AN ADAPTIVE FINITE ELEMENT EIGENVALUE SOLVER OF QUASI-OPTIMAL COMPUTATIONAL COMPLEXITY*

CARSTEN CARSTENSEN AND JOSCHA GEDICKE

Abstract. This paper presents a combined adaptive finite element method with an iterative algebraic eigenvalue solver for the Laplace eigenvalue problem of quasi-optimal computational complexity. The analysis is based on a direct approach for eigenvalue problems and allows the use of higher order conforming finite element spaces with fixed polynomial degree $k > 0$. The optimal adaptive finite element eigenvalue solver (AFEMES) involves a proper termination criterion for the algebraic eigenvalue solver and does not need any coarsening. Numerical evidence illustrates the optimal computational complexity.

August 30, 2009

1. Introduction

The eigenvalue problems for second order elliptic boundary value problems involve the discretisation error of some adaptive finite element method as well as the error left from some iterative solver for the algebraic eigenvalue problem. This paper presents the first adaptive finite element eigenvalue solver (AFEMES) of overall quasi-optimal complexity shown in the pseudocode below.

AFEMES

```
Input: Coarse triangulation T_0, initial guess (μ_0, w_0),
       parameter 0 < θ ≤ 1, 0 < ω.
δ_0 := 2√ω η_0(μ_0, w_0);
for ℓ = 0,1,...
  while ( δ_ℓ > √ω η_ℓ(μ_ℓ, w_ℓ) )
    δ_ℓ := δ_ℓ/2;
    [μ_ℓ, w_ℓ] := LAES(T_ℓ, μ_ℓ, w_ℓ, δ_ℓ);
  end
  T_{ℓ+1} := Refine(T_ℓ,Mark(T_ℓ, θ, η_ℓ(μ_ℓ, w_ℓ)));
  δ_{ℓ+1} := 2√ω η_ℓ(μ_ℓ, w_ℓ); μ_{ℓ+1} := μ_ℓ; w_{ℓ+1} := w_ℓ;
end
Output: Sequence of triangulations T_ℓ and
        inexact discrete eigenpairs (μ_ℓ, w_ℓ).
```

2000 *Mathematics Subject Classification.* 65N15, 65N25, 65N30.

Key words and phrases. eigenvalue, adaptive finite element method, convergence rates, complexity.

*Supported by the DFG Research Center MATHEON "Mathematics for key technologies", the WCU program through KOSEF (R31-2008-000-10049-0), and the DFG graduate school BMS "Berlin mathematical school" in Berlin.

The adaptive mesh refinement via subroutines `Mark` and `Refine` is well-established in the finite element community [Dör96, BDD04, Ste07, BS08, CKNS08] while `LAES` represents any state-of-the-art iterative eigensolver well-established in the numerical linear algebra community. The pseudocode gives one possible error balance of the two error sources of quasi-optimal complexity. The works on convergence [GG09, Sau07, GMZ08, CG08] as well as on quasi-optimal convergence [DXZ08] of adaptive mesh refinement for the eigenvalue problem do assume unrealistically the exact knowledge of algebraic eigenpairs. While a second optimality result for linear symmetric operator eigenvalue problems [DRSZ08] is based on coarsening. Assuming a saturation assumption, [Ney02, MM09] present combined adaptive finite element and linear algebra algorithms.

As a simple model problem for a symmetric, elliptic eigenvalue problem consider the following eigenvalue problem of the Laplace operator: Seek a non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times H^1_0(\Omega;\mathbb{R}) \cap H^2_{loc}(\Omega;\mathbb{R})$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

for an open bounded Lipschitz domain $\emptyset \neq \Omega \subset \mathbb{R}^n$, $n = 2, 3$. It is well known, that problem (1.1) has countable infinite many solutions with positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

For simplicity this paper restricts to the case, that the eigenvalue of interest λ is a simple eigenvalue, hence its algebraic and geometric multiplicity equals one.

The weak problem seeks for a non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times V := \mathbb{R} \times H^1_0(\Omega;\mathbb{R})$ with $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V.$$

The bilinear forms $a(.,.)$ and $b(.,.)$ are defined by

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \text{and} \qquad b(u, v) := \int_\Omega uv \, dx$$

and induce the norms $|||.||| := |.|_{H^1(\Omega)}$ on V and $\|.\| := \|.\|_{L^2(\Omega)}$ on $L^2(\Omega)$. The conforming finite element space of order $k \in \mathbb{N}$ for the triangulation $\mathcal{T}_\ell$ is defined by

$$\mathcal{P}_k(\mathcal{T}_\ell) := \left\{ v \in H^1(\Omega) : \forall T \in \mathcal{T}_\ell,\, v_T \text{ is polynomial of degree} \leq k \right\}$$

Let $V_\ell := \mathcal{P}_k(\mathcal{T}_\ell) \cap V$ denote the finite-dimensional subspace of fixed order $k > 0$. The corresponding discrete eigenvalue problem reads: Seek a non-trivial eigenpair $(\lambda_\ell, u_\ell) \in \mathbb{R} \times V_\ell$ with $b(u_\ell, u_\ell) = 1$ and

$$a(u_\ell, v_\ell) = \lambda b(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell.$$

In this paper the quasi-optimal convergence for the model problem (1.1) is first shown for exact algebraic eigenvalue solutions without using the inner node property: Suppose that (λ_ℓ, u_ℓ) is a discrete eigenpair to the continuous eigenpair (λ, u). Let $u \in \mathcal{A}_s$, $\mathcal{A}_s$ denotes an approximation space, and $(\mathcal{T}_\ell)_\ell$

be a sequence of nested regular triangulations. Then (λ_ℓ, u_ℓ) converges quasi-optimal,

$$|||u - u_\ell|||^2 + |\lambda - \lambda_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}.$$

Throughout this paper, standard notations on Sobolev and Lebesgue spaces are used. The notation $a \lesssim b$ abbreviates the inequality $a \leq Cb$ with a constant $C > 0$ which does not depend on the mesh-size. Finally, $|\mathcal{T}_\ell|$ denotes the cardinality of $\mathcal{T}_\ell$. In contrast to [DXZ08] the proofs are based on the eigenvalue formulation and not on a relation to its corresponding source problem. Hence, no additional oscillations arising from the corresponding source problem have to be treated. In a second step this result is extended to the case of inexact algebraic eigenvalue solutions: Suppose (λ, u) with $u \in \mathcal{A}_s$ is an exact eigenpair and (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ corresponding discrete eigenpairs on level ℓ and $\ell + 1$. Let the iterative approximations (μ_ℓ, w_ℓ) on $\mathcal{T}_\ell$ and $(\mu_{\ell+1}, w_{\ell+1})$ on $\mathcal{T}_{\ell+1}$ satisfy

$$\begin{aligned} |||u_{\ell+1} - w_{\ell+1}|||^2 + |\lambda_{\ell+1} - \mu_{\ell+1}| &\leq \omega\eta_\ell^2(\mu_\ell, w_\ell), \\ |||u_\ell - w_\ell|||^2 + |\lambda_\ell - \mu_\ell| &\leq \omega\eta_\ell^2(\mu_\ell, w_\ell), \end{aligned}$$

for sufficiently small $\omega > 0$. Then, the iterative solutions μ_ℓ and w_ℓ converge optimally up to positive constants,

$$|||u - w_\ell|||^2 + |\lambda - \mu_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}.$$

Finally, this result leads to an optimal combined adaptive finite element and numerical linear algebra algorithm (AFEMES) which does not need any coarsening or saturation assumption.

The outline of this paper is as follows. Section 2 concerns the basic structure of the standard adaptive finite element method (AFEM) for eigenvalue problems. Section 3 presents some algebraic and analytic properties for the model problem (1.1). The discrete reliability of a residual type error estimator is shown in Section 4 together with the standard reliability and efficiency. In Section 5 a contraction property for the quasi-error up to higher-order terms leads to quasi-optimal convergence for the AFEM under the usual assumption that the mesh-size is sufficiently small and that the algebraic sub-problems are solved exactly. Relaxing this last assumption in Section 6, the results for quasi-optimal convergence are extended to the case of inexact algebraic eigenvalue solutions. These relaxed results are in Section 7 combined with an quasi-optimal iterative eigenvalue solver and thus lead to the combined AFEM and iterative algebraic eigenvalue solver AFEMES with quasi-optimal computational complexity. The numerical experiments of Section 8 show empirical optimal convergence rates for the case of exact algebraic eigenvalue solvers and higher-order finite element methods. An implementation of the proposed optimal AFEMES for linear finite elements shows numerically quasi-optimal complexity for sufficiently small bulk parameters.

2. Adaptive Finite Element Eigenvalue Solver

The adaptive finite element method computes a sequence of discrete subspaces

$$V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_\ell \subset V$$

using local refinement of the underlying mesh of the domain Ω. The corresponding sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ consists of nested regular triangulations. The AFEM consists of the following loop

$$\text{Solve} \to \text{Estimate} \to \text{Mark} \to \text{Refine}.$$

Solve. Given a mesh $\mathcal{T}_\ell$ on level ℓ the step Solve computes the stiffness matrix A_ℓ and the mass matrix B_ℓ and solves the finite-dimensional generalised algebraic eigenvalue problem

$$A_\ell x_\ell = \lambda_\ell B_\ell x_\ell$$

with

$$u_\ell = \sum_{k=1}^{\dim(V_\ell)} x_k \varphi_k, \qquad V_\ell = \operatorname{span}\{\varphi_1, \ldots, \varphi_{\dim(V_\ell)}\}.$$

Practically, these discrete eigenvalue problems are solved inexact using iterative algebraic eigenvalue solvers. In this paper the linear algebraic eigenvalue solver (LAES), used as a 'black box' iterative solver in the optimal algorithm AFEMES, is assumed to be any iterative eigenvalue solver of optimal complexity. E.g., for any given tolerance $\varepsilon > 0$, the LAES computes some eigenpair $(\mu_{\ell,m}, w_{\ell,m})$ of the generalised algebraic eigenvalue problem from a close enough initial guess $(\mu_{\ell,0}, w_{\ell,0})$ such that

$$|||u_\ell - w_{\ell,m}|||^2 + |\lambda_\ell - \mu_{\ell,m}| \leq \varepsilon^2$$

in at most, up to a multiplicative constant,

$$\max\left\{1, \log(\varepsilon^{-1} |||u_\ell - w_{\ell,0}|||)\right\} |\mathcal{T}_\ell|$$

arithmetic operations. For example, the preconditioned inverse iteration (PINVIT), for computing the smallest eigenpair, is known to be of optimal complexity, as soon as the preconditioner is close enough to the exact inverse [KN09].

Estimate. The error in the eigenfunction or eigenvalue of interest is estimated based on the solution (λ_ℓ, u_ℓ) of the underlying algebraic eigenvalue problem

$$\eta_\ell^2(\lambda_\ell, u_\ell) := \sum_{T \in \mathcal{T}_\ell} \eta_\ell(\lambda_\ell, u_\ell; T)^2 + \sum_{E \in \mathcal{E}_\ell} \eta_\ell(\lambda_\ell, u_\ell; E)^2.$$

Mark. Based on the refinement indicators, edges and elements are marked for refinement in a bulk criterion [Dör96] such that $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ is an (almost) minimal set of marked edges with

$$\theta \eta_\ell^2(\lambda_\ell, u_\ell) \leq \eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\ell),$$
$$\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\ell) := \sum_{T \in \mathcal{M}_\ell \cap \mathcal{T}_\ell} \eta_\ell^2(\lambda_\ell, u_\ell; T) + \sum_{E \in \mathcal{M}_\ell \cap \mathcal{E}_\ell} \eta_\ell^2(\lambda_\ell, u_\ell; E)$$

for a bulk parameter $0 < \theta \leq 1$. This is done in a greedy algorithm which marks edges and elements with larger contributions.

Refine. In this step of the AFEM loop, the mesh is refined locally corresponding to the set $\mathcal{M}_\ell$ of marked edges and elements. Once an element is selected for refinement, all of it's edges will be refined. In order to preserve the quality of the mesh, i.e., the maximal angel condition or its equivalents, additionally edges have to be marked by the closure algorithm before refinement. For each triangle let one edge be the uniquely defined reference edge $E(T)$. The closure algorithm computes a superset $\overline{\mathcal{M}_\ell} \supset \mathcal{M}_\ell$ such that

$$\big\{E(T) : T \in \mathcal{T}_\ell \quad \text{with} \quad \mathcal{E}_\ell(T) \cap \overline{\mathcal{M}}_\ell \neq \emptyset \quad \text{or} \quad T \cap \overline{\mathcal{M}}_\ell \neq \emptyset\big\} \subseteq \overline{\mathcal{M}}_\ell.$$

In other words, once a edge of a triangle or itself is marked for refinement, its reference edge $E(T)$ is among them. A similar refinement algorithm for $n = 3$ based on bisection and the concept of reference edges can be found in [AMP00].

