

Checking dissipativity of behavioral systems using para-Hermitian matrix polynomials *

Tobias Brüll[†]

January 21, 2010

Abstract

The behavior approach and the problem of dissipativity have both been introduced and studied extensively by Willems et al., see [14, 18, 19, 21]. However, a computationally feasible method to check dissipativity is missing. Current methods will mostly rely on symbolic representations of rational functions. We will discuss a new characterization for linear systems in behavior form that allows to check dissipativity via the solution of a para-Hermitian, polynomial eigenvalue problem. Thus, we can employ standard methods of n^3 complexity.

Key words. Popov function, structured eigenvalue problem, generalized eigenvalue problem, polynomial eigenvalue problem, dissipativity, passivity, contractivity, purely imaginary eigenvalue, even pencil, even polynomial, cyclo-dissipativity, behavior approach, frequency-domain, para-Hermitian polynomial

1 Introduction

In this paper we will generalize the result [4, Theorem 1] to higher order systems given in behavior form. Also we will include more general types of cost functionals, i.e., we will consider general quadratic cost functionals.

The problem of checking passivity is a special case of the dissipativity-checking-problem, since passivity is usually defined to be dissipativity with respect to a special cost functional, see, e.g., [3, Section 5.9.1].

In the literature the problem of checking passivity is well known [11, 16]. There, however, most often the equivalent problem of positive definiteness of the Popov function along the imaginary axis, i.e., the problem of positive realness, is considered (compare Theorem 36). This approach is not feasible for systems with a larger number of states, since symbolic representations of rational matrices are hard to handle numerically, even for rational matrices of moderate size and degree, i.e., even for

*This research is supported by the DFG Research Center MATHEON in Berlin.

[†]Institut für Mathematik, TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany, E-mail: bruell@math.tu-berlin.de

systems with a moderate number of internal states. Our new approach generalizes a technique introduced in [7], although it uses a slightly different structure.

The computation of the Popov function from a system which is given in kernel representation is not a trivial task and the approach presented in this paper also avoids this problem. In our approach the kernel representation of the system is sufficient to construct a matrix polynomial for that an eigenvalue problem has to be solved. In the first-order case the QZ-Algorithm can be used to solve the eigenvalue problem, although the structure preserving methods presented in [5, 6, 15] are more appropriate for this problem. For higher-order systems one has to employ linearization. For the structure preserving methods a structure preserving linearization is necessary.

Notation used in this text are summed up in Tables 1 and 2.

Table 1: Notation - 1/2

S^+	for an arbitrary set $S \subset \mathbb{C}$; denotes $\{z \in S : \operatorname{Re}\{z\} > 0\}$
S^-	for an arbitrary set $S \subset \mathbb{C}$; denotes $\{z \in S : \operatorname{Re}\{z\} < 0\}$
$\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n)$	$\{z : \mathbb{R} \rightarrow \mathbb{C}^n \mid z \text{ is infinitely often differentiable}\}$
$\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^n)$	$\{z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^n) \mid z \text{ has compact support}\}$
\mathcal{C}^∞	$\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$
\mathcal{C}_c^∞	$\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$
$\mathbb{C}[\lambda]$	the ring of polynomials with coefficients in \mathbb{C}
$\mathbb{C}(\lambda)$	the field of rational functions with coefficients in \mathbb{C}
$\mathbb{C}[\lambda]^{p,q}$	a p-by-q matrix with polynomial entries
$\mathbb{C}(\lambda)^{p,q}$	a p-by-q matrix with entries from the field of the rational functions
polynomial matrix	an element of $\mathbb{C}[\lambda]^{p,q}$, i.e., a matrix with polynomial entries
matrix polynomial	the same as a polynomial matrix, since one can also look at a polynomial matrix as a polynomial that has matrices as coefficients
rational matrix	an element of $\mathbb{C}(\lambda)^{p,q}$, i.e., a matrix with entries from the field of rational functions
$\mathfrak{P}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$ is a rational matrix; denotes the set of poles of R , i.e., the set of all $\lambda \in \mathbb{C}$ where at least one entry of R has a pole
$\mathfrak{D}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$ is a rational matrix; denotes the set $\mathbb{C} \setminus \mathfrak{P}(R)$, i.e., the set of all λ such that $R(\lambda)$ is a well defined matrix in $\mathbb{C}^{p,q}$ and thus the domain of definition of R
$\operatorname{rank}_{\mathbb{C}(\lambda)}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the rank of R over the field $\mathbb{C}(\lambda)$
$\operatorname{kernel}_{\mathbb{C}(\lambda)}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the kernel of R over the field $\mathbb{C}(\lambda)$ which is a subset of $\mathbb{C}(\lambda)^q$
$\operatorname{range}_{\mathbb{C}(\lambda)}(R)$	where $R \in \mathbb{C}(\lambda)^{p,q}$; denotes the range of R over the field $\mathbb{C}(\lambda)$ with is a subset of $\mathbb{C}(\lambda)^p$

Table 2: Notation - 1/2

$\text{rank}(R(\lambda))$	where $R \in \mathbb{C}(\lambda)^{p,q}$ and $\lambda \in \mathfrak{D}(R)$; denotes the rank of $R(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\text{kernel}(R(\lambda))$	where $R \in \mathbb{C}(\lambda)^{p,q}$ and $\lambda \in \mathfrak{D}(R)$; denotes the kernel of $R(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\text{range}(R(\lambda))$	where $R \in \mathbb{C}(\lambda)^{p,q}$ and $\lambda \in \mathfrak{D}(R)$; denotes the range of $R(\lambda) \in \mathbb{C}^{p,q}$ in the usual way
$\text{diag}(\alpha_1, \dots, \alpha_r)$	where $\alpha_1, \dots, \alpha_r$ are scalars; denotes the r-by-r diagonal matrix which has the scalars $\alpha_1, \dots, \alpha_r$ on the diagonal and is zero everywhere else
$z^{(i)}$	the i -th derivative of the function z
$P\left(\frac{d}{dt}\right)z$	where $P \in \mathbb{C}[\lambda]^{p,q}$ has the form $P(\lambda) = \sum_{i=0}^d \lambda^i P_i$ and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$; denotes the function $\sum_{i=0}^d P_i z^{(i)}$
$\mathbb{C}[\lambda, \mu]$	the set of all two-variable-polynomials, i.e., all $\phi(\lambda, \mu) = \sum_{i,j=0}^n \phi_{i,j} \lambda^i \mu^j,$ where $n \in \mathbb{N}$ and $\phi_{i,j} \in \mathbb{C}$ for $i, j = 0, 1, \dots, n$
$\mathbb{C}[\lambda, \mu]^{p,q}$	a p-by-q matrix with two-variable-polynomial entries or equivalently a two-variable-polynomial with matrix coefficients, i.e., $\Phi(\lambda, \mu) = \sum_{i,j=0}^n \Phi_{i,j} \lambda^i \mu^j,$ where $n \in \mathbb{N}$ and $\Phi_{i,j} \in \mathbb{C}^{p,q}$ for $i, j = 0, 1, \dots, n$
Δ_ℓ^q	where $\ell, q \in \mathbb{N}$; denotes the polynomial given by $\Delta_\ell^q(\lambda) := \begin{bmatrix} (\lambda)^0 I_q \\ (\lambda)^1 I_q \\ \vdots \\ (\lambda)^\ell I_q \end{bmatrix} \in \mathbb{C}[\lambda]^{q(\ell+1),q}$
$\Delta_\ell z$	where $\ell \in \mathbb{N}$ and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$; denotes the function $\Delta_\ell z := \begin{bmatrix} z \\ z^{(1)} \\ \vdots \\ z^{(\ell)} \end{bmatrix} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q(\ell+1)}),$ and thus we have $\Delta_\ell z = \Delta_\ell^q \left(\frac{d}{dt} \right) z$

We will start by reviewing some basic facts to make the paper self contained. This includes basic results about

- rational and polynomial matrices in Section 2,
- the behavior approach in Section 3,
- the frequency-domain in Section 4,
- para-Hermitian matrices in Section 5,
- dissipativity in Section 6.

With these preliminaries we are ready to state the main result in Section 7. In Section 8 we will specialize the main result to first order systems to see how to derive some well known results. In Section 10 we will briefly interpret the main result for second order systems. Finally, in Section 11 some conclusions are drawn and an outlook is presented.

2 Polynomial and rational matrices

In this section we will develop some basic properties of polynomial and rational matrices.

First of all, we can immediately see that $\mathbb{C}[\lambda]^{p,q} \subset \mathbb{C}(\lambda)^{p,q}$ and that for any matrix polynomial $P \in \mathbb{C}[\lambda]^{p,q}$ we have $\mathfrak{D}(P) = \mathbb{C}$ and $\mathfrak{P}(P) = \emptyset$. Also we see that for every $R \in \mathbb{C}(\lambda)^{p,q}$ the set $\mathfrak{P}(R)$ can only be a finite set since R has a finite number of entries and each entry (which is a rational function) can only have a finite number of poles (because the poles are the zeros of a non-zero polynomial).

Lemma 1. *Let $R \in \mathbb{C}(\lambda)^{p,p}$ and let there exist a $\lambda_0 \in \mathfrak{D}(R)$ such that $R(\lambda_0) \in \mathbb{C}^{p,p}$ is an invertible matrix. Then R is an invertible rational matrix.*

Proof. Set $d := \det R \in \mathbb{C}(\lambda)$ and deduce that $d(\lambda_0) = \det R(\lambda_0) \neq 0$. This shows that $d \neq 0$ is not the zero function and thus R is invertible. \square

Lemma 2. *Let $R \in \mathbb{C}(\lambda)^{p,p}$ be an invertible rational matrix. Then there exists a finite set $\sigma(R)$ such that the following properties hold:*

1. $p = \text{rank}_{\mathbb{C}(\lambda)}(R) = \text{rank}(R(\lambda))$ for all $\lambda \in \mathfrak{D}(R) \setminus \sigma(R)$.
2. $p = \text{rank}_{\mathbb{C}(\lambda)}(R) > \text{rank}(R(\lambda))$ for all $\lambda \in \sigma(R)$.

Proof. Set $d := \det R \in \mathbb{C}(\lambda)$ and deduce that $\mathfrak{P}(d) = \mathfrak{P}(R)$ by using the Leibniz formula for determinants. Set

$$\sigma(R) := \{\lambda \in \mathfrak{D}(d) = \mathfrak{D}(R) \mid d(\lambda) = 0\}$$

and observe that in this case $\sigma(R)$ is finite, due to the fundamental theorem of algebra and since $d \neq 0$. Further we see that for every $\lambda \in \mathfrak{D}(R) \setminus \sigma(R)$ we have $0 \neq d(\lambda) = \det R(\lambda)$ and thus that $R(\lambda)$ is invertible from which 1. follows. For 2. we see that for every $\lambda \in \sigma(R)$ we have $0 = d(\lambda) = \det R(\lambda)$ and thus that $R(\lambda)$ is not invertible. \square

The following Lemma 3 provides an alternative definition of the rank of a rational matrix which is frequently used in the literature.

Lemma 3. *Let $R \in \mathbb{C}(\lambda)^{p,q}$. Then we have*

$$r := \text{rank}_{\mathbb{C}(\lambda)}(R) = \max_{\lambda \in \mathfrak{D}(R)} \text{rank}(R(\lambda)). \quad (1)$$

Proof. Using Gaussian elimination reduce the matrix R into echelon form over $\mathbb{C}(\lambda)$ as

$$R = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q,$$

where $P \in \mathbb{C}(\lambda)^{p,p}$ and $Q \in \mathbb{C}(\lambda)^{q,q}$ are invertible rational matrices. Let $\sigma(P)$ and $\sigma(Q)$ be defined as in Lemma 2 and choose any

$$\lambda_0 \in (\mathfrak{D}(P) \setminus \sigma(P)) \cap (\mathfrak{D}(Q) \setminus \sigma(Q)),$$

i.e., such that $P(\lambda_0)$ and $Q(\lambda_0)$ are well defined invertible matrices over \mathbb{C} . Since

$$R(\lambda_0) = P(\lambda_0) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q(\lambda_0),$$

we conclude that $\lambda_0 \in \mathfrak{D}(R)$ and further that

$$\max_{\lambda \in \mathfrak{D}(R)} \text{rank}(R(\lambda)) \geq \text{rank}(R(\lambda_0)) = \text{rank}\left(P(\lambda_0) \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q(\lambda_0)\right) = r.$$

To finish the proof assume to the contrary that $\max_{\lambda \in \mathfrak{D}(R)} \text{rank}(R(\lambda)) > r$, i.e., let there be a $\lambda_1 \in \mathfrak{D}(R)$ with $\text{rank}(R(\lambda_1)) > r$. Then there exist invertible matrices $P_1 \in \mathbb{C}^{p,p}$ and $Q_1 \in \mathbb{C}^{q,q}$ such that

$$R(\lambda_1) = P_1 \begin{bmatrix} I_{r+s} & 0 \\ 0 & 0 \end{bmatrix} Q_1,$$

where $s \in \mathbb{N}$, $s \geq 1$. Define the rational matrix $\tilde{R}_{11} \in \mathbb{C}(\lambda)^{r+s, r+s}$ through the relation

$$P_1^{-1} R(\lambda) Q_1^{-1} =: \begin{bmatrix} \tilde{R}_{11}(\lambda) & \tilde{R}_{12}(\lambda) \\ \tilde{R}_{21}(\lambda) & \tilde{R}_{22}(\lambda) \end{bmatrix}.$$

Then clearly $\tilde{R}_{11}(\lambda_1) = I_{r+s}$, $\tilde{R}_{12}(\lambda_1) = 0$, $\tilde{R}_{21}(\lambda_1) = 0$, and $\tilde{R}_{22}(\lambda_1) = 0$. By Lemma 1 this shows that \tilde{R}_{11} is an invertible rational matrix of dimension $r+s$ which implies that

$$r = \text{rank}_{\mathbb{C}(\lambda)}(R) \geq r+s > r,$$

which is a contradiction. \square

Definition 4. A square rational matrix $R \in \mathbb{C}(\lambda)^{p,p}$ is called *unimodular* if its determinant is a non-zero constant, i.e., there exists a non-zero constant $c \in \mathbb{C} \setminus \{0\}$ such that

$$\det R(\lambda) = c,$$

for all $\lambda \in \mathfrak{D}(R)$.

Lemma 5. *A polynomial, unimodular matrix $P \in \mathbb{C}[\lambda]^{p,p}$ is invertible over $\mathbb{C}(\lambda)$ and its inverse is again a polynomial, unimodular matrix.*

Proof. Invertibility follows since the determinant is non-zero. We can see that the inverse (which we know exists in $\mathbb{C}(\lambda)^{p,p}$) is also a polynomial matrix by using the adjoint formula of the inverse. To see that the inverse is unimodular, note that

$$1 = \det I = \det(PP^{-1}) = \det(P)\det(P^{-1}),$$

which implies that the determinant of P^{-1} is also a constant. \square

Theorem 6. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$. Then there exist unimodular matrices $S \in \mathbb{C}[\lambda]^{p,p}$ and $T \in \mathbb{C}[\lambda]^{q,q}$ such that*

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T, \quad (2)$$

where $d_1, \dots, d_r \in \mathbb{C}[\lambda]$ with $d_i \neq 0$ for $i = 1, \dots, r$ and d_{i+1} divides d_i for $i = 1, \dots, r-1$.

Proof. The proof is quite simple and a completely self contained presentations can be found in [10, Chapter S1.1] and another in [9, p.141, Theorem 3]. \square

The canonical form in (2) is called Smith form.

The following Lemma transfers Lemma 2 from invertible rational matrices to rectangular polynomial matrices.

Lemma 7. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ a matrix polynomial. Then there exists a finite set $\sigma(P)$ such that the following properties hold:*

1. $\text{rank}_{\mathbb{C}(\lambda)}(P) = \text{rank}(P(\lambda))$ for all $\lambda \notin \sigma(P)$,
2. $\text{rank}_{\mathbb{C}(\lambda)}(P) > \text{rank}(P(\lambda))$ for all $\lambda \in \sigma(P)$.

Proof. Let the Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

For a scalar non-zero polynomial $d \in \mathbb{C}[\lambda]$ we define $\sigma(d)$ to be the finite set of roots of d . With this define the set $\sigma(P)$ through

$$\sigma(P) := \bigcup_{i=1, \dots, r} \sigma(d_i).$$

Then for $\lambda_0 \notin \sigma(P)$ we see that

$$\text{diag}(d_1(\lambda_0), \dots, d_r(\lambda_0)) \in \mathbb{C}^{r,r}$$

is an invertible matrix (over \mathbb{C}). From Lemma 5 we obtain that also $S(\lambda_0)$ and $T(\lambda_0)$ are invertible. Thus we have that

$$\text{rank}(P(\lambda_0)) = \text{rank} \left(\begin{bmatrix} \text{diag}(d_1(\lambda_0), \dots, d_r(\lambda_0)) & 0 \\ 0 & 0 \end{bmatrix} \right) = r = \text{rank}_{\mathbb{C}(\lambda)}(P).$$

Analogously, we see that for $\lambda_0 \in \sigma(P)$ we have

$$\text{rank}(P(\lambda_0)) = \text{rank} \left(\begin{bmatrix} \text{diag}(d_1(\lambda_0), \dots, d_r(\lambda_0)) & 0 \\ 0 & 0 \end{bmatrix} \right) < r = \text{rank}_{\mathbb{C}(\lambda)}(P),$$

since in this case for at least one $i = 1, \dots, r$ we have that $d_i(\lambda_0) = 0$. \square

Definition 8. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and define the set $\sigma(P)$ as in Lemma 7. Then the elements of $\sigma(P)$ are called *eigenvalues of P* . For $\lambda \in \sigma(P)$ the quantity

$$g(\lambda) := \text{rank}_{\mathbb{C}(\lambda)}(P) - \text{rank}(P(\lambda))$$

is called *geometric multiplicity of λ* .

Remark 9. In the theory of matrix polynomials it is common practice to distinguish between *finite* and *infinite* eigenvalues, see [1, Section 3]. Usually, a matrix polynomial $P \in \mathbb{C}[\lambda]^{p,p}$ which is regular over $\mathbb{C}(\lambda)$ is said to have the infinite eigenvalue if in the (unique) representation

$$P(\lambda) = \sum_{i=0}^{\pi} \lambda^i P_i,$$

with $P_i \in \mathbb{C}^{p,p}$ and $P_{\pi} \neq 0$ we have that P_{π} is a singular matrix (over \mathbb{C}). This approach cannot be easily generalized to rectangular matrix polynomials.

Another approach is to use the so called homogeneous representation of matrix polynomials. This approach is described in [1] and can be generalized to rectangular matrix polynomials in the following way. Consider the homogeneous representation of a matrix polynomial $P \in \mathbb{C}[\lambda]^{p,q}$ given by

$$P(c, s) = \sum_{i=0}^{\pi} c^{\pi-i} s^i P_i,$$

with $(c, s) \in \mathbb{C}^2 \setminus \{0\}$. Then P is said to have the eigenvalue $(c_0, s_0) \in \mathbb{C}^2 \setminus \{0\}$ if

$$\text{rank}(P(c_0, s_0)) < \text{rank}_{\mathbb{C}(\lambda)}(P(1, \lambda)) = \text{rank}_{\mathbb{C}(\lambda)}(P(\lambda, 1)).$$

Further, if $c_0 \neq 0$ the tuple (c_0, s_0) is identified with the finite eigenvalue $\lambda_0 := \frac{s_0}{c_0}$ and if $c_0 = 0$ the tuple (c_0, s_0) is identified with the infinite eigenvalue $\lambda_0 := \infty$.

