Path regularity and explicit truncation order for BSDE with drivers of quadratic growth *

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Abstract

We consider backward stochastic differential equations with drivers of quadratic growth (qgBSDE). We prove several statements concerning path regularity and stochastic smoothness of the solution processes of the qgBSDE, in particular we prove an extension of Zhang's path regularity theorem to the quadratic growth setting. We give explicit convergence rates for the difference between the solution of a qgBSDE and its truncation, filling an important gap in numerics for qgBSDE. We give an alternative proof of second order Malliavin differentiability for BSDE with drivers that are Lipschitz continuous (and differentiable), and then derive the same result for qgBSDE.

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1 Introduction

Backward Stochastic differential equations (BSDE) have been receiving much attention in the last 15 years, due to their central significance in optimization problems for instance in stochastic finance, and more generally in stochastic control theory. A particularly important class, BSDE with drivers of quadratic growth, for example arise in the context of utility optimization problems with exponential utility functions, or alternatively in questions related to risk minimization for the entropic risk measure. BSDE provide the genuinely stochastic approach of control problems which find their analytical expression in the Hamilton-Jacobi-Bellman formalism. BSDE with drivers of this type keep being a source of intensive research.

As for Monte-Carlo methods to simulate random processes, numerical schemes for BSDE provide a robust method for simulating and approximating solutions of control problems. Much has been done in recent years to create schemes for BSDE with Lipschitz continuous drivers (see [BT04] or [Eli06] and references therein). So far BSDE with drivers of quadratic growth resisted attempts to allow such schemes, which was the main motivation for this paper.

If the driver is Lipschitz continuous, following [BT04], the strategy to prove convergence of a numerical discretization combines two ingredients: regularity of the trajectories of the control

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process, and a convenient a priori estimate for the solution. The regularity result we refer to is obtained in [Zha01]. It allows to establish the convergence order for the approximation of the control process.

Our approach for the case of drivers with quadratic growth contains an essential modification of [BT04].

In a first step, we extend the path regularity result for the control process to the setting of qgBSDE. The methods we apply to achieve this goal rely crucially on the power of the stochastic calculus of variations. If (Y, Z) is the solution pair of a BSDE, it is well known that the trace of the first Malliavin derivative allows a description of Z by the formula $D_t Y_t = Z_t$, which in turn allows estimates of Z in the sup norm, provided an extra continuity result is established. To describe path regularity of Z efficiently, one also needs estimates of the Malliavin derivative of Z in the sup norm, whence second order Malliavin derivatives of Y are needed and add to the complexity of the treatment. Nonetheless we derive path regularity without assuming that Z is bounded.

In the second step of our approach, we truncate the quadratic growth part of the driver to fall back into the setting of Lipschitz continuous drivers. We are able to explicitly capture the convergence rate for the solutions of the truncated BSDE as a function of the truncation height. Combining the error estimate for the truncation with the ones for the discretizations in any existent numerical scheme for BSDE with Lipschitz continuous drivers, we find a numerical approximation for quadratic growth BSDE.

This result is new to the best of our knowledge. The truncation procedure, however, does not look like the most efficient solution one hopes for. The main drawback of the approach resides in the running times of the numerical algorithm. Roughly, if K is the truncation dependent Lipschitz constant, the time step h of the partition for the usual numerical discretization has to satisfy $e^K h < 1$ modulo some multiplicative constant which results from the use of Gronwall's inequality. So if the truncation height increases, h will have to become small very quickly, which computationally is a rather inconvenient fact.

The paper is organized as follows. In the introductory Section 2 we recall some of the well known results concerning SDE and BSDE. In section 3 we establish some estimates concerning a special class of BSDE, and in Section 4 we establish the second order Malliavin differentiability of solutions of Lipschitz BSDE and qgBSDE. These results are used in Section 5 to state and prove several regularity results for the trajectories of the solution processes. In Section 6 we discuss convergence rates of solutions of truncated BSDE to those related to BSDE with drivers of quadratic growth.

2 Preliminaries

2.1 Spaces and Notation

We work on a canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a d-dimensional Wiener process $W = (W^1, \cdots, W^d)$ restricted to the time interval $[0, T]$, and we denote by $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ its natural filtration enlarged in the usual way by the $\mathbb{P}$-zero sets. We shall need the following operators, and auxiliary spaces of functions and stochastic processes: let $p \geq 2, m, n, d \in \mathbb{N}$, $\mathbb{Q}$ a probability measure on $(\Omega, \mathcal{F})$. We use the symbol $\mathbb{E}^{\mathbb{Q}}$ for the expectation with respect to $\mathbb{Q}$, and omit the superscript for the canonical measure $\mathbb{P}$. For vectors $x = (x^1, \cdots, x^m)$ in Euclidean space $\mathbb{R}^m$ we write $|x| = (\sum_{i=1}^m (x^i)^2)^{\frac{1}{2}}$. We denote further

- $C_b^k(\mathbb{R}^m)$ the set of k-times differentiable real valued maps defined on $\mathbb{R}^m$ with bounded partial derivatives up to order k, and $C_b^\infty(\mathbb{R}^m) = \cap_{k\geq 1} C_b^k(\mathbb{R}^m)$; We omit the subscript b

to denote the same set but without the boundedness assumptions.

- $B_n^{m\times d}$ the set of all functions $h:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{m\times d}$ for which there is a constant C such that for all $t\in[0,T]$ we have $|h(t,x)|\le C(1+|x|)$ and $x\mapsto h(t,x)$ is differentiable with bounded Lipschitz derivative;

- $L^p(\mathbb{R}^m;\mathbb{Q})$ the space of $\mathcal{F}_T$-measurable random variables $X:\Omega\mapsto\mathbb{R}^m$, normed by $\|X\|_{L^p}=\mathbb{E}^{\mathbb{Q}}[\,|X|^p]^{\frac{1}{p}}$; L^∞ the space of bounded random variables;

- $\mathcal{S}^p(\mathbb{R}^m)$ the space of all measurable processes $(Y_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by $\|Y\|_{\mathcal{S}^p}=\mathbb{E}[\Big(\sup_{t\in[0,T]}|Y_t|\Big)^p]^{\frac{1}{p}}$; $\mathcal{S}^\infty(\mathbb{R}^m)$ the space of bounded measurable processes;

- $\mathcal{H}^p(\mathbb{R}^m,\mathbb{Q})$ the space of all progressively measurable processes $(Z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m$ normed by $\|Z\|_{\mathcal{H}^p}=\mathbb{E}^{\mathbb{Q}}[\Big(\int_0^T|Z_s|^2\mathrm{d}s\Big)^{p/2}]^{\frac{1}{p}}$;

- $BMO(\mathbb{Q})$ or $BMO_2(\mathbb{Q})$ the space of square integrable martingales Φ with $\Phi_0=0$ and we set
$$\|\Phi\|^2_{BMO(\mathbb{Q})}=\sup_\tau\Big\|\,\mathbb{E}^{\mathbb{Q}}\big[\langle\Phi\rangle_T-\langle\Phi\rangle_\tau|\mathcal{F}_\tau\big]\Big\|_\infty<\infty,$$
where the supremum is taken over all stopping times $\tau\in[0,T]$.

- $\mathbb{D}^{k,p}(\mathbb{R}^d)$ and $\mathbb{L}_{k,d}(\mathbb{R}^d)$ are the spaces of Malliavin differentiable random variables and processes, see subsection 2.2.

If there is no ambiguity about the underlying spaces or measures, we also omit them as arguments in the function spaces defined above.

To denote stochastic integral processes of the Wiener process on $[0,T]$, according to Paul-André Meyer, we write
$$Z*W=\int_0^{\cdot}Z_s\mathrm{d}W_s,\qquad\text{with } Z\in\mathcal{H}^2.$$

Constants appearing in inequalities of our proofs will for simplicity be denoted by C, although they may change from line to line.

2.2 Malliavin Calculus

We shall use techniques of the stochastic calculus of variations. To this end, we use the following notation. For more details, we refer the reader to [Nua95]. Let $\mathcal{S}$ be the space of random variables of the form
$$\xi=F\Big((\int_0^T h_s^{1,i}\mathrm{d}W_s^1)_{1\le i\le n},\cdots,(\int_0^T h_s^{d,i}\mathrm{d}W_s^d)_{1\le i\le n})\Big),$$
where $F\in C_b^\infty(\mathbb{R}^{n\times d})$, $h^1,\cdots,h^n\in L^2([0,T];\mathbb{R}^d)$, $n\in\mathbb{N}$. To simplify notation, assume that all h^j are written as row vectors. For $\xi\in\mathcal{S}$, we define $D=(D^1,\cdots,D^d):\mathcal{S}\to L^2(\Omega\times[0,T])^d$ by
$$D_\theta^i\xi=\sum_{j=1}^n\frac{\partial F}{\partial x_{i,j}}\Big(\int_0^T h_t^1\mathrm{d}W_t,\ldots,\int_0^T h_t^n\mathrm{d}W_t\Big)h_\theta^{i,j},\quad 0\le\theta\le T,\quad 1\le i\le d,$$
and for $k\in\mathbb{N}$ its k-fold iteration by
$$D^{(k)}=(D^{i_1}\cdots D^{i_k})_{1\le i_1,\cdots,i_k\le d}.$$

For $k \in \mathbb{N}, p \geq 1$ let $\mathbb{D}^{k,p}$ be the closure of $\mathcal{S}$ with respect to the norm

$$\|\xi\|_{k,p}^p = \mathbb{E}\Big[\|\xi\|_{L^p}^p + \sum_{i=1}^k \||D^{(k)]}\xi|\|_{(\mathcal{H}^p)^i}^p\Big].$$

$D^{(k)}$ is a closed linear operator on the space $\mathbb{D}^{k,p}$. Observe that if $\xi \in \mathbb{D}^{1,2}$ is $\mathcal{F}_t$-measurable then $D_\theta \xi = 0$ for $\theta \in (t,T]$. Further denote $\mathbb{D}^{k,\infty} = \cap_{p>1}\mathbb{D}^{k,p}$.