Proposition 2.1 (Boundedness of Closure, [BDD04, Ste08])**.** *Let $\mathcal{T}_{\ell+1}$ be a refinement of $\mathcal{T}_\ell$, obtained using the refinement algorithm and closure. Suppose $\mathcal{T}_0$ is the initial coarse triangulation, then it holds*

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{L-1} |\mathcal{M}_\ell|. \quad \square$$

After the closure algorithm is applied one of the following refinement rules is applicable, namely no refinement, green refinement, blue left or blue right refinement and bisec3 refinement depicted in Figure 2.1.

Proposition 2.2 (Overlay, [Ste07, CKNS08])**.** *For the smallest common refinement $\mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell$ of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\ell$ it holds*

$$|\mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell| - |\mathcal{T}_\ell| \leq |\mathcal{T}_\varepsilon| - |\mathcal{T}_0|. \quad \square$$

3. Algebraic Properties

This section summarises some known and some new algebraic properties of the model problem (1.1), such as the relation between the eigenvalue error

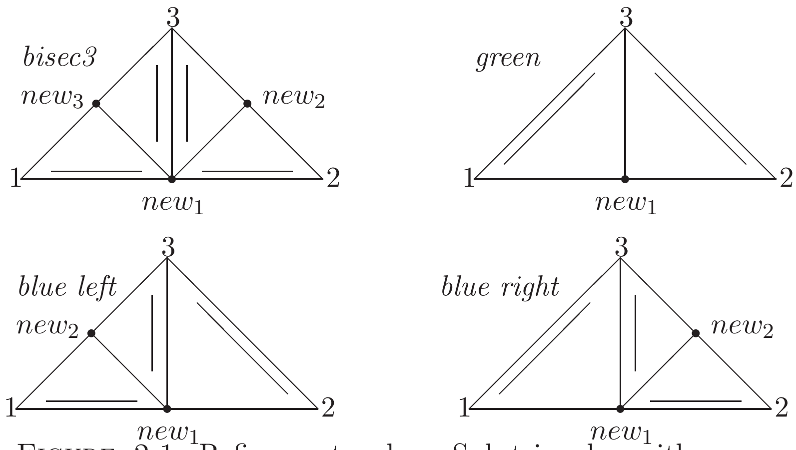

Figure 2.1. Refinement rules: Sub-triangles with corresponding reference edges depicted with a second edge.

and the error with respect to the norms $|||.|||$ and $\|.\|$ [SF73]

$$|||u-u_\ell|||^2 = \lambda\|u-u_\ell\|^2 + \lambda_\ell - \lambda. \tag{3.1}$$

Suppose that $(\lambda, u_\ell) \in \mathbb{R}\times V_\ell$ and $(\lambda_{\ell+m}, u_{\ell+m}) \in \mathbb{R}\times V_{\ell+m}$ are discrete eigenpairs approximating the continuous eigenpair $(\lambda, u) \in \mathbb{R}\times V$ on the levels ℓ and $\ell+m$.

Lemma 3.1 (Quasi-orthogonality)**.** *Let $\mathcal{T}_{\ell+m}$ be a refinement of the triangulation $\mathcal{T}_\ell$ for some level ℓ such that $V_\ell \subset V_{\ell+m}$ or even in the limit $V_{\ell+m} = V$ such that $u_{\ell+m} = u$ and $\lambda_{\ell+m} = \lambda$. Then, for $e_\ell := u - u_\ell$ and $e_{\ell+m} := u - u_{\ell+m}$, the quasi-orthogonality holds*

$$\begin{aligned} |||u_{\ell+m}-u_\ell|||^2 &= |||e_\ell|||^2 - |||e_{\ell+m}|||^2 - \lambda\|e_\ell\|^2 \\ &\qquad + \lambda\|e_{\ell+m}\|^2 + \lambda_{\ell+m}\|u_{\ell+m}-u_\ell\|^2. \end{aligned}$$

Proof. Since $\mathcal{T}_{\ell+m}$ is a refinement of $\mathcal{T}_\ell$, (3.1) implies

$$|||u_{\ell+m}-u_\ell|||^2 = \lambda_{\ell+m}\|u_{\ell+m}-u_\ell\|^2 + \lambda_\ell - \lambda_{\ell+m}.$$

Hence,

$$\begin{aligned} |||u_{\ell+m}-u_\ell|||^2 &= \lambda_{\ell+m}\|u_{\ell+m}-u_\ell\|^2 + \lambda_\ell - \lambda - (\lambda_{\ell+m}-\lambda) \\ &= |||e_\ell|||^2 - |||e_{\ell+m}|||^2 - \lambda\|e_\ell\|^2 + \lambda\|e_{\ell+m}\|^2 \\ &\qquad + \lambda_{\ell+m}\|u_{\ell+m}-u_\ell\|^2. \qquad \square \end{aligned}$$

Let the residual $\mathrm{Res}_\ell \in V^*$ be defined by

$$\mathrm{Res}_\ell(v) := \lambda_\ell b(u_\ell, v) - a(u_\ell, v) \quad \text{for all } v \in V.$$

Notice that $V_\ell \subset \ker(\mathrm{Res}_\ell)$.

Lemma 3.2. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$, then it holds*

$$|||u_{\ell+m}-u_\ell||| \le |||Res_\ell|||_{V^*_{\ell+m}} + \frac{(\lambda_{\ell+m}+\lambda_\ell)}{2}\frac{\|u_{\ell+m}-u_\ell\|^2}{|||u_{\ell+m}-u_\ell|||}.$$

Proof. Elementary algebraic manipulations, together with the assumption that $V_\ell \subset V_{\ell+m}$, show

$$\begin{aligned} |||u_{\ell+m}-u_\ell|||^2 &= \lambda_\ell b(u_\ell, u_{\ell+m}-u_\ell) - a(u_\ell, u_{\ell+m}-u_\ell) \\ &\qquad + a(u_{\ell+m}, u_{\ell+m}-u_\ell) - \lambda_\ell b(u_\ell, u_{\ell+m}-u_\ell) \\ &= \mathrm{Res}_\ell(u_{\ell+m}-u_\ell) + (\lambda_{\ell+m}+\lambda_\ell)(1-b(u_{\ell+m}, u_\ell)) \\ &\le |||\mathrm{Res}_\ell|||_{V^*_{\ell+m}} |||u_{\ell+m}-u_\ell||| + \frac{(\lambda_{\ell+m}+\lambda_\ell)}{2}\|u_{\ell+m}-u_\ell\|^2. \qquad \square \end{aligned}$$

The remaining part of this section is devoted to show that the second term on the right hand side in Lemma 3.2 is of higher-order, namely

$$\|u_{\ell+m}-u_\ell\| \lesssim \|h_\ell\|^r_{L^\infty(\Omega)} |||u_{\ell+m}-u_\ell|||.$$

Here and throughout this paper, $h_\ell \in \mathcal{P}_0(\mathcal{T}_\ell)$ is the piecewise constant mesh-size function with $h_\ell|_T := \mathrm{diam}(T)$ for $T \in \mathcal{T}_\ell$ and $0 < r \le 1$ depends on the regularity of the solution of the corresponding boundary value problem. The first part follows the argumentation as in [SF73] for the case $u_{\ell+m} = u$. The second part exploits regularity of the corresponding boundary value problem together with the Aubin-Nitsche technique. Let $G_\ell^{\ell+m} : V_{\ell+m} \to V_\ell$ denote the Galerkin projection onto V_ℓ such that

$$a(u_{\ell+m} - G_\ell^{\ell+m} u_{\ell+m}, v_\ell) = 0 \quad \text{for all } v_\ell \in V_\ell.$$

Note that the case $V = V_{\ell+m}$ with Galerkin projection $G_\ell : V \to V_\ell$ is included. Suppose the k-th eigenvalue $\lambda = \lambda_k$ is simple. Let the initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$ be sufficiently small such that there exist two separation constants ρ and $\rho_{\ell+m}$, which satisfy

$$\frac{\lambda_k}{\lambda_{\ell,j} - \lambda_k} \leq \rho \quad \text{for all } j \neq k, 1 \leq j \leq \dim(V_\ell),$$
$$\frac{\lambda_{\ell+m,k}}{\lambda_{\ell,j} - \lambda_{\ell+m,k}} \leq \rho_{\ell+m} \quad \text{for all } j \neq k, 1 \leq j \leq \dim(V_\ell).$$

In the case of a multiple eigenvalue λ_k there still exists such separation constants for sufficiently small $\|h_0\|_{L^\infty(\Omega)}$.

Lemma 3.3. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m}$, then for the Galerkin projections $G_\ell^{\ell+m} : V_{\ell+m} \to V_\ell$ and $G_\ell : V \to V_\ell$ it holds*

$$\|u_{\ell+m} - u_\ell\| \leq 2(1 + \rho_{\ell+m})\|u_{\ell+m} - G_\ell^{\ell+m} u_{\ell+m}\|,$$
$$\|u - u_\ell\| \leq 2(1 + \rho)\|u - G_\ell u\|.$$

Proof. Note that for the Galerkin projection it holds

$$(\lambda_{\ell,j} - \lambda_{\ell+m,k}) b(G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,j}) = \lambda_{\ell+m,k} b(u_{\ell+m,k} - G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,j}).$$

Since $u_{\ell,1}, \ldots, u_{\ell,N_\ell}$, for $N_\ell = \dim V_\ell$, forms an orthogonal basis for V_ℓ, the Galerkin projection of $u_{\ell+m,k}$ can be written as

$$G_\ell^{\ell+m} u_{\ell+m,k} = \sum_{j=1}^{N_\ell} b(G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,j}) u_{\ell,j}.$$

Let $\beta := b(G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,k})$ be the coefficient for $j = k$ in the previous formula. Because of the orthogonality of the discrete eigenfunctions $u_{\ell,1}, \ldots, u_{\ell,N_\ell}$, it holds

$$\begin{aligned}
\|G_\ell^{\ell+m} u_{\ell+m,k} - \beta u_{\ell,k}\|^2 &= \sum_{\substack{j=1\\ j\neq k}}^{N_\ell} b(G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,j})^2 \\
&= \sum_{\substack{j=1\\ j\neq k}}^{N_\ell} \left(\frac{\lambda_{\ell+m,k}}{\lambda_{\ell,j} - \lambda_{\ell+m,k}}\right)^2 b(u_{\ell+m,k} - G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,j})^2 \\
&\leq \rho_{\ell+m}^2 \sum_{\substack{j=1\\ j\neq k}}^{N_\ell} b(u_{\ell+m,k} - G_\ell^{\ell+m} u_{\ell+m,k}, u_{\ell,j})^2 \\
&\leq \rho_{\ell+m}^2 \|u_{\ell+m,k} - G_\ell^{\ell+m} u_{\ell+m,k}\|^2.
\end{aligned}$$

The triangle inequality shows that

$$\|u_{\ell+m,k}\| - \|u_{\ell+m,k} - \beta u_{\ell,k}\| \leq \|\beta u_{\ell,k}\| \leq \|u_{\ell+m,k}\| + \|u_{\ell+m,k} - \beta u_{\ell,k}\|.$$

Since the eigenfunctions are normalized to one this implies

$$|\beta - 1| \leq \|u_{\ell+m,k} - \beta u_{\ell,k}\|.$$

Hence,

$$\begin{aligned}\|u_{\ell+m,k} - u_{\ell,k}\| &\leq \|u_{\ell+m,k} - \beta u_{\ell,k}\| + \|(\beta-1)u_{\ell,k}\| \\ &\leq 2\|u_{\ell+m,k} - \beta u_{\ell,k}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_{\ell+m,k} - u_{\ell,k}\| &\leq 2\|u_{\ell+m,k} - G_\ell^{\ell+m} u_{\ell+m,k}\| + 2\|G_\ell^{\ell+m} u_{\ell+m,k} - \beta u_{\ell,k}\| \\ &\leq 2(1+\rho_{\ell+m})\|u_{\ell+m,k} - G_\ell^{\ell+m} u_{\ell+m,k}\|.\end{aligned}$$

The second inequality follows analogously since $V_\ell \subset V$. □

Remark 3.4. Lemma 3.3 implies

$$\begin{aligned}\|e_{\ell+m}\| &\leq 2(1+\rho)\|u - G_{\ell+m} u\| \\ &\leq 2(1+\rho)\|u - G_\ell u\| \leq 2(1+\rho)\|e_\ell\|.\end{aligned}$$

With (3.1) and the min-max principle, it follows

$$|||e_{\ell+m}||| \lesssim |||e_\ell||| \quad \text{and} \quad \lambda \leq \lambda_{\ell+m} \leq \lambda_\ell.$$

Lemma 3.5. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$. Suppose the corresponding boundary value problem to* (1.1) *is H^{1+r}-regular, i.e., $u_{\ell+m} \in H^{1+r}(\Omega) \cap V$. Then it holds*

$$\|u_{\ell+m} - u_\ell\| \leq \sigma(h_\ell)|||u_{\ell+m} - u_\ell|||$$

with

$$\sigma(h_\ell) := 2(1+\rho_{\ell+m})C_{reg}\|h_\ell\|^r_{L^\infty(\Omega)}\|\lambda u\|_{H^{1+r}(\Omega)}$$

for some $0 < r \leq 1$.

