LUR'E EQUATIONS AND EVEN MATRIX PENCILS

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Abstract. In this work we consider the so-called Lur'e matrix equations that arise e.g. in model reduction and linear-quadratic infinite time horizon optimal control. We characterize the set of solutions in terms of deflating subspaces of even matrix pencils. In particular, it is shown that there exist solutions which are extremal in terms of definiteness. It is shown how these special solutions can be constructed deflating subspaces of even matrix pencils.

Key words. Lur'e equations, Riccati equations, optimal control, spectral factorization, deflating subspaces, even matrix pencils

1. Introduction.

For given matrices $A, Q \in \mathbb{C}^{n,n}$ with $Q = Q^*$ and $B, C \in \mathbb{C}^{n,m}$, $R \in \mathbb{C}^{m,m}$, we consider *Lur'e equations*

$$\begin{aligned} A^*X + XA + Q &= K^*K, \\ XB + C &= K^*L, \\ R &= L^*L, \end{aligned} \tag{1.1}$$

that have to be solved for $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n} \times \mathbb{C}^{p,m}$ with $X = X^*$ and p as small as possible. Equations of type (1.1) were first introduced by A.I. Lur'e [26] in 1951 (see [5] for an historical overview) and play a fundamental role in systems theory, e.g. since properties like dissipativity of linear systems can be characterized via their solvability [1, 3, 4, 37]. This type of equations moreover appears in the infinite time horizon linear-quadratic optimal control problem [10, 11, 12, 38, 39] as well as in balancing-related model reduction [9, 20, 27, 28, 29]. In the case where R is invertible, the matrices K and L can be eliminated by obtaining the algebraic Riccati equation

$$A^*X + XA - (XB + C)R^{-1}(XB + C)^* + Q = 0. \tag{1.2}$$

It is well-known [25, 39] that solvability criteria and the construction of solutions can be constructed via consideration of certain eigenspaces of the Hamiltonian matrix

$$\mathcal{A}_H = \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^* \\ C^*R^{-1}C - Q & -(A - BR^{-1}C)^* \end{bmatrix}. \tag{1.3}$$

Simple arithmetical considerations lead to the fact that $X \in \mathbb{C}^{n,n}$ solves (1.2) if and only if $\operatorname{im}[\, I_n \,,\, X \,]^*$ is an invariant subspace of $\mathcal{A}_H$. A well-known sufficient (but not necessary) criterion for the solvability of (1.2) is the absence of eigenvalues of $\mathcal{A}_H$ on the imaginary axis. Numerical and theoretical difficulties appear in the characterization of solvability of (1.2) when $\mathcal{A}_H$ has eigenvalues on the imaginary axis. Roughly speaking, this corresponds to the case where (1.2) is "on the border to unsolvability".

Another essential difficulty in the analysis of (1.1) is the possible non-invertibility of R. In this case, neither the algebraic Riccati equation nor the Hamiltonian matrix

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$\mathcal{A}_H$ can be formulated. Instead utilizing $\mathcal{A}_H$, one may consider the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -sI + A & B \\ sI + A^* & Q & C \\ B^* & C^* & R \end{bmatrix} \tag{1.4}$$

and characterize the solvability of (1.1) via *deflating subspaces* of (1.4), i.e., a generalization of the concept of invariant subspaces to matrix pencils[18]. Note that in the case where R is invertible, simple row and column transformations to $s\mathcal{E} - \mathcal{A}$ lead to $sI - \mathcal{A}_H$. However, the non-invertibility of R causes some additional problems, since then we are at least in one of the following situations:

(i) The pencil $s\mathcal{E} - \mathcal{A}$ has *eigenvalues at infinity* (see Section 2 for a definition).

(ii) The pencil $s\mathcal{E} - \mathcal{A}$ is *singular*, i.e. $\det(s\mathcal{E} - \mathcal{A}) = 0$ for all $s \in \mathbb{C}$.

Besides presenting equivalent solvability criteria for Lur'e equations, the main contribution of this work is the characterization of the set of solutions of (1.1) in terms of deflating subspaces of (1.4).

Let us briefly give an overview about the present state of research in the field of Lur'e equations. One of the first works dealing with this problem are [36, 37]. Under the assumption that (A, B) is *controllable*, i.e., $\operatorname{rank}[\, sI - A \,,\, B\,] = n$ for all $s \in \mathbb{C}$, the solvability of the Lur'e equations is shown to be equivalent to the positive semidefiniteness of the so-called *spectral density function*

$$\Phi(i\omega) = \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & C \\ C^* & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix} \tag{1.5}$$

for all $\omega \in \mathbb{R}$.

The works [23, 35] give an iterative technique for the elimination of variables corresponding to $\ker R$. After a finite number of steps this leads to a Riccati equation. This also gives an equivalent solvability criterion that is obtained by the feasibility of this iteration.

In [24, 34], the matrix R is slightly perturbed by εI_m for some $\varepsilon > 0$. Then by using the invertibility of $R + \varepsilon I$, the corresponding perturbed Lur'e equations are now equivalent to a Riccati equation. It is shown that certain corresponding solutions X_ε then converge to a solution of (1.1).

The matrix pencil approach to the solution of Lur'e equations is considered in [5, 6, 7, 13, 14, 15, 16, 17, 19, 21, 22]. The works [21, 22] give a complete characterization of the eigenstructure of so-called *extended Hamiltonian matrix pencils*, that can be obtained by simple row and column transformations to the matrix pencil in (1.4). The connection is highlighted to equations of the form

$$\begin{aligned} A^*X + XA + Q &= F^*RF, \\ XB + C &= F^*R, \end{aligned} \tag{1.6}$$

that have to be solved for $(X, F) \in \mathbb{C}^{n,n} \times \mathbb{C}^{m,n}$. This type can be considered as a special case of (1.1), since every Lur'e equation that has a solution (X, K, L) with $\operatorname{im} L \subset \operatorname{im} K$ can be expressed by a system of type (1.6). However, this is by far not fulfilled by every Lur'e equation.

The works [15, 16, 17, 19] consider the case where the matrix

$$\begin{bmatrix} Q & C \\ C^* & R \end{bmatrix}$$

is positive semi-definite and its rank equals to m, the matrix pencil (1.4) is regular and the pair (A, B) is controllable. It is shown that under these assumptions, a particular solution can be constructed via the consideration of certain deflating subspaces of a matrix pencil closely related to (1.4).

Our approach to Lur'e equations is mostly related to [5, 6, 7, 14, 13]. In particular, the works [5, 6, 7] consider the relation between deflating subspaces of associated matrix pencils and the solutions of a slight generalization of (1.1) and some a posteriori conditions on deflating subspaces are given such that a solution of the Lur'e equations can be constructed from these. We aim to give a priori conditions on the deflating subspaces such that they lead to a solution of (1.1).

The works [14, 13] consider the closely related problem of *spectral factorization*, that is, the construction of a rational function $\Psi : i\mathbb{R} \to \mathbb{C}^{p,m}$ such that the spectral density function fulfills $\Phi(i\omega) = \Psi^*(i\omega)\Psi^*(i\omega)$. As a byproduct, [13] gives equivalent criteria for the solvability of (1.1) under some weak controllability conditions of (A, B). The construction of solutions from the pencil (1.4) consists of an iterative elimination of the "critical" deflating subspaces. The work [14] considers spectral factorization on the basis of minimal descriptor realizations $\Phi(i\omega) = \mathfrak{B}^*(i\omega\mathfrak{E} - \mathfrak{A})^{-1}\mathfrak{B} + \mathfrak{R}$ for some square matrix pencil $s\mathfrak{E} - \mathfrak{A}$ fulfilling $\mathfrak{E} = -\mathfrak{E}^*$ and $\mathfrak{A}^* = \mathfrak{A}$. Characterizations for the positive semi-definiteness of the spectral density function in terms of the eigenstructure of the pencil

$$\begin{bmatrix} s\mathfrak{E} - \mathfrak{A} & \mathfrak{B} \\ \mathfrak{B}^* & \mathfrak{R} \end{bmatrix} \tag{1.7}$$

Though this work is not directly dealing with equations of the form (1.1), we can also gainfully use its techniques for the analysis of Lur'e equations.

Finally we remark that for the case of positive definite R, the work [30] shows that, under some slight extra conditions on the controllability of (A, B), there exist some particular solutions X_- and X_+ of (1.2) which are, in terms of definiteness, below (resp. above) all Hermitian X solving the *algebraic Riccati inequality*, i.e., the expression on the left hand side of (1.2) is positive semi-definite. It is furthermore shown that there, in terms of definiteness, the set of solutions is between two extremal solutions. The extremal solutions are shown to express optimal energies of certain linear-quadratic optimal control problems on the real half axis [36]. If the Hamiltonian matrix $\mathcal{A}_H$ has no eigenvalues on the imaginary axis, it is known from [25, 30] that these solutions correspond to the eigenspaces belonging to the eigenvalues in the positive (resp. negative) complex half-plane. We also generalize these facts to Lur'e equations.

We finally remark that the pencil (1.4) has the special property being *even*, that is $\mathcal{E}$ is skew-Hermitian and $\mathcal{A}$ is Hermitian. Matrix pencils of this type are intensively considered in [31, 32, 33], where especially the eigenstructure is analyzed and even canonical forms of Jordan-type are introduced.

The paper is organized as follows. Section 2 introduces the notation and contains some required control and matrix theoretic background, in particular a normal form for even matrix pencils is introduced. In Section 3 we collect some criteria being equivalent to the solvability of Lur'e equations. The solutions set of Lur'e equations is characterized in Section 4 by means of deflating subspaces of the associated even matrix pencil. Section 5 deals with the existence and characterization of extremal solutions.

2. Control and Matrix Theoretic Preliminaries. Throughout the paper real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, the open left and right half-planes by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. The symbol i stands for the imaginary unit and $i\mathbb{R}$ denotes the imaginary axis. By $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\overline{z}$ we denote the real part, imaginary part and, respectively, the conjugate transpose of $z \in \mathbb{C}$. Natural numbers excluding and including 0 are denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The spaces of $n \times m$ complex matrices are denoted by $\mathbb{C}^{n,m}$, and the set of invertible and complex $n \times n$ matrices by $\mathrm{Gl}_n(\mathbb{C})$. The matrices A^T and A^* denote, respectively, the transpose and the conjugate transpose of $A \in \mathbb{C}^{n,m}$, and $A^{-T} = (A^{-1})^T$, $A^{-*} = (A^{-1})^*$. We denote by $\mathrm{rank}(A)$ the rank, by $\mathrm{im}\, A$ the image, by $\ker A$ the kernel, by $\sigma(A)$ the spectrum of a matrix A. For Hermitian matrices $P, Q \in \mathbb{C}^{n,n}$, we write $P > Q$ ($P \geq Q$) if $P - Q$ is positive (semi-)definite. For a Hermitian matrix $P \in \mathbb{C}^{n,n}$ the *inertia*, i.e., triple consisting of the numbers of positive, negative and zero eigenvalues are denoted by $\mathrm{In}(P) \in \mathbb{N}_0^3$.
For a rational matrix-valued function $\Phi : \mathbb{C}\backslash D \to \mathbb{C}^{n,m}$, where $D \subset \mathbb{C}$ is the finite set of poles, we define the normal rank by $\mathrm{normalrank}\, \Phi = \max_{s \in \mathbb{C}\backslash D} \mathrm{rank}\, \Phi(s)$.
With $A_i \in \mathbb{C}^{n_i,m_i}$ with $m_i, n_i \in \mathbb{N}_0$ for $i = 1, \ldots, k$, we denote the block diagonal matrix by $\mathrm{diag}(A_1, \ldots, A_k)$. In particular, we set $\mathrm{diag}(A_1, 0_{0,m_2}) = [\, A_1\,,\, 0_{n_1,m_2}\,]$. An identity matrix of order n is denoted by I_n or simply by I. The zero $n \times m$ ($n \times n$) matrix is denoted by $0_{n,m}$ (resp. 0_n) or simply by 0. Moreover, for $k \in \mathbb{N}$ we introduce the following special matrices $J_k, M_k, N_k \in \mathbb{R}^{k,k}$, $K_k, L_k \in \mathbb{R}^{k-1,k}$ with

$$J_k = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \qquad K_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad L_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},$$

$$M_k = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{bmatrix}, \quad N_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Definition 2.1. *Let $sE - A$ be a matrix pencil with $E, A \in \mathbb{R}^{m,n}$. Then $sE - A$ is called* regular *if $m = n$ and* $\mathrm{normalrank}(sE - A) = n$.