We also need Malliavin's calculus for smooth stochastic processes with values in $\mathbb{R}^m$. For $k \in \mathbb{N}, p \geq 1$, denote by $\mathbb{L}_{k,p}(\mathbb{R}^m)$ the set of $\mathbb{R}^m$-valued progressively measurable processes $u = (u^1, \cdots, u^m)$ on $[0,T] \times \Omega$ such that

i) For Lebesgue a.a. $t \in [0,T]$, $u(t,\cdot) \in (\mathbb{D}^{k,p})^m$;

ii) $[0,T] \times \Omega \ni (t,\omega) \mapsto D^{(k)}u(t,\omega) \in (L^2([0,T]^{1+k}))^{d\times n}$ admits a progressively measurable version;

iii) $\|u\|_{k,p} = \Big[\, \||u|\|_{\mathcal{H}^p}^p + \sum_{i=1}^k \| D^i u \|_{(\mathcal{H}^p)^{1+i}}^p \,\Big]^{\frac{1}{p}} < \infty$

For instance, for a process $X \in \mathbb{L}_{2,2}(\mathbb{R})$ we have

$$\|X\|_{1,2}^2 = \mathbb{E}\Big[\int_0^T |X_t|^2 \mathrm{d}t + \int_0^T\int_0^T |D_\theta X_t|^2 \mathrm{d}\theta \mathrm{d}t\Big],$$

$$\|X\|_{2,2}^2 = \|X\|_{1,2}^2 + \mathbb{E}\Big[\int_0^T\int_0^T\int_0^T |D_{\theta_1}D_{\theta_2}X_t|^2 \mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}t\Big].$$

Note that Jensen's inequality gives for all $p \geq 2$

$$\mathbb{E}\Big[\Big(\int_0^T\int_0^T |D_u X_t|^2 \mathrm{d}u\,\mathrm{d}t\Big)^{\frac{p}{2}}\Big] \leq T^{p/2-1}\int_0^T \|D_u X\|_{\mathcal{H}^p}^p \mathrm{d}u.$$

This inequality is very useful since techniques used to deal with BSDE don't allow a direct estimate of the left hand side, but easily give access to the right hand side.

For the sake of completeness we state a result from [Nua95] that we will use in this work. (One needs an $L^2([0,T])$-valued version of this Lemma for Theorem 4.1.)

Lemma 2.1 (Lemma 1.2.3 in [Nua95]). *Let $\{F_n,\, n \geq 1\}$ be a sequence of random variables in $\mathbb{D}^{1,2}$ that converges to F in $L^2(\Omega)$ and such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\,\|DF_n\|_{\mathcal{H}^2}] < \infty.$$

Then F belongs to $\mathbb{D}^{1,2}$, and the sequence of derivatives $\{DF_n,\, n \in \mathbb{N}\}$ converges to DF in the weak topology of $L^2(\Omega \times [0,T])$.

2.3 Some results on BMO martingales

BMO martingales play a key role for a priori estimates needed in our sensitivity analysis of solutions of BSDE. For details about their theory we refer the reader to [Kaz94].

If Φ is a square integrable martingale, the martingale representation theorem yields a square integrable process ϕ such that $\Phi_t = \int_0^t \phi_s \mathrm{d}W_s, t \in [0,T]$. Hence the $BMO(\mathbb{Q})$ norm can be alternatively expressed as

$$\sup_{\tau\in[0,T]} \mathbb{E}^{\mathbb{Q}}\Big[\int_\tau^T \phi_s^2 \mathrm{d}s|\mathcal{F}_\tau\Big] < \infty.$$

As an easy consequence, if $\Phi \in BMO$ then $\int H\mathrm{d}\Phi \in BMO$ for any bounded adapted process H.

Lemma 2.2 (Properties of BMO martingales). *Let Φ be a BMO martingale. Then we have:*

1) The stochastic exponential $\mathcal{E}(\Phi)$ is uniformly integrable.

2) There exists a number $r > 1$ such that $\mathcal{E}(\Phi_T) \in L^r$. This property follows from the Reverse Hölder inequality*. The maximal r with this property can be expressed in terms of the BMO norm of Φ.*

3) For probability measures $\mathbb{P}$ and $\mathbb{Q}$ satisfying $\mathbb{Q} = \mathcal{E}(\Phi_T) \cdot \mathbb{P}$, and for $\Phi \in BMO(\mathbb{P})$, the process $\hat{\Phi} = \Phi - \langle \Phi \rangle$ is a $BMO(\mathbb{Q})$ martingale.

4) Energy inequalities imply the inclusion $BMO \subset \mathcal{H}^p$ for all $p \geq 1$. More precisely, for $\Phi = \int_0^{\cdot} \phi_s \mathrm{d}s \in BMO$ with BMO norm C, and $p \geq 1$ the following estimate holds

$$\mathbb{E}[\Big(\int_0^T |\phi_s|^2 \mathrm{d}s \Big)^p] \leq 2p!(4C^2)^p. \tag{1}$$

2.4 The setting and its assumptions

For functions b, σ, g and f, for $x \in \mathbb{R}^m$ and a d-dimensional Brownian motion W we intend to study the solution processes of the following system of forward-backward stochastic differential equations (with generators of quadratic growth (qgFBSDE)). For $t \in [0, T]$ they are given by

$$X_t = x + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \tag{2}$$

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}W_s + \int_t^T f(s, \Theta_s)\mathrm{d}s, \tag{3}$$

with $\xi = g(X_T)$ and $\Theta_s = (X_s, Y_s, Z_s)$.

For the functions figuring in the above system of equations we hierarchically order the properties they will be assumed to satisfy.

HX0 There is a constant K such that $b, \sigma_i : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m, 1 \leq i \leq d$, are uniformly Lipschitz continuous with Lipschitz constant K, and $b(\cdot, 0)$ and $\sigma_i(\cdot, 0), 1 \leq i \leq d$, are bounded by K.

HX1 Hypothesis **HX0** holds. For any $0 \leq t \leq T$ the functions $b(t, \cdot), \sigma_i(t, \cdot), 1 \leq i \leq d$, are differentiable and uniformly Lipschitz with Lipschitz constant K independent of t. In other words, $\sigma \in B_m^{m \times d}$ and $b \in B_m^{m \times 1}$. There exists a positive constant c such that

$$y^T \sigma(t, x)\sigma^T(t, x)y \geq c|y|^2, \qquad x, y \in \mathbb{R}^m, \ t \in [0, T].$$

HX2 Hypothesis **HX1** holds. There exists a positive constant K such that $b(t, \cdot) \in C_b^2(\mathbb{R}^m)$ and $\sigma(t, \cdot) \in C_b^2(\mathbb{R}^{m \times d})$ with second derivatives bounded by K.

HY0 There is a constant $M \in \mathbb{R}^+$ such that $g : \mathbb{R}^m \to \mathbb{R}$ is bounded by M, hence $|\xi| \leq M$. $f : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an adapted measurable function, continuous in (y, z), for which there exists $M \in \mathbb{R}_+$ such that $|f(\cdot, \cdot, y, z)| \leq M(1 + |y| + |z|^2)$, for $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

HY1 Hypothesis **HY0** holds. f is differentiable in (x,y,z) and there exists $M \in \mathbb{R}_+$ such that

$$\begin{aligned}|\nabla_x f(t,x,y,z)| &\leq M(1+|y|+|z|^2),\\ |\nabla_y f(t,x,y,z)| &\leq M,\\ |\nabla_z f(t,x,y,z)| &\leq M(1+|z|).\end{aligned}$$

$g : \mathbb{R}^m \to \mathbb{R}$ is a Lipschitz differentiable function satisfying $|\nabla g| \leq M$.

HY2 Hypothesis **HY1** holds, $g \in C_b^2(\mathbb{R}^m)$. The driver f is twice differentiable with continuous second order derivatives. There exists an adapted process $(K_t)_{0\leq t\leq T}$ belonging to $\mathcal{S}^{2p}(\mathbb{R})$ for all $p \geq 1$ such that for any $t \in [0,T]$ all second order derivatives of f at $(t,\Theta_t) = (t,X_t,Y_t,Z_t)$ are a.s. dominated by K_t.

2.5 Some results on SDE

We recall the results on SDE known from the literature that are relevant for this work. We state our assumptions in the multidimensional setting. However, for ease of notation we present some formulas in the one dimensional case.[1]

Theorem 2.3 (Moment estimates for SDE). *Assume that* **HX0** *holds. Then (2) has a unique solution and the following moment estimates hold: for any $p \geq 2$ there exists a constant $C > 0$, depending only on T, K and p such that for any $x \in \mathbb{R}^m, s,t \in [0,T]$*

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^p] \leq C\mathbb{E}\Big[|x|^p + \int_0^T \big(|b(t,0)|^p + |\sigma(t,0)|^p\big)\mathrm{d}t\Big], \tag{4}$$

$$\mathbb{E}[\sup_{s\leq u\leq t} |X_u - X_s|^p] \leq C\mathbb{E}\Big[|x|^p + \sup_{0\leq t\leq T}\big\{|b(t,0)|^p + |\sigma(t,0)|^p\big\}\Big]\,|t-s|^{p/2}. \tag{5}$$

Furthermore, given two different initial conditions $x, x' \in \mathbb{R}^m$ and denoting the respective solutions of (2) by X^x and $X^{x'}$, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t^x - X_t^{x'}|^p\Big] \leq C|x-x'|^p. \tag{6}$$

Theorem 2.4 (Classical differentiability). *Assume* **HX1** *holds. Then the solution process X of (2) as a function of the initial condition $x \in \mathbb{R}^m$ is differentiable and satisfies for $t \in [0,T]$*

$$\nabla X_t = I_m + \int_0^t \nabla b(X_s)\nabla X_s \mathrm{d}s + \int_0^t \nabla\sigma(X_s)\nabla X_s \mathrm{d}W_s, \tag{7}$$

where I_m denotes the $m \times m$ unit matrix. Moreover, ∇X_t as an $m \times m$-matrix is invertible for any $t \in [0,T]$. Its inverse $(\nabla X_t)^{-1}$ satisfies an SDE and for any $p \geq 2$ there are positive constants C_p and c_p such that

$$\|\nabla X\|_{\mathcal{S}^p} + \|(\nabla X)^{-1}\|_{\mathcal{S}^p} \leq C_p \tag{8}$$

and

$$\mathbb{E}\Big[\sup_{s\leq u\leq t} |(\nabla X_u) - (\nabla X_s)|^p + \sup_{s\leq u\leq t} |(\nabla X_u)^{-1} - (\nabla X_s)^{-1}|^p\Big] \leq c_p\,|t-s|^{p/2}. \tag{9}$$

[1] For a beautiful presentation of this subsection's Theorems we point the reader to [Eli06].