Proof. Since $u_{\ell+m} \in H^{1+r}(\Omega) \cap V$, the following convergence estimate holds for the Galerkin projection $G_\ell^{\ell+m}$ and any solution $z_{\ell+m}$ of the corresponding dual boundary value problem

$$\|z_{\ell+m} - G_\ell^{\ell+m} z_{\ell+m}\|_{H^1(\Omega)} \leq C_{\mathrm{reg}}\|h_\ell\|^r_{L^\infty(\Omega)}\|\lambda u\|_{H^{1+r}(\Omega)}$$

for some $0 < r \leq 1$ [BS08, Theorem 14.3.3]. Under the assumption, that the problem is H^{1+r}-regular, the Aubin-Nitzsche duality technique leads to

$$\begin{aligned}\|u_{\ell+m} - G_\ell^{\ell+m} u_{\ell+m}\| &\leq C_{\mathrm{reg}}\|h_\ell\|^r_{L^\infty(\Omega)}\|\lambda u\|_{H^{1+r}(\Omega)}|||u_{\ell+m} - G_\ell^{\ell+m} u_{\ell+m}||| \\ &\leq C_{\mathrm{reg}}\|h_\ell\|^r_{L^\infty(\Omega)}\|\lambda u\|_{H^{1+r}(\Omega)}|||u_{\ell+m} - u_\ell|||.\end{aligned}$$

Lemma 3.3 finishes the proof for both cases $V_{\ell+m} \subset V$ and $V_{\ell+m} = V$. □

Lemma 3.6. *Let $\mathcal{T}_{\ell+m}$ be a refinement of $\mathcal{T}_\ell$ such that $V_\ell \subset V_{\ell+m} \subseteq V$. For sufficiently small initial mesh-size $\|h_0\|_{L^\infty(\Omega)}$ there exists a constant $C_0 > 0$ depending only on $\mathcal{T}_0$ such that $1 \leq \kappa(h_\ell) < C_0$ with*

$$|||u_{\ell+m} - u_\ell||| \leq \kappa(h_\ell)|||Res_\ell|||_{V^*_{\ell+m}} \quad \textit{and} \quad \lim_{\|h_\ell\|_{L^\infty(\Omega)} \to 0} \kappa(h_\ell) = 1.$$

Proof. Suppose that $\|h_\ell\|_{L^\infty(\Omega)}$ is sufficiently small such that

$$\delta_\ell := 2C_{\mathrm{reg}}^2 \|\lambda u\|^2_{H^{1+r}(\Omega)}(\lambda_{\ell+m} + \lambda_\ell)(1+\rho_{\ell+m})^2 \|h_\ell\|^{2r}_{L^\infty(\Omega)} \ll 1.$$

Then Lemma 3.2, Lemma 3.3 together with Lemma 3.5 lead to

$$|\!|\!| u_{\ell+m} - u_\ell |\!|\!| \leq (1-\delta_\ell)^{-1} |\!|\!| \mathrm{Res}_\ell |\!|\!|_{V^*_{\ell+m}}.$$

Notice that $\kappa(h_\ell) := (1-\delta_\ell)^{-1} \to 1$ as the maximal mesh-size tends to zero and $C_0 := (1-\delta_0)^{-1}$. □

4. A Posteriori Error Estimator

This section establishes the discrete reliability and recalls the reliability and efficiency of the standard residual-based error estimator [DPR03, DXZ08, GG09, GMZ08]. Let $p_\ell := \nabla u_\ell$ denote the discrete gradient and $\mathcal{E}_\ell$ the set of inner edges of $\mathcal{T}_\ell$. For $E \in \mathcal{E}_\ell$ let $T_+, T_- \in \mathcal{T}_\ell$ be the two neighbouring triangles such that $E = T_+ \cap T_-$. The jump of the discrete gradient p_ℓ along an inner edge $E \in \mathcal{E}_\ell$ in normal direction ν_E, pointing from T_+ to T_-, is defined by $[p_\ell]\cdot\nu_E := \big(p_\ell|_{T_+} - p_\ell|_{T_-}\big)\cdot \nu_E$. Then the residual error estimator is defined by

$$\eta^2_\ell(\lambda_\ell, u_\ell) := \sum_{T\in\mathcal{T}_\ell} \eta_\ell(\lambda_\ell, u_\ell; T)^2 + \sum_{E\in\mathcal{E}_\ell} \eta_\ell(\lambda_\ell, u_\ell; E)^2$$

with $n = 2, 3$ and

$$\eta_\ell(\lambda_\ell, u_\ell; T)^2 := |T|^{2/n} \|\lambda_\ell u_\ell + \mathrm{div}(p_\ell)\|^2_{L^2(T)},$$
$$\eta_\ell(\lambda_\ell, u_\ell; E)^2 := |E|^{1/(n-1)} \|[p_\ell]\cdot\nu_E\|^2_{L^2(E)}.$$

Note that the Scott-Zhang quasi-interpolation operator $J : V \to V_\ell$ is a projection $J(v_\ell) = v_\ell$ for all $v_\ell \in V_\ell$. In addition, it is locally a L^2-projection onto (n-1)-dimensional edges or faces. Therefore each node is assigned any edge or face which contains it. Edge-basis functions are interpolated on their edge and element-basis functions are interpolated over the interior of their element. The element and edge patches Ω_T and Ω_E are displayed in Figure 4.1. In the following, the Scott-Zhang quasi-interpolation operator is restricted to $V_{\ell+m}$ for a refined triangulation $\mathcal{T}_{\ell+m}$ of $\mathcal{T}_\ell$. If it is possible, each nodal-basis function is assigned an edge of the boundary or an edge which is not refined. Thus the homogeneous boundary values are preserved. Let v_ℓ denote the Scott-Zhang interpolant of $v_{\ell+m}$ in V_ℓ. Then for all elements $T \in \mathcal{T}_\ell$ and all edges $E \in \mathcal{E}_\ell$ that are not refined it holds $v_{\ell+m}|_T = v_\ell|_T$ and $v_{\ell+m}|_E = v_\ell|_E$. Since the finite overlay of the patches Ω_T and Ω_E the approximation property holds

$$\sum_{T\in\mathcal{T}_\ell} |T|^{-1/n} \|v_{\ell+m} - v_\ell\|_{L^2(T)} + \sum_{E\in\mathcal{E}_\ell} |E|^{-1/(2n-2)} \|v_{\ell+m} - v_\ell\|_{L^2(E)} \lesssim |\!|\!| v_{\ell+m} |\!|\!|.$$

Lemma 4.1 (Discrete Reliability)**.** *For sufficiently small $\|h_0\|_{L^\infty(\Omega)}$ let (λ_ℓ, u_ℓ) be a discrete eigenpair on level ℓ and $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell$ be any set of edges and elements. Suppose the refinement algorithm of Section 2 computes the refined mesh $\mathcal{T}_{\ell+m}$, then it holds*

$$|\!|\!| Res_\ell |\!|\!|_{V^*_{\ell+m}} \lesssim \eta_\ell(\lambda_\ell, u_\ell; \mathcal{M}_\ell).$$

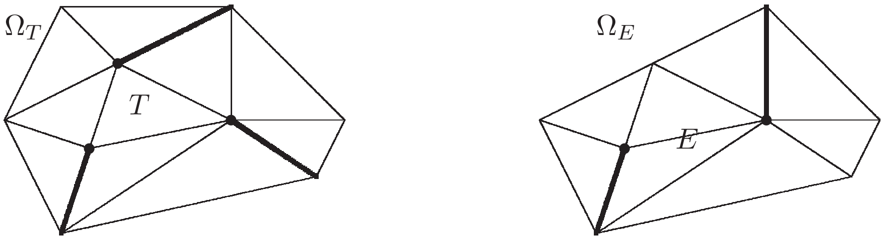

FIGURE 4.1. Patches for the Scott-Zhang interpolation operator.

Proof. Let v_ℓ denote the Scott-Zhang interpolant of $v_{\ell+m} \in V_{\ell+m}$ in V_ℓ. For all common elements $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+m}$ and all common edges $E \in \mathcal{E}_\ell \cap \mathcal{E}_{\ell+m}$ it holds $v_\ell|_T = v_{\ell+m}|_T$ and $v_\ell|_E = v_{\ell+m}|_E$. Hence,

$$\begin{aligned}
&\mathrm{Res}_\ell(v_{\ell+m}) = \mathrm{Res}_\ell(v_{\ell+m} - v_\ell) = \lambda_\ell b(u_\ell, v_{\ell+m} - v_\ell) - a(u_\ell, v_{\ell+m} - v_\ell)\\
&\lesssim \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+m}} |T|^{1/n} \|\mathrm{div}(p_\ell) + \lambda_\ell u_\ell\|_{L^2(T)} \||T|^{-1/n}(v_{\ell+m} - v_\ell)\|_{L^2(T)}\\
&\qquad + \sum_{E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+m}} |E|^{1/(2n-2)} \|[p_\ell]\cdot\nu_E\|_{L^2(E)} \||E|^{-1/(2n-2)}(v_{\ell+m} - v_\ell)\|_{L^2(E)}\\
&\lesssim \eta_\ell(\lambda_\ell, u_\ell; \mathcal{M}_\ell) |||v_{\ell+m}|||.
\end{aligned}$$

For sufficiently small $\|h_0\|_{L^\infty(\Omega)}$, Lemma 3.6 finishes the proof. $\square$

Lemma 4.2. *For sufficiently small* $\|h_0\|_{L^\infty(\Omega)}$ *it holds*

$$|||Res_\ell|||_{V^*} \lesssim \eta_\ell(\lambda_\ell, u_\ell) \lesssim |||e_\ell|||.$$

Proof. The first inequality can be proven as Lemma 4.1. For the second inequality, Durán et al. [DPR03] showed the local lower bound for piecewise linear finite element functions using the bubble-function technique. In the case of higher-order finite elements the arguments of the proof remain the same as in the linear case except that $\mathrm{div}(p_\ell)$ can be non-zero. Thus the local discrete inverse inequality $|\omega_E|^{1/n}\|\mathrm{div}(p_\ell)\|_{L^2(\omega_E)} \lesssim \|\nabla e_\ell\|_{L^2(\omega_E)}$ has to be applied additionally. Therefore it holds the local lower bound

$$\begin{aligned}
|\omega_E|^{1/n}\|\lambda_\ell u_\ell + \mathrm{div}(p_\ell)\|_{L^2(\omega_E)} + |E|^{1/(2n-2)}\|[p_\ell]\cdot\nu_E\|_{L^2(E)}&\\
\lesssim \|\nabla e_\ell\|_{L^2(\omega_E)} + |\omega_E|^{1/n}\|\lambda u - \lambda_\ell u_\ell\|_{L^2(\omega_E)}&
\end{aligned}$$

for the edge patch $\omega_E := T_+ \cup T_-$, for $T_\pm \in \mathcal{T}_\ell$ with $E = T_+ \cap T_-$. The global version reads

$$\eta_\ell^2(\lambda_\ell, u_\ell) \lesssim |||e_\ell|||^2 + \|h_\ell\|^2_{L^\infty(\Omega)}\|\lambda u - \lambda_\ell u_\ell\|^2.$$

As shown in [CG08], some elementary algebra in the spirit of Lemma 3.1 shows

$$\|\lambda u - \lambda_\ell u_\ell\|^2 = (\lambda_\ell - \lambda)^2 + \lambda\lambda_\ell\|e_\ell\|^2.$$

Equation (3.1) yields $(\lambda_\ell - \lambda)^2 \leq |||e_\ell|||^4$ and $\lambda\lambda_\ell\|e_\ell\|^2 \leq \lambda_\ell|||e_\ell|||^2$. Since λ_ℓ is bounded by λ_0 it holds

$$\eta_\ell(\lambda_\ell, u_\ell) \lesssim |||e_\ell|||$$

even for larger mesh-sizes $\|h_\ell\|_{L^\infty(\Omega)} \lesssim 1$. $\square$

Remark 4.3. Lemma 3.6, Lemma 4.1 and Lemma 4.2 show that there exist two constants $0 < C_{\mathrm{rel}}$ and $0 < C_{\mathrm{eff}}$ such that

$$\begin{aligned} \eta_\ell(\lambda_\ell, u_\ell)/C_{\mathrm{eff}} \le \|e_\ell\| \le C_{\mathrm{rel}}\eta_\ell(\lambda_\ell, u_\ell),\\ \|u_{\ell+m} - u_\ell\| \le C_{\mathrm{rel}}\eta_\ell(\lambda_\ell, u_\ell; \mathcal{M}_\ell). \end{aligned}$$

5. Quasi-Optimal Convergence

This section is devoted to the quasi-optimal convergence analysis of the adaptive eigenvalue computation based on exact solutions of the algebraic eigenvalue problems. At first the approximation class A_s is defined and its properties are described. Lemma 5.2 shows an estimator reduction which is used in the proof of the contraction property in Lemma 5.3. The contraction property and the bulk criterion are key arguments in the proof of the quasi-optimality in Theorem 5.4.