However, in this paper we are not concerned with the infinite eigenvalue, as it does not seem to play a role for the results presented. Thus, if we speak of eigenvalues in the following we always refer to finite eigenvalues.

Note, that a unimodular matrix can have no eigenvalues, since at such an eigenvalue the determinant would vanish which contradicts the assumption that the determinant is a non-zero constant.

Theorem 10. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$. Then there exist polynomial matrices $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ with the properties*

1. $PU = 0$,
2. $\text{rank}_{\mathbb{C}(\lambda)}(PV) = r$,
3. $\text{rank}_{\mathbb{C}(\lambda)}(U) = \text{rank}(U(\lambda)) = q - r$ for all $\lambda \in \mathbb{C}$,
4. $\text{rank}_{\mathbb{C}(\lambda)}(V) = \text{rank}(V(\lambda)) = r$ for all $\lambda \in \mathbb{C}$,
5. $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular.

This especially means that all the matrices U , V , and $\begin{bmatrix} U & V \end{bmatrix}$ have no eigenvalues.

Proof. Let the Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

Partition the inverse of T (which is again a polynomial unimodular matrix due to Lemma 5) according to the block structure of the diagonal matrix in the Smith form as

$$T^{-1} =: \begin{bmatrix} V & U \end{bmatrix},$$

i.e., such that V has r columns and U has $q - r$ columns. Then clearly also $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular since it can be obtained from T^{-1} through a column permutation, which is itself a unimodular transformation, and 5. is proved. Also we see that 1. holds, since

$$PU = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TU = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TT^{-1} \begin{bmatrix} 0 \\ I_{q-r} \end{bmatrix} = 0.$$

Property 2. holds since analogously

$$PV = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) \\ 0 \end{bmatrix},$$

and all $d_i \neq 0$ for $i = 1, \dots, r$. Parts 3. and 4. follow since $\begin{bmatrix} U & V \end{bmatrix}$ has full rank (over $\mathbb{C}(\lambda)$) and due to the unimodularity also the matrix $\begin{bmatrix} U(\lambda) & V(\lambda) \end{bmatrix}$ has full rank (over \mathbb{C}) for every $\lambda \in \mathbb{C}$. \square

Theorem 10 motivates the following Definition.

Definition 11. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$. Then the rational matrix $U \in \mathbb{C}(\lambda)^{q,q-r}$ and the polynomial matrix $V \in \mathbb{C}[\lambda]^{q,r}$ are called *kernel matrix* and *co-kernel matrix* of P , resp., if they fulfill the following properties

1. $PU = 0$,
2. $\text{rank}_{\mathbb{C}(\lambda)}(PV) = r$,
3. $\text{rank}_{\mathbb{C}(\lambda)}(U) = \text{rank}(U(\lambda)) = q - r$ for all $\lambda \in \mathfrak{D}(U)$,
4. $\text{rank}_{\mathbb{C}(\lambda)}(V) = \text{rank}(V(\lambda)) = r$ for all $\lambda \in \mathbb{C}$,
5. $\begin{bmatrix} U & V \end{bmatrix}$ is unimodular.

Theorem 10 shows that for every matrix polynomial there exists a kernel and co-kernel matrix, where the kernel matrix is even a polynomial. The co-kernel matrix has to be a polynomial matrix for the proof of Theorem 44 to work. The kernel matrix is allowed to be a rational function, because for regular first order state-space systems we will later present a kernel matrix in explicit form, see (24), which happens to be a rational matrix.

Lemma 12. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ be kernel and co-kernel matrices of P . Let a Smith form (2) of P be given by*

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T,$$

and partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q - r$ columns, partitioned analogously to the Smith form.

Then, there exists a unimodular rational matrix $U_2 \in \mathbb{C}(\lambda)^{q-r,q-r}$ with $\mathfrak{D}(U) = \mathfrak{D}(U_2)$, an unimodular polynomial matrix $V_1 \in \mathbb{C}[\lambda]^{r,r}$, and a polynomial matrix $V_2 \in \mathbb{C}[\lambda]^{q-r,r}$ such that

$$U = T_2 U_2, \text{ and } V = T_1 V_1 + T_2 V_2 = T \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

If U is in addition a polynomial matrix, then the matrix U_2 is also polynomial.

Proof. Set $\tilde{U} = TU$ and $\tilde{V} = TV$. Since T is a unimodular polynomial matrix we clearly have that \tilde{V} is a polynomial matrix and that $\mathfrak{D}(\tilde{U}) = \mathfrak{D}(U)$. Partition

$$\tilde{U} =: \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{matrix} r \\ q-r \end{matrix} \quad \text{and} \quad \tilde{V} =: \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{matrix} r \\ q-r \end{matrix},$$

i.e., such that V_1 and U_2 are rectangular. Taking the Smith form into consideration we first find that

$$\begin{aligned} 0 = S^{-1}PU &= \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TU \\ &= \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} \tilde{U} = \begin{bmatrix} \text{diag}(d_1, \dots, d_r) U_1 \\ 0 \end{bmatrix}, \end{aligned}$$

and thus that $U_1 = 0$, since $\text{diag}(d_1, \dots, d_r)$ is invertible. This implies that U_2 is invertible because \tilde{U} has full rank $q - r$. Since by assumption U has no eigenvalues

also $\tilde{U} = TU$ has no eigenvalues. This implies that U_2 has no eigenvalues, which means that U_2 is unimodular. For \tilde{V} , on the other hand, we find that

$$\begin{aligned} r = \text{rank}_{\mathbb{C}(\lambda)}(PV) &= \text{rank}_{\mathbb{C}(\lambda)}(S^{-1}PV) \\ &= \text{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TV\right) \\ &= \text{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \text{diag}(d_1, \dots, d_r) V_1 \\ 0 \end{bmatrix}\right), \end{aligned}$$

and thus that V_1 is invertible since, again, $\text{diag}(d_1, \dots, d_r)$ is invertible. In matrix notation we find that since both T and $\begin{bmatrix} U & V \end{bmatrix}$ are unimodular also the block anti-diagonal matrix

$$\begin{bmatrix} 0 & V_1 \\ U_2 & V_2 \end{bmatrix} = [\tilde{U} \quad \tilde{V}] = T \begin{bmatrix} U & V \end{bmatrix},$$

is unimodular. Since

$$\det \begin{bmatrix} 0 & V_1 \\ U_2 & V_2 \end{bmatrix} = \det V_1 \cdot \det U_2,$$

this implies unimodularity of V_1 since the unimodularity of U_2 has already been shown. \square

The following Lemma 13 justifies that one may speak of a kernel matrix and a co-kernel matrix independently, i.e., for $P \in \mathbb{C}[\lambda]^{p,q}$ one can for example speak of a kernel matrix U without referring to an accompanying co-kernel matrix, as suggested by Definition 11.

Lemma 13. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ together be a kernel and co-kernel matrix of P (as in Definition 11) and independently of that, let also $\tilde{U} \in \mathbb{C}(\lambda)^{q,q-r}$ and $\tilde{V} \in \mathbb{C}[\lambda]^{q,r}$ together be a kernel and co-kernel matrix of P . Then we have that also U and \tilde{V} together are a kernel and co-kernel matrix of P as well as we have that \tilde{U} and V together are a kernel and co-kernel matrix of P .*

Proof. Since the other properties of Definition 11 are clear it is sufficient to show that the composed matrices $\begin{bmatrix} U & \tilde{V} \end{bmatrix}$ and $\begin{bmatrix} \tilde{U} & V \end{bmatrix}$ are unimodular. Let a Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T,$$

and partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q - r$ columns. Using the previous Lemma 12 we obtain the existence of unimodular U_2, V_1, \tilde{U}_2 , and \tilde{V}_1 and polynomial matrices V_2 and \tilde{V}_2 such that

$$\begin{aligned} U &= T_2 U_2, & V &= T_1 V_1 + T_2 V_2, \\ \tilde{U} &= T_2 \tilde{U}_2, & \tilde{V} &= T_1 \tilde{V}_1 + T_2 \tilde{V}_2. \end{aligned}$$

Thus we obtain that

$$\det [U \quad \tilde{V}] = \det [T_1 \quad T_2] \begin{bmatrix} 0 & \tilde{V}_1 \\ U_2 & \tilde{V}_2 \end{bmatrix},$$

which proves unimodularity of $[U \quad \tilde{V}]$ since U_2 and \tilde{V}_1 are unimodular. Unimodularity of $[\tilde{U} \quad V]$ follows analogously from

$$\det [\tilde{U} \quad V] = \det [T_1 \quad T_2] \begin{bmatrix} 0 & V_1 \\ \tilde{U}_2 & V_2 \end{bmatrix},$$

and the proof is finished. \square

In the proofs of Lemma 41 and Theorem 44 the following two Lemmas 14 and 15 are needed. However, we prove them in this section since they fit thematically in here.

Lemma 14. *Let $Q \in \mathbb{C}[\lambda]^{p,r}$ be a matrix polynomial with full column rank, i.e., with $\text{rank}_{\mathbb{C}(\lambda)}(Q) = r$. Then we have the following.*

1. *In general, there exists a (potentially rational) left inverse $X \in \mathbb{C}(\lambda)^{r,p}$ of Q , i.e., a rational matrix X such that $XQ = I_r$, with the property*

$$\mathfrak{D}(X) = \mathbb{C} \setminus \sigma(Q).$$

2. *If Q has no eigenvalues then there exists a polynomial left inverse $X \in \mathbb{C}[\lambda]^{r,p}$ of Q , i.e., a polynomial matrix X such that $XQ = I_r$.*

Proof. Consider the Smith canonical form of Q which takes the form

$$Q = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) \\ 0 \end{bmatrix} T,$$

since we have assumed that $\text{rank}_{\mathbb{C}(\lambda)}(Q) = r$. For part 1. remember that $\sigma(Q) = \bigcup_{i=1}^r \sigma(d_i)$. Setting

$$X := T^{-1} \begin{bmatrix} \text{diag}\left(\frac{1}{d_1}, \dots, \frac{1}{d_r}\right) & 0 \end{bmatrix} S^{-1}$$

we obtained a left inverse of Q . That we have

$$\mathfrak{D}(X) = \mathbb{C} \setminus \bigcup_{i=1}^r \sigma(d_i) = \mathbb{C} \setminus \sigma(Q),$$

finally follows from the unimodularity of S and T and thus the unimodularity of T^{-1} and S^{-1} . For part 2. partition S and its inverse S^{-1} through the relations

$$S =: [S_1 \quad S_2], \text{ and} \quad S^{-1} =: \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} \begin{matrix} r \\ p-r \end{matrix},$$

such that $S_1 \in \mathbb{C}[\lambda]^{p,r}$ and $S_2 \in \mathbb{C}[\lambda]^{p,p-r}$. Since $I_p = S^{-1}S$ we readily obtain that $\tilde{X}S_1 = I_r$. Since we assume that Q has no eigenvalues, we find that w.l.o.g. $\text{diag}(d_1, \dots, d_r) = I_r$. Define $X := T^{-1}\tilde{X}$ and observe that with this

$$XQ = T^{-1}\tilde{X} \begin{bmatrix} S_1 & S_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} T = T^{-1}XS_1T = I_r.$$

We have that $X = T^{-1}\tilde{X}$ is a polynomial since T^{-1} is a polynomial (remember that T is unimodular) and also \tilde{X} is a polynomial. \square

Lemma 15. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and set $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$. Let $V \in \mathbb{C}[\lambda]^{q,r}$ be a co-kernel matrix of P . Then we have*

$$\sigma(PV) = \sigma(P)$$

Proof. Let the Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T,$$

and partition the inverse of T as $T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with T_1 having r columns and T_2 having $q - r$ columns. Using Lemma 12 we obtain the existence of a unimodular polynomial matrix V_1 and polynomial matrix V_2 such that $V = T_1V_1 + T_2V_2$. Thus we see that

$$PV = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TT^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) \\ 0 \end{bmatrix} V_1,$$

with V_1 being a unimodular polynomial matrix. This means that we have a Smith form of PV and we can obtain the eigenvalues through

$$\sigma(PV) = \bigcup_{i=1}^r \sigma(d_i) = \sigma(P),$$

which proves the claim. \square

3 The behavior approach to systems theory

With the preliminaries from Section 2 we are ready to introduce some results concerning the behavior approach. All the results can also be found in [14], where a slightly different notation is used.

Definition 16. Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a polynomial matrix. Then we call

$$\mathfrak{B}(P) := \left\{ z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \mid P \left(\frac{d}{dt} \right) z = 0 \right\} = \text{kernel}_{\mathcal{C}^\infty} \left(P \left(\frac{d}{dt} \right) \right)$$

the *behavior* of P and

$$\mathfrak{B}_c(P) := \mathfrak{B}(P) \cap \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q) = \text{kernel}_{\mathcal{C}_c^\infty} \left(P \left(\frac{d}{dt} \right) \right)$$

the compact behavior of P . The elements of $\mathfrak{B}(P)$ and $\mathfrak{B}_c(P)$ are called *trajectories* of P .

The following two Lemmas 17 and 18 can be seen as an adaption of [20, Proposition 4.1] to differential equations and the proofs are in principle taken from there.

Lemma 17. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix according to Definition 11 and Theorem 10. Let $a \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ be arbitrary. Then*

$$z := U \left(\frac{d}{dt} \right) a \in \mathfrak{B}(P),$$

i.e., z is a trajectory of P .

Proof. Let $P(\lambda) = \sum_{i=0}^{\pi} \lambda^i P_i$ and $U(\lambda) = \sum_{j=0}^{\nu} \lambda^j U_j$. Then by using the fact that U is a kernel matrix of P and equating coefficients in

$$0 = P(\lambda)U(\lambda) = \sum_{i=0}^{\pi} \sum_{j=0}^{\nu} \lambda^{i+j} P_i U_j = \sum_{k=0}^{\pi+\nu} \lambda^k \sum_{i+j=k} P_i U_j$$

we find that $\sum_{i+j=k} P_i U_j = 0$. Thus for the function z we obtain

$$\begin{aligned} P \left(\frac{d}{dt} \right) z(t) &= P \left(\frac{d}{dt} \right) U \left(\frac{d}{dt} \right) a(t) = P \left(\frac{d}{dt} \right) \sum_{j=0}^{\nu} U_j a^{(j)}(t) \\ &= \sum_{i=0}^{\pi} P_i \left(\frac{d}{dt} \right)^i \sum_{j=0}^{\nu} U_j a^{(j)}(t) = \sum_{i=0}^{\pi} \sum_{j=0}^{\nu} P_i U_j a^{(i+j)}(t) \\ &= \sum_{k=0}^{\pi+\nu} \left(\sum_{i+j=k} P_i U_j \right) a^{(k)}(t) = 0, \end{aligned}$$

which means that $z \in \mathfrak{B}(P)$. □

Lemma 18. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P . Then*

$$\mathfrak{B}_c(P) = \text{range}_{\mathcal{C}^\infty} \left(U \left(\frac{d}{dt} \right) \right).$$

Proof. The inclusion " \supset " follows directly from Lemma 17, since if in the statement of Lemma 17 the function a is a function with compact support, so is z .

For the inclusion " \subset " let $z \in \mathfrak{B}_c(P)$ be arbitrary and let the Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

Partition the inverse of T as $T^{-1} = [T_1 \quad T_2]$ with T_1 having r columns and T_2 having $q - r$ columns. Using Lemma 12 we obtain the existence of a unimodular polynomial

U_2 such that $U = T_2 U_2$. Since $z \in \mathfrak{B}_c(P) = \mathfrak{B}_c(S^{-1}P)$ and by defining the functions $z_1 \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^r)$, $z_2 \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$, and $\tilde{z} \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q)$, through

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \tilde{z}(t) := T \left(\frac{d}{dt} \right) z(t),$$

we also see that

$$\begin{aligned} 0 = S^{-1} \left(\frac{d}{dt} \right) P \left(\frac{d}{dt} \right) z(t) &= \begin{bmatrix} \text{diag} \left(d_1 \left(\frac{d}{dt} \right), \dots, d_r \left(\frac{d}{dt} \right) \right) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \text{diag} \left(d_1 \left(\frac{d}{dt} \right), \dots, d_r \left(\frac{d}{dt} \right) \right) z_1(t) \\ 0 \end{bmatrix}. \end{aligned}$$

Further defining the functions $z_i \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$ for $i = 1, \dots, r$ by partitioning

$$\begin{bmatrix} z_1(t) \\ \vdots \\ z_r(t) \end{bmatrix} := z_1(t)$$

we obtain that $0 = d_i \left(\frac{d}{dt} \right) z_i(t)$ for all $t \in \mathbb{R}$ and all $i = 1, \dots, r$. If d_i is a non-zero constant we immediately see that this implies that $z_i \equiv 0$ (remember that all d_i are assumed to be non-zero). If, however, d_i is another non-zero polynomial this means that the scalar-valued function z_i satisfies a linear ordinary differential equation (of order potentially higher than one). The fact that z_i has compact support gives us the initial condition $z_i(R) = 0$ (where $R \in \mathbb{R}$ is small enough or large enough). From the theory of linear ordinary differential equations (reducing the system to first order and writing down the explicit solution formula) we again see that $z_i \equiv 0$. Thus, it follows that $z_1 \equiv 0$ and we deduce that

$$z(t) = T^{-1} \left(\frac{d}{dt} \right) \tilde{z}(t) = \begin{bmatrix} T_1 \left(\frac{d}{dt} \right) & T_2 \left(\frac{d}{dt} \right) \end{bmatrix} \begin{bmatrix} 0 \\ z_2(t) \end{bmatrix} = T_2 \left(\frac{d}{dt} \right) z_2(t).$$

Since $U = T_2 U_2$, with U_2 being a unimodular polynomial also, U_2^{-1} is a unimodular polynomial and we have $U U_2^{-1} = T_2$. Setting $\alpha(t) := U_2^{-1} \left(\frac{d}{dt} \right) z_2(t)$ we see that $\alpha \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$, since $z_2 \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ and we finally get

$$z(t) = T_2 \left(\frac{d}{dt} \right) z_2(t) = U \left(\frac{d}{dt} \right) U_2^{-1} \left(\frac{d}{dt} \right) z_2(t) = U \left(\frac{d}{dt} \right) \alpha(t),$$

which finishes the proof. \square

Loosely speaking, Lemmas 17 and 18 show that the compact behavior of a linear system given by a polynomial matrix $P \in \mathbb{C}[\lambda]^{p,q}$ is completely specified through the range of one of its polynomial kernel matrices. This is the reason why in [14] the polynomial matrix P is called kernel representation and a kernel matrix U of P is called image representation of a system.

Lemma 19 will be used to connect the kernel matrix to the Laplace transformation.

Lemma 19. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P . Let $Z : \mathbb{C} \rightarrow \mathbb{C}^q$ be a continuous function such that*

$$0 = P(\lambda)Z(\lambda),$$

for $\lambda \in \mathbb{C}$. Then, there exists a function $\alpha : \mathfrak{D}(U) \rightarrow \mathbb{C}^{q-r}$ such that

$$Z(\lambda) = U(\lambda)\alpha(\lambda),$$

for all $\lambda \in \mathfrak{D}(U)$. If in addition U is a polynomial kernel matrix we furthermore have that α is continuous.