A pencil $sE - A$ is called even *if $E = -E^*$ and $A = A^*$.*

Many properties of a matrix pencil can be characterized in terms of the *Weierstraß canonical form (WCF)*.

Type	Size	$\mathcal{C}_j(s)$	Parameters
W1	$k_j \times k_j$	$(s-\lambda)I_{k_j} - N_{k_j}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}$
W2	$k_j \times k_j$	$sN_{k_j} - I_{k_j}$	$k_j \in \mathbb{N}$
W3	$(k_j - 1) \times k_j$	$sK_{k_j} - L_{k_j}$	$k_j \in \mathbb{N}$
W4	$k_j \times (k_j - 1)$	$sK_{k_j}^T - L_{k_j}^T$	$k_j \in \mathbb{N}$

Table 2.1
Block types in Weierstraß canonical form

THEOREM 2.2. *[18] For a matrix pencil $sE - A$ with $E, A \in \mathbb{C}^{n,m}$, there exist matrices $U_l \in \mathrm{Gl}_n(\mathbb{C})$, $U_r \in \mathrm{Gl}_m(\mathbb{C})$, such that*

$$U_l(sE - A)U_r = \mathrm{diag}(\mathcal{C}_1(s), \ldots, \mathcal{C}_k(s)), \tag{2.1}$$

where each of the pencils $\mathcal{C}_j(s)$ is of one of the types presented in Table 2.1.
The numbers λ appearing in the blocks of type W1 are called the (generalized) eigenvalues *of $sE-A$. Blocks of type W2 are said to be corresponding to infinite eigenvalues.*

It is shown in [18] that the WCF is unique up to permutation of the indices $i = 1, \ldots, k$. Since each block of type W3 (W4) leads to an additional column (resp. row) rank deficiency of 1, the regularity of a pencil is equivalent the absence of blocks of type W3 and W4 in its WCF.
In the following, we review a special modification of the WCF from [32] for even matrix pencils, the *even Weierstraß canonical form (EWCF)*. This form is achieved by a congruence transform of $sE - A$ and therefore preserves the evenness. Note that, intrinsically, in [32] pencils $sE - A$ with Hermitian E, A were considered. The corresponding result for even pencils simply follows by a replacement of E with iE. In [31, 33], a further canonical form for the eigendecomposition of real even matrix pencils is introduced that is also preserving realness. Note that this form can be employed to derive real versions of the results in this paper.

Type	Size	$\mathcal{D}_j(s)$	Parameters
E1	$2k_j \times 2k_j$	$\begin{bmatrix} 0_{k_j,k_j} & (\lambda-s)I_{k_j}-N_{k_j} \\ (\overline{\lambda}+s)I_{k_j}-N_{k_j}^T & 0_{k_j,k_j} \end{bmatrix}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}^+$
E2	$k_j \times k_j$	$\epsilon_j((-is-\mu)J_{k_j} + M_{k_j})$	$k_j \in \mathbb{N}, \mu \in \mathbb{R}$, $\epsilon_j \in \{-1, 1\}$
E3	$k_j \times k_j$	$\epsilon_j(isM_{k_j} + J_{k_j})$	$k_j \in \mathbb{N}$, $\epsilon_j \in \{-1, 1\}$
E4	$(2k_j-1)\times$ $(2k_j-1)$	$\begin{bmatrix} 0_{k_j-1,k_j-1} & -sK_{k_j} + L_{k_j} \\ sK_{k_j}^T + L_{k_j}^T & 0_{k_j,k_j} \end{bmatrix}$	$k_j \in \mathbb{N}$

TABLE 2.2
Block types in even Weierstraß canonical form

THEOREM 2.3. *[32] For an even matrix pencil $sE - A$ with $E, A \in \mathbb{C}^{n,n}$, there exists a matrix $U \in \mathrm{Gl}_n(\mathbb{C})$ such that*

$$U^*(sE - A)U = \mathrm{diag}(\mathcal{D}_1(s), \ldots, \mathcal{D}_k(s)), \tag{2.2}$$

where each of the pencils $\mathcal{D}_j(s)$ is of one of the types presented in Table 2.2.
The numbers ε_j in the blocks of type E2 and E3 are called the block signatures.
The appearance of block of type E1 shows that generalized eigenvalues $\lambda \notin i\mathbb{R}$ occur in pairs $(\lambda, -\overline{\lambda})$. The blocks of type E2 and E3 respectively correspond to the purely imaginary and infinite eigenvalues. The additional sign parameter is contained which is basically due to the fact that for a fixed $\lambda \in i\mathbb{R}$ the congruence transformation with U preserves the inertia of the Hermitian matrix $\lambda E - A$. Blocks of type E4 consist of

a combination of blocks that are equivalent to those of type W3 and W4.
We now classify the inertia of the matrices $\mathcal{D}_i(i\omega)$ in dependence of the corresponding parameters and $\omega \in \mathbb{R}$.

PROPOSITION 2.4. *[14]*
a) If $\mathcal{D}_i$ is of type E1, then for all $\omega \in \mathbb{R}$ holds

$$\mathrm{In}(\mathcal{D}_j(i\omega)) = [\, k_j \,,\, 0 \,,\, k_j \,].$$

b) If $\mathcal{D}_i$ is of type E2 and k_j is even, then

$$\mathrm{In}(\mathcal{D}_j(i\omega)) = \begin{cases} [\, k_j/2 \,,\, 0 \,,\, k_j/2 \,], & \text{if } \mu \neq \omega, \\ [\, k_j/2-1 \,,\, 1 \,,\, k_j/2-1 \,] + \mathrm{In}(\epsilon_j), & \text{if } \mu = \omega. \end{cases}$$

c) If $\mathcal{D}_i$ is of type E2 and k_j is odd, then

$$\mathrm{In}(\mathcal{D}_j(i\omega)) = \begin{cases} [\, (k_j-1)/2 \,,\, 0 \,,\, (k_j-1)/2 \,] + \mathrm{In}(\epsilon_j(\omega-\mu)), & \text{if } \mu \neq \omega, \\ [\, (k_j-1)/2 \,,\, 1 \,,\, (k_j-1)/2 \,], & \text{if } \mu = \omega. \end{cases}$$

d) If $\mathcal{D}_i$ is of type E3 and k_j is even, then for all $\omega \in \mathbb{R}$ holds

$$\mathrm{In}(\mathcal{D}_j(i\omega)) = [\, k_j/2 \,,\, 0 \,,\, k_j/2 \,].$$

e) If $\mathcal{D}_i$ is of type E3 and k_j is odd, then for all $\omega \in \mathbb{R}$ holds

$$\mathrm{In}(\mathcal{D}_j(i\omega)) = [\, (k_j-1)/2 \,,\, 0 \,,\, (k_j-1)/2 \,] + \mathrm{In}(\varepsilon_j).$$

f) If $\mathcal{D}_i$ is of type E4, then for all $\omega \in \mathbb{R}$ holds

$$\mathrm{In}(\mathcal{D}_j(i\omega)) = [\, k_j \,,\, 1 \,,\, k_j \,].$$

DEFINITION 2.5. *A subspace $\mathcal{V} \subset \mathbb{C}^N$ is called* (right) deflating subspace *for the pencil $sE - A$ with $E, A \in \mathbb{C}^{M,N}$ if for a matrix $V \in \mathbb{C}^{N,k}$ with full column rank and $\operatorname{im} V = \mathcal{V}$, there exists an $l \leq k$ and matrices $W \in \mathbb{C}^{M,l}$, $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{l,k}$ with*

$$EV = W\widetilde{E}, \quad AV = W\widetilde{A}. \tag{2.3}$$

Equivalently, equation (2.3) can be formulated as

$$(sE - A)V = W(s\widetilde{E} - \widetilde{A}), \tag{2.4}$$

where s has to be treated as a formal variable.

DEFINITION 2.6. *Let the matrices*

$$\mathcal{E} = \begin{bmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0_m \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} 0_n & I_n & 0_{n,m} \end{bmatrix} \in \mathbb{R}^{n,2n+m}$$

be given. Then a subspace $\mathcal{V} \subset \mathbb{C}^{2n+m}$ is called
- generalized isotropic *if $x^*\mathcal{E}y = 0$ for all $x, y \in \mathcal{V}$,*
- generalized Lagrangian *if $\mathcal{L}$ is generalized isotropic with* $\dim \mathcal{V} = n + m$,
- generalized disconjugate *if* $\dim(\mathcal{P}\mathcal{L}) = n$.

Note that a subspace $\mathcal{V}$ is generalized Lagrangian if and only if it is a maximal generalized isotropic space that is, $\mathcal{V}$ is generalized isotropic and every proper superspace $\mathcal{V}_L \supset \mathcal{V}$ is not generalized isotropic. It can be further shown that a generalized Lagrangian subspace $\mathcal{V}$ fulfills $\dim \mathcal{E}\mathcal{V} = n$.

Definition 2.7. *Let a pair (A, B) with $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$ be given. Then*

(i) $\lambda \in \mathbb{C}$ is called an uncontrollable mode *of (A, B), if $\operatorname{rank}[\, sI - A\, ,\, B\,] < n$;*

(ii) (A, B) is called controllable *if it does not have an uncontrollable mode;*

(iii) (A, B) is called sign controllable *if all uncontrollable modes $\lambda, \mu \in \mathbb{C}$ satisfy $\overline{\lambda} + \mu \neq 0$;*

(iv) (A, B) is called stabilizable *if all uncontrollable modes have negative real part;*

(v) (A, B) is called anti-stabilizable *if all uncontrollable modes have positive real part.*

3. Solvability of Lur'e equations. We collect some known solvability criteria. As in [13], most of the results in this work require that either the pair (A, B) is controllable or the the pair (A, B) is sign-controllable together with the regularity of the even pencil $s\mathcal{E} - \mathcal{A}$. Observing that for all $\omega \in \mathbb{R}$ with $i\omega \notin \sigma(A)$ holds

$$\begin{aligned} U^*(i\omega)(i\omega\mathcal{E} - \mathcal{A})U(i\omega) &= \begin{bmatrix} 0 & -i\omega I + A & 0 \\ i\omega I + A & Q & 0 \\ 0 & 0 & \Phi(i\omega) \end{bmatrix} \\ \text{with } U(i\omega) &= \begin{bmatrix} I & 0 & (i\omega I + A^*)^{-1}(Q(-i\omega I + A)^{-1}B - C) \\ 0 & I & -(-i\omega I + A)^{-1}B \\ 0 & 0 & I_m \end{bmatrix}, \end{aligned} \tag{3.1}$$

a comparison of the normal ranks of the blocks of the EWCF of $s\mathcal{E} - \mathcal{A}$ leads to the fact that the quantity $m - \operatorname{normalrank} \Phi$ is to the number of blocks of type E4. in particular, the pencil $s\mathcal{E} - \mathcal{A}$ is regular if and only if $\operatorname{normalrank} \Phi = m$.

In the following result, we collect several statements equivalent to the solvability of Lur'e equations.

Theorem 3.1. *Let the Lur'e equations (1.1) with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) and spectral density function Φ as in (1.5) be given. Assume that at least one of the claims*

(i) the pair (A, B) is sign-controllable and the pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) is regular;

(ii) the pair (A, B) is controllable;

holds true. Then the following statements are equivalent:

1. *There exists a solution (X, K, L) of the Lur'e equations.*
2. *There exists some Hermitian $Y \in \mathbb{C}^{n,n}$ with*

$$\begin{bmatrix} A^*Y + YA + Q & YB + C \\ B^*Y + C^* & R \end{bmatrix} \geq 0. \tag{3.2}$$

3. *For all $\omega \in \mathbb{R}$ with $i\omega \notin \sigma(A)$ holds $\Phi(i\omega) \geq 0$;*
4. *In the EWCF of $s\mathcal{E} - \mathcal{A}$, all blocks of type E2 have positive signature and even size, and all blocks of type E3 have negative sign and odd size.*
5. *In the EWCF of $s\mathcal{E} - \mathcal{A}$, all blocks of type E2 have even size, and all blocks of type E3 have negative sign and odd size.*

In particular, solutions of the Lur'e equations fulfill $(X, K, L) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,p} \times \mathbb{C}^{m,p}$ with $p = \operatorname{normalrank} \Phi$.

