k\cdot)\alpha^{(i-j)}(\cdot)\right\|_+ \\ \leq\ &\sum_{j=0}^{i} \binom{i}{j}^2 k^j \left\|\tilde{b}^{(j)}(k\cdot)\alpha^{(i-j)}(\cdot)\right\|_+ \\ =\ &\sum_{j=0}^{i} \binom{i}{j}^2 k^j \sqrt{\int_0^\infty \left|\tilde{b}^{(j)}(kt)\alpha^{(i-j)}(t)\right|^2 dt} \\ =\ &\sum_{j=0}^{i} \binom{i}{j}^2 k^j \sqrt{\int_0^{\frac{1}{k}} \left|\tilde{b}^{(j)}(kt)\alpha^{(i-j)}(t)\right|^2 dt} \\ \leq\ &\sum_{j=0}^{i} \binom{i}{j}^2 k^j B \sqrt{\int_0^{\frac{1}{k}} \left|\alpha^{(i-j)}(t)\right|^2 dt} \\ \leq\ &\sum_{j=0}^{i} \binom{i}{j}^2 k^j B \sqrt{\int_0^{\frac{1}{k}} \frac{t^{2\theta-2i+2j}}{(\theta-i+j)!^2} A^2 dt} \\ =\ &\sum_{j=0}^{i} \binom{i}{j}^2 k^j \frac{BA}{(\theta-i+j)!} \sqrt{\int_0^{\frac{1}{k}} t^{2\theta-2i+2j} dt} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{i} \binom{i}{j}^2 k^j \frac{BA}{(\theta-i+j)!} \sqrt{\frac{1}{2\theta-2i+2j+1} t^{2\theta-2i+2j+1}\Big|_0^{\frac{1}{k}}} \\
&= \sum_{j=0}^{i} \binom{i}{j}^2 \frac{k^j}{k^{\theta-i+j}\sqrt{k}} \frac{BA}{(\theta-i+j)!} \frac{1}{\sqrt{2\theta-2i+2j+1}} \\
&= \sum_{j=0}^{i} \binom{i}{j}^2 \frac{1}{k^{\theta-i}\sqrt{k}} \frac{BA}{(\theta-i+j)!} \frac{1}{\sqrt{2\theta-2i+2j+1}} \\
&= \frac{1}{k^{\theta-i}\sqrt{k}} \sum_{j=0}^{i} \binom{i}{j}^2 \frac{BA}{(\theta-i+j)!} \frac{1}{\sqrt{2\theta-2i+2j+1}} \\
&=: \frac{C}{k^{\theta-i}\sqrt{k}}
\end{aligned}
$$

by setting

$$
C := \sum_{j=0}^{i} \binom{i}{j}^2 \frac{BA}{(\theta-i+j)!} \frac{1}{\sqrt{2\theta-2i+2j+1}},
$$

which yields the assertion. □

Lemma 32. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a first order polynomial, i.e., let P take the form*

$$
P(\lambda) = \lambda P_1 + P_0,
$$

with $P_0, P_1 \in \mathbb{C}^{p,q}$. Let $\ell \in \mathbb{N}$ and $\hat{z} \in \mathfrak{B}_c(P)$ be a fixed trajectory which fulfills $\Delta_{\ell-1}\hat{z}(0) = 0$ and $P_1\hat{z}(0) = 0$. Then for any $\epsilon > 0$ there exists a trajectory $z_\epsilon \in \mathfrak{B}_c(P)$ such that $z_\epsilon(t) = \hat{z}(t)$ for all $t \leq 0$ and

$$
\|\Delta_\ell z_\epsilon\|_+ < \epsilon.
$$

Proof. Let the Kronecker canonical form of P be given by (16), i.e., let $X \in \mathbb{C}^{p,p}$ and $Y \in \mathbb{C}^{q,q}$ be nonsingular such that

$$
X(\lambda P_1 + P_0)Y = \operatorname{diag}\left(\mathcal{L}_{\zeta_1}, \ldots, \mathcal{L}_{\zeta_\mu}, \mathcal{J}_{\rho_1}, \ldots, \mathcal{J}_{\rho_r}, \mathcal{N}_{\sigma_1}, \ldots, \mathcal{N}_{\sigma_s}, \mathcal{M}_{\eta_1}, \ldots, \mathcal{M}_{\eta_v}\right).
$$

