Jordan Structures of Alternating Matrix Polynomials *

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Abstract

Alternating matrix polynomials, that is, polynomials whose coefficients alternate between symmetric and skew-symmetric matrices, generalize the notions of even and odd scalar polynomials. We investigate the Smith forms of alternating matrix polynomials, showing that each invariant factor is an even or odd scalar polynomial. Necessary and sufficient conditions are derived for a given Smith form to be that of an alternating matrix polynomial. These conditions allow a characterization of the possible Jordan structures of alternating matrix polynomials, and also lead to necessary and sufficient conditions for the existence of structure-preserving strong linearizations. Most of the results are applicable to singular as well as regular matrix polynomials.

Key words. matrix polynomial, matrix pencil, structured linearization, Smith form, Jordan form, elementary divisor, invariant factor, invariant polynomial, alternating matrix polynomial, even/odd matrix polynomial.

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1 Introduction

We investigate matrix polynomials $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0$ with coefficient matrices $A_j \in \mathbb{F}^{n\times n}$, $j = 0, \ldots, k$, $A_k \neq 0$, where $\mathbb{F}$ is an arbitrary field of characteristic different from two (denoted $\operatorname{char} \mathbb{F} \neq 2$). In particular, we consider matrix polynomials with coefficients A_j that alternate between symmetric and skew-symmetric matrices.

Definition 1.1. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial.*

(a) *P is said to be* T-even *if* $P(-\lambda) = P(\lambda)^T$.

(b) *P is said to be* T-odd *if* $P(-\lambda) = -P(\lambda)^T$.

(c) *P is called* T-alternating *if it is either T-even or T-odd.*

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The names "T-even" and "T-odd" were chosen because of the analogy with even and odd functions, while the term "T-alternating" stems from the observation that the coefficient matrices of T-even or T-odd matrix polynomials alternate between symmetric and skew-symmetric matrices: if $P(\lambda)$ is T-even then the coefficient matrices of even powers of λ are symmetric and all remaining ones are skew-symmetric, while if $P(\lambda)$ is T-odd then the coefficients of odd powers of λ are symmetric and all others are skew-symmetric. The "T" in these names emphasizes the involvement of the transposition operation (even when $\mathbb{F}$ is the field of complex numbers). For the special case $n = 1$, we often drop the "T" and speak of just even or odd scalar polynomials, or use the term alternating scalar polynomials as inclusive of both even and odd polynomials.

Alternating matrix polynomials arise in a variety of applications such as the computation of corner singularities for anisotropic elastic structures [3, 22], and the optimal control of second or higher order linear systems [23], see also [20] for further applications. In most numerical solution methods for eigenvalue problems with alternating matrix polynomials, the polynomial eigenvalue problem is first turned into a linear eigenvalue problem via linearization, defined as follows [11].

Definition 1.2. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree $k \geq 1$. A matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{kn \times kn}$ is a* linearization *of $P(\lambda)$ if there exist unimodular (i.e., with constant nonzero determinant) matrix polynomials $E(\lambda), F(\lambda)$ such that*

$$E(\lambda)L(\lambda)F(\lambda) = \left[\begin{array}{c|c} P(\lambda) & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right].$$

Linearizations preserve finite eigenvalues and their associated Jordan chains when $P(\lambda)$ is regular; however, the structure associated with the infinite eigenvalue may be altered. In order to guarantee that the Jordan structure at ∞ is also preserved, the following strengthened notion of linearization was introduced in [10], and given the name *strong linearization* in [17].

Definition 1.3. *Let $P(\lambda)$ be a matrix polynomial of degree $k \geq 1$.*

(a) *The* reversal *of $P(\lambda)$ is the polynomial*

$$\mathrm{rev}P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^i A_{k-i}\,.$$

(b) *A matrix pencil $L(\lambda)$ is called a* strong linearization *for $P(\lambda)$ if $L(\lambda)$ is a linearization for $P(\lambda)$ and $\mathrm{rev}L(\lambda)$ is a linearization for $\mathrm{rev}P(\lambda)$.*

Once a linearization has been obtained, one may employ standard numerical techniques for the solution of the corresponding eigenvalue problem [4, 12, 28]. However, alternating matrix polynomials have a special eigenvalue pairing, and for many applications it is essential that this pairing be preserved by finite precision arithmetic computations. It is therefore important that the linearization being used have the same structure, and hence the same eigenvalue pairing as the original matrix polynomial. Conditions when it is possible to find a T-even or T-odd linearization were investigated in [20, 23]. In particular, it was shown in [20] that the presence of both zero and infinite eigenvalues may lead to difficulties in the construction of such structure-preserving linearizations. The following example illustrates one of the problems that may be encountered.

Example 1.4. The quadratic T-even matrix polynomial

$$P(\lambda) = \sum_{i=0}^{2} \lambda^i A_i = \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

does not admit a T-alternating strong linearization. Indeed, see [10], any strong linearization $L(\lambda) = \lambda X + Y$ of $P(\lambda)$ must be strictly equivalent to the first companion form

$$C_1(\lambda) = \lambda \left[\begin{array}{c|c} A_2 & 0 \\ \hline 0 & I_2 \end{array}\right] + \left[\begin{array}{c|c} A_1 & A_0 \\ \hline -I_2 & 0 \end{array}\right] = \lambda \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] - \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right],$$

i.e., there must exist nonsingular matrices $Q, R \in \mathbb{F}^{n\times n}$ such that $Q \cdot L(\lambda) \cdot R = C_1(\lambda)$. Since both coefficient matrices of $C_1(\lambda)$ have rank 3, so must the matrices X and Y. However, if $L(\lambda)$ is to be T-alternating, one of X, Y must be skew-symmetric. But any skew-symmetric matrix over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$ must have even rank. Thus $P(\lambda)$ does not admit a T-alternating strong linearization.

The underlying reason for the non-existence of a T-alternating strong linearization in Example 1.4 is that the Kronecker canonical form for alternating pencils has a very special structure, which imposes restrictions on the types of Jordan blocks associated with the eigenvalues zero or infinity that may be present [27]. For matrix polynomials no Kronecker canonical form is available in general, but information about the Jordan structure can be read off from the Smith form, see, e.g., [9, 11, 18]. It is therefore quite natural to investigate the possible Smith forms for alternating matrix polynomials, and to analyze the extra properties that follow from this alternating structure.

After providing the relevant mathematical background in Section 2, we show in Section 3 that the individual invariant polynomials of any T-alternating polynomial each have an alternating structure, and then characterize all possible Smith forms of T-alternating matrix polynomials. These results lead to the derivation in Section 4 of the special Jordan structures that may occur for this class of matrix polynomials. Section 5 then considers the existence of structure-preserving strong linearizations for T-alternating matrix polynomials, and an important distinction between the odd and even degree cases is delineated. Although we present a complete resolution of the odd degree case, for even degrees we are only able to characterize which *regular* T-alternating matrix polynomials allow a structure-preserving strong linearization. (Recall that a matrix polynomial $P(\lambda)$ is said to be regular if $\det P(\lambda) \not\equiv 0$, otherwise $P(\lambda)$ is singular.) Finally, in Section 6 we revisit the question of the possible Jordan structures that may occur for T-alternating polynomials, giving an alternative derivation which provides a different insight into the results presented in this paper.

It is worth emphasizing that, with the exception of Theorem 5.5, the matrix polynomials in this paper are not assumed to be regular, so that most of the results presented here apply to singular polynomials as well.

2 Background: Tools from Matrix Theory

In this section we review some well-known tools and results from matrix theory that will be needed in the following sections. For detailed proofs, the reader may refer to standard

monographs like [9, Ch.VI], [11, Part IV], [18]. Throughout the paper we use $\mathbb{N}$ to denote the set of non-negative integers $\{0, 1, 2, \dots, \}$.

2.1 Smith form and invariant polynomials

Two $m \times n$ matrix polynomials $P(\lambda), Q(\lambda)$ are said to be *equivalent*, denoted $P(\lambda) \sim Q(\lambda)$, if there exist unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ of size $m \times m$ and $n \times n$, respectively, such that

$$E(\lambda)P(\lambda)F(\lambda) = Q(\lambda). \tag{2.1}$$

The canonical form of a matrix polynomial $P(\lambda)$ under equivalence transformations is referred to as the *Smith form* of $P(\lambda)$. This form was first developed for integer matrices by H.J.S. Smith [26] in the context of solving linear systems of Diophantine equations [19]. It was then extended by Frobenius in [8] to matrix polynomials; for a more modern treatment see, e.g., [9] or [18].

Theorem 2.1 (Smith form)**.**
Let $P(\lambda)$ be an $m \times n$ matrix polynomial over an arbitrary field $\mathbb{F}$. Then there exists $r \in \mathbb{N}$, and unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ of size $m \times m$ and $n \times n$, respectively, such that

$$E(\lambda)P(\lambda)F(\lambda) = \operatorname{diag}(d_1(\lambda), \dots, d_{\min\{m,n\}}(\lambda)) =: D(\lambda), \tag{2.2}$$

where $d_1(\lambda), \dots, d_r(\lambda)$ are monic (i.e., the highest degree terms all have coefficient 1), $d_{r+1}(\lambda), \dots, d_{\min\{m,n\}}(\lambda)$ are identically zero, and $d_j(\lambda)$ is a divisor of $d_{j+1}(\lambda)$ for $j = 1, \dots, r-1$. Moreover, $D(\lambda)$ is unique.

The r nonzero diagonal elements $d_j(\lambda)$ in the Smith form are called the *invariant polynomials* or *invariant factors* of $P(\lambda)$. They can be characterized as ratios of greatest common divisors of minors of $P(\lambda)$, as stated in Theorem 2.2. We will mostly use the term "invariant polynomials" to avoid confusion with the factors of the invariant polynomials. A variation of the notation in [13] will greatly facilitate working with minors. For an $m \times n$ matrix A, let $\alpha \subseteq \{1, \dots, m\}$ and $\beta \subseteq \{1, \dots, n\}$ be arbitrary index sets of cardinality $j \leq \min(m, n)$. Then $A_{\alpha\beta}$ denotes the $j \times j$ submatrix of A in rows α and columns β; the determinant $\det A_{\alpha\beta}$ is called the *$\alpha\beta$-minor of order j* of A. Note that A has $\binom{m}{j} \cdot \binom{n}{j}$ minors of order j.

For $d(x) \neq 0$, it is standard notation to write $d(x) \,|\, p(x)$ to mean that $d(x)$ is a divisor of $p(x)$, i.e., there exists some $q(x)$ such that $p(x) = d(x)q(x)$. Note that $d(x) \,|\, 0$ is true for any $d(x) \neq 0$. Extending this notation to a set S of scalar polynomials, we write $d \,|\, S$ to mean that $d(x)$ divides each element of S, i.e., $d(x)$ is a *common divisor* of the elements of S. The *greatest common divisor* (or GCD) of a set S containing at least one nonzero polynomial is the unique monic polynomial $g(x)$ such that $g(x) \,|\, S$, and if $d(x) \,|\, S$ then $d(x) \,|\, g(x)$. With this preparation, we can now state the next theorem.

Theorem 2.2 (Characterization of invariant polynomials)**.**
Let $P(\lambda)$ be an $m \times n$ matrix polynomial over an arbitrary field $\mathbb{F}$ with Smith form (2.2). Set $p_0(\lambda) \equiv 1$. For $1 \leq j \leq \min(m, n)$, let $p_j(\lambda) \equiv 0$ if all minors of $P(\lambda)$ of order j are zero; otherwise, let $p_j(\lambda)$ be the greatest common divisor of all minors of $P(\lambda)$ of order j. Then the number r in Theorem 2.1 is the largest integer such that $p_r(\lambda) \not\equiv 0$. Furthermore, the invariant polynomials $d_1(\lambda), \dots, d_r(\lambda)$ of $P(\lambda)$ are ratios of GCDs given by

$$d_j(\lambda) = \frac{p_j(\lambda)}{p_{j-1}(\lambda)}, \quad j = 1, \dots, r,$$

while the remaining diagonal entries of the Smith form of $P(\lambda)$ are given by

$$d_j(\lambda) = p_j(\lambda) \equiv 0\,, \quad j = r+1, \ldots, \min\{m, n\}.$$

The uniqueness of $D(\lambda)$ in the Smith form (2.2) follows from this characterization, since it can be shown that the GCD of all minors of order j is invariant under equivalence of matrix polynomials.

Theorem 2.2 serves as the foundation for the proofs of our results in Section 3 on Smith forms of alternating matrix polynomials. Both Theorem 2.1 and Theorem 2.2 can be used with greater ease for our purposes by adopting the following conventions:

(i) $d(x) \,|\, 0$ is true even when $d(x) \equiv 0$ $\big($note that there exists $q(x)$ such that $0 = 0 \cdot q(x)\big)$.

(ii) the GCD of a collection of polynomials, all of which are zero, is the zero polynomial.