For the proof, we need the following two auxiliary results.

LEMMA 3.2. *[13] Let the Lur'e equations (1.1) be given with associated even matrix pencil $s\mathcal{E}-\mathcal{A}$ as in (1.4) and let spectral density function Φ as in (1.5). Let $T_x \in \mathrm{Gl}_n(\mathbb{C})$, $T_u \in \mathrm{Gl}_m(\mathbb{C})$, $F \in \mathbb{C}^{m,n}$ and*

$$\begin{aligned} &A_F = T_x^{-1}(A+BF)T_x, && B_F = T_x^{-1}BT_u,\\ &C_F = T_x^*CT_u + T_x^*F^*RT_u, && Q_F = T_x^*(Q+CF+F^*C^*+F^*RF)T_x,\\ &R_F = T_u^*RT_u. && \end{aligned} \tag{3.3}$$

Then the Lur'e equations

$$\begin{aligned} A_F^*X_F + X_FA_F + Q_F &= K_F^*K_F,\\ XB_F + C_F &= K_F^*L_F,\\ R_F &= L_F^*L_F \end{aligned} \tag{3.4}$$

with associated even matrix pencil $s\mathcal{E}_F - \mathcal{A}_F$ and spectral density function $\Phi_F(i\omega)$ have the following properties:
a) For

$$U_F = \begin{bmatrix} T_x^{-*} & 0 & 0\\ 0 & T_x & 0\\ 0 & FT_x & T_u \end{bmatrix}, \tag{3.5}$$

holds $s\mathcal{E}_F - \mathcal{A}_F = U_F^*(s\mathcal{E}-\mathcal{A})U_F$.
b) For

$$\Theta_F(i\omega) = I + F(i\omega I - A_F)^{-1}BT_u \tag{3.6}$$

holds $\Phi_F(i\omega) = \Theta_F^*(i\omega)\Phi(i\omega)\Theta_F(i\omega)$.
c) (X,K,L) solves (1.1) if and only if $(X_F,K_F,L_F) = (T_x^*XT_x, (K+LF^*)T_x, LT_u)$ solves (3.4).

LEMMA 3.3. *[14] Let an even matrix pencil $sE-A$ be given. Then the following two statements are equivalent.*
1. In the EWCF, all blocks of type E2 have even size and positive sign, and all blocks of type E3 have odd size and negative sign.
2. There exist $n,c,d \in \mathbb{N}$ and a function $a:\mathbb{R}\to\mathbb{N}$ which is zero except for a finite set of values of ω, such that $\mathrm{In}(iE) = [\,n\,,\,d+c\,,\,n\,]$ and for all $\omega\in\mathbb{R}$ holds $\mathrm{In}(i\omega E - A) = [\,n+c-a(\omega)\,,\,d+a(\omega)\,,\,n\,]$.
In the case where the above statements are fulfilled, $a(\omega)$ corresponds to the blocks of type E2 with $\omega=\mu$, c is the number of blocks of type E3 and d is the number of blocks of type E4.

Now we show Theorem 3.1.

Proof. The equivalence between 1., 2. and 3. as well as $p = \mathrm{normalrank}\,\Phi$ is shown in [13].

3.⇒4. First assume that A has no eigenvalues on the imaginary axis. Then the spectral density function Φ is defined and positive semi-definite on the whole imaginary axis. Defining $d = \mathrm{normalrank}\,\Phi$, the rationality of Φ implies that there exists some function $a:\mathbb{R}\to\mathbb{N}$ that is zero except for a finite set of values and $\mathrm{rank}\,\Phi(i\omega) = d - a(\omega)$. Defining $c = m-d$ and applying the particular congruence transformation (3.1) to $s\mathcal{E}-\mathcal{A}$, the preservation of inertia yields that $s\mathcal{E}-\mathcal{A}$ fulfills the statement b) of Lemma 3.3 which then implies 4.. It remains to include the case where A has imaginary eigenvalues. Sign-controllability of (A,B) in particular implies the absence of uncontrollable modes on the imaginary axis. Hence there exists some $F\in\mathbb{C}^{m,n}$ such that $A_F=A+BF$ fulfills $\sigma(A_F)\cap i\mathbb{R}=\emptyset$[39]. Further, for $T_x=I_n$, $T_u=I_m$, defining the matrices B_F, C_F, Q_F and R_F as in (3.3) and considering the Lur'e equations (3.4), Lemma 3.2 implies that $\Phi_F(i\omega)=\Theta_F^*(i\omega)\Phi(i\omega)\Theta_F(i\omega)$ and $i\omega\mathcal{E}_F-\mathcal{A}_F=U_F^*(i\omega\mathcal{E}-\mathcal{A})U_F$ with U_F and $\Theta_F(i\omega)$ as in (3.5) and (3.6). The preservation of inertia now implies that we are in the situation of the first part of the proof and the desired result can be directly concluded.

4.⇒5. This statement is trivial.

5.⇒3. Let $\Theta_1=i\mathbb{R}\cap\sigma(A)$ and define Θ_2 as the set of generalized eigenvalues of $s\mathcal{E}-\mathcal{A}$ on the imaginary axis. Consider $\omega\in\mathbb{R}$ such that $i\omega\notin\Theta_1\cup\Theta_2$. Consider the EWCF of and $s\mathcal{E}-\mathcal{A}$ let d_3,d_4 be the number of of blocks of type E3 and E4, respectively. Writing $\mathcal{D}_j(s)=sE_j-A_j$ for accordant matrices E_j,A_j, a comparison of the rank deficiency of $\mathcal{E}$ with those of the matrices $E_j:j=1,\ldots,k$, we get that $m=d_3+d_4$. Proposition 2.4 then implies that $\mathrm{In}(i\mathcal{E}-\mathcal{A})=[\,n+d_3\,,\,d_4\,,\,n\,]$. Relation (3.1) and the preservation of inertia then implies $\Phi(i\omega)\geq 0$ with $\operatorname{rank}\Phi(i\omega)=d_3$. Now the continuity of Φ on $i\mathbb{R}\backslash\Theta_2$ and the the finiteness of both Θ_1 and Θ_2 implies that $\Phi(i\omega)\geq 0$ for all $\omega\in\mathbb{R}\backslash\Theta_2$. □

The definiteness relation (3.2) belongs to the type of *linear matrix inequalities (LMIs)* [8] that often arises in stability analysis of linear systems. The non-emptiness of the solution set of an LMI is called *feasibility*. It can be seen that a Hermitian matrix X is a part of a solution of the Lur'e equations (1.1) if it solves the LMI (3.2) with the additional property that the rank of the matrix on the left hand side of the LMI (3.2) equals to p.

The equivalence between 4. and 5. consequences that blocks of type E2 automatically have positive sign, if 5. is fulfilled. An analogous assertion for blocks of type E3 does unfortunately not hold true. For instance, consider the 1×1 matrices $A=-1$, $B=-C=R=1$ and $Q=0$.

Note that sign-controllability of (A,B) together with $\Phi(i\omega)\geq 0$ for all $\omega\in\mathbb{R}$ is in general not sufficient for the solvability of the Lur'e equations (1.1). A counterexample can be found in [13]. It is however shown in [13] that the feasibility of the LMI together with sign-controllability of (A,B) implies the solvability of Lur'e equations.

In [14], the assertions 4.-6. were considered for the pencil (1.7) that corresponds to a minimal descriptor realization of Φ. We employed the same technique of inertia comparison for the proof of our more general result.

4. Construction of Solutions via Deflating Subspaces. In this part we present solvability criteria and a parametrization of the solution set of Lur'e equations in terms of deflating subspaces of the associated even matrix pencil.

Theorem 4.1. *Let the associated even matrix pencil $s\mathcal{E}-\mathcal{A}$ as in (1.4) be given and let the spectral density function as in (1.5) satisfy $\Phi(i\omega)\geq 0$ for all $\omega\in\mathbb{R}$ with $i\omega\notin\sigma(A)$. Moreover, let $p=\operatorname{normalrank}\Phi$. Then the following two statements are equivalent:*

1. *For the Hermitian $X\in\mathbb{C}^{n,n}$ there exist $K\in\mathbb{C}^{p,n}$, $L\in\mathbb{C}^{p,m}$ such that (X,K,L) is a solution of (1.1).*

2. There exist $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$, $W_\mu, W_x \in \mathbb{C}^{n,n+p}$, $W_u \in \mathbb{C}^{m,n+p}$ and $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n+p,n+m}$ such that
- *$X = V_\mu V_x^-$ for some arbitrary right inverse V_x^- of V_x;*
- *$\mathcal{V} = \mathrm{im}[\, V_\mu^T \,,\, V_x^T \,,\, V_u^T \,]^T$ is generalized Lagrangian and disconjugate;*
- *For $V = [\, V_\mu^T \,,\, V_x^T \,,\, V_u^T \,]^T$ and $W = [\, W_\mu^T \,,\, W_x^T \,,\, W_u^T \,]^T$ holds*

$$(s\mathcal{E} - \mathcal{A})V = W(s\widetilde{E} - \widetilde{A}). \tag{4.1}$$

Lemma 4.2. *Let the associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) be given and let the spectral density function as in (1.5) satisfy $\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $i\omega \notin \sigma(A)$. Let $p = \mathrm{normalrank}\,\Phi$ and assume that the Hermitian matrix $X \in \mathbb{C}^{n,n}$ fulfills*

$$\mathrm{rank} \begin{bmatrix} A^*X + XA + Q & C + XB \\ C^* + B^*X & R \end{bmatrix} = p. \tag{4.2}$$

Then there exist $K \in \mathbb{C}^{p,n}$, $L \in \mathbb{C}^{p,m}$ such that (X, K, L) is a solution of the Lur'e equations (1.1).