Definition 5.1 (Approximation class)**.** *For an initial triangulation $\mathcal{T}_0$ and for $s > 0$ let the approximation class be defined by*

$$\mathcal{A}_s := \left\{ u \in H_0^1(\Omega) : |u|_{\mathcal{A}_s} := \sup_{\varepsilon > 0} \varepsilon \inf_{\mathcal{T}_\varepsilon : \|u - u_\varepsilon\| \le \varepsilon} (|\mathcal{T}_\varepsilon| - |\mathcal{T}_0|)^s < \infty \right\}.$$

The infimum is taken over all refinements $\mathcal{T}_\varepsilon$ of $\mathcal{T}_0$ computed by the refinement algorithm of Section 2 with $\|u - u_\varepsilon\| \le \varepsilon$.

Notice that $\mathcal{A}_s$ contains all eigenfunctions that can be approximated within pre-described tolerance $\varepsilon > 0$, $\|u - u_\varepsilon\| \le \varepsilon$, on a triangulation $\mathcal{T}_\varepsilon$ with $|\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \le \varepsilon^{-1/s}|u|_{\mathcal{A}_s}^{1/s}$. For uniform refinement classical a priori estimates show that for $0 < r \le 1$, $H^{1+r}(\Omega) \cap V \subset \mathcal{A}_{r/n}$, but the class obtains much more functions which motivates the use of adaptivity. Due to [Ste07] an equivalent formulation, similar to that of [CKNS08], reads

$$\mathcal{A}_s := \left\{ u \in H_0^1(\Omega) : \sup_{N \in \mathbb{N}} N^s \inf_{\mathcal{T}_\varepsilon : |\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \le N} \|u - u_\varepsilon\| < \infty \right\}.$$

In the following the marking strategy of Section 2 is a key argument in the proofs.

Lemma 5.2. *Let (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ be discrete eigenpairs on the levels ℓ and $\ell + 1$ to the continuous eigenpair (λ, u), then there exist some $\Lambda > 0$, such that, for all levels $\ell \ge 0$ and $0 < \theta \le 1$, it holds*

$$\eta_{\ell+1}(\lambda_{\ell+1}, u_{\ell+1}) \le \sqrt{(1 - \theta(1 - 2^{-1/(n-1)}))}\eta_\ell(\lambda_\ell, u_\ell) + \Lambda\|u_{\ell+1} - u_\ell\|.$$

Proof. As in the proof of [CG08, Lemma 5.3], Young's inequality, some discrete inverse inequalities and the bulk criterion of Section 2 lead to

$$\begin{aligned} \eta_{\ell+1}^2(\lambda_{\ell+1}, u_{\ell+1}) \le (1 + \delta)(1 - \theta(1 - 2^{-1/(n-1)}))\eta_\ell^2(\lambda_\ell, u_\ell)\\ + \Lambda^2(1 + 1/\delta)\|u_{\ell+1} - u_\ell\|^2 \end{aligned}$$

for any $0 < \delta$ from Young's inequality, $0 < \theta \le 1$ bulk parameter and $0 < \Lambda$ from application of various discrete inverse inequalities. Thereby the factor

$2^{-1/(n-1)}$ results from at least one bisection of refined elements or edges. The choice

$$\delta = \frac{\Lambda \| u_{\ell+1} - u_\ell \|}{\sqrt{(1-\theta(1-2^{-1/(n-1)}))}\eta_\ell(\lambda_\ell, u_\ell)}$$

proves the assertion. □

Lemma 5.3 (Contraction property)**.** *Let (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ be discrete eigenpairs on the levels ℓ and $\ell+1$ approximating the same continuous eigenpair (λ, u) and let the mesh-size $\|h_\ell\|_{L^\infty(\Omega)}$ be sufficiently small, then there exist constants $0 < \varrho < 1$ and $\gamma > 0$, such that, for all $\ell = 0, 1, 2, \ldots$, it holds*

$$\gamma\eta_{\ell+1}^2(\lambda_{\ell+1}, u_{\ell+1}) + \| u - u_{\ell+1} \|^2 \le \varrho\left(\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + \| u - u_\ell \|^2\right). \tag{5.1}$$

Proof. The proof of [CG08, Theorem 5.5] shows

$$\begin{aligned}\gamma\eta_{\ell+1}^2(\lambda_{\ell+1}, u_{\ell+1}) + \| u - u_{\ell+1} \|^2 &\le \varrho\left(\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + \| u - u_\ell \|^2\right)\\ &\quad + 9\lambda_\ell\left(\|u - u_\ell\|^2 + \|u - u_{\ell+1}\|^2\right).\end{aligned}$$

Remark 3.4 reads

$$\|e_{\ell+1}\|^2 \le 4(1+\rho)^2\|e_\ell\|^2,$$

where ρ depends of the distance of the eigenvalue λ to all others as in Lemma 3.3. This results in

$$\begin{aligned}\gamma\eta_{\ell+1}^2(\lambda_{\ell+1}, u_{\ell+1}) + \| u - u_{\ell+1} \|^2 &\le \varrho\left(\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + \| u - u_\ell \|^2\right)\\ &\quad + 37(1+\rho)^2\lambda_\ell\|u - u_\ell\|^2.\end{aligned}$$

Lemma 3.5 shows

$$\|u - u_\ell\|^2 \lesssim \|h_0\|_{L^\infty(\Omega)}^{2r}\eta_\ell^2(\lambda_\ell, u_\ell).$$

Hence, for sufficiently small mesh-size $\|h_0\|_{L^\infty(\Omega)} \ll 1$, it follows (5.1) with some constant $\varrho < 1$. □

Theorem 5.4. *Suppose that (λ_ℓ, u_ℓ) is an exact discrete eigenpair to the continuous eigenpair (λ, u) with $u \in \mathcal{A}_s$. Then λ_ℓ and u_ℓ from the AFEM converge quasi-optimal in the sense that*

$$\| e_\ell \|^2 + |\lambda - \lambda_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}.$$

Proof. First it is shown that for a set $\mathcal{M}_\ell$ of marked edges and elements from the marking strategy of Section 2, based on the bulk criterion and $\eta_\ell(\lambda_\ell, u_\ell)$, and a bulk parameter $\theta > 0$, it holds

$$|\mathcal{M}_\ell| \lesssim \| e_\ell \|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}.$$

Suppose $\mathcal{T}_{\ell+\varepsilon}$ is any refinement of $\mathcal{T}_\ell$ such that

$$\| e_{\ell+\varepsilon} \| \le \rho \| e_\ell \|$$

for some $0 < \rho < 1$. Suppose that $\|h_\ell\|_{L^\infty(\Omega)}$ and θ are sufficiently small,

$$0 < \theta \le \frac{(1-\rho^2)}{C_{\text{rel}}^2 C_{\text{eff}}^2} - \lambda\sigma(h_\ell)^2,$$

where $\sigma(h_\ell)$ from Lemma 3.5 tends to zero as $\|h_\ell\|_{L^\infty(\Omega)} \to 0$. Using the efficiency estimates of Remark 4.3 together with the quasi-orthogonality of Lemma 3.1 yields

$$\begin{aligned}(1-\rho^2)\eta_\ell^2(\lambda_\ell,u_\ell)/C_{\text{eff}}^2 &\le (1-\rho^2)\|e_\ell\|^2 \le \|e_\ell\|^2 - \|e_{\ell+\varepsilon}\|^2\\ &= \|u_{\ell+\varepsilon}-u_\ell\|^2 + \lambda\|e_\ell\|^2 - \lambda\|e_{\ell+\varepsilon}\|^2 - \lambda_{\ell+\varepsilon}\|u_{\ell+\varepsilon}-u_\ell\|^2.\end{aligned}$$

Let $\mathcal{M}_\varepsilon := (\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+\varepsilon}) \cup (\mathcal{E}_\ell\backslash\mathcal{E}_{\ell+\varepsilon})$. The combination of Lemma 3.6, together with Lemma 4.1 and Lemma 3.5 yields

$$\begin{aligned}(1-\rho^2)\eta_\ell^2(\lambda_\ell,u_\ell)/C_{\text{eff}}^2 &\le C_{\text{rel}}^2\eta_\ell^2(\lambda_\ell,u_\ell;\mathcal{M}_\varepsilon) + \lambda\|e_\ell\|^2\\ &\le C_{\text{rel}}^2\eta_\ell^2(\lambda_\ell,u_\ell;\mathcal{M}_\varepsilon) + \lambda\sigma(h_\ell)^2C_{\text{rel}}^2\eta_\ell^2(\lambda_\ell,u_\ell).\end{aligned}$$

Therefore $\mathcal{M}_\varepsilon$ satisfies the bulk criterion. Since $\mathcal{M}_\ell$ is the set with almost minimal cardinality that fulfils the bulk criterion, it holds

$$|\mathcal{M}_\ell| \lesssim |\mathcal{M}_\varepsilon| \lesssim |\mathcal{T}_{\ell+\varepsilon}| - |\mathcal{T}_\ell|.$$

Let $\mathcal{T}_\varepsilon$ be an optimal mesh with smallest cardinality such that

$$\|e_\varepsilon\| \le \rho\|e_\ell\|.$$

The definition of the approximation space $\mathcal{A}_s$ shows that

$$|\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \le \rho^{-1/s}\|e_\ell\|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}.$$

Let $\mathcal{T}_{\ell+\varepsilon}$ be the smallest common refinement of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_\ell$. Then the overlay estimate yields

$$|\mathcal{M}_\ell| \lesssim |\mathcal{T}_{\ell+\varepsilon}| - |\mathcal{T}_\ell| = |\mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell| - |\mathcal{T}_\ell| \le |\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \lesssim \|e_\ell\|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}.$$

This and the boundedness of closure in Lemma 2.1 yield

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{L-1}|\mathcal{M}_\ell| \lesssim |u|_{\mathcal{A}_s}^{1/s}\sum_{\ell=0}^{L-1}\|e_\ell\|^{-1/s}.$$

The efficiency estimate of Remark 4.3 yields

$$\gamma\eta_\ell^2(\lambda_\ell,u_\ell) + \|u-u_\ell\|^2 \le \left(1+\gamma C_{\text{eff}}^2\right)\|u-u_\ell\|^2.$$

Thus,

$$\|u-u_\ell\|^{-1/s} \le \left(1+\gamma C_{\text{eff}}^2\right)^{1/(2s)}\left(\gamma\eta_\ell^2(\lambda_\ell,u_\ell) + \|u-u_\ell\|^2\right)^{-1/(2s)}.$$

Lemma 5.3 leads to

$$\left(\gamma\eta_\ell^2(\lambda_\ell,u_\ell) + \|u-u_\ell\|^2\right)^{-1} \le \varrho\left(\gamma\eta_{\ell+1}^2(\lambda_{\ell+1},u_{\ell+1}) + \|u-u_{\ell+1}\|^2\right)^{-1}.$$

Exploiting the reliability of the estimator and a geometric series argument yields that $|\mathcal{T}_L| - |\mathcal{T}_0|$ is, up to a multiplicative constant, bounded by

$$\begin{aligned}&|u|_{\mathcal{A}_s}^{1/s}\left(1+\gamma C_{\text{eff}}^2\right)^{1/(2s)}\left(\gamma\eta_L^2(\lambda_L,u_L) + \|u-u_L\|^2\right)^{-1/(2s)}\sum_{\ell=1}^{L}\varrho^{\ell/(2s)}\\ &\lesssim |u|_{\mathcal{A}_s}^{1/s}\left(\frac{1+\gamma C_{\text{eff}}^2}{1+\gamma/C_{\text{rel}}^2}\right)^{1/(2s)}(1-\varrho^{1/(2s)})^{-1}\|u-u_L\|^{-1/s}.\end{aligned}$$

Equation (3.1) proves the second statement of Theorem 5.4. $\square$

6. Quasi-Optimal Convergence for Inexact Algebraic Solutions

This section contributes to the fact that in practise the underlying algebraic eigenvalue problems are solved inexact using iterative algebraic eigenvalue solvers. A relationship between the error estimator in the exact algebraic solution and any approximation to it is established in Lemma 6.1. As in the case of exact algebraic solutions, the contraction property in Lemma 6.2 and local optimality in Lemma 6.3 lead to the global quasi-optimality in Theorem 6.4.

Lemma 6.1. *Let $v_\ell, w_\ell \in V_\ell$ be arbitrary, not necessary eigenfunctions, but normalised with $\|v_\ell\| = \|w_\ell\| = 1$ and $\lambda, \mu \in \mathbb{R}^+$ arbitrary positive real numbers, then it holds*

$$|\eta_\ell(\lambda, v_\ell) - \eta_\ell(\mu, w_\ell)|^2 \leq C\left(\|\!|v_\ell - w_\ell|\!\|^2 + |\lambda - \mu|^2\right)$$

for a constant $0 < C$ independent of the mesh size $\|h_\ell\|_{L^\infty(\Omega)}$.