These conventions, which streamline the notions of divisibility and GCD, greatly simplify the proofs of later results. Observe, however, that the *quotient* $\frac{0}{0}$ remains undefined despite $0 \,|\, 0$ being a true statement. The first convention allows us to propagate the successive divisibility property of the invariant polynomials $d_j(\lambda)$ in (2.2) all the way down the diagonal of the Smith form, while the second unites the specification of all the polynomials $p_j(\lambda)$ in Theorem 2.2 with $j > 0$ as GCDs of $j \times j$ minors, irrespective of whether there is a nonzero $j \times j$ minor or not. Note that the second convention is consistent with the first, since when S contains only zero polynomials, then $g(x) \equiv 0$ is the unique polynomial such that $g \,|\, S$, and if $d \,|\, S$ then $d \,|\, g$. Our conventions are also consistent with the ring-theoretic characterization of the GCD of a set S containing at least one nonzero polynomial as the unique monic generator of the smallest ideal containing S. Since the polynomial ring $\mathbb{F}[x]$ is a principal ideal domain, the existence of such a generator is guaranteed. When S consists of only zero polynomials, then the smallest ideal containing S is $\{0\}$, and its (unique) generator is the zero polynomial. Thus by our convention the GCD of a collection S of polynomials corresponds in all cases to a generator of the smallest ideal containing S.

Since we are considering matrix polynomials over (almost) arbitrary fields $\mathbb{F}$, it is important to know what effect the choice of field has on the Smith form. In particular, if $P(\lambda)$ is a matrix polynomial over $\mathbb{F}$, does its Smith form change if we view $P(\lambda)$ as a matrix polynomial over a larger field $\widetilde{\mathbb{F}}$, as now a larger class of unimodular transformations $\widetilde{E}(\lambda)$ and $\widetilde{F}(\lambda)$ are admitted in (2.2)? Observe that the Smith form over the smaller field $\mathbb{F}$ remains a Smith form over the larger field $\widetilde{\mathbb{F}}$. Uniqueness of the Smith form over $\widetilde{\mathbb{F}}$ now implies that expanding $\mathbb{F}$ to $\widetilde{\mathbb{F}}$ cannot affect the Smith form. Since this observation will be needed later, we state it as a lemma:

Lemma 2.3. *Suppose $P(\lambda)$ is a matrix polynomial over the field $\mathbb{F}$, and $\mathbb{F} \subseteq \widetilde{\mathbb{F}}$ is any field extension. Then the Smith form of $P(\lambda)$ as a polynomial over $\mathbb{F}$ is exactly the same as the Smith form of $P(\lambda)$ as a polynomial over $\widetilde{\mathbb{F}}$.*

2.2 Elementary divisors and Jordan structures

Whenever the field $\mathbb{F}$ is algebraically closed, the invariant polynomials of the Smith form (2.2) can be uniquely represented as products of powers of linear factors

$$d_i(\lambda) = (\lambda - \lambda_{i,1})^{\alpha_{i,1}} \cdots (\lambda - \lambda_{i,k_i})^{\alpha_{i,k_i}}, \qquad i = 1, \ldots, r,$$

where $\lambda_{i,1}, \ldots, \lambda_{i,k_i} \in \mathbb{F}$ are distinct and $\alpha_{i,1}, \ldots, \alpha_{i,k_i}$ are positive integers. Then the factors $(\lambda - \lambda_{i,j})^{\alpha_{i,j}}$, $j = 1, \ldots, k_i$, $i = 1, \ldots, r$ are called the *elementary divisors* of $P(\lambda)$. Note that some polynomials $(\lambda - \lambda_0)^\alpha$ may occur multiple times as elementary divisors of $P(\lambda)$, because they may appear as factors in distinct invariant polynomials $d_{i_1}(\lambda)$ and $d_{i_2}(\lambda)$. A list of all the elementary divisors may therefore, in general, include some repetitions.

We make the convention that the elementary divisors of a matrix polynomial $P(\lambda)$ over a general field $\mathbb{F}$ are just those of $P(\lambda)$ viewed as a polynomial over the algebraic closure $\overline{\mathbb{F}}$. While this convention differs from the one used, for example, in [9], it yields the greatest simplicity in the statement of later results in this paper. Note also the consistency of this convention with the result of Lemma 2.3.

In the particular case of matrices and matrix pencils, elementary divisors are closely related to the Jordan blocks in the corresponding Jordan or Kronecker canonical form. Indeed, for a matrix $A \in \mathbb{C}^{n\times n}$, each elementary divisor $(\lambda - \lambda_0)^\alpha$ of the matrix pencil $\lambda I - A$ corresponds to a Jordan block of size $\alpha \times \alpha$ associated with the eigenvalue λ_0 of A. Thus the Smith form of the pencil $\lambda I - A$ can be used to deduce the Jordan canonical form of the matrix A [9]. For example, if $\lambda I - A$ has the Smith form $D(\lambda)$ in (2.3), then A is similar to the Jordan matrix $\mathcal{J}$ in (2.3).

$$D(\lambda) = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \lambda - 2 & & \\ & & & & (\lambda-2)^2 & \\ & & & & & (\lambda-2)^2(\lambda-3) \end{bmatrix}, \quad \mathcal{J} = \left[\begin{array}{c|cc|cc} 2 & & & & \\ \hline & 2 & 1 & & \\ & & 2 & & \\ \hline & & & 2 & 1 \\ & & & & 2 \\ \hline & & & & 3 \end{array}\right] \tag{2.3}$$

Elementary divisors also display the Jordan structure of regular matrix polynomials $P(\lambda)$ of degree greater than one. Recall from [11] that if a polynomial $P(\lambda)$ over $\mathbb{F} = \mathbb{C}$ is regular, then a *Jordan chain* corresponding to an eigenvalue $\lambda_0 \in \mathbb{C}$ of $P(\lambda)$ is a sequence of vectors $(x_0, x_1, \ldots, x_{\beta-1})$, $x_0 \neq 0$, such that

$$\sum_{j=0}^{i} \frac{1}{j!} P^{(j)}(\lambda_0) x_{i-j} = 0, \qquad i = 0, \ldots, \beta - 1.$$

Here $P^{(j)}(\lambda)$ denotes the jth derivative of $P(\lambda)$ with respect to λ. A set of Jordan chains for the eigenvalue λ_0

$$(x_{m,0}, \ldots, x_{m,\beta_m-1}), \quad m = 1, \ldots, \ell$$

is said to be *canonical* if $x_{1,0}, x_{2,0}, \ldots, x_{\ell,0}$ are linearly independent (note that these are eigenvectors of $P(\lambda)$ associated with λ_0), and if $\beta_1 + \cdots + \beta_\ell$ is the multiplicity of λ_0 as a zero of $\det P(\lambda)$. In this case, the pair of matrices $(X, \mathcal{J})$ with

$$X = [x_{1,0}, \ldots, x_{1,\beta_1-1},\ x_{2,0}, \ldots, x_{2,\beta_2-1},\ \ldots,\ x_{\ell,0}, \ldots, x_{\ell,\beta_\ell-1}], \tag{2.4}$$

$$\mathcal{J} = \operatorname{diag}\big(\mathcal{J}_{\beta_1}(\lambda_0), \ldots, \mathcal{J}_{\beta_\ell}(\lambda_0)\big), \tag{2.5}$$

is called a *Jordan pair* of $P(\lambda)$ corresponding to λ_0. Here $\mathcal{J}_\beta(\lambda_0)$ denotes the Jordan block of size $\beta \times \beta$ associated with λ_0. The relation of Jordan pairs and elementary divisors corresponding to λ_0 is then the following: if $(\beta_1, \ldots, \beta_\ell)$ is the list of partial multiplicities of

λ_0 in the Jordan pair $(X, \mathcal{J})$ and if $(\alpha_1, \ldots, \alpha_k)$ is the list of the degrees of all the elementary divisors of $P(\lambda)$ corresponding to λ_0, then $\ell = k$ and $(\beta_1, \ldots, \beta_\ell)$ is a permutation of $(\alpha_1, \ldots, \alpha_k)$.

By definition [11], the Jordan structure of $P(\lambda)$ at $\lambda_0 = \infty$ corresponds to the Jordan structure at $\lambda_0 = 0$ of the reversal of $P(\lambda)$. Thus the Jordan structure of $P(\lambda)$ at ∞ can be read off from the Smith form of $\operatorname{rev} P(\lambda)$, or more precisely, from the elementary divisors of $\operatorname{rev} P(\lambda)$ corresponding to $\lambda_0 = 0$. We will refer to those as the *infinite elementary divisors* of $P(\lambda)$. In particular, if $P(\lambda)$ is regular, then a pair $(X_\infty, \mathcal{J}_\infty)$ is called an *infinite Jordan pair* of $P(\lambda)$ if it is a Jordan pair of $\operatorname{rev} P(\lambda)$ associated with the eigenvalue $\lambda_0 = 0$.

In the remainder of this paper we focus on elementary divisors rather than on Jordan chains. This allows us to consider regular and singular matrix polynomials on an equal footing, as well as to deal with all degrees in a uniform way. From now on, then, we will use the phrase "Jordan structure of $P(\lambda)$" in the following sense.

Definition 2.4 (Jordan structure of a matrix polynomial)**.**
For an $m \times n$ matrix polynomial $P(\lambda)$ over the field $\mathbb{F}$, the Jordan Structure of $P(\lambda)$ *is the collection of all the finite and infinite elementary divisors of $P(\lambda)$, including repetitions, where $P(\lambda)$ is viewed as a polynomial over the algebraic closure $\overline{\mathbb{F}}$.*

2.3 Compound matrices and their properties

Compound matrices will be an important ingredient in the proofs of our main results. We recall here the definition and some key properties, as well as prove some new results on the compounds of structured matrix polynomials, and on the invariant polynomials of compounds of general matrix polynomials. For further information, see for example [13, Sect.0.8], [21, Ch.I, Sect.2.7], or [24, Sect.2 and 28].

Definition 2.5 (Compound Matrices)**.**
Let A be an $m \times n$ matrix and let $r \leq \min(m, n)$ be a positive integer. Then the rth compound matrix (*or the rth* adjugate) *of A, denoted $\mathcal{C}_r(A)$, is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose (α, β)-entry is the $r \times r$ minor $\det A_{\alpha\beta}$ of A. Here, the index sets $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$ of cardinality r are ordered lexicographically.*

Observe that we always have $\mathcal{C}_1(A) = A$ and, if A is square, $\mathcal{C}_n(A) = \det A$. The key properties of $\mathcal{C}_r(A)$ that we need are collected in the following theorem.

Theorem 2.6 (Properties of compound matrices)**.**
Let $A \in \mathbb{F}^{m \times n}$ and let $r \leq \min(m, n)$ be a positive integer. Then

(a) $\mathcal{C}_r(A^T) = \big(\mathcal{C}_r(A)\big)^T$;

(b) $\mathcal{C}_r(\mu A) = \mu^r \mathcal{C}_r(A)$, *where* $\mu \in \mathbb{F}$;

(c) $\det \mathcal{C}_r(A) = (\det A)^\beta$, *where* $\beta = \binom{n-1}{r-1}$, *provided that* $m = n$;

(d) $\mathcal{C}_r(AB) = \mathcal{C}_r(A)\mathcal{C}_r(B)$, *provided that* $B \in \mathbb{F}^{n \times p}$ *and* $r \leq \min(m, n, p)$.

(e) *If $A \in \mathbb{F}^{n \times n}$ is a diagonal matrix, then $\mathcal{C}_r(A)$ is also diagonal.*

These properties can now be used to prove that the compounds of structured matrix polynomials also have structure:

Corollary 2.7 (Compounds of structured matrix polynomials)**.**

(a) *The rth compound of a T-even polynomial is T-even.*

(b) *The rth compound of a T-odd polynomial is* $\begin{cases} T\text{-even when } r \text{ is even,} \\ T\text{-odd when } r \text{ is odd.} \end{cases}$

(c) *The rth compound of a unimodular polynomial is unimodular.*

(d) *The rth compounds of equivalent matrix polynomials are equivalent, i.e.,*
$P(\lambda) \sim Q(\lambda) \;\Rightarrow\; \mathcal{C}_r\big(P(\lambda)\big) \sim \mathcal{C}_r\big(Q(\lambda)\big)$.

Proof. Parts (a), (b), and (c) follow immediately from the corresponding properties in Theorem 2.6 applied to the definitions of T-even, T-odd, and unimodular matrix polynomials. For part (d) suppose that $P(\lambda)$ and $Q(\lambda)$ are equivalent, so that $E(\lambda)P(\lambda)F(\lambda) = Q(\lambda)$ for some unimodular $E(\lambda)$ and $F(\lambda)$. Then from Theorem 2.6(d) we have

$$\mathcal{C}_r\big(E(\lambda)\big) \cdot \mathcal{C}_r\big(P(\lambda)\big) \cdot \mathcal{C}_r\big(F(\lambda)\big) \;=\; \mathcal{C}_r\big(Q(\lambda)\big)\,.$$

Since $\mathcal{C}_r\big(E(\lambda)\big)$ and $\mathcal{C}_r\big(F(\lambda)\big)$ are unimodular by part (c), we see that equivalences of matrix polynomials "lift" to equivalences of their rth compounds. □

Are the invariant polynomials of $\mathcal{C}_r\big(P(\lambda)\big)$ related to those of $P(\lambda)$? The next lemma establishes a simple result along these lines.