Proof. Let $\omega \in \mathbb{R}$ such that $i\omega \notin \sigma(A)$ and $\mathrm{rank}\,\Phi(i\omega) = p$. Then we have

$$\begin{aligned} \Phi(i\omega) &= \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & C \\ C^* & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*X + XA + Q & C + XB \\ C^* + B^*X & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix} \\ &\quad - B^*(-i\omega I - A^*)^{-1}(A^*X + XA)(i\omega I - A)^{-1}B \\ &\quad - B^*(-i\omega I - A^*)^{-1}XB - B^*X(i\omega I - A)^{-1}B. \end{aligned}$$

Due to

$$\begin{aligned} &- B^*(-i\omega I - A^*)^{-1}(A^*X + XA)(i\omega I - A)^{-1}B \\ =& B^*(-i\omega I - A^*)^{-1}((-i\omega I - A^*)X + X(i\omega I - A))(i\omega I - A)^{-1}B \\ =& B^*X(i\omega I - A)^{-1}B + B^*(-i\omega I - A^*)^{-1}XB, \end{aligned}$$

the above expression for $\Phi(i\omega)$ reduces to

$$\Phi(i\omega) = \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*X{+}XA{+}Q & C{+}XB \\ C^*{+}B^*X & R \end{bmatrix} \begin{bmatrix} (i\omega I - A)^{-1}B \\ I_m \end{bmatrix}. \tag{4.3}$$

Now assume that X is not a part of a solution of the Lur'e equations, i.e., there exist $p_P, p_N \in \mathbb{N}$ with $p_P + p_N = p$ and $p_N > 0$ such that

$$\begin{bmatrix} A^*X + XA + Q & C + XB \\ C^* + B^*X & R \end{bmatrix} = \begin{bmatrix} M_{11}^* & M_{21}^* \\ M_{12}^* & M_{22}^* \end{bmatrix} \begin{bmatrix} I_{p_P} & 0 \\ 0 & -I_{p_N} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

for some matrices $M_{11} \in \mathbb{C}^{p_P,n}$, $M_{12} \in \mathbb{C}^{p_P,m}$, $M_{21} \in \mathbb{C}^{p_N,n}$, $M_{22} \in \mathbb{C}^{p_N,m}$. Define the functions $\mathbf{G}_1(i\omega) = M_{11}(i\omega I - A)^{-1}B + M_{12}$, $\mathbf{G}_2(i\omega) = M_{21}(i\omega I - A)^{-1}B + M_{22}$. Then we have

$$\Phi(i\omega) = \mathbf{G}_1^*(i\omega)\mathbf{G}_1(i\omega) - \mathbf{G}_2^*(i\omega)\mathbf{G}_2(i\omega).$$

Then Theorem 3.1 implies $\mathbf{G}_1^*(i\omega)\mathbf{G}_1(i\omega) - \mathbf{G}_2^*(i\omega)\mathbf{G}_2(i\omega)$ which is only fulfilled when $\operatorname{im}\mathbf{G}_2^*(i\omega) \subset \operatorname{im}\mathbf{G}_1^*(i\omega)$. Hence, we can estimate

$$p = \operatorname{rank}\Phi(i\omega) \leq \operatorname{rank}\mathbf{G}_2(i\omega) \leq p_P < p,$$

which is a contradiction. □

Proof of Theorem 4.1. The implication 1. ⇒ 2. follows from

$$\begin{bmatrix} 0 & -sI+A & B \\ sI+A^* & Q & C \\ B^* & C^* & R \end{bmatrix} \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -X & K^* \\ 0 & L^* \end{bmatrix} \begin{bmatrix} -sI+A & B \\ K & L \end{bmatrix}. \tag{4.4}$$

Now we show that 2. implies 1.: Since $\mathcal{V} = \operatorname{im}[\, V_\mu^T \,,\, V_x^T \,,\, V_u^T \,]^T$ is generalized Lagrangian and disconjugate, there exists some $T \in \mathrm{Gl}_{n+m}(\mathbb{C})$ and a Hermitian matrix $X \in \mathbb{C}^{n,n}$ such that

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} T = \begin{bmatrix} X & 0 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Since, by construction, $X = V_\mu V_x^-$ for some arbitrary right inverse V_x^- of V_x, it is sufficient to show that X is indeed a part of a solution of (1.1).
Define $[\, s\tilde{E}_1 - \tilde{A}_1 \,,\, s\tilde{E}_2 - \tilde{A}_2 \,] = (s\tilde{E} - \tilde{A})T$ where $\tilde{E}_1, \tilde{A}_1 \in \mathbb{C}^{n,n}$, $\tilde{E}_2, \tilde{A}_2 \in \mathbb{C}^{n,m}$. Then (4.1) implies

$$\begin{bmatrix} -sI+A & B \\ Q+A^TX+sX & C \\ B^TX+C^T & R \end{bmatrix} = \begin{bmatrix} W_\mu \\ W_x \\ W_u \end{bmatrix} \begin{bmatrix} s\tilde{E}_1+\tilde{A}_1 & s\tilde{E}_2+\tilde{A}_2 \end{bmatrix}.$$

Then we have $W_\mu\tilde{E}_2 = 0$, $W_x\tilde{E}_2 = 0$, $W_u\tilde{E}_2 = 0$ and thus

$$\begin{bmatrix} -sI+A & B \\ Q+A^TX+sX & C \\ B^TX+C^T & R \end{bmatrix} = \begin{bmatrix} W_\mu \\ W_x \\ W_u \end{bmatrix} \begin{bmatrix} s\tilde{E}_1+\tilde{A}_1 & \tilde{A}_2 \end{bmatrix}.$$

Due to $-I = W_\mu\tilde{E}_1$, we have $\operatorname{rank} W_\mu = n$ and thus, exists some $W \in \mathrm{Gl}_{n+m}(\mathbb{C})$ such that $W_\mu W = [\, I_n \,,\, 0_{n,p} \,]$. Defining

$$\begin{bmatrix} s\tilde{E}_{11}+\tilde{A}_{11} & \tilde{A}_{12} \\ s\tilde{E}_{21}+\tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = W^{-1}\begin{bmatrix} s\tilde{E}_1+\tilde{A}_1 & \tilde{A}_2 \end{bmatrix}$$

and $[\, W_{x1} \,,\, W_{x2} \,] = W_xW$, $[\, W_{u1} \,,\, W_{u2} \,] = W_uW$, we now have

$$\begin{bmatrix} -sI+A & B \\ Q+A^TX+sX & C \\ B^TX+C^T & R \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ W_{x1} & W_{x2} \\ W_{u1} & W_{u2} \end{bmatrix} \begin{bmatrix} s\tilde{E}_{11}+\tilde{A}_{11} & \tilde{A}_{12} \\ s\tilde{E}_{21}+\tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}.$$

A comparison of coefficients yields $\tilde{E}_{11} = -I$, $\tilde{A}_{11} = A$, $\tilde{A}_{12} = B$ and $W_{u1} = W_{u2}\tilde{E}_{21}$, $X = -W_{x1} + W_{x2}\tilde{E}_{21}$. Thus, for $A_{21} = \tilde{A}_{21} + \tilde{E}_{21}\tilde{A}_{11} \in \mathbb{C}^{p,n}$ and $A_{22} = \tilde{A}_{22} + \tilde{E}_{21}B \in \mathbb{C}^{p,m}$ we get

$$\begin{bmatrix} -sI+A & B \\ Q+A^TX+sX & C \\ B^TX+C^T & R \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -X & W_{x2} \\ 0 & W_{u2} \end{bmatrix} \begin{bmatrix} -sI+A & B \\ A_{21} & A_{22} \end{bmatrix}.$$

This especially leads to

$$\begin{bmatrix} A^TX+XA+Q & C^T+B^TX \\ XB+C & R \end{bmatrix} = \begin{bmatrix} W_{x2} \\ W_{u2} \end{bmatrix} \begin{bmatrix} A_{21} & A_{22} \end{bmatrix},$$

i.e., the rank of the matrix on the left hand side is bounded from above by p. We can now infer from Lemma 4.2 that X is a part of a solution of the Lur'e equations. ∎

In the remaining part of this section we further characterize the deflating subspaces with the properties as stated in Theorem 4.1.

First we classify the generalized Lagrangian deflating subspaces of $s\mathcal{E}-\mathcal{A}$ in terms of the EWCF.

THEOREM 4.3. *Let the associated even matrix pencil $s\mathcal{E}-\mathcal{A}$ as in (1.4) be given. For blocks $\mathcal{D}_j(s)$, $j=1,\ldots,k$ as presented in Table 2.2, let (2.2) be the EWCF of $s\mathcal{E}-\mathcal{A}$. Further assume that all blocks of type E2 have even and all blocks of type E3 have odd size. Consider the partitioning $U=[\,U_1\,,\ldots,\,U_k\,]$ according to the block structure of the EWCF. Then a matrix $V\in\mathbb{C}^{2n+m,n+m}$ has a generalized Lagrangian image and satisfies (4.1) for some $W\in\mathbb{C}^{2n+m,n+p}$, $\widetilde{E},\widetilde{A}\in\mathbb{C}^{n+p,n+m}$ if*

$$V=\begin{bmatrix} V_1 & \ldots & V_k \end{bmatrix} \quad \text{for } V_j=U_jZ_j, \tag{4.5}$$

where

$$Z_j=\begin{cases} \text{either } [\,I_{k_j}\,,\,0_{k_j}\,]^T \text{ or } [\,0_{k_j}\,,\,I_{k_j}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E1,} \\ [\,I_{k_j/2}\,,\,0_{k_j/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E2,} \\ [\,I_{(k_j-1)/2}\,,\,0_{(k_j+1)/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E3,} \\ [\,I_{k_j}\,,\,0_{k_j+1}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E4.} \end{cases}$$

Proof. First we show that $\operatorname{im}V$ with the matrix V as in (4.5) is indeed generalized Lagrangian. Comparing the rank of $\mathcal{E}$ with the ranks of the matrices $\mathcal{E}_j$ in the blocks $\mathcal{D}_j(s)=s\mathcal{E}_j-\mathcal{A}_j$, we can conclude that m equals to the number of blocks of type E3 and E4, we can conclude that $\operatorname{rank}V=n+m$. Furthermore, by the definition of the EWCF, we have

$$sV_l^*\mathcal{E}V_j-V_l^*\mathcal{A}V_j=\begin{cases} \mathcal{D}_j(s), & \text{if } l=j, \\ 0 & \text{if } l\neq j, \end{cases}$$

and thus

$$V^*\mathcal{E}V=\sum_{j=0}^{k} Z_j^*V_j^*\mathcal{E}V_jZ_j.$$

The construction of Z_j for each block type moreover leads to $Z_j^*V_j^*\mathcal{E}V_jZ_j=0$ for $j=1,\ldots,k$, i.e., $\operatorname{im}V$ is generalized Lagrangian. The fact that $\operatorname{im}V$ is a deflating subspace follows from the equation

$$(s\mathcal{E}-\mathcal{A})V=(T^{-*}Z_\ell)\cdot\operatorname{diag}(Z_1^*\mathcal{D}_1(s)Z_1,\ldots,Z_k^*\mathcal{D}_k(s)Z_k),$$

where $Z_\ell=\operatorname{diag}(Z_{\ell,1},\ldots,Z_{\ell,k})\in\mathbb{C}^{n+m,p}$ with

$$Z_{\ell,j}=\begin{cases} [\,I_{k_j}\,,\,I_{k_j}\,]^T-Z_j, & \text{if } \mathcal{D}_j \text{ is of type E1,} \\ [\,0_{k_j/2}\,,\,I_{k_j/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E2,} \\ [\,0_{(k_j-1)/2}\,,\,I_{(k_j+1)/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E3,} \\ [\,0_{k_j+1}\,,\,I_{k_j}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E4.} \end{cases}$$

□

Theorem 4.3 shows how a generalized Lagrangian deflating subspace can be constructed from the transformation leading to EWCF. For a generalization of the equations (1.1), the work [6] gives equivalent criteria on the matrices A, B, C, Q and R such that all generalized Lagrangian deflating subspaces of some associated even matrix pencil are disconjugate. This is called a *set of complete solutions* and leads to conditions which, for the case that we treat in this work, correspond to controllability of (A, B).

Under the assumption that the LMI (3.2) is feasible, we now give additional a priori criteria on a generalized Lagrangian deflating subspace, in particular on the choice of the matrices Z_j corresponding to the blocks of type E1, such that generalized disconjugacy is guaranteed as well. The proof needs a couple of technical lemmas and is left to the appendix to preserve clarity.

THEOREM 4.4. *Let the Lur'e equations (1.1) with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) be given and assume that the LMI (3.2) is feasible. Moreover, let a generalized Lagrangian space* $\operatorname{im} V$ *with* $V \in \mathbb{C}^{2n+m,n+m}$ *be given such that (4.1) holds true for some* $W \in \mathbb{C}^{2n+m,n+p}$, $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{n+p,n+m}$. *Furthermore, assume that for all generalized eigenvalues* λ *of the pencil* $s\widetilde{E} - \widetilde{A}$, *the number* $-\overline{\lambda}$ *is not an uncontrollable mode of* (A, B). *Then* $\operatorname{im} V$ *is generalized disconjugate.*

By Theorem 4.3, there is a certain freedom in the construction of the generalized Lagrangian deflating subspace, namely by either choosing the "first half" or the "second half" of the space corresponding to blocks of type E1. By Theorem 4.4, we get that - in order to also guarantee generalized disconjugacy - we have to incorporate the first k_j columns if λ is an uncontrollable mode and the last k_j columns if $-\overline{\lambda}$ is an uncontrollable mode. This criterion therefore implicitly contains sign-controllability of (A, B). In particular, for the case of solvable Lur'e equations, we have that a generalized Lagrangian subspace is automatically generalized disconjugate if the pair (A, B) is controllable.