Theorem 2.5 (Malliavin Differentiability). *Under* **HX1**, $X \in \mathbb{L}_{1,2}$ *and its Malliavin derivative admits a version* $(u,t) \mapsto D_u X_t$ *satisfying for* $0 \le u \le t \le T$ *the SDE*

$$D_u X_t = \sigma(X_u) + \int_u^t \nabla b(X_s) D_u X_s \mathrm{d}s + \int_u^t \nabla \sigma(X_s) D_u X_s \mathrm{d}W_s.$$

Moreover, for any $p \ge 2$ *there is a constant* $C_p > 0$ *such that for* $x \in \mathbb{R}^m, 0 \le v \le u \le t \le s \le T$

$$\|D_u X\|_{\mathcal{S}^p}^p \le C_p(1+|x|^p), \tag{10}$$

$$\mathbb{E}[\,|D_u X_t - D_u X_s|^p] \le C_p(1+|x|^p)|t-s|^{\frac{p}{2}}, \tag{11}$$

$$\|D_u X - D_v X\|_{\mathcal{S}^p}^p \le C_p(1+|x|^p)|u-v|^{\frac{p}{2}}. \tag{12}$$

By Theorem 2.4, we have the representation

$$D_u X_t = \nabla X_t (\nabla X_u)^{-1} \sigma(X_u) \mathbf{1}_{[0,u]}(t), \qquad \text{for all } u,t \in [0,T].$$

If **HX2** *holds, then* $DX \in \mathbb{L}_{1,2}$. *For all* $v,u,t \in [0,T]$, $D_v D_u X_t$ *admits a version which solves for* $0 \le v \le u \le t \le T$

$$\begin{aligned} D_v D_u X_t = {} & \nabla \sigma(X_u) D_v X_u + \nabla \sigma(X_v) D_u X_v \\ & + \int_u^t \Big[\nabla b(X_s) D_v D_u X_s + \Delta b(X_s) D_v X_s D_u X_s\Big]\mathrm{d}s \\ & + \int_u^t \Big[\nabla \sigma(X_s) D_v D_u X_s + \Delta \sigma(X_s) D_v X_s D_u X_s\Big]\mathrm{d}W_s. \end{aligned}$$

Furthermore, there exists a continuous version of $(D_v D_u X_t)_{v,u,t\in[0,T]}$ *and for all* $0 \le v,u \le T$ *we have*

$$\|D_u D_v X\|_{\mathcal{S}^p}^p \le C_p(1+|x|^{2p}).$$

For $0 \le v, v' \le u, u' \le t \le T$ *we have*

$$\|D_v D_u X - D_{v'} D_{u'} X\|_{\mathcal{S}^p}^p \le C_p(|v-v'|^{\frac{p}{2}} + |u-u'|^{\frac{p}{2}}).$$

2.6 Results on BSDE with drivers of quadratic growth

We next collect some results on qgBSDE. For their original versions or more information, we refer to [Kob00], [AIdR07], [BC07] and [dR09].

Theorem 2.6 (Properties of qgBSDE). *Under* **HY0**, **HX0**, *the system (2)-(3) has a unique solution* $(X,Y,Z) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times \mathcal{H}^2$. *The norms of* Y *and* Z *depend only on* T, K, M *as given by assumption* **HY0**.

The martingale $Z * W$ *belongs to the space of BMO martingales, and hence* $Z \in \mathcal{H}^p$ *for all* $p \ge 2$. *The following estimate holds*[2]*:*

$$\|Z * W\|_{BMO} \le \frac{4+6M^2T)}{3M^2} \exp\Big\{6M\|\xi\|_{L^\infty} + MT\Big\} < \infty.$$

Remark 2.7. Following Part 2 of Lemma 2.2, we define a pair $(\bar{r},\bar{q})$ such that $1/\bar{r} + 1/\bar{q} = 1$ and $\mathcal{E}(Z*W) \in L^{\bar{r}}$.

In the following, when discussing BMO martingales, an appearing exponent $\bar{r}$ will always be used in this sense.

[2]This inequality follows from applying Itô's formula to e^{aY_t+bt} with an appropriate choice of a and b.

For more properties about BMO martingales in the setting of BSDE with drivers of quadratic growth we refer to Lemma 2.1 in [AIdR07].

The two differentiability results we now present can be found in [dR09]. These results are natural extensions of results proved in [AIdR07] or [BC07]. For further details, comments and complete proofs we refer to [dR09].

Theorem 2.8 (Classical differentiability). *Suppose that* **HX1** *and* **HY1** *hold. Then for all $p \geq 2$ the solution processes (X^x, Y^x, Z^x) of the system (2)-(3) with initial vector $x \in \mathbb{R}^m$ for the forward component belongs to $\mathcal{S}^p \times \mathcal{S}^p \times \mathcal{H}^p$. The application $\mathbb{R}^m \ni x \mapsto (X^x, Y^x, Z^x) \in \mathcal{S}^p(\mathbb{R}^m) \times \mathcal{S}^p(\mathbb{R}^1) \times \mathcal{H}^p(\mathbb{R}^d)$ is differentiable. The derivatives of X satisfy (7) while the the derivatives of (Y, Z) satisfy the linear BSDE*

$$\nabla Y_t^x = \nabla g(X_T^x)\nabla X_T^x - \int_t^T \nabla Z_s^x \mathrm{d}W_s + \int_t^T \langle \nabla f(s, \Theta_s^x), \nabla \Theta_s^x \rangle \mathrm{d}s. \tag{13}$$

If **HX2** *and* **HY2** *hold, then there exists a version of the solution $\Omega \times [0,T] \times \mathbb{R}^m \ni (\omega, t, x) \mapsto (X_t^x, Y_t^x, Z_t^x)(\omega) \in \mathbb{R}^m \times \mathbb{R}^1 \times \mathbb{R}^d$, such that for almost all ω, X^x and Y^x are continuous in time and continuously differentiable in x.*

Theorem 2.9 (Malliavin differentiability). *Suppose that* **HX1** *and* **HY1** *hold. Then the solution processes (X, Y, Z) of system (2)-(3) verify*

- *for any $0 \leq t \leq T$, $x \in \mathbb{R}^m$ we have $(Y_t, Z_t) \in \mathbb{L}_{1,2} \times \left(\mathbb{L}_{1,2}\right)^d$. X satisfies the statement of Theorem 2.5, and a version of $(D_u Y_t, D_u Z_t)_{0 \leq u,t \leq T}$ satisfies*

$$\begin{aligned} &D_u Y_t = 0, \qquad D_u Z_t = 0, \qquad t < u \leq T, \\ &D_u Y_t = \nabla g(X_T) D_u X_T + \int_t^T \langle \nabla f(s, \Theta_s), D_u \Theta_s \rangle \mathrm{d}s - \int_t^T D_u Z_s \mathrm{d}W_s, \qquad t \in [u, T]. \end{aligned} \tag{14}$$

  Moreover, $(D_t Y_t)_{0 \leq t \leq T}$ defined by the above equation is a version of $(Z_t)_{0 \leq t \leq T}$.

- *the following representation holds for any $0 \leq u \leq t \leq T$ and $x \in \mathbb{R}^m$*

$$D_u Y_t = \nabla_x Y_t (\nabla_x X_u)^{-1} \sigma(u, X_u), \quad a.s., \tag{15}$$

$$Z_t = \nabla_x Y_t (\nabla_x X_t)^{-1} \sigma(s, X_t), \quad a.s.. \tag{16}$$

3 Inequalities for BSDE with stochastic Lipschitz conditions

In this section we look closely at BSDE with drivers that satisfy Lipschitz conditions with random Lipschitz constants. Our interest in this problem is motivated by the following observation. If we formally differentiate the driver of our original BSDE, we see that the essential term Z^2 produces a term of the form ZDZ. In this term we may consider the factor Z as a random growth rate of the factor DZ.

Let ζ be a random variable and f a measurable function. We consider the BSDE

$$U_t = \zeta - \int_t^T V_s \mathrm{d}W_s + \int_t^T f(\omega, s, U_s, V_s)\mathrm{d}s, \qquad t \in [0, T]. \tag{17}$$

We state a set of assumptions for ζ and f. For $p \geq 1$ we stipulate

(HA1) ζ is $\mathcal{F}_T$-adapted random variable and $\zeta \in L^{2p}(\mathbb{R})$.

(HA2) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is product measurable and for all $u, u' \in \mathbb{R}$ and $v, v' \in \mathbb{R}^d$ there exists a constant $M > 0$ and a positive predictable process H such that

$$|f(t,u,v) - f(t,u',v')| \leq M|u-u'| + H_t|v-v'|, \tag{18}$$

and such that $H * W$ is a BMO martingale.

(HA3) $f(t,0,0)$ is a measurable $(\mathcal{F}_t)$-adapted process satisfying $\mathbb{E}[\big(\int_0^T |f(s,0,0)|\mathrm{d}s\big)^p] < \infty$ for all $p \geq 1$.

Moreover, we assume that (U,V) is a solution of BSDE (17), and the constant $\bar{r}$ is related to the BMO martingale $H * W$ as in Remark 2.7.

3.1 Moment estimates for BSDE with random Lipschitz constant

For the study of sensitivity properties of solutions of qgFBSDE, as seen in [AIdR07] or [BC07], it is convenient to consider BSDE with random Lipschitz constants. The moment estimates for this type of BSDE one can find in the two cited papers still leave space for improvements. A weakness of the results of [AIdR07], owed to the techniques used, is the lack of an estimate for $\|U\|_{\mathcal{S}^2}$. We next state an extended moment estimate, obtained by using ideas of [BC07].