Lemma 2.8 (First two invariant polynomials of the rth compound of $P(\lambda)$)**.**
Suppose the Smith form of an $n \times n$ matrix polynomial $P(\lambda)$ is

$$S(\lambda) \;=\; \operatorname{diag}\big(p_1(\lambda)\,,\,\ldots\,,\,p_{r-1}(\lambda)\,,\,p_r(\lambda)\,,\,p_{r+1}(\lambda)\,,\,\ldots\,,\,p_n(\lambda)\big)\,,$$

and for $2 \le r < n$ denote the Smith form of the rth compound $\mathcal{C}_r\big(P(\lambda)\big)$ by

$$D(\lambda) = \operatorname{diag}\big(d_1(\lambda)\,,\,d_2(\lambda)\,,\ldots\,,\,d_{\binom{n}{r}}(\lambda)\big).$$

Then the first two diagonal entries of $D(\lambda)$ are given by

$$d_1(\lambda) = p_1(\lambda)\cdots p_{r-1}(\lambda)p_r(\lambda) \quad \textit{and} \quad d_2(\lambda) = p_1(\lambda)\cdots p_{r-1}(\lambda)p_{r+1}(\lambda)\,.$$

Proof. $\mathcal{C}_r\big(P(\lambda)\big)$ is equivalent to $\mathcal{C}_r\big(S(\lambda)\big)$ by Corollary 2.7(d), so $\mathcal{C}_r\big(P(\lambda)\big)$ and $\mathcal{C}_r\big(S(\lambda)\big)$ have the same Smith form. $\mathcal{C}_r\big(S(\lambda)\big)$ is diagonal by Theorem 2.6(e), but the successive diagonal entries of $\mathcal{C}_r\big(S(\lambda)\big)$ may not have all the divisibility properties of a Smith form, so $\mathcal{C}_r\big(S(\lambda)\big) \neq D(\lambda)$ in general. However, the first two diagonal entries of $\mathcal{C}_r\big(S(\lambda)\big)$ and $D(\lambda)$ will always coincide, as we now prove. Letting

$$\mathcal{C}_r\big(S(\lambda)\big) \;=\; \operatorname{diag}\Big(q_1(\lambda)\,,\,q_2(\lambda)\,,\,\ldots\,,\,q_{\binom{n}{r}}(\lambda)\Big),$$

it is easy to see from the definition of $\mathcal{C}_r$ that each $q_j(\lambda)$ is a product of r diagonal entries of $S(\lambda)$. Since these products are arranged lexicographically, we know that

$$q_1(\lambda) = p_1(\lambda)\cdots p_{r-1}(\lambda)p_r(\lambda) \quad \text{and} \quad q_2(\lambda) = p_1(\lambda)\cdots p_{r-1}(\lambda)p_{r+1}(\lambda).$$

By the divisibility property of invariant polynomials together with the conventions established after Theorem 2.2, we have $p_j(\lambda) \,|\, p_{j+1}(\lambda)$ for $j = 1, \ldots, n-1$, and so

$$q_1 = \gcd\big(\{q_1\,,\, q_2\,,\, q_3\,,\, \ldots\,,\, q_{\binom{n}{r}}\}\big)\,, \qquad q_2 = \gcd\big(\{q_2\,,\, q_3\,,\, \ldots\,,\, q_{\binom{n}{r}}\}\big)\,,$$

and the product $q_1 q_2$ is the GCD of all 2×2 minors of $\mathcal{C}_r\big(S(\lambda)\big)$. These relations remain valid even when $P(\lambda)$ is singular, by the aforementioned conventions. Applying Theorem 2.2 to $\mathcal{C}_r\big(S(\lambda)\big)$ we get

$$d_1(\lambda) = \gcd\big(\{q_1\,,\, q_2\,,\, q_3\,,\, \ldots\,,\, q_{\binom{n}{r}}\}\big) = q_1(\lambda),$$

and hence $d_1(\lambda) = p_1(\lambda) \cdots p_{r-1}(\lambda) p_r(\lambda)$. If $q_2 = 0$, then the GCD of all 2×2 minors of $\mathcal{C}_r\big(S(\lambda)\big)$ is zero, and Theorem 2.2 gives $d_2 = 0 = q_2$. Otherwise, if q_2 is nonzero, then q_1 is also nonzero by the divisibility properties among the $p_j(\lambda)$'s, and Theorem 2.2 says

$$d_2(\lambda) = \frac{q_1 q_2}{q_1} = q_2(\lambda).$$

Thus, in all cases, $d_2(\lambda) = p_1(\lambda) \cdots p_{r-1}(\lambda) p_{r+1}(\lambda)$ as desired. □

Note that with some small modifications, this argument can be used to establish a similar result for $m \times n$ matrix polynomials that are not square.

3 Smith forms of alternating matrix polynomials

We now turn to the task of characterizing all the possible Smith forms for T-alternating (i.e., T-even or T-odd) matrix polynomials. Keep in mind that throughout Sections 3.1, 3.2, and 3.3 all matrix and scalar polynomials are over an arbitrary field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$. The situation when $\mathbb{F}$ has characteristic 2 will be briefly considered in Section 3.4. For convenience, we introduce the parity function ε of T-alternating polynomials.

Definition 3.1. *The* parity $\varepsilon(P)$ *of a T-alternating matrix polynomial $P(\lambda)$ is defined by*

$$\varepsilon(P) = \begin{cases} +1 & \text{if } P(\lambda) \text{ is } T\text{-even} \\ -1 & \text{if } P(\lambda) \text{ is } T\text{-odd}, \end{cases}$$

so that $P(-\lambda) = \varepsilon(P) \cdot P(\lambda)^T$.

The next result lists some useful (and easily verified) elementary properties of alternating scalar polynomials.

Lemma 3.2. *Let $p(\lambda), q(\lambda)$ be alternating scalar polynomials.*

(a) *If $p(\lambda)$ is even (odd), then only the coefficients associated with even (odd) powers of λ can be nonzero.*

(b) *The product $p(\lambda)q(\lambda)$ is even (odd) if p and q have the same (different) parity.*

3.1 GCDs of alternating polynomials

Theorem 2.2, which connects invariant polynomials and certain GCDs, will be a fundamental element in the proof of our characterization of all possible Smith forms for alternating matrix polynomials. Consequently, some further results on GCDs arising from such matrix polynomials are needed.

Lemma 3.3 (Subset Collapsing Lemma for GCD's)**.**
Suppose S is a finite set of scalar polynomials, and $S = S_1 \cup S_2$ for some nonempty subsets S_1 and S_2 (not necessarily disjoint). Then $\gcd(S) = \gcd\big(S_1 \cup \{\gcd(S_2)\}\big)$.

Proof. For a polynomial d, we clearly have

$$\big(d \,|\, S\big) \quad \text{if and only if} \quad \big(d \,|\, S_1 \text{ and } d \,|\, S_2\big) \quad \text{if and only if} \quad \big(d \,|\, S_1 \text{ and } d \,|\, \gcd(S_2)\big)\,.$$

Thus, the set of all common divisors of S is identical to the set of all common divisors of $S_1 \cup \{\gcd(S_2)\}$. Hence $\gcd(S)$ and $\gcd\big(S_1 \cup \{\gcd(S_2)\}\big)$ must be the same. □

Lemma 3.4 (Determinant of T-alternating polynomials)**.**
Let $P(\lambda)$ be an $n \times n$ matrix polynomial.

(a) *If $P(\lambda)$ is T-even, then the scalar polynomial* $\det P(\lambda)$ *is even;*

(b) *If $P(\lambda)$ is T-odd, then the scalar polynomial* $\det P(\lambda)$ *is* $\begin{cases} \text{even} & \text{if } n \text{ is even}, \\ \text{odd} & \text{if } n \text{ is odd}. \end{cases}$

Proof. Since $P(\lambda) = \varepsilon(P)P^T(-\lambda)$, we have $\det P(\lambda) = \big(\varepsilon(P)\big)^n \det P(-\lambda)$, from whence the desired results immediately follow. □

Lemma 3.5 (Even/oddness in polynomial division)**.**
Suppose that the scalar polynomial $p(\lambda)$ is divided by $d(\lambda) \neq 0$ with $\deg d \leq \deg p$ *to get*

$$p(\lambda) = d(\lambda)q(\lambda) + r(\lambda), \quad \deg r < \deg d.$$

If p and d are alternating (not necessarily with the same parity), then the quotient q and the remainder r are also alternating. Moreover, the three polynomials $p(\lambda)$, $r(\lambda)$, and $d(\lambda)q(\lambda)$ all have the same parity.

Proof. Starting with $p(\lambda) = d(\lambda)q(\lambda) + r(\lambda)$, it follows from $p(-\lambda) = d(-\lambda)q(-\lambda) + r(-\lambda)$ that $\varepsilon(p)p(\lambda) = \varepsilon(d)d(\lambda)q(-\lambda) + r(-\lambda)$ and hence $p(\lambda) = \varepsilon(p)\varepsilon(d)d(\lambda)q(-\lambda) + \varepsilon(p)r(-\lambda)$. Then by the uniqueness of quotient and remainder

$$\begin{aligned} q(\lambda) &= \varepsilon(p)\varepsilon(d)\, q(-\lambda)\,, \\ r(\lambda) &= \varepsilon(p)\, r(-\lambda)\,. \end{aligned}$$

Thus both $q(\lambda)$ and $r(\lambda)$ are alternating, and $p(\lambda)$, $r(\lambda)$, and $d(\lambda)q(\lambda)$ all have the same parity $\varepsilon(p)$. □

Lemma 3.6 (GCD of alternating scalar polynomials)**.**
Let S be a finite set of alternating scalar polynomials. Then $\gcd(S)$ *is also an alternating polynomial.*

Proof. Since the zero polynomial is alternating (it is both odd and even), the result holds trivially if S contains only zero polynomials. Otherwise, since retaining only the nonzero polynomials does not change the GCD, we may assume without loss of generality that S does not contain the zero polynomial. The proof now proceeds by induction on the size of S, starting with the case when S consists of two nonzero alternating polynomials f and g. The standard Euclidean algorithm to compute $\gcd(\{f, g\})$ generates a sequence of remainders by polynomial division, the last nonzero one of which is the desired GCD. But Lemma 3.5 shows that each of the remainders in this sequence is alternating; hence so is the GCD.

Now suppose the assertion holds for all sets of n alternating polynomials, and consider an arbitrary set $S = \{p_1(\lambda), \ldots, p_{n+1}(\lambda)\}$ consisting of $n+1$ alternating polynomials. By the induction hypothesis $\widehat{d}(\lambda) = \gcd(\{p_1(\lambda), \ldots, p_n(\lambda)\})$ is alternating, and by Lemma 3.3 $\gcd(S) = \gcd(\{\widehat{d}(\lambda), p_{n+1}(\lambda)\})$. Hence $\gcd(S)$ is alternating by the argument of the preceding paragraph. □

Lemma 3.7. *For an arbitrary scalar polynomial $q(\lambda)$, let $S = \{q(\lambda), q(-\lambda)\}$. Then* $\gcd(S)$ *is an alternating polynomial.*

Proof. Let $d(\lambda) = \gcd(S)$. If $q(\lambda) = 0$, then $d(\lambda) = 0$ and we are done, since the zero polynomial is alternating. Otherwise, consider the ideal $\mathcal{I} = \{p(\lambda) = a(\lambda)q(\lambda) + b(\lambda)q(-\lambda) : a(\lambda), b(\lambda) \in \mathbb{F}[\lambda]\} \subseteq \mathbb{F}[\lambda]$ generated by S. Now the ring of polynomials $\mathbb{F}[\lambda]$ over any field $\mathbb{F}$ is a Euclidean domain, so every ideal is principal. Hence $d(\lambda)$ may be characterized [7] as the unique monic generator of $\mathcal{I}$, or equivalently as the unique monic polynomial of minimal degree in $\mathcal{I}$. Since

$$d(\lambda) = a_0(\lambda)q(\lambda) + b_0(\lambda)q(-\lambda)$$

for some polynomials $a_0(\lambda)$ and $b_0(\lambda)$, it follows that $d(-\lambda) = a_0(-\lambda)q(-\lambda) + b_0(-\lambda)q(\lambda)$ is also an element of $\mathcal{I}$, with the same (minimal) degree as $d(\lambda)$. Clearly either $d(-\lambda)$ or $-d(-\lambda)$ is monic, so $d(\lambda) = \pm d(-\lambda)$ must be alternating. □

Recall that $A_{\alpha\beta}$ denotes the $j \times j$ submatrix of an $m \times n$ matrix A in rows α and columns β where $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$ are sets of cardinality $j \leq \min(m, n)$. Submatrices of A and A^T are easily seen to satisfy the basic relationship $(A^T)_{\alpha\beta} = (A_{\beta\alpha})^T$, generalizing the defining property of the transpose, i.e., $(A^T)_{ij} = A_{ji}$. When $\alpha = \beta$, then $A_{\alpha\beta}$ is a *principal* submatrix of A. When $\alpha \neq \beta$, we refer to $A_{\alpha\beta}$ and $A_{\beta\alpha}$ as a *dual pair* of submatrices of A. Determinants of dual submatrices of T-alternating polynomials turn out to be closely related, a fact that plays a key role in the following result.