Define $\zeta := \zeta_1 + \ldots + \zeta_\mu$, $\rho := \rho_1 + \ldots + \rho_r$, $\sigma := \sigma_s + \ldots + \sigma_s$, and $\eta := \eta_1 + \ldots + \eta_v$ and define $U \in \mathbb{C}[\lambda]^{q,\mu}$ through

$$
U(\lambda) := Y \begin{bmatrix} \Delta^1_{\zeta_1}(\lambda) & & 0 \\ & \ddots & \\ 0 & & \Delta^1_{\zeta_\mu}(\lambda) \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.
$$

One can easily see that such a U is a (polynomial) kernel matrix according to Definition 8. Using Lemma 14 we see that that this implies that there exists $\hat{\alpha} \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^u)$ such that $\hat{z} = U\left(\frac{d}{dt}\right)\hat{\alpha}$. Denoting the elements of $\hat{\alpha}$ by $\hat{\alpha}_i$ we find that

$$\hat{z} = Y \begin{bmatrix} \Delta_{\zeta_1}\hat{\alpha}_1 \\ \vdots \\ \Delta_{\zeta_\mu}\hat{\alpha}_\mu \\ 0_{\rho+\sigma+\eta} \end{bmatrix},$$

where $0_{\rho+\sigma+\eta}$ denotes the zero vector of size $(\rho+\sigma+\eta)$-by-1. Introduce the notation

$$\mathcal{L}_{\zeta_i} =: \lambda\mathcal{L}^1_{\zeta_i} + \mathcal{L}^0_{\zeta_i},$$

with $\mathcal{L}^1_{\zeta_i}, \mathcal{L}^0_{\zeta_i} \in \mathbb{C}^{\zeta_i,\zeta_i+1}$ to denote the left and right matrix in (17). Since we know that $P_1\hat{z}(0) = 0$ we then deduce that

$$0 = XP_1\hat{z}(0) = XP_1Y \begin{bmatrix} \Delta_{\zeta_1}\hat{\alpha}_1(0) \\ \vdots \\ \Delta_{\zeta_\mu}\hat{\alpha}_\mu(0) \\ 0_{\rho+\sigma+\eta} \end{bmatrix} = \begin{bmatrix} \mathcal{L}^1_{\zeta_1}\Delta_{\zeta_1}\hat{\alpha}_1(0) \\ \vdots \\ \mathcal{L}^1_{\zeta_\mu}\Delta_{\zeta_\mu}\hat{\alpha}_\mu(0) \\ 0_{\rho+\sigma+\eta} \end{bmatrix},$$

and thus

$$\mathcal{L}^1_{\zeta_i}\Delta_{\zeta_i}\hat{\alpha}_i(0) = 0,$$

for $i = 1, \ldots, \mu$. By using the definition of $\mathcal{L}^1_{\zeta_i}$ from (17) we see that this implies

$$\Delta_{\zeta_i-1}\hat{\alpha}_i(0) = 0, \tag{21}$$

for $i = 1, \ldots, \mu$. If $\ell > 0$, then we obtain from $\Delta_{\ell-1}\hat{z}(0) = 0$ that

$$\Delta_{\zeta_i+\ell-1}\hat{\alpha}_i(0) = 0.$$

If, however, $\ell = 0$ then we obtain no additional condition. In any case, if $\ell \geq 0$ we obtain together with (21) that we have

$$\Delta_{\zeta_i+\ell-1}\hat{\alpha}_i(0) = 0, \tag{22}$$

for $i = 1, \ldots, \mu$. Let $\tilde{b} : \mathbb{R} \to \mathbb{R}$ be a smooth transition from 1 to 0, i.e., a function such that

$$\tilde{b}(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 1 \end{cases},$$

while $\tilde{b} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is infinitely often differentiable. Define the sequence of functions $b_k : \mathbb{R} \to \mathbb{R}$ through