Proof. Let a Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T,$$

and partition the inverse of T as $T^{-1} = [T_1 \ T_2]$ with T_1 having r columns and T_2 having $q - r$ columns. Using Lemma 12 we obtain the existence of a unimodular $U_2 \in \mathbb{C}[\lambda]^{q-r,q-r}$ such that $U = T_2 U_2$ and $\mathfrak{D}(U) = \mathfrak{D}(U_2)$. Define the continuous functions $Z_1 : \mathbb{C} \rightarrow \mathbb{C}^r$, $Z_2 : \mathbb{C} \rightarrow \mathbb{C}^{q-r}$, and $\tilde{Z} : \mathbb{C} \rightarrow \mathbb{C}^q$ through

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} := \tilde{Z} := TZ,$$

and observe that this implies

$$0 = S^{-1}PZ = \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} TZ = \begin{bmatrix} \text{diag}(d_1, \dots, d_r) Z_1 \\ 0 \end{bmatrix},$$

which in turn implies that $Z_1 \equiv 0$, since $\text{diag}(d_1, \dots, d_r)$ is invertible. From this we deduce that

$$Z = T^{-1}\tilde{Z} = [T_1 \ T_2] \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = T_2 Z_2.$$

With this notation at hand define $\alpha := U_2^{-1}Z_2$. Due to the Definition, Z_2 is a continuous function. Since, U_2 is unimodular we know that $\mathfrak{D}(U_2) = \mathfrak{D}(U_2^{-1})$ and thus α can be considered to be function defined on $\mathfrak{D}(U_2^{-1}) = \mathfrak{D}(U)$. If U is also polynomial, so is U_2 (by the statement of Lemma 12) and with it U_2^{-1} , resulting in α being continuous. Finally, from the equation $U = T_2 U_2$ we find that also $U U_2^{-1} = T_2$ and conclude that

$$Z = T_2 Z_2 = U U_2^{-1} Z_2 = U \alpha,$$

which is the assertion. \square

4 Frequency-domain

Definition 20. For a function $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ we define its *two sided Laplace-transform* $Z : \mathbb{C} \rightarrow \mathbb{C}^q$ via

$$Z(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda t} z(t) dt, \quad (3)$$

for all $\omega \in \mathbb{R}$ where the integral exists. The two sided Laplace-transform is also denoted by $Z = \mathfrak{L}\{z\}$.

Lemma 21. *Let $z_1, z_2 \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q)$ be two functions with compact support. Then their two sided Laplace-transforms Z_1, Z_2 are continuous functions that are well defined for all $\lambda \in \mathbb{C}$, i.e., $Z_1(\lambda)$ and $Z_2(\lambda)$ are well defined in the complete complex plane. Furthermore, the Parseval/Plancherel identity*

$$\int_{-\infty}^{\infty} z_1^*(t)z_2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z_1^*(i\omega)Z_2(i\omega)d\omega \quad (4)$$

holds. For the derivative of a function z_1 we have the formula

$$\mathfrak{L}\{\dot{z}_1\}(\lambda) = \lambda Z_1(\lambda).$$

Proof. Since z is assumed to have compact support there has to be a $R \in \mathbb{R}^+$ such that $z(t) = 0$ for every $|t| \geq R$. Thus, for every $\lambda \in \mathbb{C}$ the integral

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} z(t)dt = \int_{-R}^R e^{-\lambda t} z(t)dt$$

is an integral of a continuous function over a compact interval which exists in the Riemann- as well as the Lebesgue-sense.

Partial integration shows that we also have

$$\begin{aligned} \mathfrak{L}\{\dot{z}_1\} &= \int_{-\infty}^{\infty} e^{-\lambda t} \dot{z}_1(t)dt \\ &= \int_{-R}^R e^{-\lambda t} \dot{z}_1(t)dt \\ &= e^{-\lambda t} \underbrace{z_1(t)}_{=0} \Big|_{t=-R}^{t=R} - \int_{-R}^R (-\lambda) e^{-\lambda t} z_1(t)dt \\ &= \lambda \int_{-R}^R e^{-\lambda t} z(t)dt = \lambda \int_{-\infty}^{\infty} e^{-\lambda t} z(t)dt = \lambda Z_1(\lambda). \end{aligned}$$

The Parseval/Plancherel identity is harder to prove, see [8, §12].

To see that Z is a continuous function let $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$ be a sequence that converges to a $\lambda \in \mathbb{C}$. Then the functions

$$z_k(t) := e^{-\lambda_k t} z(t) \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q)$$

constitute a sequence of functions which converges uniformly to the function $e^{-\lambda t} z(t)$. Thus, from basic calculus we know that

$$\begin{aligned} \lim_{k \rightarrow \infty} Z(\lambda_k) &= \lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} e^{\lambda_k t} z(t)dt = \lim_{k \rightarrow \infty} \int_{-R}^R e^{\lambda_k t} z(t)dt \\ &= \int_{-R}^R \lim_{k \rightarrow \infty} e^{\lambda_k t} z(t)dt = \int_{-R}^R e^{\lambda t} z(t)dt = Z(\lambda), \end{aligned}$$

which proves the continuity. \square

The following Lemma 22 is needed later to draw connections between the time-domain and the frequency-domain. It basically says that for a given $P \in \mathbb{C}[\lambda]^{p,q}$ one can construct trajectories with compact support which almost only consist of one predominant frequency.

Lemma 22. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r = \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P . Let $\lambda_0 \in \mathbb{C}$ and $v \in \mathbb{C}^{q-r}$ be arbitrary but fixed. Split $\lambda_0 = r_0 + i\omega_0$ into the real part $r_0 = \text{Re}\{\lambda_0\}$ and the imaginary part $\omega_0 = \text{Im}\{\lambda_0\}$. Define*

$$\tilde{\omega}_0 := \begin{cases} |\omega_0| & \text{if } \omega_0 \neq 0 \\ 1 & \text{if } \omega_0 = 0 \end{cases}.$$

Then there exists a sequence of trajectories $\{z_k\}_{k \in \mathbb{N}} = \{U \left(\frac{d}{dt}\right) v_k\}_{k \in \mathbb{N}} \subset \mathfrak{B}_c(P)$ with $v_k \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ such that the following properties are satisfied for all $k \in \mathbb{N}$.

1. $z_k(t) = U \left(\frac{d}{dt}\right) v e^{i\omega_0 t} = U(i\omega_0) v e^{i\omega_0 t}$ for all $t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right]$.
2. $z_k\left(t + \frac{2\pi k}{\tilde{\omega}_0}\right) = e^{r_0 \frac{2\pi k}{\tilde{\omega}_0}} z_0(t)$ for all $t \in \left[0, \frac{2\pi}{\tilde{\omega}_0}\right]$.
3. $z_k\left(t - \frac{2\pi k}{\tilde{\omega}_0}\right) = e^{-r_0 \frac{2\pi k}{\tilde{\omega}_0}} z_0(t)$ for all $t \in \left[-\frac{2\pi}{\tilde{\omega}_0}, 0\right]$.
4. $z_k(t) = 0$ for all $t \in \left]-\infty, -\frac{2\pi(k+1)}{\tilde{\omega}_0}\right] \cup \left[\frac{2\pi(k+1)}{\tilde{\omega}_0}, \infty\right[$.

Proof. Let $\tilde{b} : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth transition from 0 to 1, i.e., a function such that

$$\tilde{b}(t) = \begin{cases} 0 & t < -1 \\ 1 & t > 0 \end{cases},$$

with $\tilde{b} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ infinitely often differentiable. Define the sequence of functions $b_k : \mathbb{R} \rightarrow \mathbb{R}$ through

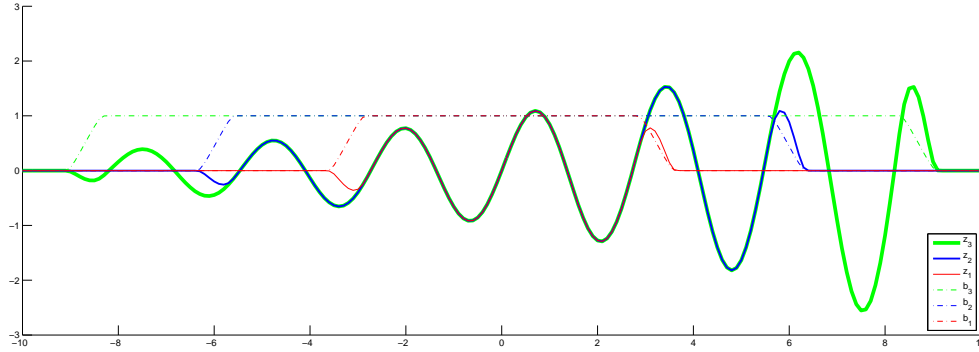
$$b_k(t) = \tilde{b}\left(t \frac{\tilde{\omega}_0}{2\pi} + k\right) \tilde{b}\left(-t \frac{\tilde{\omega}_0}{2\pi} + k\right),$$

for $k \in \mathbb{N}$ and observe that $b_k \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ is infinitely often differentiable and has the properties

$$b_k(t) = \begin{cases} 0 & \text{for } t \in \left]-\infty, -\frac{2\pi(k+1)}{\tilde{\omega}_0}\right] \cup \left[\frac{2\pi(k+1)}{\tilde{\omega}_0}, \infty\right[\\ 1 & \text{for } t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0}\right] \end{cases}. \quad (5)$$

Next, define the sequence of trajectories $z_k \in \mathfrak{B}_c(P)$ through

$$z_k(t) := U \left(\frac{d}{dt}\right) [v e^{\lambda_0 t} b_k(t)].$$



That all z_k are indeed trajectories of P thereby follows from Lemma 17. The construction implies that for $t \in \left[-\frac{2\pi k}{\bar{\omega}_0}, \frac{2\pi k}{\bar{\omega}_0}\right]$ we have

$$z_k(t) = U\left(\frac{d}{dt}\right)[ve^{\lambda_0 t}] = U(\lambda_0)ve^{\lambda_0 t},$$

and thus we have shown part 1. Part 4. follows from (5). To see part 2. we find that for all $k \in \mathbb{N}$ and $t \in \left[0, \frac{2\pi}{\bar{\omega}_0}\right]$ we have

$$\begin{aligned} z_k\left(t + \frac{2\pi k}{\bar{\omega}_0}\right) &= U\left(\frac{d}{dt}\right)\left[ve^{\lambda_0\left(t + \frac{2\pi k}{\bar{\omega}_0}\right)}b_k\left(t + \frac{2\pi k}{\bar{\omega}_0}\right)\right] \\ &= U\left(\frac{d}{dt}\right)\left[ve^{\lambda_0 t + \frac{r_0}{\bar{\omega}_0}2\pi k + i\frac{\bar{\omega}_0}{\bar{\omega}_0}2\pi k}\tilde{b}\left(t\frac{\bar{\omega}_0}{2\pi} + k + k\right)\tilde{b}\left(-t\frac{\bar{\omega}_0}{2\pi} - k + k\right)\right] \\ &= U\left(\frac{d}{dt}\right)\left[ve^{i\omega_0 t}e^{\frac{r_0}{\bar{\omega}_0}2\pi k}\underbrace{e^{i\frac{\bar{\omega}_0}{\bar{\omega}_0}2\pi k}}_{=1}\tilde{b}\left(t\frac{\bar{\omega}_0}{2\pi}\right)\tilde{b}\left(-t\frac{\bar{\omega}_0}{2\pi}\right)\right] \\ &= e^{\frac{r_0}{\bar{\omega}_0}2\pi k}U\left(\frac{d}{dt}\right)[ve^{i\omega_0 t}b_0(t)] = e^{\frac{r_0}{\bar{\omega}_0}2\pi k}z_0(t). \end{aligned}$$

Part 3. can be shown analogously and thus the proof is finished. \square

5 Para-Hermitian matrices

Consider a rational matrix $R \in \mathbb{C}(\lambda)^{p,q}$ in the form

$$R = \begin{bmatrix} r_{1,1} & \cdots & r_{1,q} \\ \vdots & & \vdots \\ r_{p,1} & \cdots & r_{p,q} \end{bmatrix},$$

where $r_{i,j} \in \mathbb{C}(\lambda)$. Then we have that

$$R^*(-\bar{\lambda}) = \begin{bmatrix} r_{1,1}(-\bar{\lambda}) & \cdots & r_{1,q}(-\bar{\lambda}) \\ \vdots & & \vdots \\ r_{p,1}(-\bar{\lambda}) & \cdots & r_{p,q}(-\bar{\lambda}) \end{bmatrix}^* = \begin{bmatrix} \overline{r_{1,1}(-\bar{\lambda})} & \cdots & \overline{r_{p,1}(-\bar{\lambda})} \\ \vdots & & \vdots \\ \overline{r_{1,q}(-\bar{\lambda})} & \cdots & \overline{r_{p,q}(-\bar{\lambda})} \end{bmatrix},$$

and one can easily see that $\overline{r_{i,j}(-\bar{\lambda})} \in \mathbb{C}(\lambda)$ is again a rational function in λ . This implies that $R^*(-\bar{\lambda})$ can be interpreted as a rational function in λ and justifies the following definition.

Definition 23. [22, Def. 2] Let $R \in \mathbb{C}(\lambda)^{p,q}$. Then we call the rational matrix $R^\sim \in \mathbb{C}(\lambda)^{q,p}$ which is defined through

$$R^\sim(\lambda) := R^*(-\bar{\lambda}),$$

the *para-Hermitian* of R . Further, R is called *para-Hermitian* if $R = R^\sim$.

Note that if $P \in \mathbb{C}[\lambda]^{p,q}$ is a polynomial matrix $P^\sim \in \mathbb{C}[\lambda]^{q,p}$ is also a polynomial. Some properties of the para-Hermitian are summed up in the following Lemma.

Lemma 24. *We have*

1. $(A^{-1})^\sim = (A^\sim)^{-1}$ for a square rational matrix A which is invertible over $\mathbb{C}(\lambda)$.
2. $(BC)^\sim = C^\sim B^\sim$ for arbitrary rational matrices B and C of proper dimension.
3. $(B^\sim)^\sim = B$ for every rational matrix B .
4. $B^\sim B$ is a para-Hermitian matrix for every rational matrix B .
5. If D is a para-Hermitian rational matrix so is $U^\sim D U$ for every rational matrix U of appropriate dimension.

Proof. The proof follows in the same way as that for the transpose of matrices with complex entries. \square

Lemma 24 especially justifies the notation $A^{-\sim} := (A^\sim)^{-1}$ to denote the para-Hermitian of the inverse of a square invertible rational matrix.

We will also need the notion of the para-Hermitian of a two-variable-polynomial matrix and the notion of para-Hermitian two-variable-polynomial matrices.

Definition 25. Let $M \in \mathbb{C}[\lambda, \mu]^{p,q}$ be a two-variable-polynomial matrix. Then we call $M^\sim \in \mathbb{C}[\lambda, \mu]^{q,p}$ defined through

$$M^\sim(\lambda, \mu) := M^*(\bar{\mu}, \bar{\lambda})$$

the *para-Hermitian* of M . Furthermore, we say that M is *para-Hermitian* if $M^\sim = M$.

Again, one can easily see that M^\sim is indeed a two-variable-polynomial in λ and μ . Note that the para-Hermitian is only defined for polynomials. To see how the para-Hermitian of two-variable-polynomial matrices and polynomial matrices are related, consider the following Lemma.

Lemma 26. *Let $M \in \mathbb{C}[\lambda, \mu]^{p,q}$ and define a polynomial matrix $R \in \mathbb{C}[\lambda]^{p,q}$ through*

$$R(\lambda) := M(-\lambda, \lambda).$$

Then, we have that $R^\sim(\lambda) = M^\sim(-\lambda, \lambda)$. Also, if $M = M^\sim$ is para-Hermitian so is $R = R^\sim$.

Proof. First, we have that

$$R^\sim(\lambda) = R^*(-\bar{\lambda}) = M^*(\bar{\lambda}, -\bar{\lambda}) = M^\sim(-\lambda, \lambda).$$

If M is also para-Hermitian, then we obtain that

$$R^\sim(\lambda) = M^\sim(-\lambda, \lambda) = M(-\lambda, \lambda) = R(\lambda),$$

and thus the assertion follows. \square

Definition 27. Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a non-zero matrix polynomial, $n \in \mathbb{N}$, and $P_i \in \mathbb{C}^{p,q}$ for $i = 0, 1, \dots, n$ such that

$$P(\lambda) = P_n \lambda^n + \dots P_1 \lambda + P_0,$$

where $P_n \neq 0$. Then we call

$$\deg(P) := n$$

the *degree of P* . For the zero matrix polynomial we set $\deg(0) := 0$.

Para-Hermitian matrices can easily be generated from bigger constant Hermitian matrices. The following Lemma 28 shows the relation between constant Hermitian matrices and para-Hermitian matrix polynomials. Lemma 29 shows the relation between constant Hermitian matrices and para-Hermitian two-variable-polynomial matrices.

Lemma 28. *Let $\ell \in \mathbb{N}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ be a Hermitian matrix. Define the matrix polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through*

$$H(\lambda) := (\Delta_\ell^q(-\bar{\lambda}))^* \tilde{H} \Delta_\ell^q(\lambda) = (\Delta_\ell^q(\lambda))^\sim \tilde{H} \Delta_\ell^q(\lambda). \quad (6)$$

Then H is para-Hermitian. On the other hand, consider an arbitrary para-Hermitian matrix polynomial $H = H^\sim \in \mathbb{C}[\lambda]^{q,q}$. Then, there exists an $\ell \in \mathbb{N}$ and $(3 \cdot \ell + 1)$ matrices $H_{0,0}, H_{i,i}, H_{i,i+1}, H_{i+1,i} \in \mathbb{C}^{q,q}$ with $i = 1, \dots, \ell$ which fulfill $H_{i,j} = H_{j,i}^$ for $|i - j| \leq 1$ and $i, j = 0, 1, \dots, \ell$ such that H can be represented in the form*

$$H(\lambda) = \begin{bmatrix} (-\bar{\lambda})^0 I_q \\ (-\bar{\lambda})^1 I_q \\ (-\bar{\lambda})^2 I_q \\ \vdots \\ (-\bar{\lambda})^\ell I_q \end{bmatrix}^* \begin{bmatrix} H_{0,0} & H_{0,1} & 0 & \cdots & 0 \\ H_{1,0} & H_{1,1} & H_{1,2} & \ddots & \vdots \\ 0 & H_{2,1} & H_{2,2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & H_{\ell-1,\ell} \\ 0 & \cdots & 0 & H_{\ell,\ell-1} & H_{\ell,\ell} \end{bmatrix} \begin{bmatrix} \lambda^0 I_q \\ \lambda^1 I_q \\ \lambda^2 I_q \\ \vdots \\ \lambda^\ell I_q \end{bmatrix}.$$

Especially, for every para-Hermitian matrix $H = H^\sim \in \mathbb{C}[\lambda]^{q,q}$ there exists an $\ell \in \mathbb{N}$ and a Hermitian matrix $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ such that (6) is fulfilled.

Proof. The first assertion follows directly from Lemma 24 since \tilde{H} can be viewed as an para-Hermitian matrix polynomial of degree 0.