Lemma 3.1. *Let (HA1)-(HA3) be satisfied, let $p \geq 1$ and $\bar{r} > 1$ such that $\mathcal{E}(H * W) \in L^{\bar{r}}(P)$. Then there exists a constant $C > 0$, depending only on p, T, M and the BMO-norm of $H * W$, such that with the conjugate exponent $\bar{q}$ of $\bar{r}$ we have*

$$\|U\|^{2p}_{\mathcal{S}^{2p}} + \|V\|^{2p}_{\mathcal{H}^{2p}} \leq C\mathbb{E}\Big[|\zeta|^{2p\bar{q}^2} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}}. \tag{19}$$

Proof. Assumption (HA2) states that the driver is Lipschitz continuous in u. We first use this hypothesis to simplify the BSDE. For $t \in [0,T]$ we define

$$a_t = \frac{f(t,U_t,V_t) - f(t,0,V_t)}{U_t}\, 1_{\{U_t \neq 0\}} \quad \text{and} \quad e_t = \exp\big\{\int_0^t a_s \mathrm{d}s\big\}.$$

Under (HA2), the process a is well defined and bounded by M. Hence e is bounded as well. For $t \in [0,T]$ we further define

$$b_t = \frac{f(t,0,V_t) - f(t,0,0)}{|V_t|^2} V_t\, 1_{\{V_t \neq 0\}}.$$

By (HA2), b is well defined and bounded in absolute value by the process H. Applying Itô's formula to $e \cdot U$ we obtain

$$e_t U_t = e_T\zeta + \int_t^T e_s\big[f(s,0,0) + b_s V_s\big]\mathrm{d}s - \int_t^T e_s V_s \mathrm{d}W_s.$$

We simplify the BSDE further by defining a new measure $\mathbb{Q}^b$ for which $W^b = W - \int_0^\cdot b_s ds$ is a $\mathbb{Q}^b$-Brownian motion. The Radon-Nikodym density of $\mathbb{Q}^b$ with respect to $\mathbb{P}$ is given by the stochastic exponential $\mathcal{E}(b * W)$. Since $|b| \leq H$ we have $\|b * W\|_{BMO} \leq \|H * W\|_{BMO}$. Hence the measure $\mathbb{Q}^b$ is indeed a probability measure. For $t \in [0,T]$ our BSDE takes the form

$$e_t U_t = e_T\zeta + \int_t^T e_s f(s,0,0)\mathrm{d}s - \int_t^T e_s V_s \mathrm{d}W^b_s. \tag{20}$$

We now proceed with moment estimates. Taking conditional expectation with respect to $\mathbb{Q}^b$, estimating by absolute values and integrating on the whole interval we obtain

$$|e_t U_t| \leq \mathbb{E}^{\mathbb{Q}^b}\Big[e_T|\zeta| + \int_0^T e_s|f(s,0,0)|\mathrm{d}s\,|\mathcal{F}_t\Big].$$

Applying Doob's moment inequality for $2p \geq 2$ we obtain a similar inequality as in this Theorem's statement, but under the measure $\mathbb{Q}^b$, i.e.

$$\|U\|^{2p}_{\mathcal{S}^{2p}(\mathbb{Q}^b)} \leq C\mathbb{E}^{\mathbb{Q}^b}\Big[|\zeta|^{2p} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p}\Big]. \tag{21}$$

If we rewrite equation (20), isolate the stochastic integral on the left hand side and take $t = 0$, use Burkholder-Davis-Gundy's inequality and remember that e is also bounded below by a positive constant thanks to (HA2), we get

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^b}\Big[\Big(\int_0^T |V_s|^2\mathrm{d}s\Big)^p\Big] &\leq c_p\mathbb{E}^{\mathbb{Q}^b}\Big[|e_T\zeta|^{2p} + \sup_{0\leq t\leq T}|e_tU_t|^{2p} + \Big(\int_0^T |e_sf(s,0,0)|\mathrm{d}s\Big)^{2p}\Big]\\ &\leq C\mathbb{E}^{\mathbb{Q}^b}\Big[|\zeta|^{2p} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p}\Big].\end{aligned}$$

For the second inequality we used (21) and the fact that $[\int_0^T |V_s|^2\mathrm{d}s]^{1/2}$ is integrable.

Summarizing the last two inequalities we get

$$\|U\|^{2p}_{\mathcal{S}^{2p}(\mathbb{Q}^b)} + \|V\|^{2p}_{\mathcal{H}^{2p}(\mathbb{Q}^b)} \leq C\mathbb{E}^{\mathbb{Q}^b}\Big[|\zeta|^{2p} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p}\Big]. \tag{22}$$

This inequality is already close to the one we have to deduce. To complete the proof, we just have to get rid of the dependence of the right hand side on $\mathbb{Q}^b$. We do this for (21), noting that for the other inequality the arguments are very similar. As mentioned before, b is dominated by H and therefore

$$\|b * W\|_{BMO} \leq \|H * W\|_{BMO}.$$

Further, part 3) of Lemma 2.2 implies that since $b*W \in BMO(\mathbb{P})$, also $(-b)*W^b \in BMO(\mathbb{Q}^b)$. Moreover, since $[\mathcal{E}(b*W)]^{-1} = \mathcal{E}\big((-b)*W^b\big)$, part 2) of the same Lemma states the existence[3] of a real number $\bar{r} > 1$ for which $\mathcal{E}(b*W) \in L^{\bar{r}}(\mathbb{P})$ and $[\mathcal{E}(b*W)]^{-1} \in L^{\bar{r}}(\mathbb{Q}^b)$. The constant $\bar{r}$ is estimated from the $BMO(\mathbb{P})$ norm of $H*W$, as indicated in Lemma 2.2.

Throughout let $D = \max\big\{\|\mathcal{E}(b*W)\|_{L^{\bar{r}}(\mathbb{P})}, \|\mathcal{E}(b*W)^{-1}\|_{L^{\bar{r}}(\mathbb{Q}^b)}\big\}$ and let $\bar{q}$ be the conjugate Hölder exponent of $\bar{r}$.

Combining (21) and Hölder's inequality, we obtain for any $p \geq 1$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\sup_{s\in[0,T]}|U_s|^{2p}] = \mathbb{E}^{\mathbb{Q}^b}\big[\mathcal{E}(b*W)^{-1}\sup_{s\in[0,T]}|U_s|^{2p}\big] \;&\leq\; D\mathbb{E}^{\mathbb{Q}^b}\big[\sup_{s\in[0,T]}|U_s|^{2p\bar{q}}\big]^{\frac{1}{\bar{q}}}\\ &\leq C_1\,D\,\mathbb{E}^{\mathbb{Q}^b}\Big[|\zeta|^{2p\bar{q}} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p\bar{q}}\Big]^{\frac{1}{\bar{q}}}\\ &= C_1\,D\,\mathbb{E}^{\mathbb{P}}\Big[\mathcal{E}(b*W)\Big(|\zeta|^{2p\bar{q}} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p\bar{q}}\Big)\Big]^{\frac{1}{\bar{q}}}\\ &\leq C_2\,D^{\frac{1+\bar{q}}{\bar{q}}}\mathbb{E}^{\mathbb{P}}\Big[|\zeta|^{2p\bar{q}^2} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}},\end{aligned}$$

[3]Here we follow the notation we stipulated in Remark 2.7.

where C_1, C_2 represent constants depending on p, M, T and the BMO norm of $(b*W)$. Similarly, with another constant C_3,

$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |V_s|^2 \mathrm{d}s\Big)^p\Big] \leq C_3\, D^{\frac{1+\bar{q}}{\bar{q}}}\, \mathbb{E}^{\mathbb{P}}\Big[|\zeta|^{2p\bar{q}^2} + \Big(\int_0^T |f(s,0,0)|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big)\Big]^{\frac{1}{\bar{q}^2}}.$$

Combining the two estimates we obtain (19). □

3.2 A priori estimates for BSDE with random Lipschitz constant

In this section, following the results of the previous one, we derive a priori inequalities which serve in the usual way to compare solutions of BSDE of the type considered obtained for different system parameters such as initial states of the forward part. This result will later be used to determine the good candidates for the derivatives of our original qgBSDE.

For each $i \in \{1, 2\}$, let ζ_i be a random variable satisfying condition (HA1) and f_i a driver function satisfying (HA2)-(HA3) with respective square integrable processes H^i such that $H^i * W \in BMO$. With this random variable and driver function we investigate the following BSDE

$$U_t^{(i)} = \zeta_i - \int_t^T V_s^{(i)} \mathrm{d}W_s + \int_t^T f_i(\omega, s, U_s^{(i)}, V_s^{(i)})\mathrm{d}s, \quad t \in [0, T]. \tag{23}$$

Lemma 3.2. *Assume that the conditions of Lemma 3.1 hold for (23). Take further $\bar{q}$ with respect to the BSDE with $i = 1$. Then we have*

$$\begin{aligned}&\|U^{(1)} - U^{(2)}\|^{2p}_{\mathcal{S}^{2p}} + \|V^{(1)} - V^{(2)}\|^{2p}_{\mathcal{H}^{2p}} \\ &\qquad \leq C\,\mathbb{E}\Big[|\zeta_1 - \zeta_2|^{2p\bar{q}^2} + \Big(\int_0^T |(f_1 - f_2)(s, U_s^{(2)}, V_s^{(2)})|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}}.\end{aligned}$$

Proof. The arguments to prove this inequality are similar to those used in the proof of Lemma 3.1. Therefore we will omit some of the already familiar details.