$$b_k(t) = \tilde{b}(kt),$$

for $k \in \mathbb{N}$ and observe that all $b_k \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ are infinitely often differentiable and we have $b_k(t) = 1$ for $t < 0$ as well as $b_k(t) = 0$ for $t > \frac{1}{k}$. Define the sequence of functions $z_k \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C}^q)$ through

$$z_k(t) := U\left(\frac{d}{dt}\right)[b_k(t)\hat{\alpha}(t)] = Y \begin{bmatrix} \Delta^1_{\zeta_1}\left(\frac{d}{dt}\right)[b_k(t)\hat{\alpha}_1(t)] \\ \vdots \\ \Delta^1_{\zeta_\mu}\left(\frac{d}{dt}\right)[b_k(t)\hat{\alpha}_\mu(t)] \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and observe that Lemma 13 implies that $z_k \in \mathfrak{B}_c(P)$ are trajectories of the system. With this and Lemma 31 we deduce that

$$\begin{aligned} \|\Delta_\ell z_k\|_+^2 &= \sum_{i=0}^{\ell} \left\| z_k^{(i)} \right\|_+^2 \\ &= \sum_{i=0}^{\ell} \left\| Y \begin{bmatrix} \left(\frac{d}{dt}\right)^i \Delta^1_{\zeta_1}\left(\frac{d}{dt}\right)[b_k(t)\hat{\alpha}_1(t)] \\ \vdots \\ \left(\frac{d}{dt}\right)^i \Delta^1_{\zeta_u}\left(\frac{d}{dt}\right)[b_k(t)\hat{\alpha}_u(t)] \end{bmatrix} \right\|_+^2 \\ &\le \|Y\|_+^2 \sum_{i=0}^{\ell}\sum_{j=1}^{u} \left\| \left(\frac{d}{dt}\right)^i \Delta^1_{\zeta_j}\left(\frac{d}{dt}\right)[b_k(t)\hat{\alpha}_j(t)] \right\|_+^2 \\ &= \|Y\|_+^2 \sum_{i=0}^{\ell}\sum_{j=1}^{u}\sum_{\xi=0}^{\zeta_j} \left\| \left(\frac{d}{dt}\right)^{i+\xi} [b_k(t)\hat{\alpha}_j(t)] \right\|_+^2 \\ &\le \|Y\|_+^2 \sum_{i=0}^{\ell}\sum_{j=1}^{u}\sum_{\xi=0}^{\zeta_j} \frac{C_{j,i,\xi}}{k^{2(\zeta_j+\ell-i-\xi)}}\frac{1}{k} \\ &\le \frac{1}{k}\|Y\|_+^2 \sum_{i=0}^{\ell}\sum_{j=1}^{u}\sum_{\xi=0}^{\zeta_j} C_{j,i,\xi} = \frac{D}{k}, \end{aligned}$$

by setting

$$D := \|Y\|_+^2 \sum_{i=0}^{\ell}\sum_{j=1}^{u}\sum_{\xi=0}^{\zeta_j} C_{j,i,\xi}.$$

Choosing k big enough proves the claim. □

Theorem 33. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a first order polynomial matrix, i.e., let P take the form*

$$P(\lambda) = \lambda P_1 + P_0,$$

with $P_0, P_1 \in \mathbb{C}^{p,q}$ *and let* $\hat{z} \in \mathfrak{B}_+(P)$ *be a fixed trajectory. Let* $\tilde{P} \in \mathbb{C}[\lambda]^{\rho,q}$ *be a matrix with*

$$\operatorname{kernel}\left(\tilde{P}\right) \subset \operatorname{kernel}(P_1),$$

where $\rho \in \mathbb{N}$ *is arbitrary. Let* $\tilde{H} \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ *be Hermitian. Then we have*

$$\begin{aligned}
& \inf_{z(t)=\hat{z}(t), t\le 0} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{z^{(j)}(0)=\hat{z}^{(j)}(0),\ j\in\mathbb{N}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ z(0)=\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ \tilde{P}z(0)=\tilde{P}\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ P_1 z(0)=P_1\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,
\end{aligned}$$

where all infima are taken over all $z \in \mathfrak{B}_+(P)$.