For the second assertion let $H \in \mathbb{C}[\lambda]^{q,q}$ be an arbitrary para-Hermitian matrix polynomial which has the form $H(\lambda) = \sum_{i=0}^{2\ell} H_i \lambda^i$, where $\ell \in \mathbb{N}$ and $H_i \in \mathbb{C}^{q,q}$ for $i = 0, 1, \dots, 2\ell$. Clearly every H can be written in this form, although if $\deg H < 2\ell$ we may have that $H_{2\ell} = 0$. Note that we may choose H to have degree less or equal to 2ℓ so that ℓ is already the ℓ mentioned in the statement of the Lemma. Equating coefficients in the identity

$$\sum_{i=0}^{2\ell} H_i \lambda^i = H(\lambda) = H^\sim(\lambda) = \left(\sum_{i=0}^{2\ell} H_i (-\bar{\lambda})^i \right)^* = \sum_{i=0}^{2\ell} (-1)^i H_i^* \lambda^i$$

reveals that $H_i = (-1)^i H_i^*$ for $i = 0, \dots, 2\ell$ and thus that $H_{2i} = H_{2i}^*$ for $i = 0, \dots, \ell$ and $H_{2i+1} = -H_{2i+1}^*$ for $i = 0, \dots, \ell - 1$. With this notation we make the definitions

$$\begin{aligned} H_{i,i} &:= (-1)^i H_{2i}, & \text{for } i = 0, \dots, \ell, \\ H_{i,i+1} &:= (-1)^{i+\frac{1}{2}} H_{2i+1} & \text{for } i = 0, \dots, \ell - 1, \\ H_{i+1,i} &:= H_{i,i+1}^* & \text{for } i = 0, \dots, \ell - 1, \end{aligned}$$

to conclude that

$$\begin{aligned} & \begin{bmatrix} (-\bar{\lambda})^0 I_q \\ (-\bar{\lambda})^1 I_q \\ (-\bar{\lambda})^2 I_q \\ \vdots \\ (-\bar{\lambda})^\ell I_q \end{bmatrix}^* \begin{bmatrix} H_{0,0} & H_{0,1} & 0 & \cdots & 0 \\ H_{1,0} & H_{1,1} & H_{1,2} & \ddots & \vdots \\ 0 & H_{2,1} & H_{2,2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & H_{\ell-1,\ell} \\ 0 & \cdots & 0 & H_{\ell,\ell-1} & H_{\ell,\ell} \end{bmatrix} \begin{bmatrix} \lambda^0 I_q \\ \lambda^1 I_q \\ \lambda^2 I_q \\ \vdots \\ \lambda^\ell I_q \end{bmatrix} \\ &= \sum_{i=0}^{\ell} (-\lambda)^i (\lambda)^i H_{i,i} + \sum_{i=0}^{\ell-1} (-\lambda)^i (\lambda)^{i+1} H_{i,i+1} + \sum_{i=0}^{\ell-1} (-\lambda)^{i+1} (\lambda)^i H_{i+1,i} \\ &= \sum_{i=0}^{\ell} (-1)^i \lambda^{2i} (-1)^i H_{2i} + \sum_{i=0}^{\ell-1} (-1)^i \lambda^{2i+1} H_{i,i+1} + \sum_{i=0}^{\ell-1} (-1)^{i+1} \lambda^{2i+1} H_{i,i+1}^* \\ &= \sum_{i=0}^{\ell} \lambda^{2i} H_{2i} + \sum_{i=0}^{\ell-1} \lambda^{2i+1} (-1)^i [H_{i,i+1} - H_{i,i+1}^*] \\ &= \sum_{i=0}^{\ell} \lambda^{2i} H_{2i} + \sum_{i=0}^{\ell-1} \lambda^{2i+1} (-1)^i [(-1)^{i+\frac{1}{2}} H_{2i+1} - (-1)^{i+\frac{1}{2}} H_{2i+1}^*] \\ &= \sum_{i=0}^{\ell} \lambda^{2i} H_{2i} + \sum_{i=0}^{\ell-1} \lambda^{2i+1} \left[\frac{1}{2} H_{2i+1} - \frac{1}{2} H_{2i+1}^* \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\ell} \lambda^{2i} H_{2i} + \sum_{i=0}^{\ell-1} \lambda^{2i+1} \left[\frac{1}{2} H_{2i+1} + \frac{1}{2} H_{2i+1} \right] \\
&= \sum_{i=0}^{\ell} \lambda^{2i} H_{2i} + \sum_{i=0}^{\ell-1} \lambda^{2i+1} H_{2i+1} = \sum_{i=0}^{\ell} \lambda^i H_i = H(\lambda),
\end{aligned}$$

which proves the claim. \square

Lemma 29. *Let $\ell \in \mathbb{N}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ be a Hermitian matrix. Define the two-variable-polynomial matrix $H \in \mathbb{C}[\lambda, \mu]^{q, q}$ through*

$$H(\lambda, \mu) := (\Delta_{\ell}^q(\bar{\lambda}))^* \tilde{H} \Delta_{\ell}^q(\mu). \quad (7)$$

Then H is para-Hermitian. On the other hand, let an arbitrary para-Hermitian two-variable-polynomial matrix $H = H^{\sim} \in \mathbb{C}[\lambda, \mu]^{q, q}$ be given. Then, there exists an $\ell \in \mathbb{N}$ and an $\tilde{H} = \tilde{H}^ \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ such that (7) is fulfilled as an equation (and not as a definition).*

Proof. First, we have that

$$\begin{aligned}
H^{\sim}(\lambda, \mu) = H^*(\bar{\mu}, \bar{\lambda}) &= \left((\Delta_{\ell}^q(\mu))^* \tilde{H} \Delta_{\ell}^q(\bar{\lambda}) \right)^* \\
&= (\Delta_{\ell}^q(\bar{\lambda}))^* \tilde{H} \Delta_{\ell}^q(\mu) = H(\lambda, \mu),
\end{aligned}$$

which shows that H as defined in (7) is para-Hermitian.

For the second assertion let H take the form $H(\lambda, \mu) = \sum_{i,j=0}^{\ell} H_{i,j} \lambda^i \mu^j$ and conclude that H being para-Hermitian implies

$$\begin{aligned}
\sum_{i,j=0}^{\ell} H_{i,j} \lambda^i \mu^j &= H(\lambda, \mu) = H^{\sim}(\lambda, \mu) = H^*(\bar{\mu}, \bar{\lambda}) \\
&= \left(\sum_{i,j=0}^{\ell} H_{i,j} \bar{\mu}^i \bar{\lambda}^j \right)^* = \sum_{i,j=0}^{\ell} H_{i,j}^* \mu^i \lambda^j.
\end{aligned}$$

By equating coefficients, this implies that

$$H_{i,j} = H_{j,i}^* \quad (8)$$

for $i, j = 0, \dots, \ell$. Define the matrix $\tilde{H} \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ through

$$\tilde{H} := \begin{bmatrix} H_{0,0} & \cdots & H_{0,\ell} \\ \vdots & & \vdots \\ H_{\ell,0} & \cdots & H_{\ell,\ell} \end{bmatrix}.$$

By using (8) we conclude that we have $\tilde{H} = \tilde{H}^*$. Also one obtains (7) (as an equation rather than a definition) through

$$(\Delta_{\ell}^q(\bar{\lambda}))^* \tilde{H} \Delta_{\ell}^q(\mu) = \begin{bmatrix} \lambda^0 I_q & \cdots & \lambda^{\ell} I_q \end{bmatrix}^* \begin{bmatrix} H_{0,0} & \cdots & H_{0,\ell} \\ \vdots & & \vdots \\ H_{\ell,0} & \cdots & H_{\ell,\ell} \end{bmatrix} \begin{bmatrix} \mu^0 I_q \\ \vdots \\ \mu^{\ell} I_q \end{bmatrix}$$

$$= \sum_{i,j}^{\ell} H_{i,j} \lambda^i \mu^j = H(\lambda, \mu),$$

which proves the claim. \square

Para-Hermitian polynomial matrices have an interesting eigenvalue symmetry.

Lemma 30. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial. Then we have*

$$\sigma(P^\sim) = -\overline{\sigma(P)}.$$

Epecially, if $p = q$ and $P = P^\sim$ is para-Hermitian then we have that

$$\sigma(P) = -\overline{\sigma(P)},$$

i.e., the spectrum of an para-Hermitian matrix is symmetric to the imaginary axis.

Proof. First note that for every unimodular matrix $T \in \mathbb{C}[\lambda]^{q,q}$ also its para-Hermitian T^\sim is unimodular. Let a Smith form (2) of P be given by

$$P = S \begin{bmatrix} \text{diag}(d_1, \dots, d_r) & 0 \\ 0 & 0 \end{bmatrix} T.$$

Then, we have that

$$\begin{aligned} P^\sim &= T^\sim \begin{bmatrix} \text{diag}(d_1, \dots, d_r)^\sim & 0 \\ 0 & 0 \end{bmatrix} S^\sim \\ &= T^\sim \begin{bmatrix} \text{diag}(d_1^\sim, \dots, d_r^\sim) & 0 \\ 0 & 0 \end{bmatrix} S^\sim, \end{aligned}$$

and since every d_i can be factored into a product of linear polynomials, it is sufficient to show that for $p(\lambda) := \lambda - a$ with $a \in \mathbb{C}$ we have $\sigma(p^\sim) = -\overline{\sigma(p)}$. Since, however,

$$(\lambda - a)^\sim = (-\bar{\lambda} - a)^* = -\lambda - \bar{a},$$

this is clearly the case. \square

The following Lemma will be needed in the next section and it shows that the differential operator $P^\sim \left(\frac{d}{dt}\right)$ is the adjoint operator of P , when considered with respect to a special scalar product.

Lemma 31. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial of the form $P(\lambda) = \sum_{i=0}^n \lambda^i P_i$, let $y \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^p)$, and $z \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^q)$. Then we have that*

$$\int_{-\infty}^{\infty} z^*(t) P^\sim \left(\frac{d}{dt}\right) y(t) dt = \int_{-\infty}^{\infty} \left(P \left(\frac{d}{dt}\right) z(t) \right)^* y(t) dt$$

Proof. Using repeated partial integration we see that for $i = 0, \dots, n$ we have

$$\begin{aligned}
 & \int_{-\infty}^{\infty} z^*(t) P_i^* y^{(i)}(t) dt \\
 &= z^*(t) P_i^* y^{(i-1)}(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \dot{z}^*(t) P_i^* y^{(i-1)}(t) dt \\
 &= (-1)^1 \dot{z}^*(t) P_i^* y^{(i-2)}(t) \Big|_{-\infty}^{\infty} + (-1)^2 \int_{-\infty}^{\infty} \ddot{z}^*(t) P_i^* y^{(i-2)}(t) dt \\
 &= \dots = (-1)^i \int_{-\infty}^{\infty} \left(z^{(i)}(t) \right)^* P_i^* y(t) dt,
 \end{aligned}$$

where we used that y and z have compact support. Using the formula $P^\sim(\lambda) = \sum_{i=0}^n \lambda^i (-1)^i P_i^*$ this implies

$$\begin{aligned}
 \int_{-\infty}^{\infty} z^*(t) P^\sim \left(\frac{d}{dt} \right) y(t) dt &= \sum_{i=0}^n (-1)^i \int_{-\infty}^{\infty} z^*(t) P_i^* y^{(i)}(t) dt \\
 &= \sum_{i=0}^n (-1)^i (-1)^i \int_{-\infty}^{\infty} \left(z^{(i)}(t) \right)^* P_i^* y(t) dt \\
 &= \int_{-\infty}^{\infty} \left(\sum_{i=0}^n P_i z^{(i)}(t) \right)^* y(t) dt
 \end{aligned}$$

i.e., the assertion. \square

Remark 32. Note that para-Hermitian matrix polynomials are sometimes also called *even* in the literature, see [12]. Indeed, the concept of even matrix polynomials generalizes the concept of para-Hermitian matrix polynomials and is particularly useful when considering the associated eigenvalue problems [15].

6 Dissipativity

In this section we prove some special cases of results which have already been shown in [21]. However, the lesser generality of the results herein seems to be justified by the greater simplicity of the presentation.

Definition 33. Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$. Then we call P *dissipative* with respect to \tilde{H} if there exists an $n \in \mathbb{N}$ and a so called *storage function* $\Theta : \mathbb{C}^{q(n+1)} \rightarrow \mathbb{R}$, i.e., a continuous function Θ such that the *dissipation inequality*

$$\Theta(\Delta_n z(t_1)) - \Theta(\Delta_n z(t_0)) \leq \int_{t_0}^{t_1} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)) dt \quad (9)$$

is fulfilled for all $t_0 \leq t_1$ and all $z \in \mathfrak{B}_c(P)$. If the dissipation inequality (9) holds for all trajectories $z \in \mathfrak{B}(P)$ we say that P is *complete dissipative* with respect to \tilde{H} .

The term on the right hand side of the dissipation inequality (9) can be viewed as a measure of the amount of energy which is supplied to the system given by P along the trajectory z in the time frame t_0 to t_1 . The function $(\Delta_\ell z(t))^* \tilde{H}(\Delta_\ell z(t))$ thus measures the amount of energy supplied to the system given by P along the trajectory z at the time point t . The left hand side of the dissipation inequality (9) can be viewed as a measure of the gain in energy which is stored in the system internally, along the trajectory z . The function $\Theta(\Delta_n z(t))$ measures the internal energy at the time point t . In other words, the dissipation inequality (9) states that the system P cannot generate energy (with energy supply measured by means of \tilde{H}), i.e., it only dissipates energy. The matrix \tilde{H} encapsulates the notion of energy which we want to impose onto the system.

Note that if we have a differentiable storage function, then we can rewrite the dissipation inequality (9) into the form

$$\frac{d}{dt}(\Theta(\Delta_n z(t))) \leq (\Delta_\ell z(t))^* \tilde{H}(\Delta_\ell z(t)), \quad (10)$$

by dividing (9) by $t_1 - t_0$, letting t_1 go to t_0 , and using the mean value theorem.

Another common notion is introduced in the following Definition.

Definition 34. Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial and let $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$. Then we call P *cyclo-dissipative* with respect to \tilde{H} if the *cyclo-dissipation inequality*

$$0 \leq \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \tilde{H}(\Delta_\ell z(t)) dt, \quad (11)$$

is fulfilled for all $z \in \mathfrak{B}_c(P)$.

Cyclo-dissipativity thus only demands that every trajectory of the system which starts at zero and comes back to zero at some time later does not generate energy. It is quite easy to see that dissipativity implies cyclo-dissipativity, see the proof of Corollary 43. In the following we will show that dissipativity and cyclo-dissipativity are indeed equivalent.

Clearly, complete dissipativity is not equivalent to (cyclo-)dissipativity although complete dissipativity implies dissipativity. In this paper we will not discuss the problem under which additional assumptions dissipativity implies complete dissipativity despite the interest of this problem. Dissipativity (and thus cyclo-dissipativity) has some very nice equivalent characterizations as we will see. We start by deriving a frequency-domain characterization.

Definition 35. Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial and let $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ be its rank. Let $U \in \mathbb{C}(\lambda)^{q, q-r}$ be a kernel matrix of P . Let $\ell \in \mathbb{N}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$. Define $H \in \mathbb{C}[\lambda]^{q,q}$ through

$$H(\lambda) := (\Delta_\ell^q(-\bar{\lambda}))^* \tilde{H}(\Delta_\ell^q(\lambda))$$

Then we call $\Pi \in \mathbb{C}(\lambda)^{q-r, q-r}$ defined through

$$\Pi := U^{\sim} H U,$$

a Popov function of P with respect to \tilde{H} .

Theorem 10 implies that for every matrix polynomial P and Hermitian matrix \tilde{H} there exists a Popov function which is polynomial.

In the following Theorem 36 cyclo-dissipativity is characterized by a frequency-domain condition. Although the proof has already been given in [21, Proposition 5.2] we repeat it here to make the paper self-contained.

Theorem 36. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial and let $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ be its rank. Let $U \in \mathbb{C}(\lambda)^{q,q-r}$ be a kernel matrix of P . Let $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ and construct from it the para-Hermitian matrix polynomial $H \in \mathbb{C}[\lambda]^{q,q}$ through*

$$H(\lambda) := (\Delta_\ell^q(-\bar{\lambda}))^* \tilde{H} (\Delta_\ell^q(\lambda)).$$

Then P is cyclo-dissipative with respect to \tilde{H} if and only if the Popov function $\Pi := U^ H U$ is positive semi-definite along the imaginary axis, i.e., we have*

$$\Pi(i\omega) \geq 0,$$

for all $\omega \in \mathbb{R}$ such that $i\omega \in \mathfrak{D}(U)$.

Proof. First, assume that $\Pi(i\omega) \geq 0$ for all $i\omega \in \mathfrak{D}(U)$ and let $z \in \mathfrak{B}_c(P)$ be arbitrary. Using the Parseval identity from Lemma 21 we find that

$$\begin{aligned} & \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)) dt \\ &= \int_{-\infty}^{\infty} \left(\Delta_\ell^q \left(\frac{d}{dt} \right) z(t) \right)^* \tilde{H} \left(\Delta_\ell^q \left(\frac{d}{dt} \right) z(t) \right) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\mathfrak{L} \left\{ \Delta_\ell^q \left(\frac{d}{dt} \right) z \right\} (i\omega) \right)^* \tilde{H} \left(\mathfrak{L} \left\{ \Delta_\ell^q \left(\frac{d}{dt} \right) z \right\} (i\omega) \right) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\Delta_\ell^q(i\omega) Z(i\omega))^* \tilde{H} (\Delta_\ell^q(i\omega) Z(i\omega)) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^*(i\omega) (\Delta_\ell^q(-i\bar{\omega}))^* \tilde{H} (\Delta_\ell^q(i\omega)) Z(i\omega) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^*(i\omega) H(i\omega) Z(i\omega) dt. \end{aligned}$$

By taking the two sided Laplace-transform of the identity $P \left(\frac{d}{dt} \right) z = 0$, using the linearity of the two sided Laplace-transform, and using Lemma 21 we obtain

$$0 = \mathfrak{L} \left\{ P \left(\frac{d}{dt} \right) z \right\} (\lambda) = \sum_{j=0}^d P_j \mathfrak{L} \left\{ z^{(j)} \right\} (\lambda) = P(\lambda) Z(\lambda).$$

Thus, Lemma 19 shows that there exists an $\alpha : \mathfrak{D}(U) \rightarrow \mathbb{C}^{q-r}$ such that $Z(\lambda) = U(\lambda)\alpha(\lambda)$ for all $\lambda \in \mathfrak{D}(U)$. Since we can divide any integral over \mathbb{R} into a finite

number of distinct intervals such that the interior of each interval does not contain any singularity of U , i.e., an element of $\mathfrak{B}(U)$ we can write (in a slightly symbolic fashion) that

$$\begin{aligned}
 \int_{-\infty}^{\infty} (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Z^*(i\omega) H(i\omega) Z(i\omega) dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha^*(i\omega) U^*(i\omega) H(i\omega) U(i\omega) \alpha(i\omega) dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha^*(i\omega) U^*(-i\bar{\omega}) H(i\omega) U(i\omega) \alpha(i\omega) dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \alpha^*(i\omega) \Pi(i\omega) \alpha(i\omega) dt \geq 0,
 \end{aligned}$$

since Π is assumed to be positive semi-definite along the imaginary axis. This means that P is cyclo-dissipative with respect to \tilde{H} .