Proposition 3.8 (GCD of minors of T-alternating polynomials)**.**
Let $P(\lambda)$ be an $n \times n$ T-alternating matrix polynomial, and let S be the set of all $j \times j$ minors of $P(\lambda)$, where $j \leq n$ is a positive integer. Then $\gcd(S)$ *is alternating.*

Proof. Since $P(\lambda)$ is T-alternating, $P^T(-\lambda) = \varepsilon(P) \cdot P(\lambda)$. Looking at the $\alpha\beta$-submatrix on each side of this equation gives $\big(P^T(-\lambda)\big)_{\alpha\beta} = \big(\varepsilon(P) \cdot P(\lambda)\big)_{\alpha\beta}$, which implies that

$$\big(P_{\beta\alpha}(-\lambda)\big)^T = \varepsilon(P) \cdot P_{\alpha\beta}(\lambda)\,. \tag{3.1}$$

With $\alpha = \beta$ we see that principal $j \times j$ submatrices of P inherit the property of being T-alternating, with the same parity as P. Hence by Lemma 3.4 each principal $j \times j$ minor of $P(\lambda)$ is alternating.

On the other hand, a non-principal minor of $P(\lambda)$ can be an arbitrary polynomial. But dual minors have the following simple relationship that can be quickly derived from (3.1):

$$\det P_{\beta\alpha}(-\lambda) = \det\big(P_{\beta\alpha}(-\lambda)\big)^T = \det\big(\varepsilon(P)\cdot P_{\alpha\beta}(\lambda)\big) = \big(\varepsilon(P)\big)^j \cdot \det P_{\alpha\beta}(\lambda).$$

Setting $q(\lambda) = \det P_{\beta\alpha}(\lambda)$, we see that the dual minor $\det P_{\alpha\beta}(\lambda)$ is just $\pm q(-\lambda)$. Thus, up to a sign, dual minors of T-alternating polynomials come in $\big\{q(\lambda), q(-\lambda)\big\}$ pairs; and by Lemma 3.7, any such pair always has a GCD that is alternating.

Finally consider the GCD of the set S of *all* $j \times j$ minors of $P(\lambda)$. Lemma 3.3 allows us to replace each dual minor pair in S by a single alternating polynomial, and since each principal $j\times j$ minor is already alternating, $\gcd(S)$ is the same as the GCD of a set consisting only of alternating polynomials. Thus $\gcd(S)$ must itself be alternating by Lemma 3.6. □

3.2 E-Smith form

We now characterize all possible Smith forms of T-even matrix polynomials over an arbitrary field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$. Some simple restrictions on the invariant polynomials of any T-even polynomial can be immediately derived from the results in Section 3.1.

Lemma 3.9. *Suppose that $D(\lambda) = \operatorname{diag}\big(d_1(\lambda), d_2(\lambda), \ldots, d_n(\lambda)\big)$ is the Smith form of the T-even $n \times n$ matrix polynomial $P(\lambda)$. Then the following statements hold:*

(a) *Each $d_\ell(\lambda)$ is alternating.*

(b) *If $P(\lambda)$ is regular, and ν is the number of indices ℓ for which the invariant polynomial $d_\ell(\lambda)$ is odd, then ν is even.*

Proof. Those $d_\ell(\lambda)$ that are zero are trivially alternating. Recall that each nonzero $d_\ell(\lambda)$, i.e., each invariant polynomial, is a ratio of GCD's of sets of minors. These GCD's are all alternating by Lemma 3.8, hence their ratios are all alternating by Lemma 3.5. Moreover, the product of $d_1(\lambda), \ldots, d_n(\lambda)$ is the same as $\det P(\lambda)$, up to a scalar multiple. Since $\det P(\lambda)$ is even by Lemma 3.4, and nonzero if $P(\lambda)$ is regular, the number of indices ℓ for which $d_\ell(\lambda)$ is odd must be even. □

Lemma 3.9(b) holds even[1] in the case of singular T-even matrix polynomials, as will be shown in Theorem 3.10. However, there are further constraints on the invariant polynomials that are less easy to anticipate. For example, consider $\widetilde{D}(\lambda) = \operatorname{diag}\big(\lambda, \lambda^3\big)$, which clearly satisfies both conditions (a) and (b) of Lemma 3.9. Surprisingly, $\widetilde{D}(\lambda)$ is not the Smith form of any T-even polynomial, because of additional restrictions on the degrees of the elementary divisors associated with $\lambda_0 = 0$. In order to conveniently state these additional constraints, we express invariant polynomials in the factored form $d(\lambda) = \lambda^\alpha p(\lambda)$, where $p(0) \neq 0$. Thus the Smith form of a *general* $n \times n$ matrix polynomial can be uniquely expressed as

$$D(\lambda) = \operatorname{diag}\big(\lambda^{\alpha_1}p_1(\lambda),\, \lambda^{\alpha_2}p_2(\lambda), \ldots,\, \lambda^{\alpha_\ell}p_\ell(\lambda),\, 0, \ldots, 0\big)\,, \quad \text{where}$$

- $\alpha_1, \ldots, \alpha_l$ are nonnegative integers satisfying $0 \le \alpha_1 \le \cdots \le \alpha_\ell$,
- $p_j(\lambda)$ is monic with $p_j(0) \neq 0$ for $j = 1, \ldots, \ell$,

[1] oddly enough

- $p_j(\lambda) \,|\, p_{j+1}(\lambda)$ for $j = 1, \ldots, \ell - 1$.

Theorem 3.10 (E-Smith form)**.** *Suppose that*

$$D(\lambda) = \operatorname{diag}\big(\lambda^{\alpha_1} p_1(\lambda),\, \lambda^{\alpha_2} p_2(\lambda), \ldots,\, \lambda^{\alpha_\ell} p_\ell(\lambda),\, 0, \ldots, 0\big)$$

*is an $n \times n$ diagonal matrix polynomial such that $0 \le \alpha_1 \le \cdots \le \alpha_\ell$ are nonnegative integers, $p_j(\lambda)$ is monic with $p_j(0) \neq 0$ for $j = 1, \ldots, \ell$, and $p_j(\lambda) \,|\, p_{j+1}(\lambda)$ for $j = 1, \ldots, \ell - 1$. Then $D(\lambda)$ is the Smith form of some $n \times n$ T-**even** matrix polynomial $P(\lambda)$ if and only if the following conditions hold:*

(1) *$p_j(\lambda)$ is even for $j = 1, \ldots, \ell$.*

(2) *If ν is the number of* **odd** *exponents among $\alpha_1, \ldots, \alpha_\ell$, then $\nu = 2m$ is an even integer. Letting $k_1 < k_2 < \cdots < k_{2m}$ be the positions on the diagonal of $D(\lambda)$ where these odd exponents α_{k_j} occur, the following properties hold:*

   (a) *adjacency-pairing of positions:*

$$k_2 = k_1 + 1\,, \quad k_4 = k_3 + 1\,, \quad \ldots \quad , \quad k_{2m} = k_{2m-1} + 1$$

   (b) *equality-pairing of odd exponents:*

$$\alpha_{k_2} = \alpha_{k_1}\,, \quad \alpha_{k_4} = \alpha_{k_3}\,, \quad \ldots \quad , \quad \alpha_{k_{2m}} = \alpha_{k_{2m-1}}\,.$$

Proof. ($\Rightarrow$): The necessity of condition (1) follows easily from Lemma 3.9. Each invariant polynomial $d_j(\lambda) = \lambda^{\alpha_j} p_j(\lambda)$ is alternating; thus each $p_j(\lambda)$ is either even or odd. But $p_j(0) \neq 0$, so each $p_j(\lambda)$ is even.

The necessity of condition (2), however, requires extensive further argument. Of course if there are no odd exponents, condition (2) is vacuously true. Observe also that if $P(\lambda)$ is regular, then Lemma 3.9(b) says that ν is even.

As a first step we show that if α_1 is odd, then $\alpha_2 = \alpha_1$. Next, the results on compound matrices from section 2.3 are used to "push" the pairing properties for odd exponents further down the diagonal of $D(\lambda)$, until no odd exponents remain. No assumption on the regularity of $P(\lambda)$ is needed for the argument.

Step 1: We show that if α_1 is odd, then $n \ge 2$, $\ell \ge 2$, and $\alpha_2 = \alpha_1$.
By Theorem 2.2 the first invariant polynomial $d_1(\lambda) = \lambda^{\alpha_1} p_1(\lambda)$ is just the GCD of all the entries of $P(\lambda)$. But every diagonal entry of any T-even matrix polynomial is even, and thus each $P(\lambda)_{ii}$ is divisible by λ^{α_1+1}. Hence there must be some *off-diagonal* entry $P(\lambda)_{ij}$ with $i > j$ of the form $\lambda^{\alpha_1} p_1(\lambda) s(\lambda)$, where $s(0) \neq 0$. (In particular, this implies $n \ge 2$.) Letting $\beta = \big\{i, j\big\}$, consider the principal submatrix

$$P(\lambda)_{\beta\beta} \;=\; \begin{bmatrix} \lambda^{\alpha_1+1} p_1(\lambda) r(\lambda) & -\lambda^{\alpha_1} p_1(\lambda) s(-\lambda) \\ \lambda^{\alpha_1} p_1(\lambda) s(\lambda) & \lambda^{\alpha_1+1} p_1(\lambda) t(\lambda) \end{bmatrix},$$

where $r(\lambda)$ and $t(\lambda)$ are even polynomials. Then

$$\det P(\lambda)_{\beta\beta} \;=\; \lambda^{2\alpha_1} p_1^2(\lambda) \Big(\lambda^2 r(\lambda) t(\lambda) + s(\lambda) s(-\lambda) \Big)\,. \tag{3.2}$$

Observe that the expression $\lambda^2 r(\lambda)t(\lambda) + s(\lambda)s(-\lambda)$ is nonzero because $s(0) \neq 0$. This means $\det P(\lambda)_{\beta\beta}$ is a nonzero polynomial, and hence the GCD $g(\lambda)$ of all 2×2 minors of $P(\lambda)$ is also nonzero, so $\ell \geq 2$.

From (3.2) we also see that $2\alpha_1$ is the highest power of λ that divides $\det P(\lambda)_{\beta\beta}$, since $p_1(0) \neq 0$ and $s(0) \neq 0$. Therefore the GCD $g(\lambda)$ contains a power of λ no higher than $2\alpha_1$. Recall from Theorem 2.2 that

$$g(\lambda) = d_1(\lambda)d_2(\lambda) = \lambda^{\alpha_1+\alpha_2} p_1(\lambda)p_2(\lambda)$$

with $p_1(0) \neq 0$ and $p_2(0) \neq 0$. Hence $\alpha_1 + \alpha_2 \leq 2\alpha_1$, or $\alpha_2 \leq \alpha_1$, yielding $\alpha_1 = \alpha_2$ as desired.

Step 2: Push forward.
We argue by contradiction. Let $\nu > 0$ and assume that the positions $k_1 < k_2 < \cdots < k_\nu$ of all the odd exponents do not satisfy condition (2). Let $r := k_{2j-1}$ be the first position where (2) fails, so that we have one of the following situations:

- $\nu = 2j - 1$ and k_{2j} does not exist;
- or k_{2j} exists, but adjacency-pairing of positions fails, i.e., $k_{2j} \neq k_{2j-1} + 1$;
- or k_{2j} exists and k_{2j-1}, k_{2j} are adjacent positions, but $\alpha_{k_{2j}} \neq \alpha_{k_{2j-1}}$.

Now the presence of an odd exponent (since $\nu > 0$) implies that $2 \leq r$ by Step 1. Also $r < n$, because $r = n$ would imply that $P(\lambda)$ is regular with $\nu = 2j - 1$; but this is impossible by Lemma 3.9(b). Thus $2 \leq r < n$.

Since $P(\lambda)$ is T-even, Corollary 2.7(a) tells us that $\mathcal{C}_r\big(P(\lambda)\big)$ is also T-even. By Lemma 2.8 the first invariant polynomial $c_1(\lambda)$ of $\mathcal{C}_r\big(P(\lambda)\big)$ is just the product of the first r invariant polynomials of $P(\lambda)$, i.e.,

$$c_1(\lambda) = \lambda^\alpha q(\lambda) = \lambda^\alpha \prod_{i=1}^{r} p_i(\lambda)\,, \qquad \alpha := \sum_{i=1}^{r} \alpha_i\,. \tag{3.3}$$

The definition of r guarantees $r \leq \ell$. Since $p_i(0) \neq 0$ for $i = 1, \ldots \ell$, we have $q(0) \neq 0$.

Now $r = k_{2j-1}$ says that the number of odd exponents among $\alpha_1, \alpha_2, \ldots, \alpha_r$ is $2j - 1$, i.e., an odd integer. It follows that $\alpha = \sum_{i=1}^{r} \alpha_i$ must be odd. We can therefore apply Step 1 to $\mathcal{C}_r\big(P(\lambda)\big)$ to conclude that its second invariant polynomial $c_2(\lambda) = \lambda^{\widetilde{\alpha}} \widetilde{q}(\lambda)$ is a *nonzero* polynomial with $\widetilde{\alpha} = \alpha$ and $\widetilde{q}(0) \neq 0$.