Proof. From the basics of the theory of sets we immediately see that

$$\begin{aligned}
& \inf_{z(t)=\hat{z}(t), t\le 0} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
\ge & \inf_{z^{(j)}(0)=\hat{z}^{(j)}(0),\ j\in\mathbb{N}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
\ge & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ z(0)=\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
\ge & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ \tilde{P}z(0)=\tilde{P}\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
\ge & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ P_1 z(0)=P_1\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt.
\end{aligned}$$

Let $z \in \mathfrak{B}_+(P)$ be arbitrary with $\Delta_{\ell-1}z(0) = \Delta_{\ell-1}\hat{z}(0)$ and $P_1 z(0) = P_1\hat{z}(0)$. We will show in the following that for every such trajectory z and every $\epsilon > 0$, there exists a trajectory $z_\epsilon \in \mathfrak{B}_+(P)$ such that

$$\int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt + \epsilon \ge \int_0^\infty (\Delta_\ell z_\epsilon(t))^* \tilde{H} (\Delta_\ell z_\epsilon(t))\, dt,$$

while at the same time $z_\epsilon(t) = \hat{z}(t)$ for all $t \le 0$. From this one obtains that

$$\inf_{z(t)=\hat{z}(t), t\le 0} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt$$

$$\leq \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ P_1 z(0)=P_1\hat{z}(0)}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,$$

and thus the claim follows. Thus, let $z \in \mathfrak{B}_+(P)$ be arbitrary with $\Delta_{\ell-1}z(0) = \Delta_{\ell-1}\hat{z}(0)$ and $P_1 z(0) = P_1\hat{z}(0)$ and let $\epsilon > 0$. Define $y := z - \hat{z}$ and notice that this implies that $\Delta_{\ell-1}y(0) = 0$ as well as $P_1 y(0) = 0$. Using Lemma 32 we find that there exists a trajectory $y_\epsilon \in \mathfrak{B}_+(P)$ such that $y_\epsilon(t) = y(t)$ for $t \leq 0$ and

$$\|\Delta_\ell y_\epsilon\|_+ < \sqrt{\|\Delta_\ell z\|_+^2 + \frac{\epsilon}{\left\|\tilde{H}\right\|_+}} - \|\Delta_\ell z\|_+ . \tag{23}$$

We introduce

$$z_\epsilon := z - y_\epsilon \in \mathfrak{B}_+(P),$$

and see that with this we have

$$z_\epsilon(t) = z(t) - y_\epsilon(t) = z(t) - y(t) = z(t) - (z(t) - \hat{z}(t)) = \hat{z}(t),$$

for $t \leq 0$. We thus obtain the inequality

$$\begin{aligned}
&\int_0^\infty (\Delta_\ell z_\epsilon(t))^* \tilde{H} (\Delta_\ell z_\epsilon(t))\, dt \\
=\ & \left\langle \tilde{H}\Delta_\ell z_\epsilon, \Delta_\ell z_\epsilon \right\rangle_+ \\
=\ & \left\langle \tilde{H}\Delta_\ell z - \tilde{H}\Delta_\ell y_\epsilon, \Delta_\ell z - \Delta_\ell y_\epsilon \right\rangle_+ \\
=\ & \left\langle \tilde{H}\Delta_\ell z, \Delta_\ell z \right\rangle_+ - 2\mathrm{Re}\left\{ \left\langle \tilde{H}\Delta_\ell z, \Delta_\ell y_\epsilon \right\rangle_+ \right\} + \left\langle \tilde{H}\Delta_\ell y_\epsilon, \Delta_\ell y_\epsilon \right\rangle_+ \\
\leq\ & \left\langle \tilde{H}\Delta_\ell z, \Delta_\ell z \right\rangle_+ + 2\left| \left\langle \tilde{H}\Delta_\ell z, \Delta_\ell y_\epsilon \right\rangle_+ \right| + \left| \left\langle \tilde{H}\Delta_\ell y_\epsilon, \Delta_\ell y_\epsilon \right\rangle_+ \right| \\
\leq\ & \left\langle \tilde{H}\Delta_\ell z, \Delta_\ell z \right\rangle_+ + 2\left\|\tilde{H}\right\|_+ \|\Delta_\ell z\|_+ \|\Delta_\ell y_\epsilon\|_+ + \left\|\tilde{H}\right\|_+ \|\Delta_\ell y_\epsilon\|_+^2 .
\end{aligned} \tag{24}$$