For the reverse direction assume to the contrary that there exists an ω_0 and a $v \in \mathbb{C}^q$ such that $0 > v^* \Pi(i\omega_0) v$. Use Lemma 22 to construct a sequence of trajectories $\{z_k\}_{k \in \mathbb{N}} \subset \mathfrak{B}_c(P)$ such that

$$\begin{aligned}
 z_k(t) &= U(i\omega_0) v e^{i\omega_0 t}, \quad \text{for } t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0} \right], \\
 z_0(t) &= z_k \left(t + \frac{2\pi k}{\tilde{\omega}_0} \right), \quad \text{for } t \in \left[0, \frac{2\pi}{\tilde{\omega}_0} \right], \\
 z_0(t) &= z_k \left(t - \frac{2\pi k}{\tilde{\omega}_0} \right), \quad \text{for } t \in \left[-\frac{2\pi}{\tilde{\omega}_0}, 0 \right], \\
 z_k(t) &= 0, \quad \text{for } t \in \left[-\infty, -\frac{2\pi(k+1)}{\tilde{\omega}_0} \right] \cup \left[\frac{2\pi(k+1)}{\tilde{\omega}_0}, \infty \right],
 \end{aligned}$$

where $\tilde{\omega}_0$ is defined as in the statement of Lemma 22. This implies that for $k \in \mathbb{N}$ and $t \in \left[-\frac{2\pi k}{\tilde{\omega}_0}, \frac{2\pi k}{\tilde{\omega}_0} \right]$ we have

$$\Delta_{\ell} z_k(t) = \Delta_{\ell}^q \left(\frac{d}{dt} \right) U(i\omega_0) v e^{i\omega_0 t} = \Delta_{\ell}^q(i\omega_0) U(i\omega_0) v e^{i\omega_0 t},$$

and thus for $k \in \mathbb{N}$ we see that using the transformation rules $\phi_k(t) = t + \frac{2\pi k}{\tilde{\omega}_0}$ and $\psi_k(t) = t - \frac{2\pi k}{\tilde{\omega}_0}$ we obtain

$$\begin{aligned}
 &\int_{-\infty}^{\infty} (\Delta_{\ell} z_k(t))^* \tilde{H} (\Delta_{\ell} z_k(t)) dt \\
 &= \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_{\ell} z_k(t))^* \tilde{H} (\Delta_{\ell} z_k(t)) dt \\
 &\quad + \int_{-\frac{2\pi(k+1)}{\tilde{\omega}_0}}^{-\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_{\ell} z_k(t))^* \tilde{H} (\Delta_{\ell} z_k(t)) dt + \int_{\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi(k+1)}{\tilde{\omega}_0}} (\Delta_{\ell} z_k(t))^* \tilde{H} (\Delta_{\ell} z_k(t)) dt
 \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_\ell^q(i\omega_0)U(i\omega_0)ve^{i\omega_0 t})^* \tilde{H}(\Delta_\ell^q(i\omega_0)U(i\omega_0)ve^{i\omega_0 t}) dt \\
 &\quad + \int_{\psi_k(-\frac{2\pi}{\tilde{\omega}_0})}^{\psi_k(0)} (\Delta_\ell z_k(t))^* \tilde{H}(\Delta_\ell z_k(t)) dt + \int_{\phi_k(0)}^{\phi_k(\frac{2\pi}{\tilde{\omega}_0})} (\Delta_\ell z_k(t))^* \tilde{H}(\Delta_\ell z_k(t)) dt \\
 &= \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} e^{-i\omega_0 t} v^* U^*(i\omega_0) (\Delta_\ell^q(i\omega_0))^* \tilde{H}(\Delta_\ell^q(i\omega_0)) (U(i\omega_0)) v e^{i\omega_0 t} dt \\
 &\quad + \int_{-\frac{2\pi}{\tilde{\omega}_0}}^0 \dot{\psi}_k(t) (\Delta_\ell z_k(\psi_k(t)))^* \tilde{H}(\Delta_\ell z_k(\psi_k(t))) dt \\
 &\quad + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} \dot{\phi}_k(t) (\Delta_\ell z_k(\phi_k(t)))^* \tilde{H}(\Delta_\ell z_k(\phi_k(t))) dt \\
 &= v^* \Pi(i\omega_0) v \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} dt \\
 &\quad + \int_{-\frac{2\pi}{\tilde{\omega}_0}}^0 (\Delta_\ell z_0(t))^* \tilde{H}(\Delta_\ell z_0(t)) dt + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} (\Delta_\ell z_0(t))^* \tilde{H}(\Delta_\ell z_0(t)) dt \\
 &= \frac{4\pi k}{\tilde{\omega}_0} v^* \Pi(i\omega_0) v + c,
 \end{aligned}$$

by setting

$$c := \int_{-\frac{2\pi}{\tilde{\omega}_0}}^0 (\Delta_\ell z_0(t))^* \tilde{H}(\Delta_\ell z_0(t)) dt + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} (\Delta_\ell z_0(t))^* \tilde{H}(\Delta_\ell z_0(t)) dt.$$

Clearly, c is a constant which does not depend on k .

All in all we have shown that (under the assumption that there exists an $\omega_0 \in \mathbb{R}$ such that $v^* \Pi(i\omega_0) v < 0$) there exists a sequence of trajectories of the system with compact support $z_k \in \mathfrak{B}_c(P)$ and a $c \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} (\Delta_\ell z_k(t))^* \tilde{H}(\Delta_\ell z_k(t)) dt = \frac{4\pi k}{\tilde{\omega}_0} v^* \Pi(i\omega_0) v + c.$$

Thus, there exists a $k_0 \in \mathbb{N}$ such that

$$\int_{-\infty}^{\infty} (\Delta_\ell z_{k_0}(t))^* \tilde{H}(\Delta_\ell z_{k_0}(t)) dt < 0,$$

which contradicts the assumption of dissipativity. \square

Theorem 36 can be summarized in the following words. A system is cyclo-dissipative if and only if a Popov function of it is positive semi-definite along the

imaginary axis. In this case every Popov function is positive semi-definite along the imaginary axis.

We first need some Lemmas before we can state the main theorems of this section.

Lemma 37. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $H = H^\sim \in \mathbb{C}[\lambda]^{q,q}$ for some $\ell \in \mathbb{N}$. Let $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ be polynomial kernel and co-kernel matrices of P . Then the following are equivalent*

1. *The Popov-function $\Pi := U^\sim H U$ is positive semi-definite along the imaginary axis.*
2. *There exists a $D = D^\sim \in \mathbb{C}[\lambda]^{q,q}$ which is positive semi-definite along the imaginary axis, such that $U^\sim H U = U^\sim D U$.*

Proof. First assume that 2. holds and let $i\omega \in i\mathbb{R}$ be arbitrary. Then also

$$U^\sim(i\omega)H(i\omega)U(i\omega) = U^*(-i\bar{\omega})D(i\omega)U(i\omega) = U^*(i\omega)D(i\omega)U(i\omega) \geq 0,$$

since $D(i\omega) \geq 0$ by assumption. For the converse direction, set $W := [U \ V]^{-1} \in \mathbb{C}[\lambda]^{q,q}$, and define the matrix $D \in \mathbb{C}[\lambda]^{q,q}$ through

$$D := W^\sim \begin{bmatrix} U^\sim H U & 0 \\ 0 & 0 \end{bmatrix} W.$$

Then we have that

$$WU = [U \ V]^{-1} U = \begin{bmatrix} I_{q-r} \\ 0 \end{bmatrix},$$

and thus

$$U^\sim D U = (WU)^\sim \begin{bmatrix} U^\sim H U & 0 \\ 0 & 0 \end{bmatrix} WU = \begin{bmatrix} I_{q-r} \\ 0 \end{bmatrix}^* \begin{bmatrix} U^\sim H U & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{q-r} \\ 0 \end{bmatrix} = U^\sim H U.$$

Let $i\omega \in i\mathbb{R}$ be arbitrary and observe that

$$D(i\omega) = W^*(i\omega) \begin{bmatrix} U^*(i\omega)H(i\omega)U(i\omega) & 0 \\ 0 & 0 \end{bmatrix} W(i\omega) \geq 0,$$

which means that D is positive semi-definite along the imaginary axis. \square

Using the construction in Lemma 37 we can not guarantee that the degree of D is bounded by the degree of H as shown in the following example.

Example 38. Consider the polynomial

$$P(\lambda) := \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{=:P_1} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:P_0} \in \mathbb{C}[\lambda]^{2,3}$$

together with

$$\tilde{H} := \begin{bmatrix} 1 & & \\ & -2 & \\ & & 1 \end{bmatrix}.$$

In this case a polynomial kernel and co-kernel matrix are given by

$$U(\lambda) := \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}, \text{ and } V(\lambda) := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and thus a Popov function is

$$\Pi(\lambda) := U^\sim(\lambda)\tilde{H}U(\lambda) = [1 \quad -\lambda \quad \lambda^2] \begin{bmatrix} 1 & & \\ & -2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = 1 + 2\lambda^2 + \lambda^4.$$

For $\omega \in \mathbb{R}$ we obtain

$$\Pi(i\omega) = 1 - 2\omega^2 + \omega^4 = (\omega - 1)^2(\omega + 1)^2 \geq 0,$$

i.e., the Popov function is positive semi-definite along the imaginary axis. We set

$$W(\lambda) := [U \quad V]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ -\lambda^2 & 0 & 1 \end{bmatrix},$$

and with this

$$D(\lambda) := W^\sim(\lambda) \begin{bmatrix} \Pi(\lambda) & & \\ & 0 & \\ & & 0 \end{bmatrix} W(\lambda) = \begin{bmatrix} \Pi(\lambda) & & \\ & 0 & \\ & & 0 \end{bmatrix},$$

as in the proof of Lemma 37. This shows that with the construction of Lemma 37 we have in general that $\deg D \gg \deg H := \deg \tilde{H} = 0$, since \tilde{H} is a constant matrix (polynomial). Of course, there may be another D available with a lower degree. This is indeed the case. Define

$$\tilde{D} := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then we have

$$\begin{aligned} U^\sim(\lambda)\tilde{D}U(\lambda) &= [1 \quad -\lambda \quad \lambda^2] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} \\ &= [1 \quad -\lambda \quad \lambda^2] \begin{bmatrix} 1 + \lambda^2 \\ 0 \\ 1 + \lambda^2 \end{bmatrix} \\ &= 1 + \lambda^2 + \lambda^2 + \lambda^4 = \Pi(\lambda) = U^\sim(\lambda)\tilde{H}U(\lambda), \end{aligned}$$

i.e., there exists a polynomial $D(\lambda) = \tilde{D}$ of the same degree as $H(\lambda) = \tilde{H}$ such that $U^\sim\tilde{D}U = U^\sim HU$.

The following Lemma makes a connection between positive semi-definite matrices and para-Hermitian matrix polynomials that are positive semi-definite along the imaginary axis.

Lemma 39. *Let $D = D^\sim \in \mathbb{C}[\lambda]^{q,q}$. Then we have the following.*

1. *If D has the representation $D(\lambda) = (\Delta_m^q(-\bar{\lambda}))^* \tilde{D}(\Delta_m^q(\lambda))$, for some $m \in \mathbb{N}$ and $\tilde{D} = \tilde{D}^* \in \mathbb{C}^{q(m+1),q(m+1)}$ with $\tilde{D} \geq 0$ then D is positive semi-definite along the imaginary axis.*
2. *If D is positive semi-definite along the imaginary axis then there exists an $m \in \mathbb{N}$ and a $\tilde{D} = \tilde{D}^* \in \mathbb{C}^{q(m+1),q(m+1)}$ such that $\tilde{D} \geq 0$ and D has the representation $D(\lambda) = (\Delta_m^q(-\bar{\lambda}))^* \tilde{D}(\Delta_m^q(\lambda))$.*

Proof. For 1. assume that $\tilde{D} \geq 0$ and obtain that

$$\begin{aligned} D(i\omega) &= (\Delta_m^q(-i\bar{\omega}))^* \tilde{D}(\Delta_m^q(i\omega)) \\ &= (\Delta_m^q(i\omega))^* \tilde{D}(\Delta_m^q(i\omega)) \\ &= \begin{bmatrix} (i\omega)^0 I_q \\ (i\omega)^1 I_q \\ \vdots \\ (i\omega)^m I_q \end{bmatrix}^* \tilde{D} \begin{bmatrix} (i\omega)^0 I_q \\ (i\omega)^1 I_q \\ \vdots \\ (i\omega)^m I_q \end{bmatrix} \geq 0, \end{aligned}$$

i.e., that D is positive semi-definite along the imaginary axis. For part 2. let $F \in \mathbb{C}[\lambda]^{q,q}$ be such that $D = F^\sim F$, i.e., let F be a polynomial Youla-factor of D , which exists, since D is assumed to be positive semi-definite along the imaginary axis, see [22, Theorem 2 and Corollary 2] for the proof. Thus, since F is a polynomial, say of degree $m \in \mathbb{N}$, it can be written in the form

$$F(\lambda) = \tilde{F} \Delta_m^q(\lambda),$$

where $\tilde{F} \in \mathbb{C}^{q,q(m+1)}$ consists of the concatenated coefficient matrices of F . With this we obtain that

$$F^\sim(\lambda) = F^*(-\bar{\lambda}) = \left(\tilde{F}(\Delta_m^q(-\bar{\lambda})) \right)^* = (\Delta_m^q(-\bar{\lambda}))^* \tilde{F}^*,$$

and setting $\tilde{D} := \tilde{F}^* \tilde{F}$ shows that we have

$$D(\lambda) = F^\sim(\lambda)F(\lambda) = (\Delta_m^q(-\bar{\lambda}))^* \tilde{F}^* \tilde{F}(\Delta_m^q(\lambda)) = (\Delta_m^q(-\bar{\lambda}))^* \tilde{D}(\Delta_m^q(\lambda)),$$

with $\tilde{D} = \tilde{F}^* \tilde{F} \geq 0$ which finishes the proof. \square

The following example shows that part 2. of Lemma 39 is not true for every representation, i.e., it shows that there exist $\tilde{D} = \tilde{D}^*$ which are not positive semi-definite although they induce a para-Hermitian matrix polynomial through the identity $D(\lambda) = (\Delta_m^q(-\bar{\lambda}))^* \tilde{D}(\Delta_m^q(\lambda))$ which is positive semi-definite along the imaginary axis.

Example 40. Let $q = 1$, $m = 1$, and the matrix \tilde{D} be given by

$$\tilde{D} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly \tilde{D} is not positive semi-definite. The associated para-Hermitian matrix polynomial, however, is given by

$$D(\lambda) = (\Delta_1^1(-\bar{\lambda}))^* \tilde{D} (\Delta_1^1(\lambda)) = \begin{bmatrix} 1 & -\lambda \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda - \lambda = 0,$$

which is positive semi-definite along the imaginary axis.

The following result combines proofs of the results [21, Theorem 3.1] and [17, Theorem 4.3].

Lemma 41. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ with $r = \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P . Let $\tilde{M} = \tilde{M}^* \in \mathbb{C}^{q(m+1),q(m+1)}$, define the para-Hermitian two-variable-polynomial $M \in \mathbb{C}[\lambda, \mu]^{q,q}$ through $M(\lambda, \mu) := (\Delta_m^q(\bar{\lambda}))^* \tilde{M} (\Delta_m^q(\mu))$, and with this introduce $\Phi \in \mathbb{C}[\lambda, \mu]^{q-r,q-r}$ as $\Phi(\lambda, \mu) := U^*(\bar{\lambda})M(\lambda, \mu)U(\mu)$, which is also a para-Hermitian two-variable-polynomial. Then the following conditions are equivalent:*

1. We have that

$$\int_{-\infty}^{\infty} (\Delta_m z(t))^* \tilde{M} (\Delta_m z(t)) dt = 0,$$

for all $z \in \mathfrak{B}_c(P)$.

2. We have that $\Phi(-\lambda, \lambda) = 0$ for all $\lambda \in \mathbb{C}$.

3. There exists a para-Hermitian two-variable-polynomial $N = N^\sim \in \mathbb{C}[\lambda, \mu]^{q,q}$ such that

$$\Phi(\lambda, \mu) = (\lambda + \mu)U^*(\bar{\lambda})N(\lambda, \mu)U(\mu),$$

for all $\lambda, \mu \in \mathbb{C}$.

4. There exists an integer $n \in \mathbb{N}$ and a Hermitian matrix $\tilde{N} = \tilde{N}^* \in \mathbb{C}^{q(n+1),q(n+1)}$ such that

$$\frac{d}{dt} \left[(\Delta_n z(t))^* \tilde{N} (\Delta_n z(t)) \right] = (\Delta_m z(t))^* \tilde{M} (\Delta_m z(t)),$$

for all $z \in \mathfrak{B}_c(P)$.

Proof. We proof the result in the order 1. \Rightarrow 2. \Rightarrow 3. \Rightarrow 4. \Rightarrow 1. First assume that 1. holds and we want to show 2. Therefore, assume to the contrary there exists a $\lambda_0 \in \mathbb{C}$ such that $\Phi(-\lambda_0, \lambda_0) \neq 0$. Then there exist $v, w \in \mathbb{C}^{q-r}$ such that $w^* \Phi(-\lambda_0, \lambda_0) v \neq 0$. Split $\lambda_0 = r_0 + i\omega_0$ into its real part $r_0 = \text{Re}\{\lambda_0\}$ and imaginary part $\omega_0 = \text{Im}\{\lambda_0\}$. Let $\{z_k\}_{k \in \mathbb{N}} = \left\{ U \left(\frac{d}{dt} \right) v_k \right\}_{k \in \mathbb{N}} \subset \mathfrak{B}_c(P)$ with $v_k \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ be such that the

properties 1. - 4. of Lemma 22 are fulfilled. Also let $\{y_k\}_{k \in \mathbb{N}} = \left\{U\left(\frac{d}{dt}\right)w_k\right\}_{k \in \mathbb{N}} \subset \mathfrak{B}_c(P)$ with $w_k \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ be such that the properties 1. - 4. of Lemma 22 are fulfilled with v replaced by w and λ_0 replaced by $-\bar{\lambda}_0$ in a straightforward fashion. Since $-\bar{\lambda}_0 = -(r_0 - i\omega_0) = -r_0 + i\omega_0$, the imaginary part of λ_0 and $-\bar{\lambda}_0$ do not differ and thus also the definitions of $\tilde{\omega}_0$ in Lemma 22 do not differ for both sequences of trajectories. Using 1., the definition of $\tilde{\omega}_0$ from Lemma 22, and the transformation rules $\phi_k(t) = t + \frac{2\pi k}{\tilde{\omega}_0}$ and $\psi_k(t) = t - \frac{2\pi k}{\tilde{\omega}_0}$ we find that

$$\begin{aligned}
 0 &= \int_{-\infty}^{\infty} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt \\
 &= \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt + \int_{-\frac{2\pi(k+1)}{\tilde{\omega}_0}}^{-\frac{2\pi k}{\tilde{\omega}_0}} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt \\
 &\quad + \int_{\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi(k+1)}{\tilde{\omega}_0}} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt \\
 &= \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} \left(\Delta_m^q(-\bar{\lambda}_0)U(-\bar{\lambda}_0)w e^{-\bar{\lambda}_0 t} \right)^* \tilde{M}(\Delta_m^q(\lambda_0)U(\lambda_0)v e^{\lambda_0 t}) dt \\
 &\quad + \int_{\psi_k\left(-\frac{2\pi}{\tilde{\omega}_0}\right)}^{\psi_k(0)} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt \\
 &\quad + \int_{\phi_k(0)}^{\phi_k\left(\frac{2\pi}{\tilde{\omega}_0}\right)} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt \\
 &= \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} w^* \Phi(-\lambda_0, \lambda_0) v e^{(\lambda_0 - \lambda_0)t} dt \\
 &\quad + \int_{-\frac{2\pi}{\tilde{\omega}_0}}^0 \dot{\psi}_k(t) \left(\Delta_m y_k\left(t - \frac{2\pi k}{\tilde{\omega}_0}\right) \right)^* \tilde{M}\left(\Delta_m z_k\left(t - \frac{2\pi k}{\tilde{\omega}_0}\right)\right) dt \\
 &\quad + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} \dot{\phi}_k(t) \left(\Delta_m y_k\left(t + \frac{2\pi k}{\tilde{\omega}_0}\right) \right)^* \tilde{M}\left(\Delta_m z_k\left(t + \frac{2\pi k}{\tilde{\omega}_0}\right)\right) dt \\
 &= w^* \Phi(-\lambda_0, \lambda_0) v \int_{-\frac{2\pi k}{\tilde{\omega}_0}}^{\frac{2\pi k}{\tilde{\omega}_0}} dt \\
 &\quad + \int_{-\frac{2\pi}{\tilde{\omega}_0}}^0 \left(e^{r_0 \frac{2\pi k}{\tilde{\omega}_0}} \Delta_m y_0(t) \right)^* \tilde{M}\left(e^{-r_0 \frac{2\pi k}{\tilde{\omega}_0}} \Delta_m z_0(t) \right) dt \\
 &\quad + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} \left(e^{-r_0 \frac{2\pi k}{\tilde{\omega}_0}} \Delta_m y_0(t) \right)^* \tilde{M}\left(e^{r_0 \frac{2\pi k}{\tilde{\omega}_0}} \Delta_m z_0(t) \right) dt,
 \end{aligned}$$

where for the last identity we do not have to forget, that the real part of $-\bar{\lambda}_0$ is the

negative of the real part of λ_0 . Using that

$$\overline{e^{r_0 \frac{2\pi k}{\tilde{\omega}_0}}} e^{-r_0 \frac{2\pi k}{\tilde{\omega}_0}} = e^{(r_0 - r_0) \frac{2\pi k}{\tilde{\omega}_0}} = e^0 = 1,$$

and defining

$$c := \int_{-\frac{2\pi}{\tilde{\omega}_0}}^0 (\Delta_m y_0(t))^* \tilde{M}(\Delta_m z_0(t)) dt + \int_0^{\frac{2\pi}{\tilde{\omega}_0}} (\Delta_m y_0(t))^* \tilde{M}(\Delta_m z_0(t)) dt,$$

which is independent of $k \in \mathbb{N}$ finally shows that

$$0 = \int_{-\infty}^{\infty} (\Delta_m y_k(t))^* \tilde{M}(\Delta_m z_k(t)) dt = \frac{4\pi k}{\tilde{\omega}_0} w^* \Phi(-\lambda_0, \lambda_0) v + c, \quad (12)$$

for all $k \in \mathbb{N}$. This is a contradiction, since we assumed that $w^* \Phi(-\lambda_0, \lambda_0) v \neq 0$ and we only have to try two different k 's in the last identity (12) to find that it does not hold for all $k \in \mathbb{N}$.