Define $\delta U = U^{(1)} - U^{(2)}$, $\delta V = V^{(1)} - V^{(2)}$, $\delta\zeta = \zeta_1 - \zeta_2$ and $\delta f(t, u, v) = (f_1 - f_2)(t, u, v)$. Then to simplify the BSDE define a and b by

$$\begin{aligned}a_t &= \frac{f_1(s, U_t^{(1)}, V_t^{(1)}) - f_1(s, U_t^{(2)}, V_t^{(1)})}{U_t^{(1)} - U_t^{(2)}} \mathbf{1}_{\{U_t^{(1)} \neq U_t^{(2)}\}}, \\ b_t &= \frac{f_1(s, U_t^{(2)}, V_t^{(1)}) - f_1(s, U_t^{(2)}, V_t^{(2)})}{|V_t^{(1)} - V_t^{(2)}|^2} (V_t^{(1)} - V_t^{(2)}) \mathbf{1}_{\{V_t^{(1)} \neq V_t^{(2)}\}},\end{aligned}$$

$t \in [0, T]$. We arrive at an equation similar to (20) given by:

$$e_t \delta U_t = e_T \delta\zeta + \int_t^T [e_s\, \delta f(s, U_s^{(2)}, V_s^{(2)})]\, \mathrm{d}s - \int_t^T \delta V_s \mathrm{d}W_s^b,$$

with $W^b = W - \int_0^{\cdot} b_s ds$. Define $\mathbb{Q}^b$ with respect to b as before. Now we may proceed as in the proof of Lemma 3.1. The existence of the integral of $\delta f(s, U_s^{(2)}, V_s^{(2)})$ is justified by observing that we can dominate δf using our assumptions and also because Lemma 3.1 is applicable to each individual BSDE. The result follows. □

Without prior knowledge of the form of f_1 and f_2 the right hand side of the Lemma's inequality cannot be treated further. In the following result we assume that the drivers satisfy a stochastic linearity property. Then the increment in the drivers can be further estimated.

Corollary 3.3. *Assume that the conditions of Lemma 3.2 are satisfied, and furthermore that for each $i \in \{1,2\}$ the driver is linear, i.e. it satisfies*

$$f_i(\cdot,t,u,v) = \alpha_i(\cdot,s) + \beta_i(\cdot,t)u + \langle \gamma_i(\cdot,t), v\rangle,$$

with $(\alpha_i, \beta_i, \gamma_i)$ adapted random processes belonging to $\mathcal{H}^{2p}(\mathbb{R}) \times \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ for any $p \geq 1$. Moreover, we assume that β_i is bounded and that $(\gamma_i * W) \in BMO$. Then

$$\begin{aligned}
&\|U^{(1)} - U^{(2)}\|^{2p}_{\mathcal{S}^{2p}} + \|V^{(1)} - V^{(2)}\|^{2p}_{\mathcal{H}^{2p}} \\
&\quad \leq C\Big\{\mathbb{E}\Big[|\zeta_1 - \zeta_2|^{2p\bar{q}^2} + \Big(\int_0^T |\alpha_1(s) - \alpha_2(s)|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}} \\
&\qquad + \mathbb{E}\Big[\Big(\int_0^T |\beta_1(s) - \beta_2(s)|\mathrm{d}s\Big)^{4p\bar{q}^2} + \Big(\int_0^T |(\gamma_1(s) - \gamma_2(s)|^2\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{2\bar{q}^2}}\Big\}
\end{aligned}$$

Proof. Starting with the inequality of Lemma 3.2, and injecting the new assumptions, we obtain

$$\begin{aligned}
&\|U^{(1)} - U^{(2)}\|^{2p}_{\mathcal{S}^{2p}} + \|V^{(1)} - V^{(2)}\|^{2p}_{\mathcal{H}^{2p}} \\
&\quad \leq C\mathbb{E}\Big[|\zeta_1 - \zeta_2|^{2p\bar{q}^2} + \Big(\int_0^T |\alpha_1(s) - \alpha_2(s)|\mathrm{d}s\Big)^{2p\bar{q}^2} \\
&\qquad + \Big(\sup_{0\leq t\leq T} |U_t^{(2)}|\Big)^{2p\bar{q}^2}\Big(\int_0^T |\beta_1(s) - \beta_2(s)|\mathrm{d}s\Big)^{2p\bar{q}^2} \\
&\qquad\quad + \Big(\int_0^T |V_s^{(2)}|^2\mathrm{d}s\Big)^{p\bar{q}^2}\Big(\int_0^T |(\gamma_1(s) - \gamma_2(s)|^2\mathrm{d}s\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}}.
\end{aligned}$$

The moment estimates of Lemma 3.1 ensure that $\|U^{(2)}\|_{\mathcal{S}^{4p\bar{q}^2}}$ and $\|V^{(2)}\|_{\mathcal{H}^{4p\bar{q}^2}}$ are finite. Hence a simple application of Hölder's inequality yields the desired result. □

4 Second order Malliavin differentiability

We now give sufficient conditions on our system of stochastic equations which ensure that the solution processes are twice Malliavin differentiable.

4.1 The main result

Theorem 4.1. *Assume* **HX2** *and* **HY2** *hold. Then the solution process $\Theta = (X, Y, Z)$ of the qgFBSDE (2)-(3) is twice Malliavin differentiable, i.e. for each $u \in [0,T]$ and $i \in \{1,\dots,d\}$ the processes $(D^i_u Y_t, D^i_u Z_t) \in \mathbb{L}_{1,2} \times (\mathbb{L}_{1,2})^d$. A version of $\{(D^j_v D^i_u Y_t, D^j_v D^i_u Z_t); 0 \leq v \leq u \leq t \leq T\}$ with $0 \leq j, i \leq d$ satisfies*

$$\begin{aligned}
D^j_v D^i_u Y_t = D^j_v D^i_u \xi &- \int_t^T D^j_v D^i_u Z_s \mathrm{d}W_s \\
&+ \int_t^T \Big[(D^j_v \Theta_s)^T \big[Hf\big](s,\Theta_s) D^i_u \Theta_s + \langle \nabla f(s,\Theta_s), D^j_v D^i_u \Theta_s\rangle\Big]\mathrm{d}s,
\end{aligned} \tag{24}$$

where $\big[Hf\big]$ is the Hessian matrix of the function f and $\xi = g(X_T)$. Considered as a BSDE, (24) admits a unique solution.

Moreover $\{D_t D_u Y_t; 0 \leq u \leq t \leq T\}$ is a version of $\{D_u Z_t; 0 \leq u \leq t \leq T\}$.

By Theorem 2.5, condition **HX2** already implies that $X \in \mathbb{D}^{2,p}$. Therefore one only needs to prove the Malliavin differentiability of (DY, DZ).

4.2 Strategy of the proof

The main problem in proving the variational differentiability of equation (14) is given by the growth of $\nabla_z f(\cdot, z)$ in z. **HY1** states that $\nabla_z f(\cdot, z)$ is dominated by $C(1 + |z|)$. Considering (14) as a BSDE with solution process (DY, DZ) leads to interpreting the influence of $\nabla_z f(\cdot, z)$ in the driver as a random Lipschitz constant. We aim at using the same strategy of proof as in [AIdR07]: we approximate the BSDE (14) by truncating the random Lipschitz constant, and then use Lemma 2.1 to obtain variational differentiability in the limit. Threfore we mainly have to establish the conditions of Lemma 2.1.

4.2.1 A differentiable truncation family for the identity function

So we introduce a family functions $(h_n)_{n\in\mathbb{N}}$ that truncate the identity function on the real line, and that will be used to truncate the z variable in $\nabla f(\cdot, z)$ in the driver term of (14). We choose $h_n : \mathbb{R} \to \mathbb{R}$ continuously differentiable with the following properties:

- $h_n(z) \to z$ locally uniformly; $|h_n(z)| \leq |z|$ and $|h_n(z)| \leq n + 1$, $z \in \mathbb{R}$; moreover

$$h_n(z) = \begin{cases} (n+1) & , z > n+2, \\ z & , |z| \leq n, \\ -(n+1) & , z < -(n+2). \end{cases} \tag{25}$$

- h_n' is bounded by 1, and $h_n' \to 1$ locally uniformly.

We remark that such sequence of functions exists. The above requirements are for instance consistent with

$$h_n(z) = \begin{cases} \big(-n^2 + 2nx - x(x-4)\big)/4 & , z \in [n, n+2], \\ \big(n^2 + 2nx + x(x+4)\big)/4 & , z \in [-(n+2), -n]. \end{cases}$$

4.2.2 The family of truncated FBSDE and results concerning them

Recall the notation $\Theta = (X, Y, Z)$ for the solution of system (2)-(3), the driver of BSDE (14) with terminal condition $\xi = g(X_T)$, where g is a bounded differentiable function and **HX1** is satisfied. For $n \in \mathbb{N}$ take the sequence $(h_n)_{n\in\mathbb{N}}$ defined in (25) and define the sequence of approximate drivers $F^n : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by

$$F^n(t, x, u, v) = \big\langle \nabla_x f(t, \Theta_t), x \big\rangle + \nabla_y f(t, \Theta_t)u + \big\langle \nabla_z f\big(t, X_t, Y_t, h_n(Z_t)\big), v \big\rangle. \tag{26}$$

The advantage of approximating the driver in this way is a technical one: we can make use of the well known Θ and its properties, and do not have to deal with approximations of Θ and its Malliavin derivatives at the same time.

For $i \in \{1, \cdots, d\}$, $0 \leq u \leq t \leq T$ and $n \in \mathbb{N}$ consider the following BSDE

$$U^n_{u,t} = D^i_u \xi + \int_t^T F^n(s, \Xi^n_{u,s}) \mathrm{d}s - \int_t^T V^n_{u,s} \mathrm{d}W_s, \quad \Xi^n_{u,s} = (D^i_u X_s, U^n_{u,s}, V^n_{u,s}), \tag{27}$$

where $D^i\xi$, $D^i X$ denote the first Malliavin derivatives of ξ and X respectively.

In the following Lemma we state existence, uniqueness and Malliavin differentiability of the solution processes of BSDE (27). The Lemma's proof will result from a theorem formulated in the Appendix, where all the hypotheses, variants of the hypotheses employed in Theorem 4.1, are formulated. To avoid repetitions, we do not formulate them here again.

Lemma 4.2 (2nd order Malliavin diff. of Lipschitz BSDE). *For each $n \in \mathbb{N}$, (27) possesses a unique solution (U^n, V^n) in $\mathcal{H}^{2p}([0,T]\times[0,T]) \times \mathcal{H}^{2p}([0,T]\times[0,T])$ for $p \geq 1$.*

Furthermore for $0 \leq u \leq t \leq T$ the random variables $(U^n_{u,t}, V^n_{u,t})$ are Malliavin differentiable and for any $j \in \{1, \cdots, d\}$ a version of $\{(D^j_v U^n_{u,t}, D^j_v V^n_{u,t}); 0 \leq v \leq u \leq t \leq T\}$ satisfies

$$\begin{aligned} D^j_v U^n_{u,t} = D^j_v D^i_u \xi - \int_t^T D^j_v V^n_{u,s} \mathrm{d}W_s \\ + \int_t^T \Big[(D^j_v F^n)(s, \Xi^n_{u,s}) + \langle (\nabla F^n)(s, \Xi^n_{u,s}), D^j_v \Xi^n_{u,s} \rangle \Big] \mathrm{d}s, \end{aligned} \tag{28}$$

with $\Xi^n_{u,s} = (D^i_u X_s, U^n_{u,s}, V^n_{u,s})$ and $D^j_v \Xi^n_{u,s} = (D^j_v D^i_u X_s, D^j_v U^n_{u,s}, D^j_v V^n_{u,s}), 0 \leq v \leq u \leq s \leq T$.