Proof. Using twice the triangle inequality first for vectors and then for functions yields

$$\begin{aligned}|\eta_\ell(\lambda, v_\ell) - \eta_\ell(\mu, w_\ell)|^2 \leq &\sum_{T\in\mathcal{T}_\ell} |T|^{2/n} \|\lambda v_\ell - \mu w_\ell + \operatorname{div}(\nabla v_\ell - \nabla w_\ell)\|^2_{L^2(T)} \\ &+ \sum_{E\in\mathcal{E}_\ell} |E|^{1/(n-1)} \|[\nabla v_\ell - \nabla w_\ell]\cdot\nu_E\|^2_{L^2(E)}.\end{aligned}$$

The local discrete inverse inequality

$$|T|^{2/n}\|\operatorname{div}(\nabla v_\ell)\|^2_{L^2(T)} \lesssim \|\nabla v_\ell\|^2_{L^2(T)},$$

together with the trace inequality

$$\|v\|^2_{L^2(E)} \lesssim |E|^{-1/(n-1)}\|v\|^2_{L^2(\omega_E)} + |E|^{1/(n-1)}\|\nabla v\|^2_{L^2(\omega_E)},$$

the Poincaré inequality and the finite overlay of the patches lead to

$$\begin{aligned}|\eta_\ell(\lambda, v_\ell)-&\eta_\ell(\mu, w_\ell)|^2 \\ &\lesssim \sum_{T\in\mathcal{T}_\ell} |T|^{2/n}\|\lambda v_\ell - \mu w_\ell\|^2_{L^2(T)} + \|\nabla v_\ell - \nabla w_\ell\|^2_{L^2(T)} \\ &+ \quad \sum_{E\in\mathcal{E}_\ell} \|\nabla v_\ell - \nabla w_\ell\|^2_{L^2(\omega_E)} \\ &\lesssim \|h_\ell\|^2_{L^\infty(\Omega)}\|\lambda v_\ell - \mu w_\ell\|^2 + \|\!|v_\ell - w_\ell|\!\|^2 \\ &\lesssim (1 + \lambda^2\|h_\ell\|^2_{L^\infty(\Omega)})\|\!|v_\ell - w_\ell|\!\|^2 + |\lambda - \mu|^2.\end{aligned}$$ □

Lemma 6.2 (Contraction property for inexact algebraic solutions)**.** *Suppose that (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ are discrete eigenpairs to the continuous eigenpair (λ, u) on level ℓ and $\ell + 1$. Let (μ_ℓ, w_ℓ) and $(\mu_{\ell+1}, w_{\ell+1})$ be the corresponding iterative approximations to the discrete eigenpairs, which satisfy*

$$\begin{aligned}\|\!|u_{\ell+1} - w_{\ell+1}|\!\|^2 + |\lambda_{\ell+1} - \mu_{\ell+1}|^2 &\leq \omega\eta^2_\ell(\mu_\ell, w_\ell), \\ \|\!|u_\ell - w_\ell|\!\|^2 + |\lambda_\ell - \mu_\ell|^2 &\leq \omega\eta^2_\ell(\mu_\ell, w_\ell)\end{aligned}$$

for sufficiently small $\omega > 0$. Then, for sufficiently small mesh size $\|h_\ell\|_{L^\infty(\Omega)}$, there exists some $0 < \nu < 1$, such that the contraction property

$$\gamma\eta_\ell^2(\mu_{\ell+1}, w_{\ell+1}) + \|u - w_{\ell+1}\|^2 \le \nu\left(\gamma\eta_\ell^2(\mu_\ell, w_\ell) + \|u - w_\ell\|^2\right)$$

holds.

Proof. The assumptions and Lemma 6.1 plus Young's inequality show for any $\delta > 0$

$$\begin{aligned}
&\gamma\eta_\ell^2(\mu_{\ell+1}, w_{\ell+1}) + \|u - w_{\ell+1}\|^2 \\
&\quad\le (1+\delta)\left(\gamma\eta_\ell^2(\lambda_{\ell+1}, u_{\ell+1}) + \|u - u_{\ell+1}\|^2\right) \\
&\qquad + (1+1/\delta)\left(\gamma|\eta_\ell(\mu_{\ell+1}, w_{\ell+1}) - \eta_\ell(\lambda_{\ell+1}, u_{\ell+1})|^2 + \|u_{\ell+1} - w_{\ell+1}\|^2\right) \\
&\quad\le (1+\delta)\left(\gamma\eta_\ell^2(\lambda_{\ell+1}, u_{\ell+1}) + \|u - u_{\ell+1}\|^2\right) \\
&\qquad + (1+1/\delta)\left(\gamma C|\lambda_{\ell+1} - \mu_{\ell+1}|^2 + (1+\gamma C)\|u_{\ell+1} - w_{\ell+1}\|^2\right) \\
&\quad\le (1+\delta)\left(\gamma\eta_\ell^2(\lambda_{\ell+1}, u_{\ell+1}) + \|u - u_{\ell+1}\|^2\right) \\
&\qquad + (1+1/\delta)(1+\gamma C)\omega\eta_\ell^2(\mu_\ell, w_\ell).
\end{aligned}$$

The contraction property Lemma 5.3 and another Young's inequality yield

$$\begin{aligned}
\gamma\eta_\ell^2(\mu_{\ell+1}, w_{\ell+1}) + \|u - w_{\ell+1}\|^2 &\le (1+\delta)\varrho\left(\gamma\eta_\ell^2(\lambda_\ell, u_\ell) + \|u - u_\ell\|^2\right) \\
&\quad + (1+1/\delta)(1+\gamma C)\omega\eta_\ell^2(\mu_\ell, w_\ell) \\
&\le (1+\delta)^2\varrho\left(\gamma\eta_\ell^2(\mu_\ell, w_\ell) + \|u - w_\ell\|^2\right) \\
&\quad + (1+(1+\delta)\varrho)(1+1/\delta)(1+\gamma C)\omega\eta_\ell^2(\mu_\ell, w_\ell).
\end{aligned}$$

Any choice of $0 < \delta < \varrho^{-1/2} - 1$ results in

$$0 < \omega < \frac{\gamma - (1+\delta)^2\varrho\gamma}{(1+(1+\delta)\varrho)(1+1/\delta)(1+\gamma C)}.$$

The choice

$$0 < \nu := (1+\delta)^2\varrho + (1+(1+\delta)\varrho)(1+1/\delta)(1+\gamma C)\omega/\gamma < 1$$

finishes the proof. □

Lemma 6.3. *Let (λ, u) be an eigenpair and (λ_ℓ, u_ℓ) the corresponding discrete eigenpair with iterative approximation (μ_ℓ, w_ℓ) which satisfies*

$$\|u_\ell - w_\ell\|^2 + |\lambda_\ell - \mu_\ell|^2 \le \omega\eta_\ell^2(\mu_\ell, w_\ell)$$

for a sufficient small $\omega > 0$. Suppose that $\mathcal{M}_{\mu_\ell, w_\ell}$ is the set of marked edges and elements using the marking strategy of Section 2 based on the bulk criterion and $\eta_\ell(\mu_\ell, w_\ell)$, then for sufficiently small $\|h_\ell\|_{L^\infty(\Omega)}$ and bulk parameter $\theta > 0$ it holds

$$|\mathcal{M}_{\mu_\ell, w_\ell}| \lesssim \|u - w_\ell\|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}.$$

Proof. Let $\mathcal{T}_\varepsilon$ be the smallest partition of $\mathcal{T}_0$ such that

$$\|u - u_\varepsilon\| \le \rho\|u - w_\ell\|$$

for $0 < \rho < 1/2$. Thus, the definition of $|u|_{\mathcal{A}_s}$ yields

$$|\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \le \rho^{-1/s}\|u - w_\ell\|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}.$$

Let $\mathcal{T}_{\ell+\varepsilon} := \mathcal{T}_\ell \oplus \mathcal{T}_\varepsilon$ be the smallest common refinement of $\mathcal{T}_\ell$ and $\mathcal{T}_\varepsilon$, then it holds

$$\begin{aligned}\|u - u_{\ell+\varepsilon}\| &\le \rho\|u - w_\ell\| \le \rho\|u - u_\ell\| + \rho\|u_\ell - w_\ell\| \\ &\le \rho\|u - u_\ell\| + \rho\sqrt{\omega}\eta_\ell(\mu_\ell, w_\ell) \\ &\le \left(2\rho^2\|u - u_\ell\|^2 + 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell)\right)^{1/2}.\end{aligned}$$

This estimate proofs the following

$$\begin{aligned}(1-2\rho^2)&C_{\mathrm{eff}}^{-2}\eta_\ell^2(\lambda_\ell, u_\ell) - 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell) \\ &\le (1-2\rho^2)\|u - u_\ell\|^2 - 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell) \\ &\le \|u - u_\ell\|^2 - \|u - u_{\ell+\varepsilon}\|^2.\end{aligned}$$

Let $\mathcal{M}_\varepsilon := (\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+\varepsilon}) \cup (\mathcal{E}_\ell\backslash\mathcal{E}_{\ell+\varepsilon})$, then the quasi-orthogonality of Lemma 3.1 and the discrete reliability of Lemma 4.1 yield

$$\begin{aligned}(1-2\rho^2)C_{\mathrm{eff}}^{-2}\eta_\ell^2(\lambda_\ell, u_\ell) - 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell) &\le \|u_{\ell+\varepsilon} - u_\ell\|^2 + \lambda\|e_\ell\|^2 \\ \le C_{\mathrm{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell),&\end{aligned}$$

where $\sigma(h_\ell)$ from Lemma 3.5 tends to zero as $\|h_\ell\|_{L^\infty(\Omega)} \to 0$. Thus,

$$\begin{aligned}((1-2\rho^2)&C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)\eta(\lambda_\ell, u_\ell)^2 \\ &\le C_{rel}^2\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell).\end{aligned}$$

Lemma 6.1 together with the assumption yields

$$|\eta_\ell(\lambda_\ell, u_\ell) - \eta_\ell(\mu_\ell, w_\ell)|^2 \le C\left(\|u_\ell - w_\ell\|^2 + |\lambda_\ell - \mu_\ell|^2\right) \le C\omega\eta_\ell^2(\mu_\ell, w_\ell).$$

Therefore,

$$\begin{aligned}((1-2\rho^2)&C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)2^{-1}\eta_\ell^2(\mu_\ell, w_\ell) \\ &\le ((1-2\rho^2)C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)\eta_\ell^2(\lambda_\ell, u_\ell) \\ &\quad + ((1-2\rho^2)C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)C\omega\eta_\ell^2(\mu_\ell, w_\ell) \\ &\le C_{\mathrm{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell; \mathcal{M}_\varepsilon) + 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell) \\ &\quad + ((1-2\rho^2)C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)C\omega\eta_\ell^2(\mu_\ell, w_\ell) \\ &\le 2C_{\mathrm{rel}}^2\eta_\ell^2(\mu_\ell, w_\ell; \mathcal{M}_\varepsilon) + 2\rho^2\omega\eta_\ell^2(\mu_\ell, w_\ell) \\ &\quad + (2C_{\mathrm{rel}}^2 + (1-2\rho^2)C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)C\omega\eta_\ell^2(\mu_\ell, w_\ell).\end{aligned}$$

The choice $\|h_\ell\|_{L^\infty(\Omega)} \ll 1$ and $0 < \omega \ll 1$ shows $0 < \theta \le \Theta$ with

$$\Theta := \frac{((1-2\rho^2)C_{\mathrm{eff}}^{-2} - \lambda\sigma(h_\ell)^2C_{\mathrm{rel}}^2)(2^{-1} - C\omega) - 2(C_{\mathrm{rel}}^2C + \rho^2)\omega}{2C_{\mathrm{rel}}^2}$$

and hence the bulk criterion for the set $\mathcal{M}_\varepsilon$ based on $\eta_\ell(\mu_\ell, w_\ell)$ is satisfied. Since the set $\mathcal{M}_{\mu_\ell, w_\ell}$ has been chosen with almost minimal cardinality, the overlay estimate leads to

$$|\mathcal{M}_{\mu_\ell, w_\ell}| \lesssim |\mathcal{M}_\varepsilon| \lesssim |\mathcal{T}_{\ell+\varepsilon}| - |\mathcal{T}_\ell| \le |\mathcal{T}_\varepsilon| - |\mathcal{T}_0| \lesssim \|u - w_\ell\|^{-1/s}|u|_{\mathcal{A}_s}^{1/s}. \qquad \square$$

Theorem 6.4. *Suppose that (λ, u) with $u \in \mathcal{A}_s$ is an exact eigenpair and let (λ_ℓ, u_ℓ) and $(\lambda_{\ell+1}, u_{\ell+1})$ be the corresponding discrete eigenpairs on level*