While the correspondence between generalized Lagrangian deflating subspaces of the associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ requires the positive semi-definiteness of the spectral density function, Theorem 4.4 on the other hand assumes the feasibity of the LMI (3.2) for the criterion lsufficient for generalized disconjugacy. Note that, because of the relation (4.3), the feasibility of (3.2) is a slightly stronger condition than $\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

5. Extremal Solutions. It is known for the case were (A, B) is controllable that the solution set of the Lur'e equations has a certain structure, namely there exists a *maximal* solution (X_+, K_+, L_+) and a *minimal* solution (X_-, K_-, L_-) [2, 36]. That is, for all Hermitian solutions Y of the LMI (3.2) holds

$$X_- \leq Y \leq X_+. \tag{5.1}$$

For the case of invertible R, it was shown in [30] that (anti-)stabilizability is sufficient to the existence of a maximal (resp. minimal) solution. This solution is uniquely determined by

$$\sigma(A - BR^{-1}(B^*X + C)) \subset \mathbb{C}^- \cup i\mathbb{R}, \tag{5.2}$$

$$(\text{resp.} \quad \sigma(A - BR^{-1}(B^*X + C)) \subset \mathbb{C}^+ \cup i\mathbb{R}). \tag{5.3}$$

and is therefore called *(anti-)stabilizing solution* [25, 30]. We now aim to derive accordant results for Lur'e equations.

Roughly speaking, Theorem 4.3 and Theorem 4.4 consequence that the construction of solutions of the Lur'e equations via delating subspaces basically features a freedom in the choice of the deflating subspaces corresponding to the non-imaginary generalized eigenvalues. We show that the particular choice of the deflating subspaces corresponding to the generalized eigenvalues in $\mathbb{C}^-$ ($\mathbb{C}^+$) leads to the maximal (minimal) solution. The following two results are shown in the appendix.

THEOREM 5.1. *Let the Lur'e equations (1.1) be given with associated even matrix pencil $s\mathcal{E}-\mathcal{A}$ as is (1.4) be given. Assume that the LMI (3.2) is feasible and the pair (A,B) is stabilizable. Let $V\in\mathbb{C}^{2n+m,m}$ such that $\operatorname{im} V$ is generalized Lagrangian and disconjugate and there exists some $W\in\mathbb{C}^{2n+p,m}$ such that (4.1) holds true for some $\widetilde{E},\widetilde{A}\in\mathbb{C}^{n+p,n+m}$ with the property that all generalized eigenvalues of $s\widetilde{E}-\widetilde{A}$ have non-positive real part. Consider a partition $V=[\,V_\mu^T\,,\,V_x^T\,,\,V_u^T\,]^T$ with $V_\mu,V_x\in\mathbb{C}^{n,2n+m}$ and let $X_+=V_\mu V_x^-$ for some right inverse V_x^- of V_x. Then for all Hermitian $Y\in\mathbb{C}^+$ solving the LMI (3.2) holds*

$$Y\le X_+.$$

In regard of (4.4), a solution (X,K,L) is maximal if and only if all generalized eigenvalues of the pencil

$$\begin{bmatrix}-sI+A & B\\ K & L\end{bmatrix} \tag{5.4}$$

have non-positive real part. Note that for invertible R, this is equivalent to (5.2).

The construction of the deflating subspace leading to the maximal solution can be also performed via the help of the EWCF (2.2). If the accordantly partitioned matrix $U=[\,U_1\,,\ldots,\,U_k\,]\in\mathrm{Gl}_{2n+m}$ is the transformation leading to EWCF, then $V=[\,V_\mu^T\,,\,V_x^T\,,\,V_u^T\,]^T\in\mathbb{C}^{2n+m,n+m}$ fulfills $X_+=V_\mu V_x^-$ if (4.5) holds true for

$$Z_j=\begin{cases}[\,I_{k_j}\,,\,0_{k_j}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E1},\\ [\,I_{k_j/2}\,,\,0_{k_j/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E2},\\ [\,I_{(k_j-1)/2}\,,\,0_{(k_j+1)/2}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E3},\\ [\,I_{k_j}\,,\,0_{k_j+1}\,]^T, & \text{if } \mathcal{D}_j \text{ is of type E4}.\end{cases}$$

Note that $\operatorname{im} V$ is indeed generalized Lagrangian by Theorem 4.3. Taking into account that (A,B) is assumed to be stabilizable, i.e., all uncontrollable modes have negative real part, the generalized disconjugacy of $\operatorname{im} V$ follows by an application of Theorem 4.4 (which will be shown in the appendix).

THEOREM 5.2. *Let the Lur'e equations (1.1) be given with associated even matrix pencil $s\mathcal{E}-\mathcal{A}$ as is (1.4) be given. Assume that the LMI (3.2) is feasible and the pair (A,B) is anti-stabilizable. Let $V\in\mathbb{C}^{2n+m,m}$ such that $\operatorname{im} V$ is generalized Lagrangian and disconjugate and there exists some $W\in\mathbb{C}^{2n+p,m}$ such that (4.1) holds true for some $\widetilde{E},\widetilde{A}\in\mathbb{C}^{n+p,n+m}$ with the property that all generalized eigenvalues of $s\widetilde{E}-\widetilde{A}$ have non-negative real part. Consider a partition $V=[\,V_\mu^T\,,\,V_x^T\,,\,V_u^T\,]^T$ with $V_\mu,V_x\in\mathbb{C}^{n,2n+m}$ and let $X_+=V_\mu V_x^-$ for some right inverse V_x^- of V_x. Then for all Hermitian $Y\in\mathbb{C}^+$ solving the LMI (3.2) holds*

$$Y\ge X_+.$$

Analogous to the argumentations below Theorem 5.1, the minimality of a solution (X, K, L) can be characterized via the non-negativity of the generalized eigenvalues of the pencil (5.4). Furthermore, the anti-stabilizability of (A, B) and the results of Theorem 4.3 and Theorem 4.4 lead to the fact that the deflating subspace leading to the minimal solution can be constructed with (4.5) for

$$Z_j = \begin{cases} [\, 0_{k_j} \, , \, I_{k_j} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E1,} \\ [\, I_{k_j/2} \, , \, 0_{k_j/2} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E2,} \\ [\, I_{(k_j-1)/2} \, , \, 0_{(k_j+1)/2} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E3,} \\ [\, I_{k_j} \, , \, 0_{k_j+1} \,]^T, & \text{if } \mathcal{D}_j \text{ is of type E4.} \end{cases}$$

6. Conclusion. In this work we have studied Lur'e matrix equations. Under the assumption of either controllability or regularity of the associated even matrix pencil together with sign-controllability, equivalent criteria for the solvability of Lur'e equations are given in terms of the solvability of a linear matrix equation, the positive semi-definiteness of the spectral density function and the eigenstructure of a certain associated even matrix pencil. This associated even matrix pencil was utilized to describe the solution set. It is shown that solutions of Lur'e equations correspond to generalized Lagrangian and disconjugate deflating subspaces of the associated even matrix pencil. These particular deflating subspaces were further characterized in terms of the even Kronecker form. It is moreover shown that there exist solutions which are extremal in terms of definiteness. The corresponding deflating subspaces were particularly analyzed.

Appendix A. Some Auxiliary Results. In this section we show some technical lemmas needed to the proofs of Theorems 4.4, 5.1 and 5.2. Note that none of these lemmas require results that are presented in this paper earlier than Theorem 4.1 and we therefore do not have any mathematical redundancies.

LEMMA A.1. *Let the Lur'e equations (1.1) be given with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4). For the Hermitian matrix $Y \in \mathbb{C}^{n,n}$, define $B_Y := B$, $C_Y := C + YB^*$, $Q_Y := Q + A^*Y + YA$, $R_Y := R$ and corresponding Lur'e equations*

$$\begin{aligned} A_Y^* X_Y + X_Y A_Y + Q_Y &= K_Y^* K_Y, \\ X_Y B_Y + C_Y &= K_Y^* L_Y, \\ R_Y &= L_Y^* L_Y. \end{aligned} \tag{A.1}$$

The even matrix pencil associated to (A.1) then satisfies

$$s\mathcal{E}_Y - \mathcal{A}_Y = U_Y^* (s\mathcal{E} - \mathcal{A}) U_Y. \tag{A.2}$$

for

$$U_Y = \begin{bmatrix} I_n & Y & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \end{bmatrix}, \tag{A.3}$$

Furthermore, there is a one-to-one correspondence between the solutions (X, K, L) of (1.1) and those of via the relation $(X_Y, K_Y, L_Y) = (X - Y, K, L)$.
In particular holds that Y solves the LMI (3.2) if and only if

$$\begin{bmatrix} Q_Y & C_Y \\ C_Y^* & R_Y \end{bmatrix} \geq 0. \tag{A.4}$$

Proof. The correspondence between the associated even matrix pencils and the solutions follow by simple calculations. The result for (A.4) is a direct consequence of the definition of Q_Y, C_Y and R_Y. □

LEMMA A.2. *Let Lur'e equations (1.1) be given with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4). Further assume that*

$$\begin{bmatrix} Q & C \\ C^* & R \end{bmatrix} \geq 0 \tag{A.5}$$

holds true. Let a generalized isotropic space $\operatorname{im} V$ *for a full column rank matrix* $V \in \mathbb{C}^{2n+m,k}$ *be given such that there exist* $W \in \mathbb{C}^{2n+m,l}$ *and* $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{l,k}$ *with*

$$(s\mathcal{E} - \mathcal{A})V = W(s\widetilde{E} - \widetilde{A}).$$

Consider a row partition

$$V = [\, V_\mu\, ,\, V_x^T\, ,\, V_u^T\,]^T$$

with $V_\mu, V_x \in \mathbb{C}^{n,k}$, $V_u \in \mathbb{C}^{m,k}$. *Then* $V_\mu = 0$ *if one of the following conditions hold true:*

a) (A, B) *has no uncontrollable modes on the imaginary axis and the KCF of* $s\widetilde{E} - \widetilde{A}$ *does not contain blocks of type W4. Moreover, all generalized eigenvalues of* $s\widetilde{E} - \widetilde{A}$ *are purely imaginary.*

b) The KCF of $s\widetilde{E} - \widetilde{A}$ *only contains blocks of type W2 and W3.*

Proof. In the following, denote vectors $v_1, \ldots, v_l \in \mathbb{C}^{2n+m}$ with partition $v_i = [\, \mu_1^T\, ,\, x_i^T\, ,\, u_i^T\,]^T$ for $\mu_i, x_i \in \mathbb{C}^n$, $u_i \in \mathbb{C}^m$. Further, assume that $\operatorname{span}\{v_1, \ldots, v_l\}$ is generalized isotropic. From the fact that, without loss of generality, it can be assumed that $s\widetilde{E} - \widetilde{A}$ is in WCF, it can be seen that the above result is equivalent to the following assertions:

a) (A, B) has no uncontrollable modes on the imaginary axis and for some $\omega \in \mathbb{R}$ holds

$$i\omega\mathcal{E}v_1 = \mathcal{A}v_1, \quad \mathcal{E}(i\omega v_2 + v_1) = \mathcal{A}v_2, \ldots, \mathcal{E}(i\omega v_l + v_{l-1}) = \mathcal{A}v_l, \tag{A.6}$$

then $\mu_1 = \ldots = \mu_l = 0$.

b) If either

$$\mathcal{E}v_1 = 0, \quad \mathcal{E}v_2 = \mathcal{A}v_1, \quad \ldots \quad , \mathcal{E}v_l = \mathcal{A}v_{l-1} \tag{A.7}$$

or

$$\mathcal{E}v_1 = 0, \quad \mathcal{E}v_2 = \mathcal{A}v_1, \quad \ldots \quad , \mathcal{E}v_l = \mathcal{A}v_{l-1}, \quad \mathcal{A}v_l = 0, \tag{A.8}$$

then $\mu_1 = \ldots = \mu_l = 0$.