For clarity of exposition we add a few words about the driver of BSDE (28). Assuming $d = m = 1$ and hence omitting the superscripts i and j, and denoting $\Theta = (X, Y, Z)$ and $\Theta^n = (X, Y, h_n(Z))$, we can describe the first term inside the integral by

$$(D_v F^n)(t, \Xi^n_{u,t}) = D_v[(\nabla_x f)(t, \Theta_t)] D_u X_t + D_v[(\nabla_y f)(t, \Theta_t)] U^n_{u,t} + D_v[(\nabla_z f)(t, \Theta^n_t)] V^n_{u,t}.$$

These three terms can be further specified by

$$\begin{aligned} &D_v[(\nabla_x f)(t, \Theta_t)] D_u X_t \\ &\quad = (\nabla_{xx} f)(t, \Theta_t) D_v X_t D_u X_t + (\nabla_{xy} f)(t, \Theta_t) D_v Y_t D_u X_t + (\nabla_{xz} f)(t, \Theta_t) D_v Z_t D_u X_t, \end{aligned}$$

an analogous expression for $D_v[(\nabla_y f)]$, while the last part is given by

$$\begin{aligned} &D_v[(\nabla_z f)(t, \Theta^n_t)] V^n_{u,t} \\ &\quad = (\nabla_{zx} f)(t, \Theta^n_t) D_v X_t V^n_{u,t} + (\nabla_{zy} f)(t, \Theta^n_t) D_v Y_t V^n_{u,t} + (\nabla_{zz} f)\big(t, \Theta^n_t\big) h_n'(Z_t) D_v Z_t V^n_{u,t}. \end{aligned}$$

The second term of the driver in (28) can be expressed by

$$\begin{aligned} &\langle (\nabla F^n)(s, \Xi^n_{u,s}), D_v \Xi^n_{u,s} \rangle \\ &\quad = \nabla_x f(s, \Theta_s) D_v D_u X_s + \nabla_y f(s, \Theta_s) D_v U^n_{u,s} + \nabla_z f\big(s, X_s, Y_s, h_n(Z_s)\big) D_v V^n_{u,s}. \end{aligned}$$

To compactify notation a bit, we denote the driver component in (28) not containing $D_v U^n_{u,s}$ and $D_v V^n_{u,s}$ by

$$A^n_{v,u,s} = (D_v F^n)(s, \Xi^n_{u,s}) + (\nabla_x F^n)(s, \Xi^n_{u,s}) D_v D_u X_s, \quad v, u, s \in [0,T]. \tag{29}$$

Before giving the proof of the Theorem 4.1 we prove two helpful Lemmata.

Remark 4.3. Since $|h_n(z)|$ is dominated by $|z|$, it is clear from **HY1** that

$$\|\nabla_z f(t, X, Y, h_n(Z)) * W\|_{BMO} \leq C \|(1 + |Z|) * W\|_{BMO}.$$

Hence (see Part 2 of Lemma 2.2), there exists a $\bar{r}$ such that the stochastic exponentials related to the two BMO martingales above belong both to $L^{\bar{r}}$. We remark that $\bar{r}$ is independent of n.

Lemma 4.4. *Assume* **HX2** *and* **HY2** *hold, and that (U^n, V^n) solve BSDE (27), (DY, DZ) BSDE (14). Then, for any $p \geq 1$ we have*

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq u \leq T} \left\{ \mathbb{E}\Big[\Big(\int_0^T |D_u Y_s|^2 + |D_u Z_s|^2 \mathrm{d}s \Big)^p + \Big(\int_0^T |U^n_{u,s}|^2 + |V^n_{u,s}|^2 \mathrm{d}s \Big)^p \Big] \right\} < \infty.$$

Proof. For any $n \in \mathbb{N}, z \in \mathbb{R}$ our hypothesis gives $|h_n(z)| \leq |z|$. Hence the driver F^n of (26) satisfies the same growth conditions as the driver of BSDE (14). Therefore, one can apply the results of section 3 to either BSDE and obtain for $p \geq 1$

$$\sup_{0\leq u\leq T}\Big\{\mathbb{E}\Big[\Big(\int_0^T |D_uY_s|^2\mathrm{d}s\Big)^p + \Big(\int_0^T |D_uZ_s|^2\mathrm{d}s\Big)^p + \Big(\int_0^T |U^n_{u,s}|^2\mathrm{d}s\Big)^p + \Big(\int_0^T |V^n_{u,s}|^2\mathrm{d}s\Big)^p\Big]\Big\}$$
$$\leq C \sup_{0\leq u\leq T}\mathbb{E}\Big[\Big(|D_u\xi|^2 + \int_0^T |\nabla_x f(s,\Theta_s)D_uX_s|^2\mathrm{d}s\Big)^{p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}},$$

with $\bar{q}$ conjugate to $\bar{r}$. The results of subsection 2.5 combined with assumptions **HX2** and **HY2** yield the finiteness of the right hand side of the inequality. □

Lemma 4.5. *Assume* **HX2** *and* **HY2** *hold. For all $p \geq 1$ we have*

$$\sup_{n\in\mathbb{N}}\sup_{0\leq u,v\leq T}\mathbb{E}\Big[\,|D_vD_u\xi|^{2p} + \Big(\int_0^T |A^n_{v,u,s}|\mathrm{d}s\Big)^{2p}\Big] < \infty,$$

with $A^n, n \in \mathbb{N}$, given by (29).

Proof. To prove this result we analyze each term in more detail.

Part 1): The first term presents little difficulty, since $\xi = g(X_T)$ and X is a diffusion process. For $0 \leq u \leq v \leq T$ we have $D_vD_u\xi = D_vX_T[Hg](X_T)D_uX_T + \nabla g(X_T)D_vD_uX_T$, where $[Hg]$ is the Hessian matrix of g.

Since $g \in C^2_b$, we may use the inequality $ab \leq (a^2 + b^2)$ valid for $a, b \in \mathbb{R}$ combined with Theorem 2.5 to obtain

$$\sup_{0\leq u,v\leq T}\mathbb{E}\Big[\,|D_vD_u\xi|^{2p}\Big] \leq C\sup_{0\leq u,v\leq T}\mathbb{E}\Big[\,|D_vX_T|^{4p} + |D_uX_T|^{4p} + |D_vD_uX_T|^{2p}\Big] < \infty.$$

Part 2): We now analyze the second term, starting with the identification

$$\begin{aligned} A^n_{v,u,s} &= (D_vF^n)(s,\Xi^n_{u,s}) + (\nabla_xF^n)(s,\Xi^n_{u,s})D_vD_uX_s \\ &= (D_vF^n)(s,\Xi^n_{u,s}) + (\nabla_xf)(s,\Theta_s)D_vD_uX_s \end{aligned}$$

for $v, u, s \in [0, T]$. Now $(D_vF^n)(t, \Xi^n_{u,s})$ is composed of products of first order Malliavin derivatives of X, Y or Z and second order partial derivatives of f. Assumption **HY2** guarantees that the second order derivatives of f are dominated by a process K belonging to $\mathcal{S}^{2p}([0, T])$. Combining this with the hypothesis $|h'_n| \leq 1$ for all n we easily obtain

$$|(D_vF^n)(s,\Xi^n_{u,s})| \leq CK_s\Big\{|D_vX_s|^2 + |D_uX_s|^2 + |D_vY_s|^2 + |U^n_{u,s}|^2 + |D_vZ_s|^2 + |V^n_{u,s}|^2\Big\}.$$

Summands involving the Malliavin derivatives of X can be dealt with arguments as in *part 1)* of this proof. Furthermore,

$$\sup_{n\in\mathbb{N}}\sup_{0\leq u\leq T}\mathbb{E}\Big[\Big(\sup_{0\leq t\leq T}|K_t|\int_0^T\Big[\,|U^n_{u,s}|^2 + |V^n_{u,s}|^2 + |D_uY_s|^2 + |D_uZ_s|^2\Big]\mathrm{d}s\Big)^{2p}\Big]$$
$$\leq \|K\|^{2p}_{\mathcal{S}^{4p}}\sup_{n\in\mathbb{N}}\sup_{0\leq u\leq T}\|\,|U^n_u| + |V^n_u| + |D_uY| + |D_uZ|\,\|^{4p}_{\mathcal{H}^{8p}} < \infty.$$

The last inequality is satisfied by Lemma 4.4 and the fact that $K \in \mathcal{S}^{2p}$ for all $p \geq 1$.

We are left with the analysis of the term $(\nabla_x f)(s,\Theta_s)D_vD_uX_s$. From condition **HY1**, $\nabla_x f(s,\Theta_s)$ is dominated by $M(1+|Y_s|+|Z_s|^2)$ with a bounded process Y, and so we obtain

$$\begin{aligned}
&\sup_{0\le u,v\le T}\mathbb{E}\Big[\Big(\int_0^T|(\nabla_x f)(t,\Theta_s)D_vD_uX_s|\mathrm{d}s\Big)^{2p}\Big]\\
&\quad\le C\sup_{0\le u,v\le T}\mathbb{E}\Big[\sup_{t\in[0,T]}|D_vD_uX_t|^{2p}\Big(\int_0^T(1+|Z_s|^2)\mathrm{d}s\Big)^{2p}\Big]\\
&\quad\le C\sup_{0\le u,v\le T}\big\|D_vD_uX\big\|^{2p}_{\mathcal{S}^{4p}}\big\|1+|Z|\big\|^{4p}_{\mathcal{H}^{8p}}<\infty
\end{aligned}$$

For the last two inequalities we used Hölder's inequality, that $Z\in\mathcal{H}^{2p}$ for all $p\ge 1$ and Theorem 2.5.