But from Lemma 2.8,

$$c_2(\lambda) = \lambda^{\alpha_1+\ldots+\alpha_{r-1}+\alpha_{r+1}} p_1(\lambda) \cdots p_{r-1}(\lambda)p_{r+1}(\lambda). \tag{3.4}$$

Since $c_2(\lambda)$ is nonzero, so is $p_{r+1}(\lambda)$, and hence $r < \ell$. Because $p_i(0) \neq 0$ for $i = 1, \ldots, \ell$, it follows from (3.4) that

$$\widetilde{q}(\lambda) = p_1(\lambda) \cdots p_{r-1}(\lambda)p_{r+1}(\lambda) \quad \text{and} \quad \widetilde{\alpha} = \alpha_1 + \ldots + \alpha_{r-1} + \alpha_{r+1}.$$

But $\alpha = \widetilde{\alpha}$ now forces $\alpha_r = \alpha_{r+1}$. Thus we see that condition (2) does not fail at position $r = k_{2j-1}$, since an odd exponent at $k_{2j} = r + 1$ exists, and satisfies adjacency-pairing as well as equality-pairing. This contradiction concludes the proof of necessity.

($\Leftarrow$): Given a diagonal matrix polynomial $D(\lambda)$ satisfying the conditions of the theorem, we show how to explicitly construct a T-even matrix polynomial whose Smith form is $D(\lambda)$.

If each diagonal entry $d_i(\lambda)$ of $D(\lambda)$ is even, then $D(\lambda)$ is T-even, its Smith form is itself, and we are done. Otherwise, by hypothesis, there are an even number of odd degree polynomials $d_i(\lambda)$, and furthermore, they must satisfy the adjacency and equality-pairing properties. Together with the divisibility property this says that the odd polynomials occur in consecutive pairs of the form

$$d_{k_{2j-1}} = \lambda^\alpha p(\lambda), \quad d_{k_{2j}} = \lambda^\alpha p(\lambda) q(\lambda)$$

where α is odd, and $p(\lambda)$, $q(\lambda)$ are even polynomials satisfying $p(0) \neq 0$ and $q(0) \neq 0$. Now since q is an even polynomial, λ^2 divides $q(\lambda) - q(0)$, so $q(\lambda) - q(0) = \lambda^2 \widehat{q}(\lambda)$ for a unique even polynomial $\widehat{q}(\lambda)$. (Although α, p q, and $\widehat{q}$ depend on j, we have suppressed the index for readability.) Then the 2×2 matrix polynomial

$$R(\lambda) \;=\; \begin{bmatrix} \lambda^{\alpha+1} p(\lambda) & -\lambda^\alpha p(\lambda) \\ \lambda^\alpha p(\lambda) & \lambda^{\alpha+1} p(\lambda) \widehat{q}(\lambda)/q(0) \end{bmatrix} \tag{3.5}$$

is T-even, and $R(\lambda) \sim \operatorname{diag}\big(d_{k_{2j-1}}(\lambda), d_{k_{2j}}(\lambda)\big)$ via the equivalence

$$\begin{bmatrix} 0 & 1 \\ -q(0) & \lambda q(0) \end{bmatrix} R(\lambda) \begin{bmatrix} 1 & -\lambda \widehat{q}(\lambda)/q(0) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} d_{k_{2j-1}}(\lambda) & 0 \\ 0 & d_{k_{2j}}(\lambda) \end{bmatrix}.$$

Thus, if for each consecutive pair $\big(d_{k_{2j-1}}(\lambda), d_{k_{2j}}(\lambda)\big)$ of odd polynomials along the diagonal of $D(\lambda)$, we replace the corresponding principal 2×2 submatrix of $D(\lambda)$ by the equivalent 2×2 T-even polynomial as given by (3.5), then we obtain a (block-diagonal) $n \times n$ matrix polynomial that is T-even with Smith form $D(\lambda)$. □

3.3 O-Smith form

The story for the "O-Smith form", i.e., the characterization of all possible Smith forms of T-odd matrix polynomials, is very similar to that for the "E-Smith form"; indeed it can be reduced to "E-Smith" by a simple trick — multiplying any T-odd matrix polynomial by λ results in a T-even matrix polynomial. Thus we obtain the following result analogous to Theorem 3.10.

Theorem 3.11 (O-Smith form)**.** *Suppose that*

$$D(\lambda) = \operatorname{diag}\big(\lambda^{\alpha_1} p_1(\lambda),\, \lambda^{\alpha_2} p_2(\lambda), \ldots,\, \lambda^{\alpha_\ell} p_\ell(\lambda),\, 0, \ldots, 0\big) \tag{3.6}$$

*is an $n \times n$ diagonal matrix polynomial such that $0 \leq \alpha_1 \leq \cdots \leq \alpha_\ell$ are nonnegative integers, $p_j(\lambda)$ is monic and $p_j(0) \neq 0$ for $j = 1, \ldots, \ell$, and $p_j(\lambda) \,|\, p_{j+1}(\lambda)$ for $j = 1, \ldots, \ell - 1$. Then $D(\lambda)$ is the Smith form of some $n \times n$ T-**odd** matrix polynomial $P(\lambda)$ if and only if the following conditions hold:*

(1) *$p_j(\lambda)$ is even for $j = 1, \ldots, \ell$;*

(2) *If ν is the number of* **even** *exponents (including the exponent 0) among $\alpha_1, \ldots, \alpha_\ell$, then $\nu = 2m$ is an even integer. Letting $k_1 < k_2 < \cdots < k_{2m}$ be the positions on the diagonal of $D(\lambda)$ where these even exponents α_{k_j} occur, the following properties hold:*

(a) *adjacency-pairing of positions:*

$$k_2 = k_1 + 1\,, \quad k_4 = k_3 + 1\,, \quad \ldots \quad , \quad k_{2m} = k_{2m-1} + 1$$

(b) *equality-pairing of even exponents:*

$$\alpha_{k_2} = \alpha_{k_1}\,, \quad \alpha_{k_4} = \alpha_{k_3}\,, \quad \ldots \quad , \quad \alpha_{k_{2m}} = \alpha_{k_{2m-1}}\,.$$

Proof. ($\Rightarrow$): Let $Q(\lambda) := \lambda P(\lambda)$. Then $Q(\lambda)$ is a T-even matrix polynomial. Moreover, observe that if the Smith form of $P(\lambda)$ is

$$E(\lambda)P(\lambda)F(\lambda) = D(\lambda),$$

where $E(\lambda)$ and $F(\lambda)$ are unimodular matrix polynomials, then $E(\lambda)Q(\lambda)F(\lambda) = \lambda D(\lambda)$ is the Smith form of $Q(\lambda)$. Thus each elementary divisor λ^α of $P(\lambda)$ corresponds to an elementary divisor $\lambda^{\alpha+1}$ of $Q(\lambda)$. By invoking Theorem 3.10 on $Q(\lambda)$, the necessity of the given conditions on the Smith form of $P(\lambda)$ now follows immediately.

($\Leftarrow$): Suppose $D(\lambda)$ is a diagonal matrix polynomial satisfying the conditions of the theorem. To see that $D(\lambda)$ is the Smith form of some T-odd polynomial $P(\lambda)$, begin by observing that $\widetilde{D}(\lambda) = \lambda D(\lambda)$ satisfies the conditions of Theorem 3.10, and so is the Smith form of some T-even polynomial $Q(\lambda)$. Since the $(1,1)$-entry of the Smith form $\widetilde{D}(\lambda)$ is the GCD of the entries of $Q(\lambda)$, and this entry is clearly divisible by λ, it follows that every entry of $Q(\lambda)$ is divisible by λ, and so $Q(\lambda)$ can be factored as $Q(\lambda) = \lambda P(\lambda)$. It is now easy to see that this $P(\lambda)$ is the desired T-odd polynomial with Smith form $D(\lambda)$. □

Remark 3.12. Condition (2) of Theorem 3.11 implies that the elementary divisors λ^α associated with the eigenvalue 0 are restricted in any T-odd matrix polynomial $P(\lambda)$; for any $r > 0$, the elementary divisor λ^{2r} must occur an even number of times. But condition (2) also implies that λ^0, which is not usually viewed as an elementary divisor, must also occur an even number of times in the O-Smith form (3.6). At first this may seem somewhat strange and unexpected, but it turns out to have a simple interpretation. Consider the unimodular equivalence $E(\lambda)P(\lambda)F(\lambda) = D(\lambda)$, evaluated at $\lambda = 0$. The even multiplicity of λ^0 in $D(\lambda)$ is equivalent to the matrix $D(0)$ having an even number of nonzero entries, i.e., to $D(0)$ having even rank. But $P(0)$ in a T-odd matrix polynomial $P(\lambda)$ is a skew-symmetric matrix, so the equation $E(0)P(0)F(0) = D(0)$ simply says that any skew-symmetric matrix over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ must have even rank. This well-known fact about skew-symmetric matrices can thus be seen to be an immediate corollary and special case of Theorem 3.11, and the arguments given in this paper can be viewed as providing a new and independent proof of this fact.

3.4 T-alternating polynomials over fields of characteristic two

What about matrix polynomials over fields of characteristic two, e.g., $\mathbb{F} = \mathbb{Z}_2$? Is there any sensible notion of a T-alternating polynomial over such a field for which some version of Theorem 3.10 or 3.11 may hold? Since we have $-1 = +1$ in these fields, the conditions in Definition 1.1(a) and 1.1(b) both reduce to $P^T(\lambda) = P(\lambda)$. Not only does this render every scalar polynomial alternating, it also fails to constrain the Smith form in any way, as the condition is trivially satisfied by every diagonal matrix polynomial.

Instead we could define a T-alternating polynomial to be one whose matrix coefficients strictly alternate between symmetric and skew-symmetric matrices. But since the property $B^T = -B$ is now identical to $B^T = B$, a suitable replacement for the notion of skew-symmetry must be found. A natural candidate is the (unfortunately named) notion of "alternate matrix" [14]: an $n \times n$ B is said to be an alternate matrix if $B^T = -B$ *and* $B_{ii} = 0$ for $i = 1, \ldots, n$. It is then shown [14, Thm 6.3] that an alternate matrix over *any* field always has even rank. Thus the special case of Theorem 3.11 discussed in Remark 3.12 still holds over a field of characteristic two if skew-symmetry is replaced by alternateness. Perhaps the full E-Smith and O-Smith results might also hold over all fields if T-alternating polynomials are defined as ones whose coefficients strictly alternate between symmetric and alternate matrices.

With this re-definition of T-alternating, it turns out that many of the results on alternating matrix and scalar polynomials leading up to Theorems 3.10 and 3.11 are still true. Corollary 2.7 as well as Lemmas 3.2, 3.4, 3.5, and 3.6 all hold, albeit with somewhat more involved proofs. It is at Lemma 3.7 and the notion of dual minors in Proposition 3.8 where the argument falls apart. When char $\mathbb{F} = 2$, the polynomials $q(\lambda)$ and $q(-\lambda)$ are identical, so their GCD need not be alternating. Indeed, the following simple counterexample shows that Theorems 3.10 and 3.11 do not hold over *any* field of characteristic two, despite using this strengthened notion of T-alternating.

Example 3.13. The pencil

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda+1 \\ \lambda+1 & 0 \end{bmatrix}$$

is both T-even and T-odd over the field $\mathbb{F} = \mathbb{Z}_2$, and hence over any field of characteristic two. But the Smith form of $L(\lambda)$ is clearly $D(\lambda) = \operatorname{diag}(\lambda+1, \lambda+1)$, so none of the invariant polynomials of $L(\lambda)$ are alternating.

4 Jordan structures of T-alternating polynomials

Now that we know the possible Smith forms of alternating matrix polynomials, we can interpret these results in terms of their Jordan structures. This will allow us in Section 5 to characterize those T-alternating matrix polynomials that admit a T-alternating strong linearization. Note that for the rest of the paper, all fields $\mathbb{F}$ have char $\mathbb{F} \neq 2$.

The following factorization is fundamental for translating Smith forms into Jordan structures.

Lemma 4.1 (Factorization of scalar alternating polynomials)**.**
Let $p(\lambda)$ be a nonzero scalar alternating polynomial over an algebraically closed field $\mathbb{F}$ of characteristic not equal to 2. Then p admits the factorization

$$p(\lambda) = c\lambda^\beta \big[(\lambda - \lambda_1)(\lambda + \lambda_1)\big]^{\alpha_1} \cdots \big[(\lambda - \lambda_m)(\lambda + \lambda_m)\big]^{\alpha_m},$$

where $c \in \mathbb{F} \setminus \{0\}$, $\beta \in \mathbb{N}$, $\alpha_i \in \mathbb{N}$ with $\alpha_i > 0$, and $\lambda_1, \ldots, \lambda_m, -\lambda_1, \ldots, -\lambda_m \in \mathbb{F} \setminus \{0\}$ are distinct.