ℓ and $\ell+1$. Let the iterative approximations (μ_ℓ, w_ℓ) on $\mathcal{T}_\ell$ and $(\mu_{\ell+1}, w_{\ell+1})$ on $\mathcal{T}_{\ell+1}$ satisfy

$$\begin{aligned}\|u_{\ell+1} - w_{\ell+1}\|^2 + |\lambda_{\ell+1} - \mu_{\ell+1}| &\le \omega\eta_\ell^2(\mu_\ell, w_\ell),\\ \|u_\ell - w_\ell\|^2 + |\lambda_\ell - \mu_\ell| &\le \omega\eta_\ell^2(\mu_\ell, w_\ell)\end{aligned}$$

for sufficiently small $\omega > 0$. Then, up to positive constants, the iterative solutions μ_ℓ and w_ℓ converge optimal

$$\|u - w_\ell\|^2 + |\lambda - \mu_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}.$$

Proof. Lemma 6.3 and Proposition 2.1 yield

$$|\mathcal{T}_L| - |\mathcal{T}_0| \lesssim \sum_{\ell=0}^{L-1} |\mathcal{M}_{\mu_\ell, w_\ell}| \lesssim |u|_{\mathcal{A}_s}^{1/s} \sum_{\ell=0}^{L-1} \|u - w_\ell\|^{-1/s}.$$

The efficiency estimate of Remark 4.3 and Lemma 6.1 show

$$\begin{aligned}\eta_\ell^2(\mu_\ell, w_\ell) &\le 2\eta_\ell^2(\lambda_\ell, u_\ell) + 2C\left(\|u_\ell - w_\ell\|^2 + |\lambda_\ell - \mu_\ell|^2\right)\\ &\le 4C_{\mathrm{eff}}^2\|u - w_\ell\|^2 + (2C + 4C_{\mathrm{eff}}^2)\left(\|u_\ell - w_\ell\|^2 + |\lambda_\ell - \mu_\ell|^2\right)\\ &\le 4C_{\mathrm{eff}}^2\|u - w_\ell\|^2 + (2C + 4C_{\mathrm{eff}}^2)\omega\eta_\ell^2(\mu_\ell, w_\ell).\end{aligned}$$

Hence, for $0 < \omega < (2C + 4C_{\mathrm{eff}}^2)^{-1}$, it holds

$$\eta_\ell(\mu_\ell, w_\ell) \lesssim \|u - w_\ell\|.$$

For the other direction, notice that

$$\|u - w_\ell\| \le \|u - u_\ell\| + \|u_\ell - w_\ell\| \le C_{\mathrm{rel}}\eta_\ell(\lambda_\ell, u_\ell) + \sqrt{\omega}\eta_\ell(\mu_\ell, w_\ell),$$

implies

$$\begin{aligned}\|u - w_\ell\|^2 &\le 2C_{\mathrm{rel}}^2\eta_\ell^2(\lambda_\ell, u_\ell) + 2\omega\eta_\ell^2(\mu_\ell, w_\ell)\\ &\le \left(4C_{\mathrm{rel}}^2 + 4C_{\mathrm{rel}}^2 C\omega + 2\omega\right)\eta_\ell^2(\mu_\ell, w_\ell).\end{aligned}$$

Thus,

$$\|u - w_\ell\|^{-1/s} \lesssim \left(\gamma\eta_\ell^2(\mu_\ell, w_\ell) + \|u - w_\ell\|^2\right)^{-1/(2s)}.$$

Lemma 6.2 leads to

$$\left(\gamma\eta_\ell^2(\mu_\ell, w_\ell) + \|u - w_\ell\|^2\right)^{-1} \le \nu\left(\gamma\eta_{\ell+1}^2(\mu_{\ell+1}, w_{\ell+1}) + \|u - w_{\ell+1}\|^2\right)^{-1}.$$

A geometric series argument yields

$$\begin{aligned}|\mathcal{T}_L| - |\mathcal{T}_0| &\lesssim |u|_{\mathcal{A}_s}^{1/s}\left(\gamma\eta_L^2(\mu_L, w_L) + \|u - w_L\|^2\right)^{-1/(2s)} \sum_{\ell=1}^{L} \nu^{\ell/(2s)}\\ &\lesssim |u|_{\mathcal{A}_s}^{1/s}(1 - \nu^{1/(2s)})^{-1}\|u - w_L\|^{-1/s}.\end{aligned}$$

Since

$$\begin{aligned}|\lambda - \mu_\ell| &\le |\lambda - \lambda_\ell| + |\lambda_\ell - \mu_\ell| \le |\lambda - \lambda_\ell| + \omega\eta_\ell^2(\mu_\ell, w_\ell)\\ &\lesssim |\lambda - \lambda_\ell| + \|u - w_\ell\|^2,\end{aligned}$$

Theorem 5.4 proves the second statement of Theorem 6.4. ∎

7. Optimal Complexity

In this section the proof of the quasi-optimal computational complexity of the AFEMES is presented. The proposed algorithm combines the AFEM with some iterative algebraic eigenvalue solver. In order to prove overall quasi-optimal complexity the iterative solver needs to have a linear convergence behaviour independently of the size of the discrete problem. In other words for any $\varepsilon > 0$ the algorithm LAES has to compute an iterative solution of the algebraic eigenvalue problem $(\mu_{\ell,m}, w_{\ell,m})$ from an initial guess $(\mu_{\ell,0}, w_{\ell,0})$ such that

$$|||u_\ell - w_{\ell,m}|||^2 + |\lambda_\ell - \mu_{\ell,m}| \leq \varepsilon^2$$

in at most, up to a multiplicative constant,

$$\max\left\{1, \log(\varepsilon^{-1}|||u_\ell - w_{\ell,0}|||)\right\} |\mathcal{T}_\ell|$$

arithmetic operations.

Theorem 7.1. *For sufficiently small* $\|h_0\|_{L^\infty(\Omega)}$, $0 < \theta \ll 1$ *and* $0 < \omega \ll 1$, *the algorithm AFEMES computes from a coarse triangulation* $\mathcal{T}_0$ *and an initial guess* (μ_0, w_0) *a sequence of triangulations* $(\mathcal{T}_\ell)_\ell$ *and corresponding inexact discrete eigenpairs* (μ_ℓ, w_ℓ) *such that*

$$|||u - w_\ell|||^2 + |\lambda - \mu_\ell| \lesssim \eta_\ell^2(\mu_\ell, w_\ell) \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}$$

and the computational costs, such as CPU-time and memory requirement are quasi-optimal.

Proof. First it is shown that the while-loop is terminating after a finite number of iterations on each level. Remark, that the while-loop is executed at least once and that in further runs it holds

$$|||u_\ell - w_\ell|||^2 + |\lambda_\ell - \mu_\ell| \leq \delta_\ell^2$$

because of the previous calls of LAES. Using Lemma 6.1 yields

$$\begin{aligned}
\sqrt{\omega}\eta_\ell(\mu_\ell, w_\ell) &\geq \sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell) - \sqrt{\omega}|\eta_\ell(\mu_\ell, w_\ell) - \eta_\ell(\lambda_\ell, u_\ell)| \\
&\geq \sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell) - \sqrt{\omega C}(|||u_\ell - w_\ell||| + |\lambda_\ell - \mu_\ell|) \\
&\geq \sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell) - \delta_\ell\sqrt{\omega C}.
\end{aligned}$$

Therefore the while-loop is at least terminated on the level ℓ if

$$\delta_\ell \leq \frac{\sqrt{\omega}\eta_\ell(\lambda_\ell, u_\ell)}{1 + \sqrt{\omega C}}.$$

Due to the geometric decrease of δ_ℓ this is achieved in finitely many steps for all levels ℓ. The choice of the initial value for δ_ℓ on each level ℓ and the fact that after the while-loop terminates $\delta_\ell \leq \sqrt{\omega}\eta_\ell(\mu_\ell, w_\ell)$ shows that the conditions of Theorem 6.4 are satisfied. Thus the convergence of

$$|||u - w_\ell|||^2 + |\lambda - \mu_\ell| \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}$$

is quasi-optimal. Moreover the proof of Theorem 6.4 shows

$$|||u - w_\ell||| \lesssim \eta_\ell(\mu_\ell, w_\ell) \lesssim |||u - w_\ell|||$$

for sufficiently small $\omega > 0$. For the eigenvalue error it holds

$$\begin{aligned}|\lambda - \mu_\ell| &\le |\lambda - \lambda_\ell| + |\lambda_\ell - \mu_\ell| \\ &\le C_{\mathrm{rel}}^2 \eta_\ell^2(\lambda_\ell, u_\ell) + \delta_\ell^2 \\ &\le 2C_{\mathrm{rel}}^2 \eta_\ell^2(\mu_\ell, w_\ell) + (2C_{\mathrm{rel}}^2 C + 1)\delta_\ell^2 \\ &\le (2C_{\mathrm{rel}}^2 + (2C_{\mathrm{rel}}^2 C + 1)\omega)\eta_\ell^2(\mu_\ell, w_\ell).\end{aligned}$$

Hence,

$$|||u - w_\ell|||^2 + |\lambda - \mu_\ell| \lesssim \eta_\ell^2(\mu_\ell, w_\ell) \lesssim (|\mathcal{T}_\ell| - |\mathcal{T}_0|)^{-2s}. \tag{7.1}$$

Because of the quasi-optimal convergence and the finitely many number of iterations of the while-loop, it remains to show that Mark, Refine and LAES are of linear computational complexity. An optimal algorithm for Mark and Refine can be found in [Ste07]. In the first execution of the while-loop, except for the first level for which the costs can be bounded by a constant separately, before LAES is executed, it holds

$$|||u_\ell - w_\ell||| = |||u_\ell - w_{\ell-1}||| \le |||u - u_\ell||| + |||u - w_{\ell-1}|||.$$

Lemma 5.3 reads

$$|||u - u_\ell|||^2 \le 2\varrho\left(\gamma C_{\mathrm{eff}}^2 + 1\right)\left(|||u - w_{\ell-1}|||^2 + |||u_{\ell-1} - w_{\ell-1}|||^2\right).$$

Thus, (7.1), the termination of the while-loop on the previous level $\ell - 1$ and the initialisation of δ_ℓ yield

$$|||u_\ell - w_\ell||| \lesssim \eta_{\ell-1}(\mu_{\ell-1}, w_{\ell-1}) + \delta_{\ell-1} \lesssim \eta_{\ell-1}(\mu_{\ell-1}, w_{\ell-1}) \lesssim \delta_\ell.$$

If it is not the first evaluation of the while-loop, then

$$|||u_\ell - w_\ell||| \le 2\delta_\ell$$

because of the previous call of LAES. Thus, before any call of LAES for $\ell > 0$ it holds

$$|||u_\ell - w_\ell||| \lesssim \delta_\ell$$

which shows that LAES can be executed in linear time $O(|\mathcal{T}_\ell|)$. □

8. Numerical Experiments

The numerical experiments for $n = 2, 3$ show quasi-optimal convergence rates for linear P_1 up to fourth order P_4 finite elements. The optimal AFEMES is implemented in Matlab for linear finite elements in $2D$, using a damped version of the iterative eigenvalue solver MINIT [DRSZ08]. The implementation of the AFEM follows the ideas of [ACF99] and in an enhanced way of [FPW08]. The higher-order AFEM implementation is based on the openFFW project [BGG$^+$07]. The $3D$ mesh refinement is based on a bisection type strategy [AMP00].

In the following the optimal convergence rates for the AFEM concerning exact algebraic eigenvalue solvers is shown on one non-convex domain in $2D$ and on two different domains in $3D$, one convex and one non-convex. The figures display the eigenvalue error and the error estimators in terms of degrees of freedom $\dim(V_\ell) = N_\ell$. Note that in $2D$, $N_\ell \approx h_\ell^{-1/2}$, and in $3D$, $N_\ell \approx h_\ell^{-1/3}$. The results of the experiments for the optimal AFEMES are displayed in terms of eigenvalue error versus computational (CPU) time.