Now we inductively show these assertions.

a) Let (A.6) hold true. From $i\omega\mathcal{E}v_1 = \mathcal{A}v_1$, we get

$$\begin{bmatrix} (A - i\omega I)x_1 + Bu_1 \\ (A^* + i\omega I)\mu_1 + Qx_1 + Cu_1 \\ B^*\mu_1 + C^*x_1 + Ru_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A multiplication from the left with $[-\mu_1^*\,,\,x_1^*\,,\,u_1^*]$ yields

$$\begin{bmatrix}x_1\\u_1\end{bmatrix}^*\begin{bmatrix}Q & C\\C^* & R\end{bmatrix}\begin{bmatrix}x_1\\u_1\end{bmatrix}+2i(\omega\operatorname{Re}(x_1^*\mu_1)+\operatorname{Im}(u_1^*B\mu_1)+\operatorname{Im}(x_1^*A^*\mu_1))=0.$$

We can conclude from a comparison of real parts that the first summand vanishes. Condition (A.5) then implies that $Qx_1+Cu_1=0$ and $C^*x_1+Ru_1=0$. Hence, we have $(A^*+i\omega I)\mu_1=0$ and $B^*\mu_1=0$. This is a contradiction to the absence of uncontrollable imaginary modes.
Now assume that $\mu_1=\ldots=\mu_{i-1}=0$ and

$$\begin{bmatrix}(A-i\omega I)x_i+Bu_i\\(A^*+i\omega I)\mu_i+Qx_i+Cu_i\\B^*\mu_i+C^*x_i+Ru_i\end{bmatrix}=\begin{bmatrix}-x_{i-1}\\\mu_{i-1}\\0\end{bmatrix}=\begin{bmatrix}-x_{i-1}\\0\\0\end{bmatrix},\qquad x_{i-1}^*\mu_i=0.$$

Again multiplying the first equation from the left with $[-\mu_i^*\,,\,x_i^*\,,\,u_i^*]$, we obtain

$$\begin{bmatrix}x_i\\u_i\end{bmatrix}^*\begin{bmatrix}Q & C\\C^* & R\end{bmatrix}\begin{bmatrix}x_i\\u_i\end{bmatrix}+2i\omega(\operatorname{Re}(x_i^*\mu_i)+\operatorname{Im}(u_i^*B\mu_i)+\operatorname{Im}(x_i^*A^*\mu_i))=0.$$

and, by the same argumentation as before, we get $\mu_i=0$.

b) Let (A.6) hold true. From $\mathcal{E}v_1=0$ we trivially get $\mu_1=0$. Now assume that $\mu_1=\ldots=\mu_{i-1}=0$. Then $\mathcal{A}v_{i-1}=\mathcal{E}v_i$ gives

$$\begin{bmatrix}x_i\\-\mu_i\\0\end{bmatrix}=\begin{bmatrix}Ax_{i-1}+Bu_{i-1}\\A^*\mu_{i-1}+Qx_{i-1}+Cu_{i-1}\\B^*\mu_{i-1}+C^*x_{i-1}+Ru_{i-1}\end{bmatrix}=\begin{bmatrix}Ax_{i-1}+Bu_{i-1}\\Qx_{i-1}+Cu_{i-1}\\C^*x_{i-1}+Ru_{i-1}\end{bmatrix},\qquad x_{i-1}^*\mu_i=0.$$

and thus, a multiplication from the left with $[0_{1,n}\,,\,x_{i-1}^*\,,\,u_1^*]$ yields

$$\begin{bmatrix}x_{i-1}\\u_{i-1}\end{bmatrix}^*\begin{bmatrix}Q & C\\C^* & R\end{bmatrix}\begin{bmatrix}x_{i-1}\\u_{i-1}\end{bmatrix}=0.$$

Condition (A.5) then in particular implies $Qx_{i-1}+Cu_{u-1}=0$ and thus $\mu_i=0$.
The case for (A.8) can be shown with the same technique as in the proof of b) and are therefore omitted.

□

LEMMA A.3. *Let Lur'e equations be given and assume that (A.5) holds true and (A,B) has no uncontrollable modes on the imaginary axis. For the associated even matrix pencil $s\mathcal{E}-\mathcal{A}$ as in (1.4), let a generalized isotropic space* $\operatorname{im}V$ *with* $V\in\mathbb{C}^{2n+m,k}$ *be given such that for some* $W\in\mathbb{C}^{2n+m,l}$, *and* $\widetilde{E},\widetilde{A}\in\mathbb{C}^{l,k}$ *holds* $(s\mathcal{E}-\mathcal{A})V=W(s\widetilde{E}-\widetilde{A})$, *where* $s\widetilde{E}-\widetilde{A}$ *is a pencil whose WCF only contains blocks of types W1, W2 and W3. Consider a partition*

$$V=[\,V_\mu^T\,,\,V_x^T\,,\,V_u^T]^T.$$

Then the following holds true:
a) If all generalized eigenvalues of $s\widetilde{E}-\widetilde{A}$ *have non-positive real part, then* $V_\mu^*V_x\geq 0$.
b) If all generalized eigenvalues of $s\widetilde{E}-\widetilde{A}$ *have non-negative real part, then* $V_\mu^*V_x\leq 0$.

Proof. Since the proofs of assertions a) and b) only differ at some few places, we show a) while displaying the differences in the argumentations for b) in parentheses. Without loss of generality, assume that $s\widetilde{E} - \widetilde{A}$ is in WCF with the particular block ordering

$$W(sE - A)T = \operatorname{diag}(\mathcal{C}_1(s), \ldots, \mathcal{C}_l(s), \mathcal{C}_{l+1}(s) \ldots, \mathcal{C}_k(s)),$$

where the blocks $\mathcal{C}_1(s), \ldots, \mathcal{C}_l(s)$ correspond to the eigenvalues with negative (positive) real part and $\mathcal{C}_{l+1}(s), \ldots, \mathcal{C}_k(s)$ are the blocks of type W1 corresponding to the imaginary generalized eigenvalues and types W2 and W3. Consider an accordant partition

$$V = \begin{bmatrix} V_{\mu c} & V_{\mu 0} \\ V_{xc} & V_{x0} \\ V_{uc} & V_{u0} \end{bmatrix}, \qquad s\widetilde{E} - \widetilde{A} = \begin{bmatrix} s\widetilde{E}_c - \widetilde{A}_c & 0 \\ 0 & s\widetilde{E}_0 - \widetilde{A}_0 \end{bmatrix}.$$

From Lemma A.2, we obtain $V_{10} = 0$ and thus $V_\mu^* V_x = V_{\mu c}^* V_{xc}$. Since $s\widetilde{E}_c - \widetilde{A}_c$ is in WCF and it only contains blocks of size W1 corresponding to generalized eigenvalues with negative (positive) real part, we have $\widetilde{E}_c = I$ and $\sigma(\widetilde{A}_c) \subset \mathbb{C}^-$ ($\sigma(\widetilde{A}_c) \subset \mathbb{C}^+$). The first block column of $(s\mathcal{E} - \mathcal{A})V = W(s\widetilde{E} - \widetilde{A})$ is then equivalent to

$$\begin{bmatrix} -V_{xc} \\ V_{\mu c} \\ 0 \end{bmatrix} \widetilde{A}_c = \begin{bmatrix} AV_{xc} + BV_{uc} \\ A^* V_{\mu c} + QV_{xc} + CV_{uc} \\ B^* V_{\mu c} + C^* V_{xc} + RV_{uc} \end{bmatrix}.$$

A multiplication from the left with $[\,-V_{\mu c}^*\,,\ V_{xc}^*\,,\ V_{uc}^*\,]$ yields

$$2V_{\mu c}^* V_{xc} \widetilde{A}_c = \begin{bmatrix} V_{cc} \\ V_{uc} \end{bmatrix}^* \begin{bmatrix} Q & C \\ C^* & R \end{bmatrix} \begin{bmatrix} V_{xc} \\ V_{uc} \end{bmatrix} + V_{xc}^* A^* V_{\mu c} - V_{\mu c}^* A V_{xc} + V_{uc}^* B^* V_{\mu c} - V_{\mu c}^* B V_{uc}.$$

and thus

$$(V_{\mu c}^* V_{xc})\widetilde{A}_c + \widetilde{A}_c^* (V_{\mu c}^* V_{xc}) = \frac{1}{2} \begin{bmatrix} V_{xc} \\ V_{uc} \end{bmatrix}^* \begin{bmatrix} Q & C \\ C^* & R \end{bmatrix} \begin{bmatrix} V_{xc} \\ V_{uc} \end{bmatrix} \geq 0.$$

This is a Lyapunov equation for the Hermitian matrix $V_{\mu c}^* V_{xc}$. By the standard results for Lyapunov matrix equations [39], the facts that $\sigma(\widetilde{A}_c) \subset \mathbb{C}^-$ ($\sigma(\widetilde{A}_c) \subset \mathbb{C}^+$) and the right hand side is positive semi-definite, we can deduce $V_\mu^* V_x = V_{\mu c}^* V_{xc} \leq 0$ ($V_\mu^* V_x = V_{\mu c}^* V_{xc} \geq 0$). □

LEMMA A.4. *Let the Lur'e eqautions be given with associated even matrix pencil $s\mathcal{E} - \mathcal{A}$ as in (1.4) and assume that (A.5) holds true. Then there exist $T_x \in \mathrm{Gl}_n(\mathbb{C})$, $F \in \mathbb{C}^{n,m}$ and $T_u \in \mathrm{Gl}_m(\mathbb{C})$ such that*

$$\begin{aligned} T_x^*(Q + CF + F^*C^* + F^*RF)T_x &= \begin{bmatrix} 0_{n_1} & 0_{n_1,n_2} \\ 0_{n_2,n_1} & Q_{22} \end{bmatrix}, \\ T_u^* R T_u &= \begin{bmatrix} 0_{m_1} & 0_{m_1,m_2} \\ 0_{m_2,m_1} & R_{22} \end{bmatrix}, \\ T_x^{-1}(A + BF)T_x &= \begin{bmatrix} N & A_{12} \\ 0_{n_2,n_1} & A_{22} \end{bmatrix}, \\ T_x^{-1} B T_u &= \begin{bmatrix} E & B_{12} \\ 0_{m_2,m_1} & B_{22} \end{bmatrix}, \\ T_x^* C T_u + T_x^* F^* R T_u &= \begin{bmatrix} 0_{m_1,n_1} & 0_{m_1,n_1} \\ 0_{m_2,n_1} & C_{22} \end{bmatrix} \end{aligned} \tag{A.9}$$

for

$$N = \operatorname{diag}(N_{k_1},\ldots,N_{k_l}),\ E = \operatorname{diag}(e_{k_1}^{(k_1)},\ldots,e_{k_l}^{(k_l)}),\ \textit{where } e_j^{(j)} = [\,0\,,\ldots,\,0\,,\,1\,] \in \mathbb{C}^{j,1}$$

and some matrices $Q_{22}, R_{22}, A_{12}, A_{22}$ *of suitable dimensions. In particular,* T_x, T_u *and* F *can be chosen that the pencil*

$$s\mathcal{E}_{22} - \mathcal{A}_{22} = \begin{bmatrix} 0 & -sI - A_{22} & B_{22} \\ sI - A_{22}^* & Q_{22} & C_{22} \\ B_{22}^* & C_{22}^* & R_{22} \end{bmatrix} \tag{A.10}$$

is regular.