The Lemma's inequality follows from a combination of Part 1) and Part 2). □

We are now in a position to prove Lemma 4.2.

Proof of Lemma 4.2. We have to establish the hypotheses to hold for the application of Theorem A.1. The terminal condition is given by the Malliavin derivative of $\xi=g(X_T)$ with $g\in C_b^2$. In view of Theorems 2.4 and 2.5, conditions (A2) and (A4) are satisfied.

Given our construction, it is clear F^n is uniformly Lipschitz continuous in (y,z), since $\nabla_y f$ and $\nabla_z f(\cdot,h_n(\cdot))$ are bounded. The boundedness of $\nabla_x f(\cdot,h_n(\cdot))$ combined with the fact that $DX\in\mathcal{S}^{2p}([0,T]\times[0,T])$ enables us to conclude

$$\sup_{0\le u\le T}\mathbb{E}[\sup_{0\le t\le T}|\nabla_x f(t,\Theta_t)D_uX_t|^{2p}]<\infty,$$

and hence condition (A1) is also satisfied.

The verification of condition (A3) is also simple. F^n is continuous differentiable in (y,z). Furthermore since Y and Z are Malliavin differentiable and X is twice Malliavin differentiable, $F^n(t,D_uX_t,Y,Z)$ is also Malliavin differentiable for $0\le u\le t\le T$. The proof of the moment inequality of assumption (A3) is a consequence of Lemma 4.5. Hence we may apply Theorem A.1. □

4.3 Proof of Theorem 4.1

We are finally able to prove the main result of this section.

Proof of Theorem 4.1. To prove this result we apply Lemma 2.1. We have to show that the Lemma's assumptions are satisfied. Fix $0\le u\le t\le T$.

1) Lemma 4.2 ensures existence, uniqueness and Malliavin differentiability of each $(U^n_{u,t},V^n_{u,t})$.

2) We now prove the $\mathcal{H}^2$-convergence of $(U^n_{u,\cdot},V^n_{u,\cdot})$ to $(D_uY_\cdot,D_uZ_\cdot)$. Using Lemma 3.2 applied to the BSDE resulting from the difference $D_uY_\cdot-U^n_{u,\cdot}$ (see BSDE (14) and (27)), we have with $\Theta=(X,Y,Z)$ and $\Theta^n=(X,Y,h_n(Z))$

$$\begin{aligned}
&\sup_{0\le u\le T}\mathbb{E}\Big[\int_0^T|D_uY_s-U^n_{u,s}|^2\mathrm{d}s+\int_0^T|D_uZ_s-V^n_{u,s}|^2\mathrm{d}s\Big]\\
&\quad\le C\sup_{0\le u\le T}\mathbb{E}\Big[\Big(\int_0^T|\nabla_z f(s,\Theta_s)-\nabla_z f(s,\Theta^n_s)||V^n_{u,s}|\mathrm{d}s\Big)^{2\bar q^2}\Big]^{\frac{1}{\bar q^2}}\\
&\quad\le C\sup_{0\le u\le T}\mathbb{E}\Big[\Big(\int_0^T|V^n_{u,s}|^2\mathrm{d}s\Big)^{2\bar q^2}\Big]^{\frac{1}{2\bar q^2}}\mathbb{E}\Big[\Big(\int_0^T|\nabla_z f(s,\Theta_s)-\nabla_z f(s,\Theta^n_s)|^2\mathrm{d}s\Big)^{2\bar q^2}\Big]^{\frac{1}{2\bar q^2}},
\end{aligned}$$

where $\bar{q}$ is related to the BMO martingale $(\nabla_z f(\cdot, X, Y, Z)) * W$ as stated in subsection 2.3.

The first term in the last line is finite, uniformly in n, by Lemma 4.4. For the second term, note that by **HY1** $\nabla_z f$ is continuous and, from (25) so is the family h_n. Furthermore, both $\nabla_z f(\cdot, \cdot, z)$ and $\nabla_z f(\cdot, \cdot, h_n(z))$ are dominated by $C(1+|z|)$. Given the integrability properties of Z and the convergence of h_n to the identity function, dominated convergence yields the desired convergence result, from which the convergence of $(U^n_{u,t}, V^n_{u,t})$ to $(D_u Y_t, D_u Z_t)$ for a.e. $t \in [0,T]$ follows.

3) We prove the uniform boundedness of $\mathbb{E}\big[\|(DU^n, DV^n)\|^2_{L^2(\mathbb{D}^{1,2})}\big]$ in n.

The driver of BSDE (28) is linear. So applying Lemma 3.1, we obtain the following inequality for $n \in \mathbb{N}$

$$\|D_v U^n_u\|^{2p}_{\mathcal{S}^{2p}} + \|D_v V^n_u\|^{2p}_{\mathcal{H}^{2p}} \leq C\mathbb{E}\Big[|D_v D_u \xi|^{2p\bar{q}^2} + \Big(\int_0^T |A^n_{v,u,s}|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}},$$

where the constant $\bar{q}$ is related to the BMO martingale $\nabla_z f(\cdot, \Theta) * W$ according to Remark 2.7. Lemma 4.5 now yields

$$\sup_{n\in\mathbb{N}} \sup_{0\leq v,u\leq T} \Big\{\|D_v U^n_u\|^{2p}_{\mathcal{S}^{2p}} + \|D_v V^n_u\|^{2p}_{\mathcal{H}^{2p}}\Big\} < \infty.$$

By 1) - 3) we can now apply Lemma 2.1 and deduce the Malliavin differentiability of (DY, DZ). Arguments as the ones used in Theorem 8.4 of [AIdR07] show that $(D_v D_u Y, D_v D_u Z)$ is a solution to BSDE (24).

Uniqueness follows immediately from Lemma 3.2.

To prove the representation $D_t D_u Y_t = D_u Z_t$, one only needs to recall that for each $n, u \leq t$ we have $D_t D_u Y^n_t = D_u Z^n_t$. Since both sides converge to their respective limiting processes the equality holds true in the limit. □

5 Regularity in the time variable

With a view towards their numerical approximation, in this section we investigate regularity properties of the Malliavin derivatives of solutions of our FBSDE (2)-(3).

5.1 Continuity and bounds

Lemma 5.1. *Under* **HX1** *and* **HY1** *let* (X, Y, Z) *be the solution processes of system (2)-(3), and* (DX, DY, DZ) *their Malliavin derivatives. Then for* $u, v \in [0,T]$ *and* $p \geq 1$ *there exists* $C_p > 0$ *such that*

$$\|D_v Y - D_u Y\|^{2p}_{\mathcal{S}^{2p}} + \|D_v Z - D_u Z\|^{2p}_{\mathcal{H}^{2p}} \leq C_p |v-u|^p. \tag{30}$$

Proof. We use (14) to write for $u, v, t \in [0,T]$ a FBSDE for the difference $D_v Y_t - D_u Y_t$. For this we employ the comparison lemma 3.2, to obtain with $\xi = g(X_T)$ and for any $p \geq 1$

$$\begin{aligned}
&\|D_v Y - D_u Y\|^{2p}_{\mathcal{S}^{2p}} + \|D_v Z - D_u Z\|^{2p}_{\mathcal{H}^{2p}} \\
&\leq C\mathbb{E}\Big[\,|D_v\xi - D_u\xi|^{2p\bar{q}^2} + \Big(\int_0^T |(\nabla_x f)(s,\Theta_s)||D_v X_s - D_u X_s|\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}} \\
&\leq C\mathbb{E}\Big[\,|D_v X_T - D_u X_T|^{2p\bar{q}^2} \\
&\qquad + \sup_{0\leq t\leq T} |D_v X_t - D_u X_t|^{2p\bar{q}^2}\Big(\int_0^T (1 + |Y_s| + |Z_s|^2)\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]^{\frac{1}{\bar{q}^2}} \\
&\leq C\Big\{\|D_v X_T - D_u X_T\|^{2p}_{L^{2p\bar{q}^2}} + \|D_v X - D_u X\|^{2p}_{\mathcal{S}^{4p\bar{q}^2}}\Big\} \leq C_p|v-u|^p,
\end{aligned}$$

where $\bar{q}$ corresponds to the BMO martingale $\nabla_z f(\Theta) * W$. The last line follows from a direct application of Theorem 2.5. □

Equipped with these moment estimates we are now able to state our first main result.

Theorem 5.2 (Time continuity). *Assume* **HX1** *and* **HY1**. *Then there exists a continuous version of* $(u,t) \mapsto D_u Y_t$ *in* $\{(u,t) : 0 \le u \le t \le T.\}$ *In particular there exists a continuous version of* Z *on* $[0,T]$.

Assume **HX2** *and* **HY2**. *Then there exists a continuous version of* $(v,u,t) \mapsto D_v D_u Y_t$ *for* $0 \le v \le u \le t \le T$. *In particular there is a continuous version of* $(u,t) \mapsto D_u Z_t$ *for* $0 \le u \le t \le T$.

Proof. To make the proof simpler we assume $m = d = 1$. Under **HX1**, the results of subsection 2.5 imply the existence of continuous versions of X, ∇X, $(\nabla X)^{-1}$ and $(u,t) \mapsto D_u X_t$ for $0 \le u \le t \le T$.

A quick analysis of (13), combined with the knowledge that $(X,Y,Z) \in \mathcal{S}^{2p} \times \mathcal{S}^{\infty} \times \mathcal{H}^{2p}$ and $(\nabla X, \nabla Y, \nabla Z) \in \mathcal{S}^{2p} \times \mathcal{S}^{2p} \times \mathcal{H}^{2p}$ for all $p \ge 1$, allows one to conclude that a continuous version of ∇Y exists: the process is given by the sum of a Lebesgue and Itô integral with well behaved integrands.

In Theorem 2.9 we established $D_u Y_t = \nabla Y_t (\nabla X_u)^{-1} \sigma(X_u)$, $0 \le u \le t \le T$. Condition **HX0** ensures the continuity of σ. Hence by what has been shown there is a continuous version of $(u,t) \mapsto \big(\nabla Y_t (\nabla X_u)^{-1} \sigma(X_u)\big)$ for $0 \le u \le t \le T$. This means that $(u,t) \mapsto D_u Y_t$ has a continuous version for $0 \le u \le t \le T$.