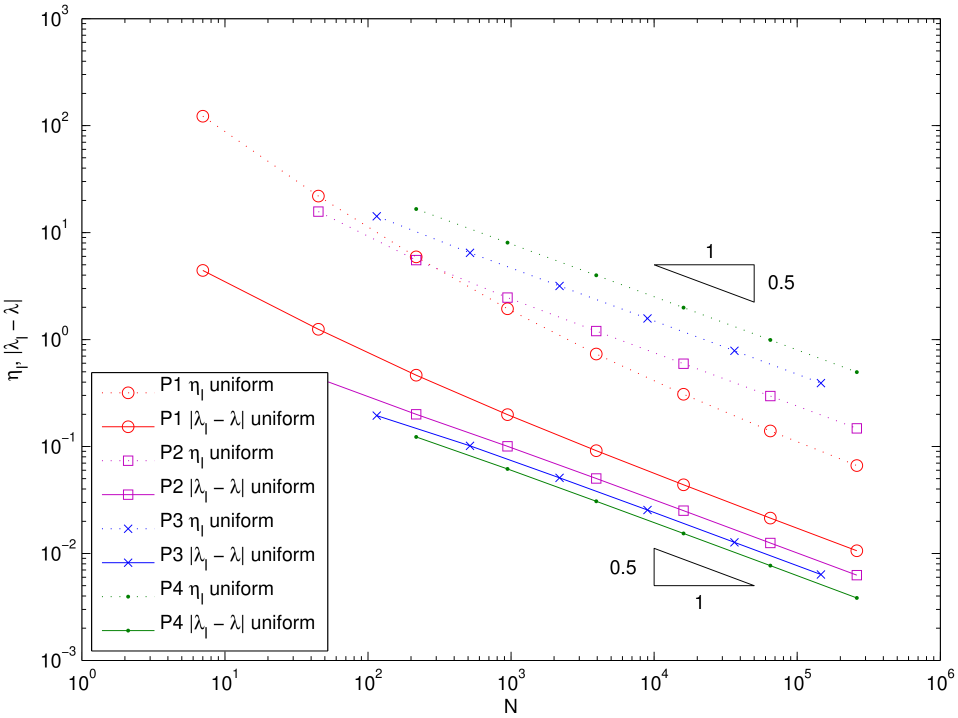

FIGURE 8.1. Eigenvalue errors and estimated errors on the slit domain for uniform meshes.

Example 8.1. Consider the two-dimensional eigenvalue problem: Seek the smallest non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times H^1(\overline{\Omega}) \cap H^2(\Omega)$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega$$

on the slit domain $\Omega = ((-1,1) \times (-1,1)) \backslash ([0,1] \times \{0\})$ with tip at the origin. An approximation of the smallest eigenvalue with high accuracy is given by

$$\lambda = 8.371329711186254.$$

In Figure 8.1 it is shown that the error estimator is numerically reliable and efficient for uniform meshes but these meshes result in suboptimal convergence rates of about $O(h^{1/2})$ due to the singularity at the origin. In contrast using adaptive refinement results in experimental optimal convergence rates of $O(h^{2k})$, $k = 1, \ldots, 4$, as shown in Figure 8.2. The error estimator is numerically reliable and efficient also for adaptive meshes.

The optimal AFEMES involves two parameters $\omega > 0$ and $0 < \theta \leq 1$ which are supposed to be sufficiently small. Figure 8.3 shows a numerically strong dependency of the convergence rate on θ. For $\theta \leq 0.2$, these rates are numerically optimal, but $\theta \ll 1$ leads to more iterations of the algebraic eigenvalue solver and thus to more computational work. Experimentally, $\theta = 0.2$ seems to be the best choice for this particular example. In contrast, different values for ω lead almost all to optimal convergence rates as depicted in Figure 8.4.

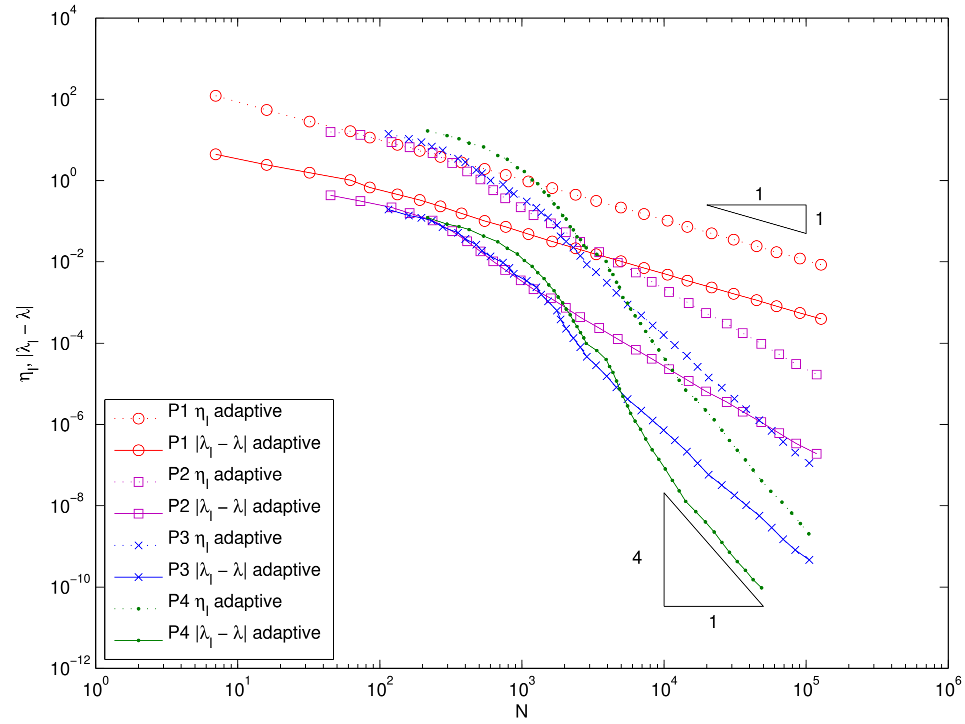

FIGURE 8.2. Eigenvalue errors and estimated errors on the slit domain for adaptive meshes.

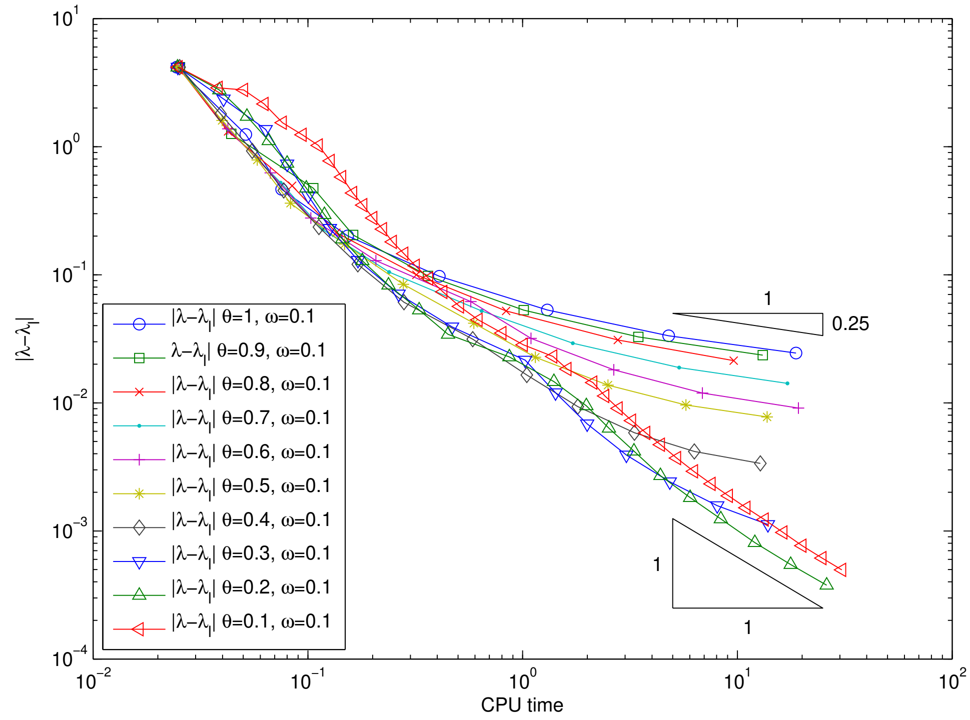

FIGURE 8.3. The optimal AFEMES for different values of $\theta = 0.1 \ldots 1$.

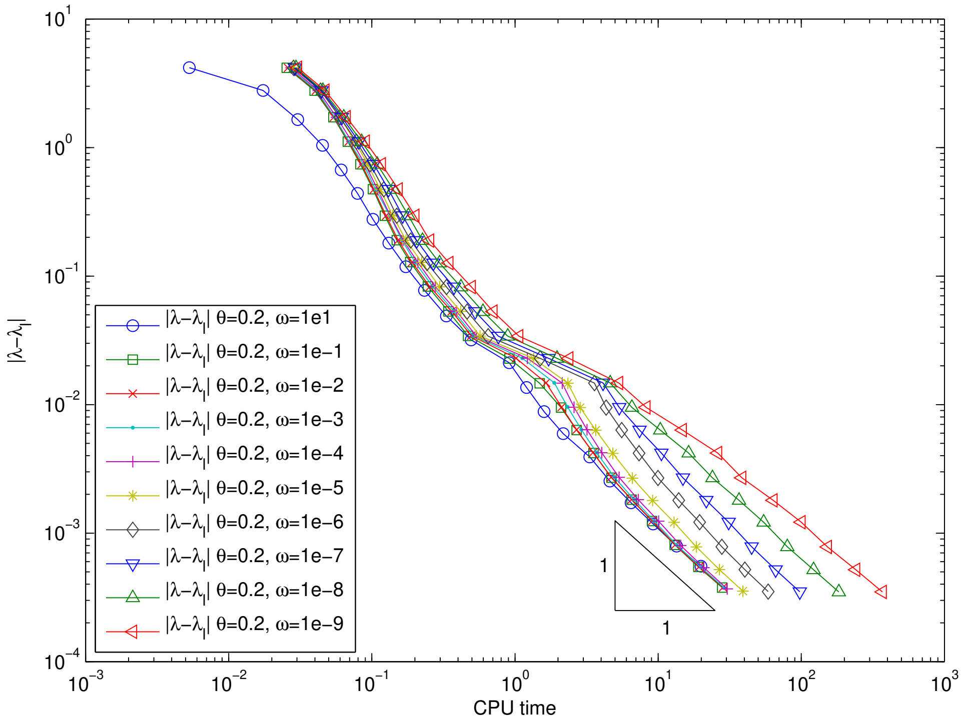

FIGURE 8.4. The optimal AFEMES for different values of $\omega = 10^{-9}, \dots, 1$.

Example 8.2. Consider the three-dimensional eigenvalue problem to compute the smallest eigenvalue λ and eigenfunction u on $\Omega = (0,1) \times (0,1) \times (0,1)$ of

$$-\Delta u = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega.$$

The exact solution reads

$$\lambda = 3\pi^2 \qquad \text{and} \qquad u(x,y,z) = \sqrt{8}\sin(\pi x)\sin(\pi y)\sin(\pi z).$$

Figure 8.5 shows optimal convergence rates for uniform meshes of $O(h^{2k})$, $k = 1, \dots, 4$. The AFEM with exact algebraic eigenvalue solutions also results in optimal convergence rates as displayed in Figure 8.6. For uniform and adaptively refined meshes the error estimator shows numerically to be reliable and efficient.

Example 8.3. Consider the three-dimensional eigenvalue problem: Seek the smallest non-trivial eigenpair $(\lambda, u) \in \mathbb{R} \times H^1(\overline{\Omega}) \cap H^2(\Omega)$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega$$

on the three-dimensional L shaped domain

$$\Omega = ((-1,1) \times (-1,1) \times (-1,1)) \backslash ((0,1) \times (0,1) \times (-1,1)).$$

An approximation with higher-order P_4 finite elements on adaptive refined meshes with about 2 million degrees of freedom yields the reference value for the first eigenvalue

$$\lambda \approx 19.509329494514329.$$

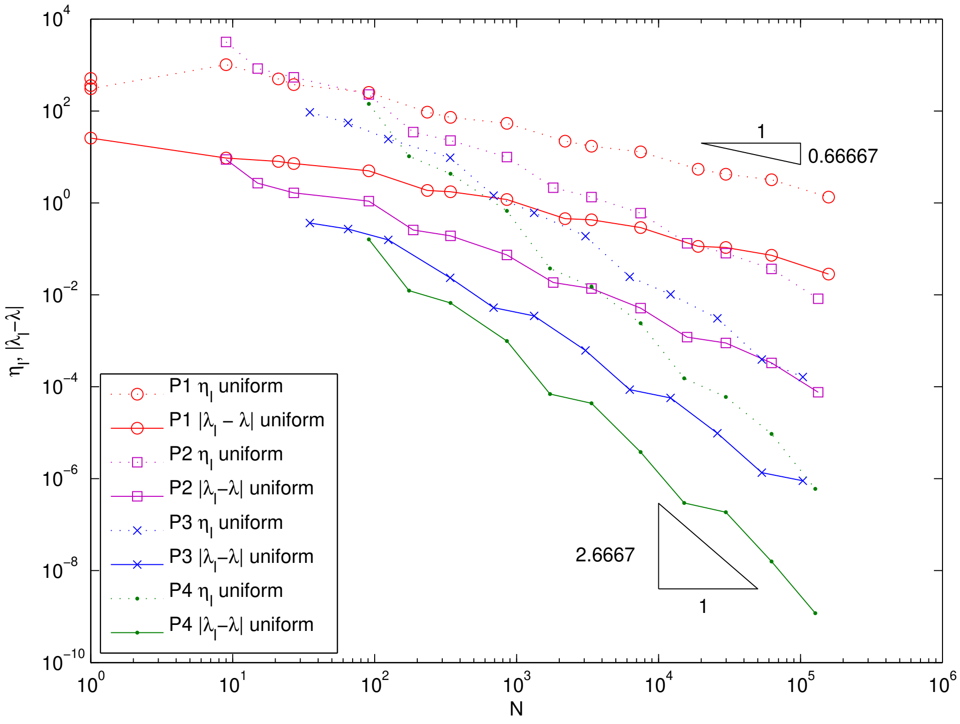

FIGURE 8.5. Eigenvalue errors and estimated errors on the cube for uniform meshes.