Proof. Consider the EWCF (2.1), partition the transformation $T = [\,T_1\,,\ldots,\,T_k\,]$ and assume that $\mathcal{D}_1$ is a block of type W4. From Lemma A.2, we know that for $Z_1 = [\,I_{k_1}\,,\,0_{k_1,k_1-1}\,]$, the matrix $V_1 = T_1 Z_1$ has the form

$$V_1 = T_1 Z_1 = [\,0_{n,k_1}^T\,,\,V_x\,,\,V_u\,]^T$$

for some $V_x \in \mathbb{C}^{n,k_1}$, $V_u \in \mathbb{C}^{m,k_3}$. In particular, there exists a full column rank matrix $W_1 \in \mathbb{C}^{2n+m,k_1-1}$ such that $(s\mathcal{E}-\mathcal{A})V_1 = W_1(-sK_{k_1}+L_{k_1})$ holds true. The relation $K_{k_1} = [\,I_{k_1}\,,\,0_{k_1,1}\,]$ implies that $V_x = [\,V_{xl}\,,\,0_{n,1}\,]$ for some matrix $V_{xl} \in \mathbb{C}^{n,k_1-1}$ with full column rank. Consider an accordant partition of $V_u = [\,V_{ul}\,,\,V_{ur}\,]$ with $V_{ul} \in \mathbb{C}^{m,k_1-1}$, $V_{ur} \in \mathbb{C}^{n,1}$. Since V_1 has full column rank, we obtain $V_{ur} \neq 0$. Now define $T_x = [\,V_{xl}\,,\,V_{xc}\,]$, $T_u = [\,V_{ur}\,,\,V_{uc}\,]$ for some $V_{xc} \in \mathbb{C}^{n-k_1}$, $V_{uc} \in \mathbb{C}^{n-k_1}$ such that $T_x \in \mathrm{Gl}_n(\mathbb{C})$, $T_u \in \mathrm{Gl}_n(\mathbb{C})$ and let $F = [\,V_{ul}\,,\,0_{m,n-k_1}\,]\,T_x^{-1}$. Partition

$$T_x^*(Q + CF + F^*C^* + F^*RF)T_x = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \qquad T_u^* R T_u = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix}$$
$$T_x^{-1}(A+BF)T_x = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad T_x^{-1} B T_u = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

First we show that

$$\begin{aligned} &A_{21} = 0, && B_{21} = 0, && Q_{11} = 0, && Q_{12} = 0, \\ &C_{11} = 0, && C_{21} = 0, && C_{12} = 0, && \\ &R_{11} = 0, && R_{12} = 0, && A_{11} = N_{k_1}, && B_{21} = e_{k_1}^{(k_1)}. \end{aligned} \tag{A.11}$$

Consider the associated even matrix pencil

$$\begin{bmatrix} 0 & 0 & -sI + A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & 0 & A_{21} & -sI + A_{22} & B_{21} & B_{22} \\ sI + A_{11}^* & A_{21}^* & Q_{11} & Q_{12} & C_{11} & C_{12} \\ A_{12}^* & sI + A_{22}^* & Q_{12}^* & Q_{22} & C_{21} & C_{22} \\ B_{11}^* & A_{21}^* & C_{11}^* & C_{21}^* & R_{11} & R_{12} \\ B_{12}^* & B_{22}^* & C_{12}^* & C_{22}^* & R_{12}^* & R_{22} \end{bmatrix}.$$

Since, by Lemma 3.2, we have $s\mathcal{E}_F - \mathcal{A}_F = U_F^*(s\mathcal{E}-\mathcal{A})U_F$ for $U_F \in \mathrm{Gl}_{2n+m}(\mathbb{C})$ as in (3.5), and, furthermore, $(s\mathcal{E}-\mathcal{A})V_1 = W_1(-sK_{k_1}+L_{k_1})$, the construction of U_F and the matrices T_x, T_u and F leads to the fact for $v(\lambda) = [\,\lambda\,,\ldots,\,\lambda^{k_1-1}\,]$ holds that

$$\begin{bmatrix} 0_n & v(\lambda) & 0_{1,n-k_1} & \lambda^{k_1} & 0_{1,m-1} \end{bmatrix}^T \in \ker(\lambda\mathcal{E}_F - \mathcal{A}_F) \quad \text{for all } \lambda \in \mathbb{C}.$$

This implies that for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} (-\lambda I + A_{11})v(\lambda) + \lambda^{k_1} B_{11} &= 0, & A_{21}v(\lambda) + \lambda^{k_1} B_{21} &= 0, \\ Q_{11}v(\lambda) + \lambda^{k_1} C_{11} &= 0, & Q_{12}^* v(\lambda) + \lambda^{k_1} C_{21} &= 0, \\ C_{11}^* v(\lambda) + \lambda^{k_1} R_{11} &= 0, & C_{12}^* v(\lambda) + \lambda^{k_1} R_{12}^* &= 0 \end{aligned}$$

and therefore the relations (A.11) have to hold true. Since the pencil $s\mathcal{E}_{22} - \mathcal{A}_{22}$ as in (A.10) is constructed from a deflation of a block of type E4 from $s\mathcal{E} - \mathcal{A}$, their generic rank deficiencies satisfy

$$\begin{aligned} & 2n_2 + m_2 - \mathrm{normalrank}(s\mathcal{E}_{22} - \mathcal{A}_{22}) \\ &= 2n + m - \mathrm{normalrank}(s\mathcal{E} - \mathcal{A}) - 1 \quad = m-p-1. \end{aligned}$$

Now a $m-p-1$-times repetition of the above deflation leads to the desired result. □

Lemma A.5. *Let the Lur'e equation (1.1) be given and assume that the associated even matrix pencil* $s\mathcal{E} - \mathcal{A}$ *as in (1.4) is regular,* (A, B) *does not have uncontrollable modes on the imaginary axis and (A.5) holds true. Let* $V = [V_\mu^T, V_x^T, V_u^T]^T \in \mathbb{C}^{2n+m,k}$ *is a matrix with full column rank such for some* $W \in \mathbb{C}^{2n+m,k}$ *and* $\widetilde{E}, \widetilde{A} \in \mathbb{C}^{k,k}$ *holds* $(s\mathcal{E} - \mathcal{A})V = W(s\widetilde{E} - \widetilde{A})$. *Further assume that* $\mathrm{im}\, V$ *is generalized isotropic and for all generalized eigenvalues* λ *of the pencil* $s\widetilde{E} - \widetilde{A}$*, the number* $-\overline{\lambda}$ *is not an uncontrollable mode of* (A, B). *Then* $\ker V_x \subset \ker V_\mu$.

Proof. Since the pencil $s\mathcal{E} - \mathcal{A}$ is regular, this also holds true for $s\widetilde{E} - \widetilde{A}$. Thus its WCF only contains blocks of type W1 and W2. Assuming without loss of generality that $\lambda\widehat{E} + \widehat{A}$ is already in WCF, we obtain

$$\mathcal{E}V\widetilde{A} = \mathcal{A}V\widetilde{E}. \tag{A.12}$$

The claim (A.5) implies that the spectral density function is pointwise positive semi-definite. From the regularity of $s\mathcal{E} - \mathcal{A}$, we further get $\mathrm{normalrank}\, \Phi = m$.

Hence there exists some $\alpha \in \mathbb{R}$ such that $i\alpha I - A$, $i\alpha\widetilde{E} - \widetilde{A}$ and $\Phi(i\alpha)$ are regular. Let $x \in \mathbb{C}^n$ such that $V_x(i\alpha\widetilde{E} - \widetilde{A})x = 0$. Relation (A.12) implies

$$0 = V_x\widetilde{A}x + AV_x\widetilde{E}x + BV_u\widetilde{E}x, \tag{A.13}$$

$$0 = -V_\mu\widehat{A}x + A^*V_\mu\widetilde{E}x + QV_x\widetilde{E}x + CV_u\widetilde{E}x, \tag{A.14}$$

$$0 = B^*V_\mu\widetilde{E}x + C^*V_x\widetilde{E}x + RV_u\widetilde{E}x. \tag{A.15}$$

A multiplication of (A.14) from the left with $x^*\widetilde{E}^*V_x^*$ yields

$$\begin{aligned} 0 &= x^*\widetilde{E}^*V_x^*V_\mu\widetilde{A}x + x^*\widetilde{E}^*V_x^*QV_x\widetilde{E}x + x^*\widetilde{E}^*V_x^*A^*V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_x^*CV_u\widetilde{E}x \\ &= x^*\widetilde{E}^*V_\mu^*V_x\widetilde{A}x + x^*\widetilde{E}^*V_x^*QX_x\widetilde{E}x + x^*\widetilde{E}^*V_x^*A^*V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_x^*CW_3\widetilde{E}x \\ &= i\alpha x^*\widetilde{E}^*V_\mu^*V_x\widetilde{E}x + x^*\widetilde{E}^*V_x^*QV_x\widetilde{E}x + x^*\widetilde{E}^*V_x^*A^*V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_x^*CV_u\widetilde{E}x \\ &= x^*\widetilde{E}^*V_x^*QV_x\widetilde{E}x + x^*\widetilde{E}^*V_x^*(i\alpha I + A^*)V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_x^*CV_u\widetilde{E}x. \end{aligned}$$

We obtain from (A.13) and $i\alpha V_x\widetilde{E}x = V_x\widetilde{A}x$ that $V_x\widetilde{E}x = (i\alpha I - A)^{-1}BV_u\widetilde{E}x$. Then we have

$$\begin{aligned}
0 &= x^*\widetilde{E}^*V_x^*QV_x\widetilde{E}x + x^*\widetilde{E}^*V_x^*(i\alpha I + A^*)V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_x^*CV_u\widetilde{E}x\\
&= x^*\widetilde{E}^*V_u^*B^*(-i\alpha I - A)^{-1}Q(i\alpha I - A)^{-1}BV_u\widetilde{E}x\\
&\quad + x^*\widetilde{E}^*V_u^*B^*(-i\alpha I - A^*)^{-1}(i\alpha I + A^*)V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_uB^*(-i\alpha I + A^*)^{-1}CV_u\widetilde{E}x\\
&= -x^*\widetilde{E}^*V_u^*B^*(i\alpha I + A)^{-1}Q(i\alpha I - A)^{-1}BV_u\widetilde{E}x\\
&\quad - x^*\widetilde{E}^*V_u^*B^*V_\mu\widetilde{E}x + x^*\widetilde{E}^*V_u^*B^*(-i\alpha I + A^*)^{-1}CV_u\widetilde{E}x.
\end{aligned}$$

Plugging the relation $B^*V_\mu\widetilde{E}x = -C^*V_x\widetilde{E}x - RV_u\widetilde{E}x$ from (A.15) into the above equation, we obtain $x^*\widetilde{E}^*V_u^*\Phi(i\alpha)V_u\widetilde{E}x = 0$. Since $\Phi(i\alpha)$ has full rank, we can infer that $V_u\widetilde{E}x = 0$ holds true. Then (A.14) reads

$$0 = -V_x\widetilde{A}x + AV_x\widetilde{E}x = -(i\alpha I - A)V_x\widetilde{E}x.$$

The invertibility of $i\alpha I - A$ now leads to $V_x\widetilde{E}x = 0$ and thus also $V_x\widetilde{A}x = 0$. Hence, we have that for all $\lambda \in \mathbb{C}$ that $V_x(\lambda\widetilde{E} - \widetilde{A})x = 0$. Hence, the space given by $\mathcal{V}_0 = \{x \in \mathbb{C}^k \,:\, (i\alpha\widetilde{E} - \widetilde{A})x \in \ker V_x\}$ is a deflating subspace of $s\widetilde{E} - \widetilde{A}$. Thus there exists some square regular pencil $\lambda\widehat{E} - \widehat{A}$ and some full column rank matrices V_0, W_0 with $\operatorname{im} V_0 = \mathcal{V}_0$ holds $(s\widetilde{E} - \widetilde{A})V_0 = W_0(s\widehat{E} - \widehat{A})$. Together with $(s\mathcal{E} - \mathcal{A})V = W(s\widetilde{E} - \widetilde{A})$, we have that

$$(s\mathcal{E} - \mathcal{A})VV_0 = WW_0(s\widehat{E} - \widehat{A}).$$

On the other hand, by definition of V_0 holds $0 = V_x(s\widetilde{E} - \widetilde{A})V_0$. Assuming without loss of generality that $s\widehat{E} - \widehat{A}$ is in KCF, this implies $V_xV_0 = 0$. Altogether, this means that the space $\operatorname{im} VV_0 \subset V$ is a deflating subspace of $s\mathcal{E} - \mathcal{A}$ and, due to $V_xV_0 = 0$, the matrix VV_0 has the form $VV_0 = [\, V_{0\mu}^T \,,\, 0 \,,\, V_{0u}^T \,]^T$. A column partition $V_{0\mu} = [\, V_{0\mu,1} \,,\, V_{0\mu,2} \,],$, $V_{0u} = [\, V_{0u,1} \,,\, V_{0u,2} \,]$ with $V_{0\mu,1} \in \mathbb{C}^{n,l_1}$ and $V_{0\mu,2} \in \mathbb{C}^{n,l_2}$, $V_{0u,1} \in \mathbb{C}^{m,l_1}$, $V_{0\mu,2} \in \mathbb{C}^{m,l_2}$ leads to

$$\mathcal{E}\begin{bmatrix} V_{0\mu,1}\\ 0\\ V_{0u,1}\end{bmatrix}\widehat{A}_f = \mathcal{A}\begin{bmatrix} V_{0\mu,1}\\ 0\\ V_{0u,1}\end{bmatrix}, \tag{A.16}$$

$$\mathcal{E}\begin{bmatrix} V_{0\mu,2}\\ 0\\ V_{0u,2}\end{bmatrix} = \mathcal{A}\begin{bmatrix} V_{0\mu,2}\\ 0\\ V_{0u,2}\end{bmatrix}\widehat{N}. \tag{A.17}$$