Next, assume that 2. holds and for a fixed $\mu \in \mathbb{C}$ define $\phi_\mu \in \mathbb{C}[\lambda]^{q-r, q-r}$ through

$$\phi_\mu(\lambda) := \Phi(\lambda, \mu),$$

i.e., view Φ as a one variable polynomial in λ . Clearly, 2. implies that $\phi_\mu(-\mu) = 0$, i.e., $-\mu$ is a root of ϕ_μ . Thus we have shown that for every (fixed) $\mu \in \mathbb{C}$ we have

$$\phi_\mu(\lambda) = (\lambda + \mu)\psi_\mu(\lambda), \quad (13)$$

where $\psi_\mu \in \mathbb{C}[\lambda]^{q-r, q-r}$. Define the function $\Psi : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ through $\Psi(\lambda, \mu) := \psi_\mu(\lambda)$, where it is at first not clear whether Ψ is a two-variable-polynomial. However, equation (13) shows that we have

$$\Phi(\lambda, \mu) = \phi_\mu(\lambda) = (\lambda + \mu)\psi_\mu(\lambda) = (\lambda + \mu)\Psi(\lambda, \mu),$$

for all $\lambda, \mu \in \mathbb{C}$ and we know that for every $\mu \in \mathbb{C}$ the function $\Psi(\lambda, \mu) = \psi_\mu(\lambda)$ is a polynomial in λ . This shows that also $\Psi \in \mathbb{C}[\lambda, \mu]^{q-r, q-r}$ is a two-variable-polynomial, since if this would not be the case also $\Phi(\lambda, \mu)$ would not be a two-variable-polynomial. To see that Ψ is para-Hermitian, assume to the contrary that there exist λ_0, μ_0 such that $\Psi(\lambda_0, \mu_0) \neq \Psi^*(\overline{\mu_0}, \overline{\lambda_0})$. Since Ψ is continuous there also has to be a neighborhood around, let us say μ_0 , such that $\Psi(\lambda_0, \mu) \neq \Psi^*(\overline{\mu}, \overline{\lambda_0})$ for all μ in this neighborhood. This shows that we can w.l.o.g. assume that $\lambda_0 \neq -\mu_0$ or $\lambda_0 + \mu_0 \neq 0$. This implies that

$$\begin{aligned} \overline{\Phi(\lambda_0, \mu_0)} &= (\lambda_0 + \mu_0)\overline{\Psi(\lambda_0, \mu_0)} \neq (\lambda_0 + \mu_0)\Psi^*(\overline{\mu_0}, \overline{\lambda_0}) \\ &= ((\overline{\mu_0} + \overline{\lambda_0})\Psi(\overline{\mu_0}, \overline{\lambda_0}))^* = \Phi^*(\overline{\mu_0}, \overline{\lambda_0}), \end{aligned}$$

i.e., that Φ is not para-Hermitian, which is a contradiction and thus we have shown that Ψ is para-Hermitian. Let $\hat{U} \in \mathbb{C}[\lambda]^{q-r, q}$ be a polynomial left inverse of U , i.e.,

a polynomial matrix such that $\hat{U}U = I_{q-r}$ which exists due to part 2. of Lemma 14. Define the two-variable-polynomial matrix $N \in \mathbb{C}[\lambda, \mu]^{q,q}$ through

$$N(\lambda, \mu) := \hat{U}^*(\bar{\lambda})\Psi(\lambda, \mu)\hat{U}(\mu).$$

Then N is para-Hermitian. Also, we see that

$$\begin{aligned} (\lambda + \mu)U^*(\bar{\lambda})N(\lambda, \mu)U(\mu) &= (\lambda + \mu)U^*(\bar{\lambda})\hat{U}^*(\bar{\lambda})\Psi(\lambda, \mu)\hat{U}(\mu)U(\mu) \\ &= (\lambda + \mu)\left(\hat{U}(\bar{\lambda})U(\bar{\lambda})\right)^* \Psi(\lambda, \mu)\hat{U}(\mu)U(\mu) \\ &= (\lambda + \mu)\Psi(\lambda, \mu) = \Phi(\lambda, \mu), \end{aligned}$$

for $\lambda, \mu \in \mathbb{C}$, which means that we have shown 3.

To see that 3. implies 4. let $n \in \mathbb{N}$ and $\tilde{N} = \tilde{N}^* \in \mathbb{C}^{q(n+1), q(n+1)}$ be such that

$$N(\lambda, \mu) = (\Delta_n^q(\bar{\lambda}))^* \tilde{N} (\Delta_n^q(\mu)),$$

where the existence of such n and \tilde{N} is obtained from Lemma 29. Let $z \in \mathfrak{B}_c(P)$ be arbitrary and $\alpha \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C}^{q-r})$ such that $z = U\left(\frac{d}{dt}\right)\alpha$ according to Lemma 18. Then we see that with $L \in \mathbb{C}[\lambda, \mu]^{q-r, q-r}$ defined by $L(\lambda, \mu) := U^*(\bar{\lambda})N(\lambda, \mu)U(\mu)$ we have

$$\begin{aligned} \frac{d}{dt} \left[(\Delta_n z(t))^* \tilde{N} (\Delta_n z(t)) \right] &= \frac{d}{dt} \left[\left(\Delta_n^q \left(\frac{d}{dt} \right) z(t) \right)^* \tilde{N} \left(\Delta_n^q \left(\frac{d}{dt} \right) z(t) \right) \right] \\ &= \frac{d}{dt} \left[\left(U \left(\frac{d}{dt} \right) \alpha(t) \right)^* N \left(\frac{d}{dt}, \frac{d}{dt} \right) \left(U \left(\frac{d}{dt} \right) \alpha(t) \right) \right] \\ &= \frac{d}{dt} \left[\alpha^*(t) L \left(\frac{d}{dt}, \frac{d}{dt} \right) \alpha(t) \right] \\ &= \dot{\alpha}^*(t) L \left(\frac{d}{dt}, \frac{d}{dt} \right) \alpha(t) + \alpha^*(t) L \left(\frac{d}{dt}, \frac{d}{dt} \right) \dot{\alpha}(t) \\ &= \alpha^*(t) \Phi \left(\frac{d}{dt}, \frac{d}{dt} \right) \alpha(t) \\ &= \left(U \left(\frac{d}{dt} \right) \alpha^*(t) \right)^* M \left(\frac{d}{dt}, \frac{d}{dt} \right) \left(U \left(\frac{d}{dt} \right) \alpha(t) \right) \\ &= (\Delta_m z(t))^* \tilde{M} (\Delta_m z(t)), \end{aligned}$$

which proves 4.

Finally, to see that 4. implies 1. note that

$$\begin{aligned} \int_{-\infty}^{\infty} (\Delta_m z(t))^* \tilde{M} (\Delta_m z(t)) dt &= \int_{-\infty}^{\infty} \frac{d}{dt} \left[(\Delta_n z(t))^* \tilde{N} (\Delta_n z(t)) \right] dt \\ &= \left[(\Delta_n z(t))^* \tilde{N} (\Delta_n z(t)) \right] \Big|_{-\infty}^{\infty} = 0, \end{aligned}$$

since $z \in \mathfrak{B}_c(P)$ has compact support. \square

The introduction of the two-variable polynomial is only needed for the previous Lemma 41. Thus, it would be desirable to prove the equivalence of parts 1. and 4. in Lemma 41 without using two-variable-polynomials, to make the notation more simple. However, we do not know such a proof.

Other equivalent conditions for cyclo-dissipativity are given in the following Theorem.

Theorem 42. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1),q(\ell+1)}$ for some $\ell \in \mathbb{N}$. Then the following are equivalent:*

1. We have

$$\int_{-\infty}^{\infty} (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)) dt \geq 0$$

for all $z \in \mathfrak{B}_c(P)$, i.e., P is cyclo-dissipative with respect to \tilde{H} .

2. There exists $m \in \mathbb{N}$ and $\tilde{D} = \tilde{D}^* \in \mathbb{C}^{q(m+1),q(m+1)}$ with $\tilde{D} \geq 0$ such that

$$\int_{-\infty}^{\infty} (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)) dt = \int_{-\infty}^{\infty} (\Delta_m z(t))^* \tilde{D} (\Delta_m z(t)) dt,$$

for all $z \in \mathfrak{B}_c(P)$.

3. There exists $n \in \mathbb{N}$ and $\tilde{N} = \tilde{N}^* \in \mathbb{C}^{q(n+1),q(n+1)}$ such that

$$\frac{d}{dt} \left((\Delta_n z(t))^* \tilde{N} (\Delta_n z(t)) \right) \leq (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)),$$

for all $z \in \mathfrak{B}_c(P)$ and all $t \in \mathbb{R}$, i.e., P admits a quadratic storage function.

Proof. To see that 1. implies 2. assume cyclo-dissipativity. Let U be a polynomial kernel matrix of P and define $H = H^{\sim} \in \mathbb{C}[\lambda]^{q,q}$ through through (6), i.e., let $H(\lambda) := (\Delta_{\ell}^q(-\bar{\lambda}))^* \tilde{H} \Delta_{\ell}^q(\lambda)$. Theorem 36 then shows that in this case the Popov-function $\Pi := U^{\sim} H U$ is positive semi-definite along the imaginary axis, i.e., $\Pi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Thus, Lemma 37 shows that there exists a $D = D^{\sim} \in \mathbb{C}[\lambda]^{q,q}$ which is positive semi-definite along the imaginary axis such that $U^{\sim} H U = U^{\sim} D U$. In this case Lemma 39 proves the existence of $\tilde{D} = \tilde{D}^* \in \mathbb{C}^{q(m+1),q(m+1)}$ such that $D(\lambda) = (\Delta_m^q(-\bar{\lambda}))^* \tilde{D} (\Delta_m^q(\lambda))$ with $\tilde{D} \geq 0$. Using Lemma 18 we obtain the existence of $\alpha \in \mathbb{R}$ such that $z = U \left(\frac{d}{dt} \right) \alpha$. Further, using Lemma 31 we finally find that

$$\begin{aligned} \int_{-\infty}^{\infty} (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)) dt &= \int_{-\infty}^{\infty} z^*(t) \left(\Delta_{\ell}^q \left(-\frac{d}{dt} \right) \right)^* \tilde{H} \Delta_{\ell}^q \left(\frac{d}{dt} \right) z(t) dt \\ &= \int_{-\infty}^{\infty} \alpha^*(t) U^* \left(\frac{d}{dt} \right) H \left(\frac{d}{dt} \right) U \left(\frac{d}{dt} \right) \alpha(t) dt \\ &= \int_{-\infty}^{\infty} \alpha^*(t) U^* \left(\frac{d}{dt} \right) D \left(\frac{d}{dt} \right) U \left(\frac{d}{dt} \right) \alpha(t) dt \\ &= \int_{-\infty}^{\infty} z^*(t) D \left(\frac{d}{dt} \right) z(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} z^*(t) \left(\Delta_{\ell}^q \left(-\frac{d}{dt} \right) \right)^* \tilde{D} \Delta_{\ell}^q \left(\frac{d}{dt} \right) z(t) dt \\
 &= \int_{-\infty}^{\infty} (\Delta_{\ell} z(t))^* \tilde{D} (\Delta_{\ell} z(t)) dt.
 \end{aligned}$$

To see that 2. implies 3. define $\tilde{m} := \max\{m, \ell\}$ and let $\hat{H} \in \mathbb{C}^{q(\tilde{m}+1), q(\tilde{m}+1)}$ be such that \hat{H} and \tilde{H} coincide in the first ℓ rows and ℓ columns and is filled up with zeros everywhere else. Define $\hat{D} \in \mathbb{C}^{q(\tilde{m}+1), q(\tilde{m}+1)}$ analogously. With this set $\tilde{M} := \hat{H} - \hat{D}$ and note that 2. implies that

$$0 = \int_{-\infty}^{\infty} (\Delta_{\tilde{m}} z(t))^* \tilde{M} (\Delta_{\tilde{m}} z(t)) dt.$$

By Lemma 41 this implies the existence of $n \in \mathbb{N}$ and $\tilde{N} = \tilde{N}^* \in \mathbb{C}^{q(n+1), q(n+1)}$ such that

$$\begin{aligned}
 \frac{d}{dt} \left[(\Delta_n z(t))^* \tilde{N} (\Delta_n z(t)) \right] &= (\Delta_{\tilde{m}} z(t))^* \tilde{M} (\Delta_{\tilde{m}} z(t)) \\
 &= (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)) - (\Delta_m z(t))^* \tilde{D} (\Delta_m z(t)) \\
 &\leq (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)),
 \end{aligned}$$

which proves that 3. holds for every $z \in \mathfrak{B}_c(P)$.

To see that 3. implies 1. let $z \in \mathfrak{B}_c(P)$ be arbitrary. Then by integration we see that for all $t_0 \leq t_1$ we have

$$\begin{aligned}
 &\int_{t_0}^{t_1} (\Delta_{\ell} z(t))^* \tilde{H} (\Delta_{\ell} z(t)) dt \\
 &\geq \int_{t_0}^{t_1} \frac{d}{dt} \left((\Delta_{\ell} z(t))^* \tilde{N} (\Delta_{\ell} z(t)) \right) dt \\
 &\geq \left((\Delta_{\ell} z(t_1))^* \tilde{N} (\Delta_{\ell} z(t_1)) \right) - \left((\Delta_{\ell} z(t_0))^* \tilde{N} (\Delta_{\ell} z(t_0)) \right),
 \end{aligned}$$

from which we obtain 1. by letting t_0 go to $-\infty$, t_1 go to ∞ , and using the property that z has compact support. \square

The problem with Theorem 42 is that we cannot bound the size of \tilde{D} and \tilde{N} by the size of \tilde{H} , compare Example 38. It should be possible to achieve such a bound by using (a more complicated) approach, which only works in the time-domain. A proof for the first-order case (i.e., $P \in \mathbb{C}[\lambda]^{p,q}$ being a first-order polynomial, $\ell = 0$, and $\tilde{H} = \tilde{H}^*$ being a constant matrix polynomial) will be discussed in a forthcoming paper.

An immediate consequence of Theorem 42 is the following Corollary.

Corollary 43. *Dissipativity (as in Definition 33) is equivalent to cyclo-dissipativity (as in Definition 34) is equivalent to the positive semi-definiteness of a Popov function along the imaginary axis (as stated in Theorem 36).*

Proof. Assuming that $P \in \mathbb{C}[\lambda]^{p,q}$ is cyclo-dissipative with respect to $\tilde{H} = \tilde{H}^* \in \mathbb{C}^{q(\ell+1), q(\ell+1)}$ we see by part 3. of Theorem 42 that there exists $n \in \mathbb{N}$ and a Hermitian matrix $\tilde{N} = \tilde{N}^* \in \mathbb{C}^{q(n+1), q(n+1)}$ such that by defining $\Theta : \mathbb{C}^{q(n+1)} \rightarrow \mathbb{R}$ through

$$\Theta(y) := y^* \tilde{N} y,$$

we find

$$\frac{d}{dt} (\Theta(\Delta_n z(t))) \leq (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)).$$

Integrating this equation from t_0 to t_1 (where we assume that $t_0 \leq t_1$) we see that

$$\Theta(\Delta_n z(t_1)) - \Theta(\Delta_n z(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt} (\Theta(\Delta_n z(t))) dt \leq \int_{t_0}^{t_1} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)) dt,$$

i.e., that P is dissipative with respect to \tilde{H} . If on the other hand P is assumed to be dissipative with respect to \tilde{H} and $z \in \mathfrak{B}_c(P)$ is an arbitrary trajectory with compact support, then we obtain from the dissipation inequality (9) and the continuity of the storage function Θ that

$$\begin{aligned} \int_{-\infty}^{\infty} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)) dt &= \lim_{t_0 \rightarrow -\infty} \lim_{t_1 \rightarrow \infty} \int_{t_0}^{t_1} (\Delta_\ell z(t))^* \tilde{H} (\Delta_\ell z(t)) dt \\ &\geq \lim_{t_1 \rightarrow \infty} \Theta(\Delta_n z(t_1)) - \lim_{t_0 \rightarrow -\infty} \Theta(\Delta_n z(t_0)) \\ &\geq \Theta\left(\lim_{t_1 \rightarrow \infty} \Delta_n z(t_1)\right) - \Theta\left(\lim_{t_0 \rightarrow -\infty} \Delta_n z(t_0)\right) \\ &= \Theta(0) - \Theta(0) = 0, \end{aligned}$$

i.e., that P is cyclo-dissipative with respect to \tilde{H} . That positive semi-definiteness of a Popov-function along the imaginary axis is also an equivalent condition follows readily from Theorem 36. \square

Corollary 43 tells us that dissipativity may be checked by considering a Popov function along the imaginary axis.