From (23) we obtain that

$$\|\Delta_\ell z\|_+^2 + \frac{\epsilon}{\left\|\tilde{H}\right\|_+} > \left(\|\Delta_\ell y_\epsilon\|_+ + \|\Delta_\ell z\|_+\right)^2 = \|\Delta_\ell y_\epsilon\|_+^2 + 2\|\Delta_\ell y_\epsilon\|_+ \|\Delta_\ell z\|_+ + \|\Delta_\ell z\|_+^2 ,$$

and thus

$$\epsilon > \left\|\tilde{H}\right\|_+ \|\Delta_\ell y_\epsilon\|_+^2 + 2\left\|\tilde{H}\right\|_+ \|\Delta_\ell y_\epsilon\|_+ \|\Delta_\ell z\|_+ .$$

Together with (24) this yields

$$\begin{aligned}
\int_0^\infty (\Delta_\ell z_\epsilon(t))^* \tilde{H} (\Delta_\ell z_\epsilon(t))\, dt &< \left\langle \tilde{H}\Delta_\ell z, \Delta_\ell z \right\rangle_+ + \epsilon \\
&= \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt + \epsilon,
\end{aligned}$$

and thus the assertion follows. □

Theorem 33 primarily shows that when considering a first order optimal control problem of the form (12) on can disregard the history of $\hat{z}$, i.e., all values of $\hat{z}(t)$ for $t < 0$; only the value of $\hat{z}$ and its derivatives at the point $t = 0$ are of interest to determine the optimal solution.

For higher order polynomials we conjecture that an analogous result holds, compare Corollary 27.

Conjecture 34. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a polynomial matrix of the form*

$$P(\lambda) = \sum_{i=0}^{\pi} \lambda^i P_i,$$

with $P_i \in \mathbb{C}^{p,q}$ for $i = 0, \ldots, \pi$ and let $\hat{z} \in \mathfrak{B}_+(P)$ be a fixed trajectory. Let $\tilde{H} \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ be Hermitian. Then we have

$$\begin{aligned}
& \inf_{z(t)=\hat{z}(t), t\leq 0} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{z^{(j)}(0)=\hat{z}^{(j)}(0),\ j\in\mathbb{N}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ z^{(j)}(0)=\hat{z}^{(j)}(0),\ 0\leq j<\pi}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt \\
= & \inf_{\substack{\Delta_{\ell-1} z(0)=\Delta_{\ell-1}\hat{z}(0) \\ P_i z^{(j)}(0)=P_i\hat{z}^{(j)}(0),\ 0\leq j<i\leq\pi}} \int_0^\infty (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t))\, dt,
\end{aligned}$$

where all infima are taken over all $z \in \mathfrak{B}_+(P)$.

To show Conjecture 34 one could first show that for every $\epsilon > 0$ and every fixed trajectory $\hat{y} \in \mathfrak{B}_+(P)$ which fulfills $\Delta_{\ell-1}\hat{y}(0) = 0$ and

$$P_i \hat{y}^{(j)}(0) = 0, \text{ for all } i, j \in \mathbb{N} \text{ with } 0 \leq j < i \leq \pi$$

there exists a trajectory $y_\epsilon \in \mathfrak{B}_+(P)$ such that $y_\epsilon(t) = \hat{y}(t)$ for all $t \leq 0$ and

$$\|\Delta_\ell y_\epsilon\|_+ < \epsilon,$$

i.e., that one can generalize Lemma 32 to higher order matrix polynomials. Then one could carry out the proof of Conjecture 34 analogously to the proof of Theorem 33.