Again employing the argumentation as at the beginning of this proof, we obtain that $V_{0u1} = 0$ and $V_{0u1}\widehat{N} = 0$. Then (A.16) implies $-V_{0\mu,1}\widehat{A}_f = A^*V_{0\mu,1}$ and $B^*V_{0\mu,1} = 0$. Assume that v_f is an eigenvector of $\widehat{A}_f$, i.e., $\widehat{A}_fv_f = \lambda v_f$. Then for $v = V_{0\mu,1}v_f$ holds $A^*v = -\lambda v$ and $B^*v = 0$. However, means that $\operatorname{im} V$ contains a generalized eigenvectors corresponding to the eigenvalue λ with the property that $-\overline{\lambda}$ is an uncontrollable mode of (A, B). This is a controdiction to the assumption.
Now we show the same statement for the matrices $V_{0\mu,2}$, $V_{0u,2}$. From $V_{0u1}\widehat{N} = 0$ and (A.17), we obtain $V_{0\mu,2} = A^*V_{0\mu,2}\widehat{N}$ and $B^*V_{0\mu,2} = 0$. In particular, we have

$$V_{0\mu,2} = A^*V_{0\mu,2}\widehat{N} = \cdots = (A^*)^{l_2}V_{0\mu,2}\widehat{N}^{l_2} = 0.$$

□

Appendix B. Proof of Theorem 4.4.

LEMMA B.1. *Theorem 4.4 holds true under the additional assumption (A.5).*

Proof. According to Lemma A.4, there exist $T_x \in \mathrm{Gl}_n(\mathbb{C})$, $T_u \in \mathrm{Gl}_m(\mathbb{C})$ and $F \in \mathbb{C}^{m,n}$ such that (A.9) holds true and the pencil $s\mathcal{E}_{22} - \mathcal{A}_{22}$ as in (A.10) is regular. Since, according to Lemma 3.2, the pencil

$$s\mathcal{E}_F - \mathcal{A}_F = \begin{bmatrix} 0 & 0 & -sI+N & A_{12} & E & B_{12} \\ 0 & 0 & 0 & -sI+A_{22} & 0 & B_{22} \\ sI+N^* & 0 & 0 & 0 & 0 & 0 \\ A_{12}^* & sI+A_{22}^* & 0 & Q_{22} & 0 & C_{22} \\ E^* & 0 & 0 & 0 & 0 & 0 \\ B_{12}^* & B_{22}^* & 0 & C_{22}^* & 0 & R_{22} \end{bmatrix}$$

satisfies $U_F^*(s\mathcal{E} - \mathcal{A})U_F = s\mathcal{E}_F - \mathcal{A}_F$ for U_F as in (3.5), we have that $V_F = T_F^{-1}V$, $W_F = T_F^{-*}W$ fulfill $(s\mathcal{E}_F - \mathcal{A}_F)V_F = W_F(s\widehat{E} - \widehat{A})$. Without loss of generality, assume that the pencil on the right hand side has the form

$$s\widehat{E} - \widehat{A} = \mathrm{diag}(s\widehat{E}_r - \widehat{A}_r, s\widehat{E}_0 - \widehat{A}_0),$$

where $s\widehat{E}_r - \widehat{A}_r$ is regular and $s\widehat{E}_0 - \widehat{A}_0$ is WCF only containing blocks of type W3.

Then a suitable column partitioning $V_F = [\, V_{Fr}\, , \, V_{F0}\,]$, $W_F = [\, W_{Fr}\, , \, W_{F0}\,]$ leads to $(s\mathcal{E}_F - \mathcal{A}_F)V_{Fr} = W_{Fr}(s\widehat{E}_r - \widehat{A}_r)$ and $(s\mathcal{E}_F - \mathcal{A}_F)V_{F0} = W_{F0}(s\widehat{E}_0 - \widehat{A}_0)$. In particular, by construction of $s\mathcal{E}_F - \mathcal{A}_F$ and the regularity of $s\mathcal{E}_{22} - \mathcal{A}_{22}$, we have $\mathrm{im}\, V_{F0} = \mathrm{im}\, 0_{n,1} \times \mathbb{C}^{n_1} \times \mathrm{im}\, 0_{n_2,1} \times \mathbb{C}^{m_1} \times \mathrm{im}\, 0_{m_2,1}$ and thus we can assume that

$$V_{F0} = \begin{bmatrix} 0_{n_1,n_1} & 0_{n_1,n_2} & I_{n_1} & 0_{n_1,n_2} & 0_{n_1,m_1} & 0_{n_1,m_2} \\ 0_{m_1,n_1} & 0_{m_1,n_2} & 0_{m_1,n_1} & 0_{m_1,n_2} & I_{m_1} & 0_{m_1,m_2} \end{bmatrix}^T.$$

The regularity of $s\mathcal{E}_{22} - \mathcal{A}_{22}$ moreover implies that an accordantly partitioned

$$V_{Fr} = [\, V_{\mu 1}^T\, , \, V_{\mu 2}^T\, , \, V_{x1}^T\, , \, V_{x2}^T\, , \, V_{u1}^T\, , \, V_{u2}^T\, ,]^T \in \mathbb{C}^{2n+m,n_1}$$

satisfies $V_{\mu 1} = 0$ and, further, its submatrix

$$V_{Fr2} = [\, V_{\mu 2}^T\, , \, V_{x2}^T\, , \, V_{u2}^T\, ,]^T \in \mathbb{C}^{2n_1+m_1,n_1}$$

fulfills $(s\mathcal{E}_{22} - \mathcal{A}_{22})V_{Fr2} = W_{Fr2}(sI_k - \widehat{A})$ for some suitable W_{Fr2}. We get from Lemma A.5 that $\ker V_{x2} \subset \ker V_{\mu 2}$. This implies that for

$$V_F = [\, V_{Fr}\, , \, V_{FW3}\,] = \begin{bmatrix} 0 & 0 & 0 \\ V_{\mu 2} & 0 & 0 \\ V_{x1} & I_{n_1} & 0 \\ V_{x2} & 0 & 0 \\ V_{u1} & 0 & I_{m_1} \\ V_{u2} & 0 & 0 \end{bmatrix}$$

holds

$$\ker \begin{bmatrix} V_{x1} & I_{n_1} & 0 \\ V_{x2} & 0 & 0 \end{bmatrix} \subset \ker V_{x2} \times \mathbb{C}^{n_1} \times \mathbb{C}^{m_1} \subset \ker V_{\mu 2} \times \mathbb{C}^{n_1} \times \mathbb{C}^{m_1} = \ker \begin{bmatrix} 0 & 0 & 0 \\ V_{\mu 2} & 0 & 0 \end{bmatrix}.$$

On the other hand, due to $V_F = T_F^{-1}V$, we have

$$T_x^{-1}V_x = \begin{bmatrix} V_{x1} & I_{n_1} & 0 \\ V_{x2} & 0 & 0 \end{bmatrix}, \quad V_\mu = T_x^* \begin{bmatrix} 0 & 0 & 0 \\ V_{\mu 2} & 0 & 0 \end{bmatrix}$$

and thus

$$\ker V_x = \ker \begin{bmatrix} V_{x1} & I_{n_1} & 0 \\ V_{x2} & 0 & 0 \end{bmatrix} \subset \ker \begin{bmatrix} 0 & 0 & 0 \\ V_{\mu 2} & 0 & 0 \end{bmatrix} = \ker V_\mu.$$

This in particular means that the matrix $V_{\mu x} = [\, V_\mu^T \,,\, V_x^T \,]^T \in \mathbb{C}^{2n,n+m}$ fulfills $\ker V_{\mu x} = \ker V_x$ and thus $\operatorname{rank} V_{\mu x} = \operatorname{rank} V_x$. On the other hand, since $\operatorname{im} V$ is generalized Lagrangian, we have $\operatorname{rank} V_{\mu x} = n$. Thus, the rank of V_x has to be n. ◻

Proof of Theorem 4.4. Partition $V = [\, V_\mu \,,\, V_x \,,\, V_u \,]$ for some $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$ and let the Hermitian matrix $Y \in \mathbb{C}^{n,n}$ be a solution of the LMI (3.2). Consider the Lur'e equations (A.1) with associated even pencil $s\mathcal{E}_Y - \mathcal{A}_Y$. By Lemma A.1, we know that

$$(s\mathcal{E}_Y - \mathcal{A}_Y)V_Y = (T_Y^* W)(s\widetilde{E} - \widetilde{A})$$

with T_Y as in (A.3) and $V_Y = [\, (V_\mu - YV_x)^T \,,\, V_x^T \,,\, V_u^T \,]^T$ and, moreover, (A.4) holds true. Lemma B.1 now implies that $\operatorname{rank} V_x = n$. ◻

Appendix C. Proof of Theorem 5.1 and Theorem 5.2. Due to a total analogy, we show the results for Theorem 5.1 and Theorem 5.2 at once. Note that the respective assertions for the minimal solutions are located in the parentheses.

Proof of Theorem 5.1 (and Theorem 5.2). Let the Hermitian matrix $Y \in \mathbb{C}^{n,n}$ be a solution of the LMI (3.2). Consider the Lur'e equations (A.1). By Lemma A.1, we know that the solutions of (1.1) and (A.1) are related by

$$(X_Y, K_Y, L_Y) = (X - Y, K, L).$$

Let $V = [\, V_\mu^T \,,\, V_x^T \,,\, V_u^T]^T$ with $V_\mu, V_x \in \mathbb{C}^{n,n+m}$, $V_u \in \mathbb{C}^{m,n+m}$ be a generalized Lagrangian and disconjugate deflating subspace such that for some $W \in \mathbb{C}^{2n+m,p}$ and an $n+p \times n+m$ pencil $s\widetilde{E} - \widetilde{A}$ whose generalized eigenvalues have non-positive (non-negative) real part holds $(s\mathcal{E} - \mathcal{A})V = W(s\widetilde{E} - \widetilde{A})$. Lemma A.1 implies that the even pencil $s\mathcal{E}_Y - \mathcal{A}_Y$ associated to the Lur'e equations (A.1) fulfills

$$(s\mathcal{E}_Y - \mathcal{A}_Y)V_Y = (T_Y^* W)(s\widetilde{E} - \widetilde{A})$$

with T_Y as in (A.3) and $V_Y = [\, (V_\mu - YV_x)^T \,,\, V_x^T \,,\, V_u^T \,]^T$. Since (A.4) is fulfilled, we can apply statement a) (b)) of Lemma A.3 to the Lur'e equations (A.1) in order to obtain that

$$0 \le V_x^*(V_\mu - YV_x) = V_x^* V_\mu - V_x^* Y V_x, \quad (0 \le V_x^*(V_\mu - YV_x) = V_x^* V_\mu - V_x^* Y V_x).$$

A multiplication from the left with $(V_x^-)^*$ and from the right with V_x^- the yields

$$\underset{(\ge)}{0 \le} (V_x^-)^* V_x^* V_\mu V_x^- - (V_x^-)^* V_x^* Y V_x V_x^- = X_+ - Y.$$

◻

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