7 The main result

Having introduced all the notation and results above we are ready to formulate the following generalization of Theorem 1 in [4] (compare section 8.2).

Theorem 44. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial and let $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ be its rank. Let $U \in \mathbb{C}(\lambda)^{q, q-r}$ be a kernel matrix of P . Let $H = H^\sim \in \mathbb{C}[\lambda]^{q,q}$ be a para-Hermitian matrix polynomial. Let*

$$\lambda_0 \in \left(\mathfrak{D}(U) \setminus \overline{-\sigma(P)} \right) \cap \left(\mathfrak{D}(U^\sim) \setminus \sigma(P) \right).$$

Then the Popov function $\Pi := U^\sim H U$ has λ_0 as an eigenvalue if and only if the para-Hermitian matrix polynomial

$$N := \begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix} \in \mathbb{C}[\lambda]^{p+q, p+q}, \quad (14)$$

has λ_0 as an eigenvalue.

Proof. Let $V \in \mathbb{C}[\lambda]^{q,r}$ be a co-kernel matrix of P . Define the block transformation matrix

$$S := \begin{bmatrix} I_p & 0 \\ 0 & [U \ V] \end{bmatrix} \in \mathbb{C}(\lambda)^{p+q, p+q}$$

and observe that with this we obtain

$$\begin{aligned} S^\sim N S &= S^\sim \begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & [U \ V] \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & \begin{bmatrix} U^\sim \\ V^\sim \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 & P U & P V \\ P^\sim & H U & H V \end{bmatrix} \\ &= \begin{bmatrix} 0 & P U & P V \\ U^\sim P^\sim & U^\sim H U & U^\sim H V \\ V^\sim P^\sim & V^\sim H U & V^\sim H V \end{bmatrix} = \begin{bmatrix} 0 & P U & P V \\ (P U)^\sim & \Pi & U^\sim H V \\ (P V)^\sim & V^\sim H U & V^\sim H V \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & P V \\ 0 & \Pi & U^\sim H V \\ V^\sim P^\sim & V^\sim H U & V^\sim H V \end{bmatrix}, \end{aligned}$$

since U is a kernel matrix of P . To eliminate the blocks below $P V$ and to the right of $(P V)^\sim$ let $X \in \mathbb{C}(\lambda)^{r,p}$ be such a matrix that $X P V = I_r$. Such a matrix exists due to part 2. in Definition 11 and Lemma 14. Then, set

$$Y := -X^\sim V^\sim H U \in \mathbb{C}(\lambda)^{p, q-r} \quad \text{and} \quad Z := -\frac{1}{2} X^\sim V^\sim H V \in \mathbb{C}(\lambda)^{p,p}$$

and with this define the block transformation matrix

$$T := \begin{bmatrix} I_p & Y & Z \\ 0 & I_{q-r} & 0 \\ 0 & 0 & I_r \end{bmatrix} \in \mathbb{C}(\lambda)^{p+q, p+q}.$$

Since we have that

$$\begin{aligned} V^\sim P^\sim Y + V^\sim H U &= -(X P V)^\sim V^\sim H U + V^\sim H U = 0, \\ Z^\sim P V + V^\sim P^\sim Z + V^\sim H V &= -\frac{1}{2} V^\sim H V - \frac{1}{2} V^\sim H V + V^\sim H V = 0 \end{aligned}$$

we can find that

$$\begin{aligned} &T^\sim S^\sim N S T \\ &= T^\sim \begin{bmatrix} 0 & 0 & P V \\ 0 & \Pi & U^\sim H V \\ V^\sim P^\sim & V^\sim H U & V^\sim H V \end{bmatrix} \begin{bmatrix} I_p & Y & Z \\ 0 & I_{q-r} & 0 \\ 0 & 0 & I_r \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 &= \begin{bmatrix} I_p & 0 & 0 \\ Y^\sim & I_{q-r} & 0 \\ Z^\sim & 0 & I_r \end{bmatrix} \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & U^\sim HV \\ V^\sim P^\sim & 0 & V^\sim P^\sim Z + V^\sim HV \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & Y^\sim PV + U^\sim HV \\ V^\sim P^\sim & 0 & Z^\sim PV + V^\sim P^\sim Z + V^\sim HV \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & (V^\sim P^\sim Y + V^\sim HU)^\sim \\ V^\sim P^\sim & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & 0 \\ V^\sim P^\sim & 0 & 0 \end{bmatrix}.
 \end{aligned} \tag{15}$$

To obtain the assertion we finally make the following observation. First, S is a unimodular matrix due to part 5. of Definition 11 and its domain of definition is given by $\mathfrak{D}(S) = \mathfrak{D}(U) \cap \mathfrak{D}(V) = \mathfrak{D}(U)$, since V is a polynomial matrix and thus $\mathfrak{D}(V) = \mathbb{C}$. Second, due to Lemma 14, we see that X can be chosen such that $\mathfrak{D}(X) = \mathbb{C} \setminus \sigma(PV)$. Using Lemma 15 we conclude that $\mathfrak{D}(X) = \mathbb{C} \setminus \sigma(P)$. This in turn implies $\mathfrak{D}(X^\sim) = \mathbb{C} \setminus \overline{-\sigma(P)}$ by Lemma 30. Third, T is a unimodular matrix since it is upper triangular with units on the diagonal and its domain of definition is given by

$$\begin{aligned}
 \mathfrak{D}(T) &= \mathfrak{D}(Y) \cap \mathfrak{D}(Z) \\
 &= (\mathfrak{D}(X^\sim) \cap \mathfrak{D}(V^\sim) \cap \mathfrak{D}(U)) \cap (\mathfrak{D}(X^\sim) \cap \mathfrak{D}(V^\sim) \cap \mathfrak{D}(V)) \\
 &= \mathfrak{D}(U) \cap \mathfrak{D}(X^\sim) = \mathfrak{D}(U) \setminus \overline{-\sigma(P)}.
 \end{aligned}$$

Fourth, the domains of definition of the para-Hermitians of S and T are given by $\mathfrak{D}(S^\sim) = \mathfrak{D}(U^\sim)$ and $\mathfrak{D}(T^\sim) = \mathfrak{D}(U^\sim) \setminus \sigma(P)$. Fifth, the identity (15) implies that

$$\begin{aligned}
 \text{rank}_{\mathbb{C}(\lambda)}(N) &= \text{rank}_{\mathbb{C}(\lambda)} \left(\begin{bmatrix} 0 & 0 & PV \\ 0 & \Pi & 0 \\ V^\sim P^\sim & 0 & 0 \end{bmatrix} \right) \\
 &= \text{rank}_{\mathbb{C}(\lambda)}(PV) + \text{rank}_{\mathbb{C}(\lambda)}(\Pi) + \text{rank}_{\mathbb{C}(\lambda)}(V^\sim P^\sim) \\
 &= 2r + \text{rank}_{\mathbb{C}(\lambda)}(\Pi),
 \end{aligned}$$

where we have used part 2. of Definition 11. Sixth, for λ_0 as in the assertion we have that $\lambda_0 \in \mathfrak{D}(S) \cap \mathfrak{D}(T)$ and thus $S(\lambda_0)$ and $T(\lambda_0)$ are well defined invertible matrices (due to the unimodularity of S and T) which together with the identity (15) implies that

$$\begin{aligned}
 \text{rank}(N(\lambda_0)) &= \text{rank} \left(\begin{bmatrix} 0 & 0 & P(\lambda_0)V(\lambda_0) \\ 0 & \Pi(\lambda_0) & 0 \\ V^\sim(\lambda_0)P^\sim(\lambda_0) & 0 & 0 \end{bmatrix} \right) \\
 &= \text{rank}(P(\lambda_0)V(\lambda_0)) + \text{rank}(\Pi(\lambda_0)) + \text{rank}(V^\sim(\lambda_0)P^\sim(\lambda_0)) \\
 &= 2r + \text{rank}(\Pi(\lambda_0)),
 \end{aligned}$$

again with the help of Lemma 15. \square

The following Corollary 45 shows that the singularities of U can be neglected when considering a Popov function along the imaginary axis.

Corollary 45. *Let $P \in \mathbb{C}[\lambda]^{p,q}$ be a matrix polynomial which has no purely imaginary eigenvalues and let $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ be its rank. Let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P , according to Theorem 10. Let $H = H^\sim \in \mathbb{C}[\lambda]^{q,q}$ be a para-Hermitian matrix polynomial. Then the Popov function $\Pi := U^\sim H U$ (which in this case is a polynomial) has the eigenvalue $i\omega_0 \in i\mathbb{R}$ if and only if the para-Hermitian polynomial*

$$\begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix}$$

has the eigenvalue $i\omega_0$.

Proof. Here we have that $\mathfrak{D}(U) = \mathbb{C}$ and due to the assumption that P has no imaginary eigenvalues also that $\sigma(P) \cap i\mathbb{R} = \emptyset$ which implies that

$$i\omega_0 \in \left(\mathfrak{D}(U) \setminus \overline{-\sigma(P)} \right) \cap (\mathfrak{D}(U^\sim) \setminus \sigma(P)),$$

for all $\omega_0 \in \mathbb{R}$. □

How can Corollary help to check dissipativity of systems given in behavior form? For this consider Figure 1 which depicts three possible Popov functions Π_1 , Π_2 , and Π_3 along the imaginary axis.

Each Popov function is represented by a single line which means that we are considering systems which are represented by polynomials $P_j \in \mathbb{C}[\lambda]^{p,q}$ such that with $r_j := \text{rank}_{\mathbb{C}(\lambda)}(P_j)$ we have $q - r_j = 1$, for $j = 1, 2, 3$ since in this case we have that the associated kernel matrices $U_j \in \mathbb{C}[\lambda]^{q,q-r_j} = \mathbb{C}[\lambda]^{q,1}$ are column vectors and thus the Popov functions

$$\Pi_j = U_j^\sim H U_j \in \mathbb{C}[\lambda],$$

for $j = 1, 2, 3$ are scalar polynomials, where $H = H^\sim \in \mathbb{C}[\lambda]^{q,q}$ is some fixed para-Hermitian polynomial. This corresponds to the single-input setting in state-space systems. Using the criterion from Theorem 36 we see that P_j is dissipative (with respect to a Hermitian matrix induced by H through Lemma 28) if and only if its Popov function Π_j is positive semi-definite along the imaginary axis. Examining Figure 1 clearly shows that P_2 is dissipative, while P_1 and P_3 are not. However, P_1 seems to be somehow close to dissipative, while P_3 is not dissipative at all. Clearly, none of the polynomials Π_1 , Π_2 , and Π_3 is the zero polynomial. When, however, we look at the rank of the Popov functions at specific points on the imaginary axis we find that the rank of Π_2 and Π_3 does not drop anywhere (from 1 to 0) on the imaginary axis, while the rank of Π_1 drops at $i\omega_1$ and $i\omega_2$, i.e., Π_1 has the imaginary eigenvalues $i\omega_1$ and $i\omega_2$, while Π_2 and Π_3 have no purely imaginary eigenvalues. Using Corollary 45 we see that to compute the values $i\omega_1$ and $i\omega_2$ we have to determine the purely imaginary eigenvalues of

$$\begin{bmatrix} 0 & P_1 \\ P_1 & H \end{bmatrix},$$

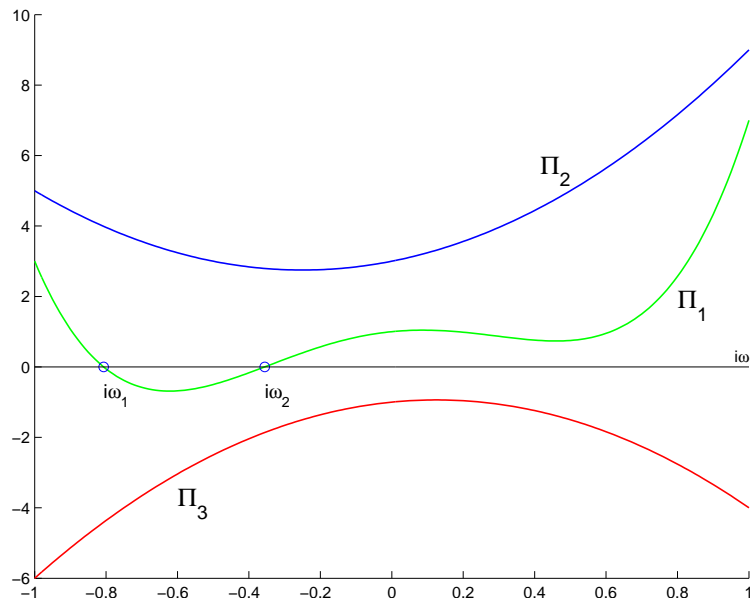


Figure 1: Three possible Popov functions along the imaginary axis

while the connected para-Hermitian matrix polynomials for P_2 and P_3 have no imaginary eigenvalues. Thus Theorem 44 can help to distinguish Π_1 from Π_2 (but not Π_2 from Π_3) without explicitly computing the kernel matrices U_j and the associated Popov functions Π_j . This is especially handy since the imaginary eigenvalues of para-Hermitian matrix polynomials are computable in a numerically stable way [6, 15]. This idea can also be generalized to systems where the kernel matrix has more than one column but then one cannot depict the Popov functions in such a convenient way.

8 Specialization to descriptor systems

Consider the state-space descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (16)$$

where $E, A \in \mathbb{R}^{\rho, n}$ are rectangular matrices, $B \in \mathbb{R}^{\rho, m}$, $x \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n)$ is called the state, $u \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^m)$ is called the input, and $y \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^p)$ is called the output. In the literature, see e.g. [3, Section 5.9], for such systems a *supply function* is frequently introduced as a quadratic function $s : \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}$ of the form

$$s(u, y) := \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \quad (17)$$

where $Q = Q^T \in \mathbb{R}^{p,p}$, $S \in \mathbb{R}^{p,m}$, and $R = R^T \in \mathbb{R}^{m,m}$. Again, as explained after Definition 33, the supply imposes a notion of energy onto the system given by the equations (16) by measuring the energy supplied to the system through its input and output ports.

Under these circumstances, system (16) is called dissipative with respect to s , if there exists a continuous function $\Theta : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the dissipation inequality

$$\Theta(x(t_1)) - \Theta(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t)) dt, \quad (18)$$

holds for all $t_0 \leq t_1$ and for all trajectories (u, x, y) of (16) which have compact support, i.e., for all triples $(u, x, y) \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^m) \times \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^n) \times \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^p)$ which fulfill (16) for all $t \in \mathbb{R}$. Note the following difference to Definition 33. In Definition 33 the storage function may depend on z (and its derivatives up to some degree) just as the term on the right hand side of (9) may depend on all components of z . In (18), however, the right hand side only depends on u and y , while Θ on the left hand side only depends on the state x . Nevertheless, it can be shown that the inequalities (9) and (18) are equivalent since we know from [17, Theorem 6.1] that every storage function Θ is a function of the state x .

Using the equation for y in (16) we can rewrite the supply to depend on the state variables (instead of the output variables) by

$$\begin{aligned} s(u(t), y(t)) &= \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(t) + Du(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C^T QC & C^T QD + C^T S \\ D^T QC + S^T C & D^T QD + D^T S + S^T D + R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &=: \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} =: \tilde{s}(x(t), u(t)), \end{aligned} \quad (19)$$

where $\tilde{Q} = \tilde{Q}^T \in \mathbb{R}^{n,n}$, $\tilde{S} \in \mathbb{R}^{n,m}$, $\tilde{R} \in \mathbb{R}^{m,m}$, and $\tilde{s} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$. Clearly, with this notation one can restate inequality (18) as

$$\Theta(x(t_1)) - \Theta(x(t_0)) \leq \int_{t_0}^{t_1} \tilde{s}(u(t), x(t)) dt, \quad (20)$$

for all $t_0 \leq t_1$ and all trajectories of $E\dot{x} = Ax + Bu$ which have compact support. Still the problem remains that the expressions on the left hand side of (20) only depend on x while the expression on the right hand side of (20) depend on x and u . We again refer the interested reader to [17, Theorem 6.1].

Introducing the notation $q := n + m$,

$$P_1 := \begin{bmatrix} E & 0 \end{bmatrix}, \quad P_0 := \begin{bmatrix} -A & -B \end{bmatrix}, \quad \tilde{H} := \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix}, \quad \text{and } z := \begin{bmatrix} x \\ u \end{bmatrix}, \quad (21)$$

we see that $F, G \in \mathbb{R}^{p,q}$, $\tilde{H} = \tilde{H}^* \in \mathbb{R}^{q,q}$, and $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and we can rewrite system (16) as the behavior system $P_1 \dot{z}(t) + P_0 z(t) = 0$ or by defining the polynomial

$$P(\lambda) := \lambda P_1 + P_0 = [\lambda E - A \quad -B], \quad (22)$$

as $P\left(\frac{d}{dt}\right)z(t) = 0$. Also the supply can be viewed as a function of the form $s : \mathbb{R}^q \rightarrow \mathbb{R}$ with

$$s(z) = z^T \tilde{H} z.$$

Let $r := \text{rank}_{\mathbb{C}(\lambda)}(P)$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ be a polynomial kernel matrix of P . Then we see from Theorem 36 and Theorem 42 that (18) is fulfilled if and only if the Popov function $\Pi \in \mathbb{C}[\lambda]^{q-r,q-r}$ given by

$$\Pi(\lambda) := U^\sim(\lambda) \tilde{H} U(\lambda)$$

is positive semi-definite along the imaginary axis. By Corollary 45 we see that to check if the Popov function Π is positive semi-definite along the imaginary axis and where it may be not be positive semi-definite, we have to consider the purely imaginary eigenvalues of the para-Hermitian polynomial (which in this case is a pencil) given by

$$\begin{aligned} \begin{bmatrix} 0 & P(\lambda) \\ P^\sim(\lambda) & \tilde{H} \end{bmatrix} &= \begin{bmatrix} 0 & \lambda E - A & -B \\ -\lambda E^T - A^T & \tilde{Q} & \tilde{S} \\ -B^T & \tilde{S}^T & \tilde{R} \end{bmatrix} \\ &= \lambda \begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^T & -\tilde{Q} & -\tilde{S} \\ B^T & -\tilde{S}^T & -\tilde{R} \end{bmatrix}. \end{aligned}$$

Since the eigenvalues of this pencil do not change under pre- and post-multiplication with invertible constant matrices we see that we can as well consider the purely imaginary eigenvalues of the para-Hermitian pencil

$$\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & P(\lambda) \\ P^\sim(\lambda) & \tilde{H} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \lambda \begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^T & \tilde{Q} & \tilde{S} \\ B^T & \tilde{S}^T & \tilde{R} \end{bmatrix}, \quad (23)$$

when we assume that the pencil

$$P(\lambda) = [\lambda E - A \quad -B]$$

has no purely imaginary eigenvalues. Note that the eigenvalues of $[\lambda E - A \quad -B]$ are sometimes also called uncontrollable eigenvalues or uncontrollable modes of the triple (E, A, B) . Thus we have to assume that (E, A, B) has no uncontrollable, purely imaginary eigenvalues so that the assumptions of Corollary 45 are fulfilled. The importance of the pencil (23) for non-regular descriptor systems has already been noticed in [13, Theorem 3.17]. Note that in contrast to [13, Theorem 3.17] the results here do not involve an index of any kind for descriptor systems. Also the matrix pencil (23) may well be singular in contrast to [13, §4], since we defined the term eigenvalue through Definition 8, which works as well for singular pencils as for regular ones.