6.1 Descriptor systems

Consider the state-space descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{25}$$

where $E, A \in \mathbb{R}^{\rho,n}$ are rectangular matrices, $B \in \mathbb{R}^{\rho,m}$, $x \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^n)$ is called the state, $u \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^m)$ is called the input, and $y \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^p)$ is called the output. In the literature, e.g., [9], for such systems the cost is frequently measured by a quadratic function $s : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ of the form

$$s(u,y) := \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{26}$$

where $Q = Q^T \in \mathbb{R}^{p,p}$, $S \in \mathbb{R}^{p,m}$, and $R = R^T \in \mathbb{R}^{m,m}$. Under these circumstances system (25) is called dissipative with respect to s, if

$$0 \leq \int_{-\infty}^{\infty} s(u(t),y(t))dt, \tag{27}$$

for all trajectories (u,x,y) of (25) which have compact support, i.e., for all triples $(u,x,y) \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}^m) \times \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}^n) \times \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}^p)$ which fulfill (25) for all $t \in \mathbb{R}$, see [15].

Using the output equation of (25) we can rewrite the cost to depend on the state variables (instead of the output variables) by

$$\begin{aligned} s(u(t),y(t)) &= \begin{bmatrix} Cx(t)+Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t)+Du(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^TQC & C^TQD + C^TS \\ D^TQC + S^TC & D^TQD + D^TS + S^TD + R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &=: \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} =: \tilde{s}(x(t),u(t)), \end{aligned} \tag{28}$$

where $\tilde{Q} = \tilde{Q}^T \in \mathbb{R}^{n,n}$, $\tilde{S} \in \mathbb{R}^{n,m}$, $R \in \mathbb{R}^{m,m}$, and $\tilde{s} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

With this we obtain the following corollaries, which are very similar to [9, 3.17 Theorem] and [9, 3.37 Theorem].

Corollary 35. *Let the system (25) be dissipative with respect to s as given by (26). Define $\tilde{Q}$, $\tilde{S}$, and $\tilde{R}$ through (28). Let $\hat{u} \in \mathcal{E}_m^+$, $\hat{x} \in \mathcal{E}_n^+$, and $\hat{\mu} \in \mathcal{E}_\rho^+$ be such that*

$$\begin{bmatrix} 0 & E & 0 \\ -E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\mu}}(t) \\ \dot{\hat{x}}(t) \\ \dot{\hat{u}}(t) \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^* & \tilde{Q} & \tilde{S} \\ B^* & \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} \hat{\mu}(t) \\ \hat{x}(t) \\ \hat{u}(t) \end{bmatrix}.$$

Then, we have

$$\int_0^\infty \begin{bmatrix} \hat{u}(t) \\ \hat{x}(t) \end{bmatrix} \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} \hat{u}(t) \\ \hat{x}(t) \end{bmatrix} dt \quad = \quad \inf_{\substack{E\dot{x}=Ax+Bu \\ x(0)=\hat{x}(0)}} \int_0^\infty \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

$$= \inf_{\substack{E\dot{x}=Ax+Bu\\ Ex(0)=E\hat{x}(0)}} \int_0^\infty \begin{bmatrix} u(t)\\ x(t)\end{bmatrix} \begin{bmatrix} \tilde{Q} & \tilde{S}\\ \tilde{S}^* & \tilde{R}\end{bmatrix} \begin{bmatrix} u(t)\\ x(t)\end{bmatrix} dt,$$

where the infimum in both cases has to be taken over all trajectories $(u,x) \in \mathcal{E}_m^+ \times \mathcal{E}_n^+$.

Proof. Define the polynomial $P \in \mathbb{C}[\lambda]^{\rho,n+m}$ through

$$P(\lambda) := \lambda \begin{bmatrix} E & 0\end{bmatrix} + \begin{bmatrix} -A & -B\end{bmatrix},$$

the Hermitian matrix $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{n+m,n+m}$ and the (constant) para-Hermitian matrix polynomial $H = H^* \in \mathbb{C}[\lambda]^{n+m,n+m}$ (i.e., matrix polynomial of degree 0) through

$$H(\lambda) := \tilde{H} := \begin{bmatrix} \tilde{Q} & \tilde{S}\\ \tilde{S}^* & \tilde{R}\end{bmatrix},$$