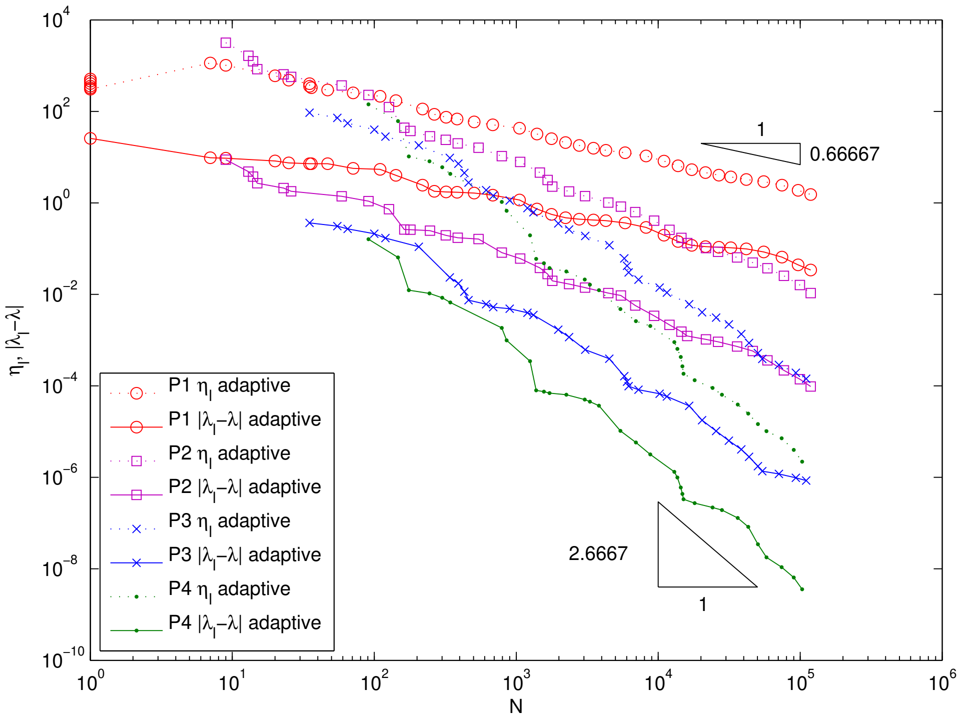

FIGURE 8.6. Eigenvalue errors and estimated errors on the cube for adaptive meshes.

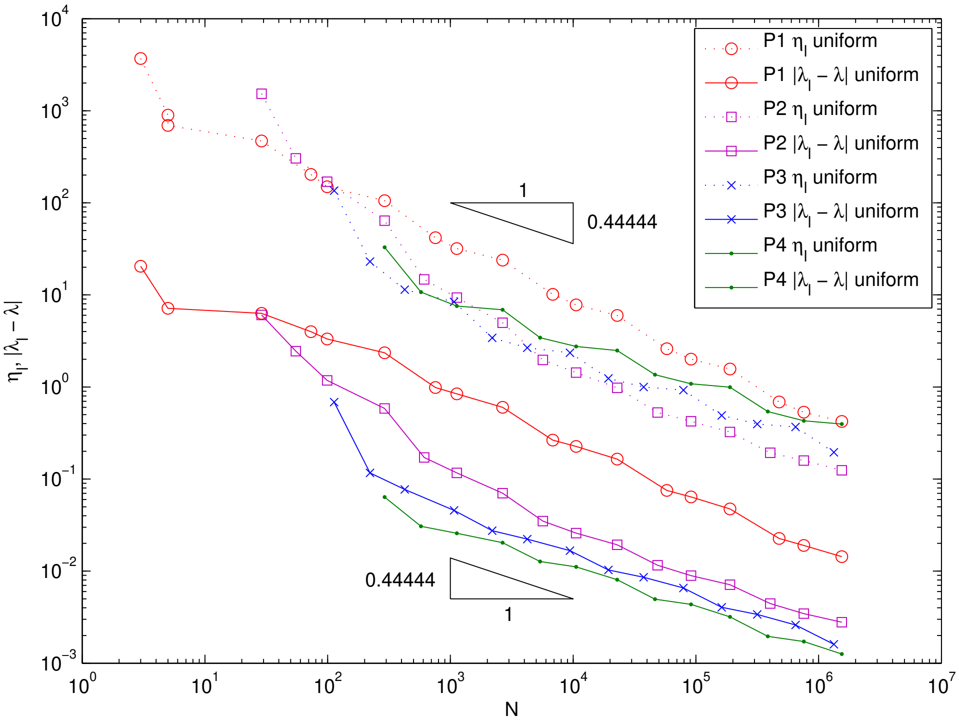

FIGURE 8.7. Eigenvalue errors and estimated errors on the three-dimensional L shaped domain for uniform meshes.

In this non-convex three-dimensional example uniform refinement results in suboptimal convergence rates as shown in Figure 8.7. Figure 8.8 shows that the AFEM with exact algebraic eigenvalue solver leads to empirically optimal convergence rates of order $O(h^{2k})$ for $k = 1, 2$. Where for $k = 3, 4$ the convergence rates are smaller than those in the convex case due to the edge-singularity at the corner. Nevertheless, the AFEM for $k = 3, 4$ results in similar or even better convergence rates than isotropic graded meshes with parameter $1/(2k)$. Therefore the convergence rates for the adaptive meshes seem to be quasi-optimal. Both for uniform and adaptively refined meshes the error estimator shows numerically to be reliable and efficient.

Acknowledgements

The work of the two authors was supported by the German Research Foundation (DFG) under C22 in the Research Center MATHEON. The work of the first author was partly supported by the WCU program through KOSEF (R31-2008-000-10049-0). The second author was also supported by the graduate school BMS.

References

[ACF99] J. Alberty, C. Carstensen, and S.A. Funken, *Remarks around 50 lines of matlab: short finite element implementation*, Numer. Algorithms **20** (1999), 117–137.

[AMP00] D. Arnold, A. Mukherjee, and L. Pouly, *Locally adapted tetrahedral meshes using bisection*, SIAM J. Sci. Comput. **22** (2000), 431–448.

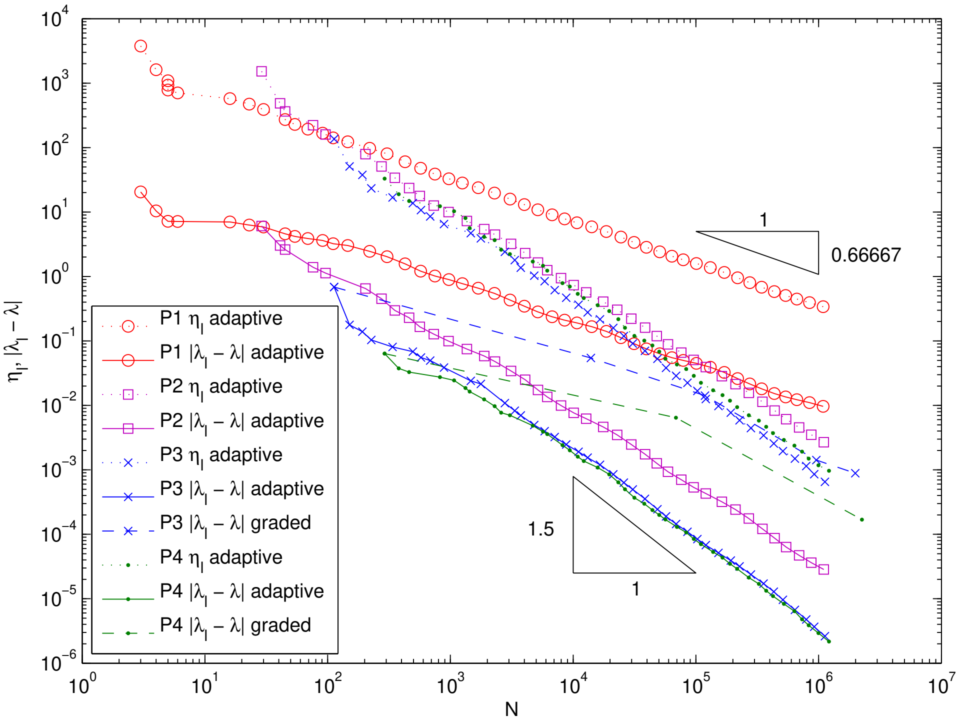

FIGURE 8.8. Eigenvalue errors and estimated errors on the three-dimensional L shaped domain for adaptive meshes.

[AO00] M. Ainsworth and J.T. Oden, *A posteriori error estimation in finite element analysis*, John Wiley & Sons, Inc., 2000.

[BDD04] P. Binev, W. Dahmen, and R. De Vore, *Adaptive finite element methods with convergence rates*, Numer. Math. **97** (2004), 219–268.

[BGG+07] A. Byfut, J. Gedicke, D. Günther, J. Reininghaus, and S. Wiedemann, *openFFW, The Finite Element Framework*, GNU General Public License v.3., 2007,
http://code.google.com/p/openffw/.

[BS08] S.C. Brenner and L.R. Scott, *The mathematical theory of finite element methods*, third ed., Texts in Applied Mathematics, Springer-Verlag, 2008.

[CG08] C. Cartensen and J. Gedicke, *An oscillation-free adaptive FEM for symmetric eigenvalue problems*, Preprint 489, DFG Research Center MATHEON, Straße des 17.Juni 136, D-10623 Berlin, 2008.

[CKNS08] J.M. Cascon, C. Kreuzer, R.H. Nochetto, and K.G. Siebert, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal. **46** (2008), no. 5, 2524–2550.

[Dör96] W. Dörfler, *A convergent adaptive algorithm for poisson's equation*, SIAM J. Numer. Anal. **33** (1996), 1106–1124.

[DPR03] R.G. Durán, C. Padra, and R. Rodríguez, *A posteriori error estimates for the finite element approximation of eigenvalue problems*, Mathematical Models and Methods in Applied Sciences **13** (2003), 1219–1229.

[DRSZ08] W. Dahmen, T. Rohwedder, R. Schneider, and A. Zeiser, *Adaptive eigenvalue computation: complexity estimates*, Numer. Math. **110** (2008), no. 3, 277–312.

[DXZ08] X. Dai, J. Xu, and A. Zhou, *Convergence and optimal complexity of adaptive finite element eigenvalue computations*, Numer. Math. **110** (2008), no. 3, 313–355.

[FPW08] S. Funken, D. Praetorius, and P. Wissgott, *Efficient implementation of adaptive P1-FEM in matlab*, Tech. report, ASC Report 19/2008, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien, 2008.

[GG09] S. Giani and I.G. Graham, *A convergent adaptive method for elliptic eigenvalue problems*, SIAM J. Numer. Anal. **47** (2009), 1067–1091.

[GMZ08] E.M. Garau, P. Morin, and C. Zuppa, *Convergence of adaptive finite element methods for eigenvalue problems*, Preprint arXiv:0803.0365v1, 2008, http://arxiv.org/abs/0803.0365v1.

[KN09] A. Knyazev and K. Neymeyr, *Gradient flow approach to geometric convergence analysis of preconditioned eigensolvers*, SIAM J. Matrix Analysis **31** (2009), 621–628.

[MM09] V. Mehrmann and A. Miedlar, *Adaptive solution of elliptic PDE-eigenvalue problems. part I: Eigenvalues*, Preprint 565, DFG Research Center Matheon, Straße des 17.Juni 136, D-10623 Berlin, 2009.

[MSZ06] D. Mao, L. Shen, and A. Zhou, *Adaptive finite element algorithms for eigenvalue problems based on local averaging type a posteriori error estimates*, Advanced in Computational Mathematics **25** (2006), 135–160.

[Ney02] K. Neymeyr, *A posteriori error estimation for elliptic eigenproblems*, Numer. Linear Algebra Appl. **9** (2002), 263–279.

[Sau07] S. Sauter, *Finite elements for elliptic eigenvalue problems in the preasymptotic regime*, Preprint 17-2007, Institut für Mathematik der Universität Zürich, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, 2007.

[SF73] G. Strang and G.J. Fix, *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, 1973.

[Ste07] R. Stevenson, *Optimality of a standard adaptive finite element method*, Foundations of Computational Mathematics **7** (2007), no. 2, 245–269.

[Ste08] ———, *The completion of locally refined simplicial partitions created by bisection*, Mathemathics of Computation **77** (2008), no. 261, 227–241.

[Ver96] R. Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Wiley and Teubner, 1996.

(C. Carstensen) Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany;
Department of Computational Science and Engineering, Yonsei University, 120–749 Seoul, Korea.

E-mail address: cc@mathematik.hu-berlin.de

(J. Gedicke) Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

E-mail address: gedicke@mathematik.hu-berlin.de