8.1 Specialization to regular descriptor systems

Assuming that the pencil $\lambda E - A$ is regular, i.e., it is invertible over $\mathbb{C}(\lambda)$, we can give an explicit representation of a kernel and a co-kernel matrix of P as defined in (22).

Lemma 46. *Let $P \in \mathbb{C}[\lambda]^{n,n+m}$ be defined through (21) and (22) and assume that the pencil $\lambda E - A$ is regular, i.e., let $\rho = n$ and let $\lambda E - A$ be invertible over $\mathbb{C}(\lambda)$. Then we have that*

$$\text{rank}_{\mathbb{C}(\lambda)}(P) = n$$

and the matrices $U \in \mathbb{C}(\lambda)^{n+m,m}$ and $V \in \mathbb{C}[\lambda]^{n+m,n}$ given by

$$U(\lambda) := \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \text{ and } V(\lambda) := \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad (24)$$

constitute a kernel and a co-kernel matrix of P with $\mathfrak{D}(U) = \mathbb{R} \setminus \sigma(\lambda E - A)$.

Proof. Since $P(\lambda) = \lambda P_1 + P_0 = \begin{bmatrix} \lambda E - A & -B \end{bmatrix}$ and $\lambda E - A$ is assumed to be invertible we have $\text{rank}_{\mathbb{C}(\lambda)}(P) = n$. U and V are matrices of proper dimension according to Definition 11. We still have to show the remaining five properties. To see 1. observe that

$$PU = \begin{bmatrix} \lambda E - A & -B \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} = B - B = 0.$$

For 2. notice that

$$\begin{aligned} \text{rank}_{\mathbb{C}(\lambda)}(PV) &= \text{rank}_{\mathbb{C}(\lambda)}\left(\begin{bmatrix} \lambda E - A & -B \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix}\right) = \text{rank}_{\mathbb{C}(\lambda)}(\lambda E - A) \\ &= n = \text{rank}_{\mathbb{C}(\lambda)}(P). \end{aligned}$$

Points 3. and 4. are obvious immediately, since both U and V have an identity matrix in one of its block rows. Note that here we have $\mathfrak{D}(U) = \mathbb{C} \setminus \sigma(\lambda E - A)$ as one can see by applying the adjoint formula of the inverse. Finally, to see 5. note that

$$\det \begin{bmatrix} U & V \end{bmatrix} = \det \begin{bmatrix} (\lambda E - A)^{-1}B & I_n \\ I_m & 0 \end{bmatrix} = \pm 1,$$

and thus $\begin{bmatrix} U & V \end{bmatrix}$ is a unimodular rational matrix according to Definition 4. \square

With the explicit representation of the kernel matrix (24) we can also give the more well known explicit representation of the Popov function as

$$\begin{aligned} \Pi(\lambda) &:= U^\sim(\lambda) H U(\lambda) = \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^\sim \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (-\bar{\lambda}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} C(-\lambda E - A)^{-1}B + D \\ I_m \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} C(\lambda E - A)^{-1}B + D \\ I_m \end{bmatrix}, \quad (25) \end{aligned}$$

or as

$$\begin{aligned} \Pi(\lambda) &:= \begin{bmatrix} B^T(-\lambda E^T - A^T)^{-1}C^T + D^T & I_m \end{bmatrix} \begin{bmatrix} Q & S \\ \tilde{S}^T & R \end{bmatrix} \begin{bmatrix} C(\lambda E - A)^{-1}B + D \\ I_m \end{bmatrix} \\ &= B^T(-\lambda E^T - A^T)^{-1}\tilde{Q}(\lambda E^T - A^T)B + B^T(-\lambda E^T - A^T)^{-1}\tilde{S} \\ &\quad + \tilde{S}^T(\lambda E^T - A^T)B + \tilde{R} \end{aligned} \quad (26)$$

depending on which representation one prefers.

We then obtain the following Corollary of 44.

Corollary 47. *Consider the system (16) with $\lambda E - A$ regular together with the supply (17). Define the Popov function $\Pi \in \mathbb{C}(\lambda)^{m,m}$ through (25). Suppose that $i\omega \in i\mathbb{R}$ is not an eigenvalue of $\lambda E - A$. Then the Popov function has the eigenvalue $i\omega$ if and only if the para-Hermitian pencil*

$$\lambda \begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^T & \tilde{Q} & \tilde{S} \\ B^T & \tilde{S}^T & \tilde{R} \end{bmatrix}, \quad (27)$$

has $i\omega$ as an eigenvalue.

Proof. Since we have $\mathfrak{D}(U) = \mathbb{C} \setminus \sigma(\lambda E - A)$ we see that in the assertion of Theorem 44 we have

$$i\omega_0 =: \lambda_0 \in \left(\mathfrak{D}(U) \setminus \overline{-\sigma(P)} \right) \cap \left(\mathfrak{D}(U^\sim) \setminus \sigma(P) \right).$$

Thus, we obtain the pencil (27) and the result by the same argument which we already performed at (23). \square

8.2 Specialization to standard systems with regular supply

With the notation introduced in Section 8 let us in the following assume that the matrix \tilde{R} as in (19) is non-singular. We obtain the following result.

Corollary 48. *Consider the system (16) with $E = I$. Let \tilde{R} as in (19) be regular. Define the Popov function $\Pi \in \mathbb{C}(\lambda)^{m,m}$ through (25). Let $i\omega \in i\mathbb{R}$ be not an eigenvalue of $\lambda I - A$. Then the matrix $\Pi(i\omega) \in \mathbb{C}^{m,m}$ has the eigenvalue 0 in the ordinary sense, i.e., there exists a $v \in \mathbb{C}^m \setminus \{0\}$ such that $\Pi(i\omega)v = 0$, if and only if the Hamiltonian matrix*

$$\mathcal{H} := \begin{bmatrix} A - B\tilde{R}^{-1}\tilde{S}^T & -B\tilde{R}^{-1}B^T \\ \tilde{S}\tilde{R}^{-1}\tilde{S}^T - \tilde{Q} & \tilde{S}\tilde{R}^{-1}B^T - A^T \end{bmatrix} \quad (28)$$

has $i\omega$ as an eigenvalue in the ordinary sense, i.e., there exists a $u \in \mathbb{C}^{2n} \setminus \{0\}$ such that $\mathcal{H}u = i\omega u$.

Proof. In this case we can see that the Popov function (25) fulfills

$$\text{rank}_{\mathbb{C}(\lambda)}(\Pi) = m,$$

by letting λ go to infinity. Thus, the Popov function has the eigenvalue $i\omega_0 \in i\mathbb{R}$ if and only if the matrix $\Pi(i\omega)$ does not have full rank which is the case if and only if there exists a $v \in \mathbb{C}^m \setminus \{0\}$ such that $\Pi(i\omega)v = 0$.

What remains to be shown is that \mathcal{H} has an eigenvalue λ in the ordinary sense, if and only if the pencil (27) with $E = I$ has the eigenvalue λ . Since under the assumptions of the Corollary, the pencil (27) is regular it has the eigenvalue λ in the sense of Definition 8 if and only if there exist $\mu, x \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ such that $[\mu^T \ x^T \ u^T] \neq 0$ and

$$\lambda \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & \tilde{Q} & \tilde{S} \\ B^T & \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}. \quad (29)$$

Using the invertibility of \tilde{R} we obtain from (29) that

$$\lambda x = Ax + Bu, \quad (30a)$$

$$-\lambda \mu = A^T \mu + \tilde{Q}x + \tilde{S}u, \quad (30b)$$

$$u = \tilde{R}^{-1}(-B^T \mu - \tilde{S}^T x). \quad (30c)$$

Substituting (30c) into (30b) and (30a) we find the set of equations

$$\begin{aligned} \lambda x &= Ax - B\tilde{R}^{-1}B^T \mu - B\tilde{R}^{-1}\tilde{S}^T x \\ &= (A - B\tilde{R}^{-1}\tilde{S}^T)x - B\tilde{R}^{-1}B^T \mu \\ -\lambda \mu &= A^T \mu + \tilde{Q}x - \tilde{S}\tilde{R}^{-1}B^T \mu - \tilde{S}\tilde{R}^{-1}\tilde{S}^T x \\ &= (\tilde{Q} - \tilde{S}\tilde{R}^{-1}\tilde{S}^T)x + (A^T - \tilde{S}\tilde{R}^{-1}B^T)\mu, \end{aligned}$$

which written in matrix form is given by

$$\lambda \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} A - B\tilde{R}^{-1}\tilde{S}^T & -B\tilde{R}^{-1}B^T \\ \tilde{S}\tilde{R}^{-1}\tilde{S}^T - \tilde{Q} & \tilde{S}\tilde{R}^{-1}B^T - A^T \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix}. \quad (31)$$

Thus we have an eigenvalue in the ordinary sense of \mathcal{H} since $[x^T \ \mu^T] \neq 0$ as one can obtain from the identity (30c). For the reverse direction let $[x^T \ \mu^T] \neq 0$ be given such that (31) holds and define u through (30c). Then we can perform all steps from above in the opposite direction to obtain (29) which means that we have an eigenvalue of the pencil (27) due to the regularity of (27). This completes the proof. \square

With proper substitutions one can obtain [4, Theorem 1] from Corollary 48.

9 Specialization to passivity

In electrical engineering often problem of passivity is considered, e.g., [2, 7].

Definition 49. Consider a regular first-order state space system with equal input and output dimension, i.e., a system of the form (16) with $\rho = n$, $\lambda E - A$ regular, and $p = m$. Such a system is called *passive* if it is dissipative (as defined at (18)) with respect to the supply function

$$s(u, y) := \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}. \quad (32)$$

With this supply function fixed and using the notation

$$G(\lambda) := C(-\lambda E - A)^{-1}B + D$$

to denote the so called *transfer function*, we obtain from (25) that the Popov function is

$$\begin{aligned} \Pi(\lambda) &= \begin{bmatrix} C(-\lambda E - A)^{-1}B + D \\ I_m \end{bmatrix}^T \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} C(\lambda E - A)^{-1}B + D \\ I_m \end{bmatrix} \\ &= G(\lambda) + G(-\lambda)^T. \end{aligned}$$

Looking at [2, Section 2.7] we find that positive semi-definiteness of $\Pi(\lambda) = G(\lambda) + G(-\lambda)^T$ along the imaginary axis as in Corollary 43 is an important property of *positive realness*. To check this property of positive realness we can use Corollary 48. Note that in (19) this case $Q = R = 0$ and $S = I_m$ and thus we get from

$$\begin{aligned} \tilde{R} &= D^T Q D + D^T S + S^T D + R = D^T + D, \\ \tilde{S} &= C^T Q D + C^T S = C^T, \\ \tilde{Q} &= C^T Q C = 0. \end{aligned}$$

This implies that Corollary 48 can only be applied in the case that $D + D^T$ is invertible, in which case one has to consider the purely imaginary eigenvalues of the Hamiltonian matrix

$$\begin{aligned} \mathcal{H} &= \begin{bmatrix} A - B\tilde{R}^{-1}\tilde{S}^T & -B\tilde{R}^{-1}B^T \\ \tilde{S}\tilde{R}^{-1}\tilde{S}^T - \tilde{Q} & \tilde{S}\tilde{R}^{-1}B^T - A^T \end{bmatrix} \\ &= \begin{bmatrix} A - B(D + D^T)^{-1}C & -B(D + D^T)^{-1}B^T \\ C^T(D + D^T)^{-1}C & C^T(D + D^T)^{-1}B^T - A^T \end{bmatrix}. \end{aligned}$$

Equivalently, checking the imaginary eigenvalues of the para-Hermitian pencil

$$\lambda \begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^T & 0 & C^T \\ B^T & C & D + D^T \end{bmatrix}, \quad (33)$$

can be performed without the assumption that $D + D^T$ is invertible and even without the assumption that $\lambda E - A$ is regular or quadratic. The meaningfulness of the purely imaginary eigenvalues of (33) is then given through Theorem 44 and not through Corollary 48.

10 Specialization to second-order systems

Consider the system

$$M\ddot{z}(t) + D\dot{z}(t) + Kz(t) = 0, \quad (34)$$

where $M, D, K \in \mathbb{R}^{p,q}$. Often M, D , and K take the form

$$M = [\tilde{M} \ 0], \quad D = [\tilde{D} \ 0], \quad K = [\tilde{K} \ -B],$$

with $p = n, q = n + m, \tilde{M}, \tilde{D}, \tilde{K} \in \mathbb{C}^{n,n}$, and $B \in \mathbb{C}^{n,m}$. Then \tilde{M}, \tilde{D} , and \tilde{K} are called the mass, damping, and stiffness matrix. This corresponds to a state-space system, where z is made up of the state x and the input u .

Suppose that the supply of such a system is measured by a function $s : \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathbb{R}$ of the form

$$s(z, \dot{z}) := \begin{bmatrix} z \\ \dot{z} \end{bmatrix}^* \begin{bmatrix} H_{zz} & H_{zv} \\ H_{zv}^* & H_{vv} \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \quad (35)$$

where $H_{zz} = H_{zz}^* \in \mathbb{R}^{q,q}$, $H_{vv} = H_{vv}^* \in \mathbb{R}^{q,q}$, and $H_{zv} \in \mathbb{R}^{q,q}$, analogously to (17). In the notation of Theorem 44 here we have

$$P(\lambda) = \lambda^2 M + \lambda D + K$$

and the para-Hermitian polynomial H is given through (6) by

$$H(\lambda) = \begin{bmatrix} I_q \\ -\lambda I_q \end{bmatrix}^* \begin{bmatrix} H_{zz} & H_{zv} \\ H_{zv}^* & H_{vv} \end{bmatrix} \begin{bmatrix} I_q \\ \lambda I_q \end{bmatrix} = H_{zz} + \lambda(H_{zv} - H_{zv}^*) - \lambda^2 H_{vv}.$$

Thus, to check dissipativity of such a system one has to consider the imaginary eigenvalues of the second-order para-Hermitian polynomial matrix

$$\begin{aligned} & \begin{bmatrix} 0 & P(\lambda) \\ P^*(-\bar{\lambda}) & H(\lambda) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \lambda^2 M + \lambda D + K \\ (((-\bar{\lambda})^2 M + (-\bar{\lambda})D + K)^* & H_{zz} + \lambda(H_{zv} - H_{zv}^*) - \lambda^2 H_{vv} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \lambda^2 M + \lambda D + K \\ \lambda^2 M^* - \lambda D^* + K^* & H_{zz} + \lambda(H_{zv} - H_{zv}^*) - \lambda^2 H_{vv} \end{bmatrix} \end{aligned}$$

which can also be considered as the matrix polynomial

$$\lambda^2 \begin{bmatrix} 0 & M \\ M^* & -H_{vv} \end{bmatrix} + \lambda \begin{bmatrix} 0 & D \\ -D^* & (H_{zv} - H_{zv}^*) \end{bmatrix} + \begin{bmatrix} 0 & K \\ K^* & H_{zz} \end{bmatrix}. \quad (36)$$

11 Conclusion and Outlook

Through Theorem 44 we obtained a method to check dissipativity with respect to general quadratic supply functions of a system given in behavior form as illustrated by Figure 1. The proof used the fact, that the polynomial (14) can be transformed to a Popov function by unimodular transformations.

We then specialized the results to state-space systems of first order which often show up in practice. A well known result has been obtained in this way, see Corollary 48. We also took a short look at how Theorem 44 can be interpreted for second order systems.

It should also be possible to devise a complete check for dissipativity, i.e., a check that does not make the assumption that the system is already at least close to dissipative (speaking in terms of Figure 1: to distinguish Π_2 from Π_3), by considering the eigenvectors corresponding to eigenvalues which are **not** on the imaginary axis (or can be moved away from the imaginary axis by an arbitrary small perturbation).

The results obtained in this paper emphasize the importance of the analysis of structured (polynomial) eigenvalues problems.

Another interesting problem is the dissipation (especially passivation) of given linear system descriptions with respect to a fixed quadratic supply function. For a given linear system description this problem poses the question, what the nearest dissipative (or passive) system is (with respect to some specified norm). In this context it seems to make sense to analyze structured perturbation theory and pseudospectra for para-Hermitian matrix polynomials.

Acknowledgments We thank Christian Mehl for pointing out the proof of the Lemma 37 and Volker Mehrmann for the continued support and for repeated comments on early versions of the manuscript.

References

- [1] S. Ahmad and R. Alam. Pseudospectra, critical points and multiple eigenvalues of matrix polynomials. *Linear Algebra Appl.*, 430(4):1171–1195, 2009.
- [2] B. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis*. Prentice-Hall, Englewood Cliffs, NJ, 1973.
- [3] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM Publications, Philadelphia, PA, 2005.
- [4] S. Boyd, V. Balakrishnan, and P. Kabamba. A bisection method for computing the H_∞ norm of a transfer matrix and related problems. *Math. Control, Signals, Sys.*, 2:207–219, 1989.
- [5] T. Brüll and V. Mehrmann. STCSSP: A FORTRAN 77 routine to compute a structured staircase form for a (skew-)symmetric/(skew-)symmetric matrix pencil. Preprint 31–2007, Institut für Mathematik, TU Berlin, 2007. <http://www.math.tu-berlin.de/preprints/>.
- [6] R. Byers, V. Mehrmann, and H. Xu. A structured staircase algorithm for skew-symmetric/symmetric pencils. *Electr. Trans. Num. Anal.*, 26:1–33, 2007.
- [7] C.K. Chu and N. Wong. A fast passivity test for descriptor systems via structure-preserving transformations of skew-hamiltonian/hamiltonian matrix pencils. In *Design Automation Conference, 2006 43rd ACM/IEEE*, pages 261–266, 0-0 2006.

- [8] O. Forster. *Analysis 3, 3. Aufl.* Vieweg, Braunschweig/Wiesbaden, 1984.
- [9] F. R. Gantmacher. *The Theory of Matrices I.* Chelsea Publishing Company, New York, NY, 1959.
- [10] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Academic Press, 1982.
- [11] Y. Liu and N. Wong. Fast sweeping methods for checking passivity of descriptor systems. In *Circuits and Systems, 2008. APCCAS 2008. IEEE Asia Pacific Conference on*, pages 1794–1797, 30 2008–Dec. 3 2008.
- [12] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.
- [13] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem.* Springer-Verlag, Berlin, 1991.
- [14] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: A Behavioral Approach.* Springer, Berlin, 1998.
- [15] C. Schröder. URV decomposition based structured methods for palindromic and even eigenvalue problems. Preprint 375, MATHEON Preprint, 2007. http://www.matheon.de/research/list_preprints.asp.
- [16] R. N. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on spectral conditions for positive realness of transfer function matrices. *IEEE Trans. Automat. Control*, 53(5):1258–1261, 2008.
- [17] H. L. Trentelman and J. C. Willems. Every storage function is a state function. *Sys. Control Lett.*, 32:249–259, 1997.
- [18] J. C. Willems. Dissipative dynamical systems, Part I: General theory. *Arch. Ration. Mech. Anal.*, 45:321–351, 1972.
- [19] J. C. Willems. Dissipative dynamical systems, Part II: Linear systems with quadratic supply rates. *Arch. Ration. Mech. Anal.*, 45:352–393, 1972.
- [20] J. C. Willems. Models for dynamics. *Dynamics Reported, Volume 2*, pages 171–269, 1989.
- [21] J. C. Willems and H. L. Trentelman. On quadratic differential forms. *SIAM J. Cont. Optim.*, 36(5):1703–1749, 1998.
- [22] D. C. Youla. On the factorization of rational matrices. *IRE Trans. Inform. Theory*, 7(3):172–189, 1961.