and set

$$\hat{z} := \begin{bmatrix} \hat{x}\\ \hat{u}\end{bmatrix}$$

With this notation we obtain

$$0 = \begin{bmatrix} 0 & P\left(\frac{d}{dt}\right)\\ P^\sim\left(\frac{d}{dt}\right) & -H\left(\frac{d}{dt}\right)\end{bmatrix} \begin{bmatrix} \hat{\mu}\\ \hat{z}\end{bmatrix} = \begin{bmatrix} 0 & -P\left(\frac{d}{dt}\right)\\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right)\end{bmatrix} \begin{bmatrix} \hat{\mu}\\ -\hat{z}\end{bmatrix},$$

from the assumption. From this we deduce that also

$$0 = \begin{bmatrix} -I & 0\\ 0 & I\end{bmatrix} \begin{bmatrix} 0 & -P\left(\frac{d}{dt}\right)\\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right)\end{bmatrix} \begin{bmatrix} \hat{\mu}\\ -\hat{z}\end{bmatrix} = \begin{bmatrix} 0 & P\left(\frac{d}{dt}\right)\\ P^\sim\left(\frac{d}{dt}\right) & H\left(\frac{d}{dt}\right)\end{bmatrix} \begin{bmatrix} \hat{\mu}\\ -\hat{z}\end{bmatrix}.$$

Using Theorem 26 we find that this implies that

$$\begin{aligned} \int_0^\infty (-\hat{z}(t))^* \tilde{H} (-\hat{z}(t))\, dt &= \inf_{\substack{z\in\mathfrak{B}_+(P)\\ z(t)=-\hat{z}(t),t\leq 0}} \int_0^\infty z^*(t)\tilde{H}z(t)dt\\ &= \inf_{\substack{-z\in\mathfrak{B}_+(P)\\ -z(t)=-\hat{z}(t),t\leq 0}} \int_0^\infty (-z(t))^* \tilde{H} (-z(t))\, dt. \end{aligned}$$

and thus also

$$\int_0^\infty \hat{z}^*(t)\tilde{H}\hat{z}(t)dt = \inf_{\substack{z\in\mathfrak{B}_+(P)\\ z(t)=\hat{z}(t),t\leq 0}} \int_0^\infty z^*(t)\tilde{H}z(t)dt.$$

We finally obtain the assertion through Theorem 33 with $\tilde{P} = \begin{bmatrix} I_n & 0\end{bmatrix}$. □

Corollary 36. *Consider the system (25) together with the cost function s as given by (26). Define* $\tilde{Q}$, $\tilde{S}$, *and* $\tilde{R}$ *through (28). Let* $\hat{u} \in \mathcal{E}_m^+$ *and* $\hat{x} \in \mathcal{E}_n^+$ *be such that* $E\dot{\hat{x}}(t) = A\hat{x}(t) + B\hat{u}(t)$ *for all* $t \in \mathbb{R}$. *Also, assume that either*

$$\int_0^\infty \begin{bmatrix} \hat{u}(t)\\ \hat{x}(t)\end{bmatrix} \begin{bmatrix} \tilde{Q} & \tilde{S}\\ \tilde{S}^* & \tilde{R}\end{bmatrix} \begin{bmatrix} \hat{u}(t)\\ \hat{x}(t)\end{bmatrix} dt = \inf_{\substack{E\dot{x}=Ax+Bu\\ x(0)=\hat{x}(0)}} \int_0^\infty \begin{bmatrix} u(t)\\ x(t)\end{bmatrix} \begin{bmatrix} \tilde{Q} & \tilde{S}\\ \tilde{S}^* & \tilde{R}\end{bmatrix} \begin{bmatrix} u(t)\\ x(t)\end{bmatrix} dt$$

or

$$\int_0^\infty \begin{bmatrix}\hat{u}(t)\\ \hat{x}(t)\end{bmatrix} \begin{bmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^* & \tilde{R}\end{bmatrix} \begin{bmatrix}\hat{u}(t)\\ \hat{x}(t)\end{bmatrix} dt \quad = \inf_{\substack{E\dot{x}=Ax+Bu\\ Ex(0)=E\hat{x}(0)}} \int_0^\infty \begin{bmatrix}u(t)\\ x(t)\end{bmatrix} \begin{bmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^* & \tilde{R}\end{bmatrix} \begin{bmatrix}u(t)\\ x(t)\end{bmatrix} dt$$

holds, where the infimum in both cases has to be taken over all trajectories $(u, x) \in \mathcal{E}_m^+ \times \mathcal{E}_n^+$*.*