Proof. Write $p(\lambda) = \lambda^\beta q(\lambda)$, where $q(0) \neq 0$. Since p is alternating, so is q; indeed q is even, since $q(0) \neq 0$. Let $\lambda_i \in \mathbb{F} \setminus \{0\}$ be a root of $q(\lambda)$. Since q is even, we see from

$q(-\lambda_i) = q(\lambda_i) = 0$ that $-\lambda_i \neq \lambda_i$ is also a root of q, thus allowing the factorization

$$q(\lambda) = (\lambda - \lambda_i)(\lambda + \lambda_i)\widetilde{q}(\lambda),$$

where, by Lemma 3.5, $\widetilde{q}(\lambda)$ is an even polynomial of degree less than that of $q(\lambda)$. Repeating this procedure a finite number of times and collating linear factors with the same roots, we achieve the desired factorization. □

Recall from Definition 2.4 that the Jordan structure of a matrix polynomial $P(\lambda)$ is the collection of all its finite and infinite elementary divisors, viewing $P(\lambda)$ as a polynomial over the algebraic closure $\overline{\mathbb{F}}$. We now describe the elementary divisors arising from a T-alternating matrix polynomial.

Theorem 4.2 (Jordan structure of T-alternating matrix polynomials)**.**
Let $P(\lambda)$ be an $n \times n$ T-alternating matrix polynomial of degree k. Then the Jordan structure of $P(\lambda)$ is comprised of elementary divisors satisfying the following pairing conditions:

(a) Nonzero elementary divisors: *if $(\lambda-\lambda_0)^{\alpha_1}, \ldots, (\lambda-\lambda_0)^{\alpha_\ell}$ are the elementary divisors associated with the eigenvalue $\lambda_0 \neq 0$, then the elementary divisors associated with the eigenvalue $-\lambda_0$ are $(\lambda+\lambda_0)^{\alpha_1}, \ldots, (\lambda+\lambda_0)^{\alpha_\ell}$.*

(b) Zero elementary divisors λ^β: *either all odd degree λ^β or all even degree λ^β occur in pairs, depending on the parity of $P(\lambda)$. Specifically,*
   (i) *if $P(\lambda)$ is T-even, then for each* odd *$\beta \in \mathbb{N}$, λ^β occurs with even multiplicity.*
   (ii) *if $P(\lambda)$ is T-odd, then for each* even *$\beta \in \mathbb{N}$, λ^β occurs with even multiplicity.*

(c) Infinite elementary divisors: *either all odd degree or all even degree elementary divisors at ∞ occur in pairs, depending on* both *the parity and the degree k of $P(\lambda)$.*
   (i) *Suppose $P(\lambda)$ and k have the* same *parity (i.e., $P(\lambda)$ is T-even and k is even, or $P(\lambda)$ is T-odd and k is odd). Then* $\operatorname{rev} P(\lambda)$ *is T-even, and for each* odd *$\gamma \in \mathbb{N}$, the infinite elementary divisor of $P(\lambda)$ of degree γ occurs with even multiplicity.*
   (ii) *Suppose $P(\lambda)$ and k have* opposite *parity (i.e., $P(\lambda)$ is T-even and k is odd, or $P(\lambda)$ is T-odd and k is even). Then* $\operatorname{rev} P(\lambda)$ *is T-odd, and for each* even *$\gamma \in \mathbb{N}$, the infinite elementary divisor of $P(\lambda)$ of degree γ occurs with even multiplicity.*

Proof. (a): Let $D(\lambda) = \operatorname{diag}\left(\lambda^{\alpha_1}p_1(\lambda), \ldots, \lambda^{\alpha_r}p_r(\lambda), 0, \ldots, 0\right)$ with $p_i(0) \neq 0, i = 1, \ldots, r$ be the Smith form of $P(\lambda)$. Then by Theorems 3.10 and 3.11 each $p_1(\lambda), \ldots, p_r(\lambda)$ is even. Thus (a) follows immediately upon applying Lemma 4.1 to $p_1(\lambda), \ldots, p_r(\lambda)$.

(b): If $P(\lambda)$ is T-even, then condition (2) from Theorem 3.10 directly translates into the desired pairing of the zero elementary divisors of odd degrees. The desired pairing for a T-odd polynomial $P(\lambda)$ follows analogously from condition (2) of Theorem 3.11.

(c): The elementary divisors of $P(\lambda)$ at ∞ correspond to the zero elementary divisors of $\operatorname{rev} P(\lambda)$. But $\operatorname{rev} P(\lambda)$ is T-alternating whenever $P(\lambda)$ is, with the same parity as $P(\lambda)$ if k is even, and opposite parity if k is odd. Then (c) follows by applying (b) to $\operatorname{rev} P(\lambda)$. □

The $(\pm)$-pairing of nonzero eigenvalues and the pairing of their corresponding elementary divisors was already known for regular complex T-alternating polynomials, see [20]. Theorem 4.2 extends this result to the singular case and arbitrary fields and also characterizes the possible Jordan structures at 0 and ∞. In particular, Theorem 4.2 recovers as

a special case the following well-known result due to Kronecker [15]; see also [27] for the structure of zero and infinite elementary divisors of T-alternating matrix pencils.

Corollary 4.3 (Jordan structure of T-alternating matrix pencils)**.**
Let $L(\lambda) = \lambda X + Y$ be an $n \times n$ T-alternating pencil. Then the Jordan structure of $L(\lambda)$ has the following properties:

(a) Nonzero elementary divisors occur in pairs: *if $(\lambda - \lambda_0)^{\alpha_1}, \ldots, (\lambda - \lambda_0)^{\alpha_\ell}$ are the elementary divisors of $L(\lambda)$ associated with $\lambda_0 \neq 0$, then the elementary divisors of $L(\lambda)$ associated with $-\lambda_0$ are $(\lambda + \lambda_0)^{\alpha_1}, \ldots, (\lambda + \lambda_0)^{\alpha_\ell}$.*

(b) *If $L(\lambda)$ is T-*even*, then the following elementary divisors occur with even multiplicity:*
   (i) *for each* odd *$\beta \in \mathbb{N}$, the elementary divisor λ^β, and*
   (ii) *for each* even *$\gamma \in \mathbb{N}$, the elementary divisor at ∞ of degree γ.*

(c) *If $L(\lambda)$ is T-*odd*, then the following elementary divisors occur with even multiplicity:*
   (i) *for each* even *$\beta \in \mathbb{N}$, the elementary divisor λ^β, and*
   (ii) *for each* odd *$\gamma \in \mathbb{N}$, the elementary divisor at ∞ of degree γ.*

It is worth noting how the results in parts (b) and (c) of Corollary 4.3 can be seen to fit together nicely, and understood more intuitively in light of the properties of the reversal operation rev. Specifically, observe that

- rev maps T-even pencils bijectively to T-odd pencils, and
- rev takes elementary divisors at 0 into elementary divisors at ∞, and vice-versa.

Hence parts (b) and (c) can be immediately deduced from each other using rev.

5 Structured linearizations for T-alternating polynomials

In light of the Jordan structure results of Section 4, we now consider the problem of determining which T-alternating polynomials have a strong linearization that is also T-alternating. Since strong linearizations preserve the elementary divisor structure of all finite and infinite eigenvalues, Theorem 4.2 and Corollary 4.3 immediately imply some necessary conditions for the existence of such a structure-preserving strong linearization.

Lemma 5.1 (Compatibility of Jordan structure for structured linearization)**.**
Let $P(\lambda)$ be a T-alternating matrix polynomial. Then it is possible for $P(\lambda)$ to have a T-alternating strong linearization only if its elementary divisors at 0 and ∞ satisfy either the conditions in Corollary 4.3(b) *for the Jordan structure of a T-even pencil, or those in Corollary* 4.3(c) *for a T-odd pencil.*

A more detailed comparison of the elementary divisor conditions in Theorem 4.2 and Corollary 4.3 reveals a fundamental dichotomy between even and odd degree. Whenever $\deg P(\lambda)$ is odd, the constraints on the Jordan structure of a T-even(odd) polynomial $P(\lambda)$ are *exactly the same* as those for a T-even(odd) pencil. Thus it appears that there should be no obstruction to the existence of a structured strong linearization when $\deg P(\lambda)$ is odd; we will consider this question in some detail in Section 5.1. However, because the pairing

of infinite elementary divisors depends on both the parity and degree of $P(\lambda)$, there can be incompatibilities between the Jordan structure of an *even* degree $P(\lambda)$ and that of every T-alternating pencil, thus precluding the possibility of any structured strong linearization for $P(\lambda)$. This insight fully explains the surprising observation in Example 1.4 of the existence of a *quadratic* T-even polynomial having *no* T-alternating strong linearization. We reconsider that example now in light of this deeper understanding.

Example 5.2. Consider again the T-even matrix polynomial

$$P(\lambda) \;=\; \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \;=\; \mathrm{diag}(\lambda^2, -1)$$

as in Example 1.4. Note that both $P(\lambda)$ and $\mathrm{rev}P(\lambda) = \mathrm{diag}(1, -\lambda^2)$ have the same Smith form $\mathrm{diag}(1, \lambda^2)$; thus $P(\lambda)$ has elementary divisor λ^2 with odd multiplicity, and also an even degree elementary divisor at ∞ with odd multiplicity, in complete accordance with Theorem 4.2. But this Jordan structure is incompatible with every T-even pencil by Corollary 4.3 b(ii), and with every T-odd pencil by Corollary 4.3 c(i). Thus we see a more fundamental reason why $P(\lambda)$ can have no T-alternating strong linearization.

The question left unanswered by Lemma 5.1 is whether compatibility of Jordan structures is also sufficient to imply the existence of a T-alternating strong linearization. We consider this question next, looking at each half of the odd/even degree dichotomy in more detail in separate sections. A more refined question concerns the existence of T-alternating linearizations that preserve all the spectral information of $P(\lambda)$, comprised not only of its finite and infinite elementary divisors, but also (when $P(\lambda)$ is singular) of its minimal indices. In this context a "good" linearization would preserve all finite and infinite elementary divisors, and also allow the minimal indices of a matrix polynomial to be recovered from those of the linearization. This topic is currently under investigation in [5, 6], and will not be addressed in this paper.

5.1 The odd degree case

We have seen that the Jordan structure of any *odd* degree T-alternating matrix polynomial $P(\lambda)$ is completely compatible with that of a T-alternating pencil of the same parity. This strongly suggests that it should be possible to construct a structure-preserving strong linearization for any such $P(\lambda)$. In this section we show how to do this, using a simple construction that works equally well for regular and singular $P(\lambda)$, over any field $\mathbb{F}$ with $\mathrm{char}\,\mathbb{F} \neq 2$. We begin this construction by developing the basic properties of a particular pencil introduced in [1], altered slightly here for the sake of simpler indexing.

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix polynomial with odd degree $k = 2m+1$, $m \geq 1$. Then from the data in P we construct a block-symmetric pencil denoted $\mathcal{S}_P(\lambda)$ as follows.

Let $T_k(\lambda)$ be the $k \times k$ symmetric tridiagonal matrix pencil

$$T_k(\lambda) \;=\; \begin{bmatrix} 0 & \lambda & & & & & & & \\ \lambda & 0 & 1 & & & & & & \\ & 1 & 0 & \lambda & & & & & \\ & & \lambda & 0 & 1 & & & & \\ & & & 1 & 0 & \ddots & & & \\ & & & & \ddots & \ddots & 1 & & \\ & & & & & 1 & 0 & \lambda & \\ & & & & & & \lambda & 0 & 1 \\ & & & & & & & 1 & 0 \end{bmatrix}_{k\times k}$$

with λ and 1 alternating down the sub-diagonal, and let $D_P(\lambda)$ be the block-$(k \times k)$ block-diagonal pencil

$$D_P(\lambda) \;=\; \begin{bmatrix} \lambda A_1 + A_0 & & & & & & \\ & 0 & & & & & \\ & & \lambda A_3 + A_2 & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \lambda A_k + A_{k-1} \end{bmatrix}_{kn\times kn},$$

where each $n \times n$ diagonal block with odd index j is $\lambda A_j + A_{j-1}$, and every diagonal block with even index is $0_{n\times n}$. Then the $kn \times kn$ companion-like pencil $\mathcal{S}_P(\lambda)$ is defined to be

$$\mathcal{S}_P(\lambda) \;:=\; D_P(\lambda) + \big(T_k(\lambda) \otimes I_n\big)\,.$$

As an illustration, here is $\mathcal{S}_P(\lambda)$ for P of degree $k = 7$:

$$\mathcal{S}_P(\lambda) \;=\; \begin{bmatrix} \lambda A_1 + A_0 & \lambda I & & & & & & \\ \lambda I & 0 & I & & & & & \\ & I & \lambda A_3 + A_2 & \lambda I & & & & \\ & & \lambda I & 0 & I & & & \\ & & & I & \lambda A_5 + A_4 & \lambda I & & \\ & & & & \lambda I & 0 & I & \\ & & & & & I & \lambda A_7 + A_6 \end{bmatrix}_{7n\times 7n}. \qquad (5.1)$$

In [1] it was shown that for any *regular* P of odd degree over the field $\mathbb{F} = \mathbb{C}$, the pencil $\mathcal{S}_P(\lambda)$ is always a strong linearization. However, this result about $\mathcal{S}_P(\lambda)$ can be significantly strengthened, using simpler and more direct arguments than in [1].

Lemma 5.3. *Let $P(\lambda)$ be any $n \times n$ matrix polynomial of odd degree, regular or singular, over an arbitrary field $\mathbb{F}$. Then $\mathcal{S}_P(\lambda)$ is a strong linearization for $P(\lambda)$.*

Proof. We first show explicitly how to reduce the pencil $\mathcal{S}_P(\lambda)$ to diag $\big(P(\lambda), I_{(k-1)n}\big)$ via unimodular transformations that are well-defined over any field, and valid without any restriction on the coefficient matrices A_i. The following observation is the key tool for this

reduction. For any $n \times n$ matrix polynomials $X(\lambda)$ and $Y(\lambda)$, the block-tridiagonal matrix polynomial

$$\begin{bmatrix} Y(\lambda) & \lambda I & \\ \lambda I & 0 & I \\ & I & X(\lambda) \end{bmatrix}$$

can be block-diagonalized to $\operatorname{diag}\left(\lambda^2 X + Y, I, I\right)$ via the unimodular transformation

$$\underbrace{\begin{bmatrix} I & \lambda X & -\lambda I \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}}_{E(\lambda)} \begin{bmatrix} Y & \lambda I & \\ \lambda I & 0 & I \\ & I & X \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 & 0 \\ \lambda X & I & -X \\ -\lambda I & 0 & I \end{bmatrix}}_{F(\lambda)} = \begin{bmatrix} \lambda^2 X + Y & & \\ & I & 0 \\ & 0 & I \end{bmatrix}. \quad (5.2)$$

The special structure of this transformation, i.e., the zero blocks in the first block column of $E(\lambda)$ and in the first block row of $F(\lambda)$, allows it to be used inductively to block-diagonalize $\mathcal{S}_P(\lambda)$, starting from the lower right and proceeding up the diagonal. An example will make the procedure clear.

Consider $\mathcal{S}_P(\lambda)$ for $k = 7$ as in (5.1). Applying the transformation (5.2) in the last three block rows and columns with $X = \lambda A_7 + A_6$ and $Y = \lambda A_5 + A_4$, we see that $\mathcal{S}_P(\lambda)$ is equivalent to

$$\begin{bmatrix} \lambda A_1 + A_0 & \lambda I & & & & & & \\ \lambda I & 0 & I & & & & & \\ & I & \lambda A_3 + A_2 & \lambda I & & & & \\ & & \lambda I & 0 & I & & & \\ & & & I & P_3(\lambda) & & & \\ & & & & & I & 0 & \\ & & & & & 0 & I & \end{bmatrix}, \quad (5.3)$$

where $P_3(\lambda) = \lambda^2 X + Y = \lambda^3 A_7 + \lambda^2 A_6 + \lambda A_5 + A_4$ is the degree 3 Horner shift of $P(\lambda)$. In general the *degree ℓ Horner shift of* $P(\lambda)$ is $P_\ell(\lambda) := \sum_{j=0}^{\ell} \lambda^{\ell-j} A_{k-j}$, so-named because it shows up in Horner's method for evaluating polynomials. Note that the Horner shifts of $P(\lambda)$ satisfy the recurrences

$$\begin{aligned} P_{\ell+1}(\lambda) &= \lambda P_\ell(\lambda) + A_{k-\ell-1} \\ \text{and} \quad P_{\ell+2}(\lambda) &= \lambda^2 P_\ell(\lambda) + \left(\lambda A_{k-\ell-1} + A_{k-\ell-2}\right). \end{aligned}$$

The fact that the right-hand side of the second recurrence is of the form $\lambda^2 X(\lambda) + Y(\lambda)$, as in the transformation (5.2), is essential to the inductive reduction of $\mathcal{S}_P(\lambda)$.

Continuing this reduction, we next apply the transformation (5.2) in block rows and columns 3, 4, and 5 with $X = P_3(\lambda)$ and $Y = \lambda A_3 + A_2$, obtaining

$$\mathcal{S}_P(\lambda) \sim \begin{bmatrix} \lambda A_1 + A_0 & \lambda I & & & & & \\ \lambda I & 0 & I & & & & \\ & I & P_5(\lambda) & & & & \\ & & & I & 0 & & \\ & & & 0 & I & & \\ & & & & & I & 0 \\ & & & & & 0 & I \end{bmatrix}. \quad (5.4)$$

A final application of transformation (5.2) in block rows and columns 1, 2, and 3 shows that $\mathcal{S}_P(\lambda) \sim \operatorname{diag}\big(P(\lambda), I_{6n}\big)$; hence $\mathcal{S}_P(\lambda)$ is a linearization for $P(\lambda)$. The reduction of $\mathcal{S}_P(\lambda)$ for the general odd degree $P(\lambda)$ proceeds inductively in a similar fashion, accumulating the coefficient matrices of $P(\lambda)$ up the diagonal into higher and higher degree Horner shifts, until finally $\operatorname{diag}\big(P(\lambda), I_{(k-1)n}\big)$ is achieved.

To see that $\mathcal{S}_P(\lambda)$ is actually a *strong* linearization for $P(\lambda)$, all that remains is to show that $\operatorname{rev}\mathcal{S}_P(\lambda) \sim \operatorname{diag}\big(\operatorname{rev}P(\lambda), I_{(k-1)n}\big)$. Clearly $\mathcal{S}_{\operatorname{rev}P}(\lambda) \sim \operatorname{diag}\big(\operatorname{rev}P(\lambda), I_{(k-1)n}\big)$ holds by applying the above result to the polynomial $\operatorname{rev}P$, so to complete the proof we only need to show that $\operatorname{rev}\mathcal{S}_P \sim \mathcal{S}_{\operatorname{rev}P}$.

First observe that

$$\begin{aligned} \operatorname{rev}\mathcal{S}_P(\lambda) &= \operatorname{rev}D_P(\lambda) + \operatorname{rev}\big(T_k(\lambda) \otimes I_n\big) \\ &= \operatorname{rev}D_P(\lambda) + \big(\operatorname{rev}T_k(\lambda) \otimes I_n\big)\,. \end{aligned}$$

Examining each term of this sum, we see that

$$\operatorname{rev}D_P(\lambda) = \begin{bmatrix} \lambda A_0 + A_1 & & & & & & \\ & 0 & & & & & \\ & & \lambda A_2 + A_3 & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \lambda A_{k-1} + A_k \end{bmatrix}$$

and

$$\operatorname{rev}T_k(\lambda) = \begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & \lambda & & & & & \\ & \lambda & 0 & 1 & & & & \\ & & 1 & 0 & \lambda & & & \\ & & & \lambda & 0 & \ddots & & \\ & & & & \ddots & \ddots & \lambda & \\ & & & & & \lambda & 0 & 1 \\ & & & & & & 1 & 0 & \lambda \\ & & & & & & & \lambda & 0 \end{bmatrix}_{k\times k},$$

which have structure very similar to that of $D_P(\lambda)$ and $T_k(\lambda)$. Indeed, we can turn $\operatorname{rev}T_k(\lambda)$ back into $T_k(\lambda)$ by reversing the order of the rows and the columns using the reverse identity $R_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{k\times k}$, i.e., $R_k \cdot \operatorname{rev}T_k(\lambda) \cdot R_k = T_k(\lambda)$. Applying this same transformation at the block level to $\operatorname{rev}D_P$ using $\widetilde{R}_k := R_k \otimes I_n$, we see that

$$\widetilde{R}_k \cdot \operatorname{rev}D_P(\lambda) \cdot \widetilde{R}_k = \begin{bmatrix} \lambda A_{k-1} + A_k & & & & & & \\ & 0 & & & & & \\ & & \lambda A_{k-3} + A_{k-2} & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \lambda A_0 + A_1 \end{bmatrix} = D_{\operatorname{rev}P}(\lambda)\,.$$

Consequently we have

$$\begin{aligned}\widetilde{R}_k \cdot \mathrm{rev}\mathcal{S}_P \cdot \widetilde{R}_k &= \widetilde{R}_k \cdot \mathrm{rev}D_P \cdot \widetilde{R}_k + \widetilde{R}_k \cdot \big(\mathrm{rev}T_k \otimes I_n\big) \cdot \widetilde{R}_k \\ &= D_{\mathrm{rev}P} + \big(T_k \otimes I_n\big) \\ &= \mathcal{S}_{\mathrm{rev}P}\,.\end{aligned}$$

Therefore $\mathrm{rev}\mathcal{S}_P \sim \mathcal{S}_{\mathrm{rev}P} \sim \mathrm{diag}\big(\mathrm{rev}P(\lambda), I_{(k-1)n}\big)$, completing the proof that $\mathcal{S}_P(\lambda)$ is a strong linearization for $P(\lambda)$. □

Although $\mathcal{S}_P(\lambda)$ is not yet the desired structure-preserving strong linearization for T-alternating polynomials, it does have a number of attractive properties that are worth noting before we proceed.

- $\mathcal{S}_P(\lambda)$ is a "companion form" – i.e., it works as a uniform template, built directly from the coefficient matrices of P, providing a strong linearization for *all* (odd degree) P, regular or singular, over any field $\mathbb{F}$.
- $\mathcal{S}_P(\lambda)$ is block-symmetric and block-tridiagonal.
- $\mathcal{S}_P(\lambda)$ is symmetric (resp., Hermitian) whenever P is.

A few small sign modifications now suffice to convert $\mathcal{S}_P(\lambda)$ into a T-alternating pencil; note that these sign modifications were done in a different way in [2]. Observe that the diagonal blocks $\lambda A_j + A_{j-1}$ of $\mathcal{S}_P(\lambda)$ are already T-alternating whenever P is, with the same parity as P. Thus only some sign-adjustment of the off-diagonal I and λI blocks is needed to make $\mathcal{S}_P(\lambda)$ be T-alternating. This can be achieved using the following diagonal "sign" matrices. Let Σ_E and Σ_O be $k \times k$ diagonal matrices with entries

$$\big(\Sigma_E\big)_{jj} = \begin{cases} 1 & \text{if } j \equiv 0 \bmod 4 \text{ or } j \equiv 1 \bmod 4 \\ -1 & \text{if } j \equiv 2 \bmod 4 \text{ or } j \equiv 3 \bmod 4\,, \end{cases}$$

$$\big(\Sigma_O\big)_{jj} = \begin{cases} 1 & \text{if } j \equiv 1 \bmod 4 \text{ or } j \equiv 2 \bmod 4 \\ -1 & \text{if } j \equiv 3 \bmod 4 \text{ or } j \equiv 0 \bmod 4\,, \end{cases}$$

and define $\mathcal{E}_P(\lambda) := \big(\Sigma_E \otimes I_n\big)\mathcal{S}_P(\lambda)$ and $\mathcal{O}_P(\lambda) := \big(\Sigma_O \otimes I_n\big)\mathcal{S}_P(\lambda)$. Then $\mathcal{E}_P$ is a T-even pencil whenever P is T-even, and $\mathcal{O}_P$ is a T-odd pencil whenever P is T-odd. As an illustration, here is $\mathcal{O}_P(\lambda)$ for P of degree 7:

$$\mathcal{O}_P(\lambda) = \begin{bmatrix} \lambda A_1 + A_0 & \lambda I & & & & & & \\ \lambda I & 0 & I & & & & & \\ & -I & -\lambda A_3 - A_2 & -\lambda I & & & & \\ & & -\lambda I & 0 & -I & & & \\ & & & I & \lambda A_5 + A_4 & \lambda I & & \\ & & & & \lambda I & 0 & I & \\ & & & & & -I & -\lambda A_7 - A_6 \end{bmatrix}.$$

Clearly $\mathcal{E}_P(\lambda)$ and $\mathcal{O}_P(\lambda)$ are strictly equivalent to $\mathcal{S}_P(\lambda)$, and hence are both strong linearizations for P. Thus we have shown the following.

Theorem 5.4. *Every T-alternating polynomial $P(\lambda)$ of* odd *degree has a T-alternating strong linearization with the same parity as P. More specifically, if $P(\lambda)$ is T-even then the pencil $\mathcal{E}_P(\lambda)$ is a T-even strong linearization, and if $P(\lambda)$ is T-odd then the pencil $\mathcal{O}_P(\lambda)$ is a T-odd strong linearization.*

5.2 The even degree case

We know already (see Example 1.4) that there are even degree T-alternating matrix polynomials that, because of Jordan structure incompatibilities, do not have any T-alternating strong linearization. If we put such cases aside, however, and consider only T-alternating matrix polynomials whose Jordan structure is compatible with at least some type of T-alternating pencil, then is that compatibility sufficient to guarantee the existence of a T-alternating strong linearization? Although we are not able to settle this question in all cases, the following theorem provides a complete resolution to the problem for any real or complex T-alternating polynomial that is *regular*.