Then, (25) is dissipative with respect to (26) and there exists a co-state function $\hat{\mu} \in \mathcal{E}_\rho^+$ *such that we have*

$$\begin{bmatrix} 0 & E & 0\\ -E^* & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}\dot{\hat{\mu}}(t)\\ \dot{\hat{x}}(t)\\ \dot{\hat{u}}(t)\end{bmatrix} = \begin{bmatrix} 0 & A & B\\ A^* & \tilde{Q} & \tilde{S}\\ B^* & \tilde{S}^* & \tilde{R}\end{bmatrix} \begin{bmatrix}\hat{\mu}(t)\\ \hat{x}(t)\\ \hat{u}(t)\end{bmatrix}.$$

Proof. Similar to the proof of Corollary 35 but one has to use Theorem 28 instead of Theorem 26. □

7 Conclusion and Outlook

In this paper we have generalized two results which are well-known for standard systems and descriptor systems (e.g., [9]) to behavior systems of potentially higher-order. These two results show that dissipativity is equivalent to the solvability of the optimal control problem. Also, the results devise an eigenvalue method to compute all solutions of the optimal control problem.

Originally, in (2) or (12), we have assumed that a special trajectory $\hat{z}$ is prescribed. However, we then have seen in Theorem 33 (at least for the first order case) that only the value of $\hat{z}(0)$ is relevant. In this case we can restate Corollary 35 and Corollary 36 without the prescribed trajectory $\hat{x}$ and instead prescribe an initial condition $\hat{x}_0 \in \mathbb{C}^n$. This would create the problem that not all initial conditions $\hat{x}_0$ are consistent with the algebraic equations in (25) and thus the optimal control problem may not be solvable which means that the infimum in (12) becomes ∞.

Instead of considering the problem (12) for a given $\hat{z} \in \mathfrak{B}(P)$ one can also consider the problem

$$\int_{-\infty}^0 (\Delta_\ell \hat{z}(t))^* \, \tilde{H} \, (\Delta_\ell \hat{z}(t)) \, dt = \inf_{\substack{z\in\mathfrak{B}_-(P)\\ z(t)=\hat{z}(t), t\geq 0}} \int_{-\infty}^0 (\Delta_\ell z(t))^* \, \tilde{H} \, (\Delta_\ell z(t)) \, dt,$$

i.e., the problem on the negative time axis. This leads to exactly the same eigenvalue problem (13) but one has to consider the eigenvalues in the strict right half plane instead of the eigenvalues in the strict left half plane. Note that the problem on the positive time axis is closely related to the so called *available storage*, whereas the problem on the negative time axis is closely related to the so called *required supply*, compare [14, Definition 3 & Definition 5] or [15, p. 359].

It is well-known that the available storage and the required supply constitute storage functions, see [14, Theorem 1 and 2]. Thus looking at Corollary 35 and

Corollary 36 in connection with Theorem 33 (or Conjecture 34) we see that the available storage and the required supply depend on the state x, i.e., one can say that the available storage and the required supply are functions of the state. It has already been noted in [13, Theorem 6.1] that every storage function is a function of the state x.

In the regular case the available storage and the required supply constitute the two extremal solutions of an algebraic Riccati equation. In the non-regular case discussed here one has to consider extremal solutions of linear matrix inequalities (also called Lur'e equations), which are generalizations of algebraic Riccati equations. The solution of such linear matrix inequalities can be performed through the determination of invariant subspaces of the pencil (15) as shown in [11]. This shows that numerical methods for the solution of para-Hermitian eigenvalue problems are needed. Such methods for the first order case are discussed, e.g., in [3, 4, 8, 12].

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