Theorem 5.5. *Let $P(\lambda)$ be a* regular $n \times n$ *T-alternating matrix polynomial of* even *degree $k \geq 2$, over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.*

(a) *If $P(\lambda)$ is T-even, then $P(\lambda)$ has*

- *a T-even strong linearization if and only if for each even $\gamma \in \mathbb{N}$, the infinite elementary divisor of degree γ occurs with even multiplicity.*
- *a T-odd strong linearization if and only if for each even $\beta \in \mathbb{N}$, the elementary divisor λ^β occurs with even multiplicity.*

(b) *If $P(\lambda)$ is T-odd, then $P(\lambda)$ has*

- *a T-even strong linearization if and only if for each odd $\beta \in \mathbb{N}$, the elementary divisor λ^β occurs with even multiplicity.*
- *a T-odd strong linearization if and only if for each odd $\gamma \in \mathbb{N}$, the infinite elementary divisor of degree γ occurs with even multiplicity.*

Proof. The necessity of the conditions in (a) and (b) follows immediately from Lemma 5.1 by comparing the results in Theorem 4.2 and Corollary 4.3. Each case describes the minimum required to ensure that the Jordan structure of the given T-alternating matrix polynomial is compatible with the Jordan structure of the desired type of strong linearization.

To establish the existence of the desired type of strong linearization, we first note that for *any* set $\mathcal{D}$ of finite and infinite elementary divisors compatible with the conditions in Corollary 4.3 for a real or complex T-even (or T-odd) pencil, there does indeed exist a regular T-even (or T-odd, respectively) matrix pencil having exactly $\mathcal{D}$ as its set of elementary divisors. This follows immediately from the canonical forms given in [27]. Thus, if the set of elementary divisors $\mathcal{D}$ of a T-alternating polynomial $P(\lambda)$ satisfies the condition of any one of the four cases of the theorem, then we know that there exists a regular T-alternating matrix pencil $L(\lambda)$ over $\mathbb{F}$ (with the indicated parity) having exactly the same elementary divisors as P.

By [9, VI.3 Corollary 1], two regular matrix polynomials of the same size are unimodularly equivalent if and only if they have the same invariant polynomials, or, equivalently, if they have the same finite elementary divisors. Therefore $L(\lambda) \sim \operatorname{diag}\big(P(\lambda), I_{(k-1)n}\big)$, and so $L(\lambda)$ is a linearization for $P(\lambda)$.

On the other hand, since $P(\lambda)$ and $L(\lambda)$ have the same infinite elementary divisors together with the same nonzero finite elementary divisors, it follows that $\operatorname{rev} P(\lambda)$ and $\operatorname{rev} L(\lambda)$ have the same finite elementary divisors. Thus $\operatorname{rev} L(\lambda) \sim \operatorname{diag}\big(\operatorname{rev} P(\lambda), I_{(k-1)n}\big)$, and therefore $L(\lambda)$ is a strong linearization for $P(\lambda)$. □

Note that this result also holds for even degree T-alternating polynomials over any algebraically closed field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$, since the canonical forms in [27] extend to any such field. It remains an open question to determine whether this result still holds for matrix polynomials $P(\lambda)$ over a general field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$, or for polynomials $P(\lambda)$ that are singular.

Observe that Theorem 5.5 allows for the possibility of structured strong linearizations of both the same and of opposite parity to that of the original T-alternating polynomial. At first glance, it may seem strange to construct a T-alternating strong linearization with parity different from that of the original matrix polynomial P, but for even degree P this may sometimes be the only possible way that such a linearization may be found. This is illustrated by the following example.

Example 5.6. Consider the T-even matrix polynomial

$$P(\lambda) = \lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 - 1 \end{bmatrix}.$$

Then $P(\lambda)$ has the finite elementary divisors $\lambda+1$ and $\lambda-1$, and a single infinite elementary divisor of degree two. By Theorem 5.5(a), $P(\lambda)$ can not have a T-even strong linearization, but it does admit a T-odd strong linearization. Indeed, we can show that the T-odd pencil

$$L(\lambda) = \lambda \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda+1 \\ 0 & 0 & \lambda-1 & 0 \end{bmatrix}.$$

is a strong linearization of $P(\lambda)$. We have that

$$\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -\frac{1}{2}(\lambda-1) & \frac{1}{2}(\lambda+1) \end{bmatrix} L(\lambda) \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \lambda+1 \\ 0 & 0 & -\frac{1}{2} & \lambda-1 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

while

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\lambda-1) & \frac{1}{2}(\lambda+1) \end{bmatrix} \underbrace{\begin{bmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda+1 \\ 0 & 0 & -\lambda+1 & 0 \end{bmatrix}}_{=\operatorname{rev}L(\lambda)} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & \lambda & 0 \\ 0 & -\frac{1}{2} & 0 & \lambda+1 \\ 0 & \frac{1}{2} & 0 & -\lambda+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & -\lambda^2+1 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & \operatorname{rev}P(\lambda) \end{bmatrix}.$$

The transformation matrix polynomials are easily checked to be unimodular.

6 E-Smith and O-Smith revisited

In this final section we reconsider the E-Smith and O-Smith theorems from a different point of view, providing an alternative proof that gives additional insight into why these theorems are true, as well as into the relationships between the two results. Note that the proofs given earlier in Sections 3.2 and 3.3 start from first principles, and then recover Kronecker's theorem on the elementary divisor structure of T-even and T-odd pencils as a corollary (Corollary 4.3). By contrast, the proof described here starts from Kronecker's pencil theorem as a known result, and then derives the elementary divisor versions of E-Smith and O-Smith in Theorem 4.2 for all T-alternating polynomials, as a *consequence* of Kronecker's theorem. The three key ingredients needed for this proof are:

- Kronecker's theorem on elementary divisors of T-even and T-odd pencils,
- the existence of the T-even and T-odd "companion forms" $\mathcal{E}_P(\lambda)$ and $\mathcal{O}_P(\lambda)$ for odd degree P described in Section 5.1,
- the "multiplying-by-λ" trick used in the proof of Theorem 3.11.

The proof begins by establishing the elementary divisor conditions in Theorem 4.2 for all *odd* degree T-alternating polynomials $P(\lambda)$. For such $P(\lambda)$, both regular as well as singular, we have shown that the two T-alternating "companion forms" $\mathcal{E}_P(\lambda)$ and $\mathcal{O}_P(\lambda)$ are strong linearizations over all fields. Linearizing a T-even $P(\lambda)$ by the T-even $\mathcal{E}_P(\lambda)$, or a T-odd $P(\lambda)$ by the T-odd $\mathcal{O}_P(\lambda)$, and applying Kronecker's pencil theorem to $\mathcal{E}_P(\lambda)$ and $\mathcal{O}_P(\lambda)$, we immediately conclude that the elementary divisors of any odd degree T-alternating $P(\lambda)$ must satisfy exactly the same conditions as those of a T-alternating pencil of the same parity.

To get the elementary divisor conditions in Theorem 4.2 for all *even* degree T-alternating polynomials, use the "multiplying-by-λ" trick. If $P(\lambda)$ is T-alternating of even degree, consider the odd degree T-alternating polynomial $Q(\lambda) := \lambda P(\lambda)$, for which Theorem 4.2 has just been shown to hold. Clearly the nonzero finite elementary divisors of P and Q are identical, so Theorem 4.2(a) holds for all even degree P. The zero elementary divisors of P and Q are simply related: λ^β is an elementary divisor of P if and only if $\lambda^{\beta+1}$ is an elementary divisor of Q. Combining this with the fact that P and Q have opposite parity, we see that Theorem 4.2(b) for all odd degree Q implies Theorem 4.2(b) for all even degree P. Finally, observe that the infinite elementary divisors of P and Q are identical, since $\operatorname{rev} P \equiv \operatorname{rev} Q$. Also note that P and $k = \deg P$ have the same (opposite) parity if and only if Q and $\deg Q$ have the same (opposite) parity. Thus Theorem 4.2(c) for all odd degree Q implies Theorem 4.2(c) for all even degree P, and the proof is complete.

This approach gives us a different insight into the Smith forms of alternating polynomials, but it has several limitations. It needs two supporting results — the existence of a structure-preserving strong linearization for all odd degree polynomials, and Kronecker's theorem which is limited to T-alternating pencils over algebraically closed fields $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$. By contrast, deriving the E-Smith and O-Smith forms from basic properties of compound matrices leads us to a direct and independent proof which applies to more general fields, works for all degrees in a uniform way, produces Kronecker's theorem as a special case, and provides a new technique that may be more widely applied.

7 Conclusion

The Smith form of T-alternating matrix polynomials over an arbitrary field $\mathbb{F}$ of characteristic different from two has been completely characterized in this paper, using a novel technique exploiting the properties of compound matrices. Knowledge of these Smith forms has then enabled us to characterize the Jordan structures of this class of structured polynomials, and thereby to recover a classical theorem of Kronecker on the elementary divisors of T-alternating pencils as a corollary. Necessary conditions for the existence of structure-preserving strong linearizations for T-alternating polynomials then follow from these Jordan structures. A detailed analysis of when these conditions are also sufficient has also been carried out, although some open questions remain.

It is natural to investigate these same issues for other important classes of structured matrix polynomials, see [16, 20, 25]. Adapting the techniques used in this paper, we have been able to characterize the Smith forms of T-palindromic matrix polynomials as well as those of skew-symmetric matrix polynomials; this work will appear in a forthcoming paper.

References

[1] E. N. Antoniou and S. Vologiannidis. A new family of companion forms of polynomial matrices. *Electron. J. Linear Algebra*, 11:78–87, 2004.

[2] E. N. Antoniou and S. Vologiannidis. Linearizations of polynomial matrices with symmetries and their applications. *Electron. J. Linear Algebra*, 15:107–114, 2006.

[3] T. Apel, V. Mehrmann, and D. Watkins. Structured eigenvalue methods for the computation of corner singularities in 3D anisotropic elastic structures. *Comp. Meth. Appl. Mech. Eng.*, 191:4459–4473, 2002.

[4] A. Bai, A. Ruhe J. Demmel, J. Dongarra, and H. van der Vorst. *Templates for the Solution of Algebraic Eigenvalue Problems*. SIAM, Philadelphia, PA, 2000.

[5] F. De Terán, F. Dopico, and D. Steven Mackey. Fiedler companion linearizations and the recovery of minimal indices. In preparation, 2009.

[6] F. De Terán, F. Dopico, and D. Steven Mackey. Linearizations of singular matrix polynomials and the recovery of minimal indices. *Electron. J. Linear Algebra*, 18:371–402, 2009. Published online, 30 July 2009.

[7] D. S. Dummit and R. M. Foote. *Abstract Algebra*. John Wiley and Sons Inc., third edition, 2004.

[8] G. Frobenius. Theorie der linearen Formen mit ganzen Coefficienten. *J. Reine Angew. Math. (Crelle)*, 86:146–208, 1878.

[9] F. R. Gantmacher. *Theory of Matrices*, volume 1. Chelsea, New York, 1959.

[10] I. Gohberg, M. A. Kaashoek, and P. Lancaster. General theory of regular matrix polynomials and band Toeplitz operators. *Integral Equations Operator Theory*, 11(6):776–882, 1988.

[11] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.

[12] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, Baltimore, 3rd edition, 1996.

[13] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[14] N. Jacobson. *Basic Algebra I*. W.H. Freeman and Co., San Francisco, 1974.

[15] L. Kronecker. Über die congruenten Transformationen der bilinearen Formen, 1874. In *Collected Works I*, pages 423–483. Chelsea, New York, 1968.

[16] P. Lancaster, U. Prells, and L. Rodman. Canonical structures for palindromic matrix polynomials. *Oper. Matrices*, 1:469–489, 2007.

[17] P. Lancaster and P. Psarrakos. A note on weak and strong linearizations of regular matrix polynomials. MIMS EPrint 2006.72, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2005.

[18] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic Press, Orlando, 2nd edition, 1985.

[19] F. Lazebnik. On systems of linear diophantine equations. *Math. Mag.*, 69(4):261–266, 1996.

[20] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[21] M. Marcus and H. Minc. *A Survey of Matrix Theory and Matrix Inequalities*. Allyn and Bacon, Boston, 1964.

[22] V. Mehrmann and D. Watkins. Structure-preserving methods for computing eigenpairs of large sparse skew-Hamiltonian/Hamiltonian pencils. *SIAM J. Sci. Comput.*, 22:1905–1925, 2001.

[23] V. Mehrmann and D. Watkins. Polynomial eigenvalue problems with Hamiltonian structure. *Electr. Trans. Num. Anal.*, 13:106–118, 2002.

[24] V. V. Prasolov. *Problems and Theorems in Linear Algebra*. American Mathematical Society, Providence, 1994.

[25] C. Schröder. *Palindromic and Even Eigenvalue Problems - Analysis and Numerical Methods*. PhD thesis, Technical University Berlin, Germany, 2008.

[26] H. J. S. Smith. On systems of linear indeterminate equations and congruences. *Phil. Trans. Roy. Soc. London*, 151:293–326, 1861.

[27] R.C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.

[28] D. S. Watkins. *The Matrix Eigenvalue Problem: GR and Krylov Subpsace methods.* SIAM Publications, Philadelphia, PA, 2007.