By Theorem 2.9, Z is a version of $t \mapsto D_t Y_t$. Hence the continuity of a version of $(u,t) \mapsto D_u Y_t$ for $0 \le u \le t \le T$ immediately implies that Z possesses a continuous version. This finishes the proof of the first statement.

For the second statement we argue in a different way.

The second Malliavin derivative of Y depends on three variables, $v,u,t \in [0,T]$. By using moment inequalities, we will show that $(v,u) \mapsto (D_v D_u Y_{t \vee u})_{0 \le t \le T}$ is continuous as a mapping to the space of continuous functions on $[0,T]$ equipped with the sup norm. By well known extensions of the Kolmogorov continuity criterion to normed vector spaces (Here we have to add a reference, e.g. Yor or...) this will establish the desired continuity of $(v,u,t) \mapsto D_v D_u Y_t$ for $0 \le v \le u \le t \le T$. To verify the inequalities, for $0 \le v \le u \le T, 0 \le v' \le u' \le T$ we will have to estimate moments of

$$\sup_{0 \le t \le T} |D_v D_u Y_t - D_{v'} D_{u'} Y_t|^p.$$

In a first step, we separate the two parameters by estimating this quantity by a constant multiple of

$$\sup_{0 \le t \le T} |D_v D_u Y_t - D_{v'} D_u Y_t|^p + \sup_{0 \le t \le T} |D_{v'} D_u Y_t - D_{v'} D_{u'} Y_t|^p.$$

In what follows, for convenience we shall only give the estimation of the first summand, remarking that the second one may be treated in a very similar way. (Give it a quick thought. Is this correct? Is the estimation of the second term really simpler?) Fix $0 \le v, v' \le u \le t \le T$. Again using the

comparison Lemma 3.2 with (24) specified to $D_vD_uY_t - D_{v'}D_uY_t$, we get for $p \geq 1$

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T}|D_vD_uY_t - D_{v'}D_uY_t|^{2p}\Big] \\
&\quad\leq C\Big\{\mathbb{E}\Big[\,|D_vD_u\xi - D_{v'}D_u\xi|^{2p\bar{q}^2} + \Big(\int_0^T\Big[\,|D_v\Theta_s - D_{v'}\Theta_s||[Hf](s,\Theta_s)||D_u\Theta_s| \\
&\qquad\qquad + |(\nabla_x f)(s,\Theta_s)||D_vD_uX_s - D_{v'}D_uX_s|\,\Big]\mathrm{d}s\Big)^{2p\bar{q}^2}\Big]\Big\} \\
&\quad\leq C\Big\{\|D_vX_T - D_{v'}X_T\|^{2p}_{L^{6p\bar{q}^2}} + \|D_vD_uX - D_{v'}D_uX\|^{2p}_{\mathcal{S}^{4p\bar{q}^2}} \\
&\qquad + \|D_vX - D_{v'}X\|^{2p}_{\mathcal{S}^{4p\bar{q}^2}} + \|D_vY - D_{v'}Y\|^{2p}_{\mathcal{S}^{4p\bar{q}^2}} + \|D_vZ - D_{v'}Z\|^{4p}_{\mathcal{H}^{8p\bar{q}^2}}\Big\} \\
&\quad\leq C_p|v - v'|^p.
\end{aligned}$$

The successive are justified in view of the growth conditions contained in the assumptions, Hölder's inequality, Theorem 2.5 and Lemma 5.1. Kolmogorov's continuity criterion for vector valued stochastic processes yields the existence of a continuous version of $(v,u,t) \mapsto D_vD_uY_t$ for $0 \leq v \leq u \leq t \leq T$, and hence by restriction also of $(u,t) \mapsto D_uZ_t$ for $0 \leq u \leq t \leq T$. □

Theorem 5.3 (Bounds). *Assume that* **HX1** *and* **HY1** *hold. Then for all* $p \geq 1$

$$\mathbb{E}\Big[\sup_{0\leq u\leq t\leq T}|D_uY_t|^{2p}\Big] < \infty. \tag{31}$$

In particular

$$\|Z\|_{\mathcal{S}^{2p}} \ < \infty. \tag{32}$$

Let **HX2** *and* **HY2** *be satisfied. Then for all* $p \geq 1$

$$\sup_{0\leq u\leq T}\mathbb{E}[\sup_{0\leq t\leq T}|D_uZ_t|^{2p}] < \infty. \tag{33}$$

Proof. As we have seen in Theorem 5.2, a continuous version of $(u,t) \mapsto D_uY_t$ is given by $\nabla_xY_t(\nabla_xX_u)^{-1}\sigma(u,X_u)$. Hence we may estimate

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\leq u\leq t\leq T}|D_uY_t|^{2p}\Big] &\leq \mathbb{E}\Big[\sup_{0\leq t\leq T}|\nabla_xY_t|^{2p}\sup_{0\leq u\leq T}\big\{|(\nabla_xX_u)^{-1}\sigma(u,X_u)|^{2p}\big\}\Big] \\
&\leq \mathbb{E}\Big[\sup_{0\leq t\leq T}|\nabla_xY_t|^{6p}\Big]^{\frac{1}{3}}\mathbb{E}\Big[\sup_{0\leq u\leq T}|(\nabla_xX_u)^{-1}|^{6p}\Big]^{\frac{1}{3}}\mathbb{E}\Big[\sup_{0\leq u\leq T}|\sigma(X_u)|^{6p}\Big]^{\frac{1}{3}} \\
&< \infty
\end{aligned}$$

The last line follows from the fact that ∇Y, $(\nabla X)^{-1}$ and X all belong to $\mathcal{S}^{2p}$ for all $p \geq 1$ (see (4), (8) and Theorem 2.8). This concludes the first part of the proof. The second claim follows as a special case of the first by identifying u and t.

For the third statement, note that the proof of Theorem 4.1 (see also the proof of A.1) yields

$$\sup_{0\leq v,u\leq T}\mathbb{E}\Big[\sup_{0\leq t\leq T}|D_vD_uY_t|^{2p}\Big] < \infty, \quad p \geq 1.$$

By the continuity result of Theorem 5.2 we may choose $u = t$ to obtain

$$\sup_{0\leq u\leq T}\mathbb{E}\Big[\sup_{0\leq t\leq T}|D_uZ_t|^{2p}\Big] < \infty, \quad p \geq 1.$$

□

5.2 A path regularity theorem

In the previous subsection we deduced the continuity property of Z and estimated moments of its sup over the interval $[0,T]$. Here we aim at providing a Kolmogorov continuity type estimate for Z. The inequality we will obtain will imply an improvement of the well known path regularity result of [Zha01].

Let Π be the collection of all partitions of the interval $[0,T]$ by finite families of real numbers. Particular partitions will be denoted by $\pi = \{t_i : 0 = t_0 < \ldots < t_N = T\}$ with $N \in \mathbb{N}$. We define the mesh size of partition π as $\Delta^\pi = \Delta = \max_{0\le i\le N} |t_{i+1} - t_i|$.

For reference purposes and before approaching the path regularity theorem we recall an elementary inequality: for real numbers $a_i, 1 \le i \le n$, and $p \ge 1$ we have

$$\sum |a_i|^p \le \Big(\sum |a_i|\Big)^p. \tag{34}$$

We start by stating an auxiliary lemma:

Lemma 5.4. *Assume* **HX0** *and* **HY0**. *Then for the solutions of BSDE (2)-(3) and for any $p \ge 2$ there exists a pair of constants A_p, C_p depending on T, M and p such that*

$$\mathbb{E}[\sup_{s\le u\le t} |Y_u - Y_s|^p] \le C_p\Big\{A_p|t-s|^p + \mathbb{E}\Big[\Big(\int_s^t |Z_v|^2 \mathrm{d}v\Big)^p + \Big(\int_s^t |Z_v|^2\mathrm{d}v\Big)^{p/2}\Big]\Big\}. \tag{35}$$

Proof. First estimate increments of Y by the sum of a Lebesgue and Ito integral provided by (3), maximize in $s \le u \le t$, and apply Doob's and Burkholder-Davis-Gundy's inequalities to the martingale part to obtain for $p \ge 2$

$$\mathbb{E}[\sup_{s\le u\le t} |Y_u - Y_s|^p] \le C_p\mathbb{E}\Big[\Big(\int_s^t |f(v, X_v, Y_v, Z_v)|\mathrm{d}v\Big)^p + \Big(\int_s^t Z_v^2\mathrm{d}v\Big)^{\frac{p}{2}}\Big].$$

Next use the growth condition valid for f, i.e. $|f(\cdot,\cdot,y,z)| \le M(1+|y|+|z|^2)$ together with the fact that Y is bounded, to obtain the claimed result. □

Let us now state our path regularity Theorem.

Theorem 5.5 (Path regularity). *Under* **HX1** *and* **HY1**, *the FBSDE system (2)-(3) has a unique solution $(X,Y,Z) \in \mathcal{S}^{2p} \times \mathcal{S}^\infty \times \mathcal{H}^{2p}$ for all $p \ge 1$. Moreover, the following holds true:*

i) For $p \ge 2$ there exists a constant $C_p > 0$ such that for $0 \le s \le t \le T$ we have

$$\mathbb{E}[\sup_{s\le u\le t} |Y_u - Y_s|^p] \le C_p|t-s|^{\frac{p}{2}}.$$

ii) For all $p \ge 1$ there exists a constant $C_p > 0$ such that for any partition π of $[0,T]$ with mesh size Δ

$$\sum_{i=0}^{N-1} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 dt\Big)^p\Big] \le C_p\Delta^p.$$

Under **HX2** *and* **HY2**, *we further have:*

iii) For all $p \ge 2$ there exists a constant $C_p > 0$ such that for $0 \le s \le t \le T$

$$\mathbb{E}[\sup_{s\le u\le t} |Z_u - Z_s|^p] \le C_p|t-s|^{\frac{p}{2}}.$$

In particular, the process Z has a continuous modification. (Why prove this once again?)

Proof. Part i): Under the hypotheses valid we can make use of Theorem 5.3. In fact, combining (32) with (35) we get

$$\begin{aligned}\mathbb{E}[\sup_{s\le u\le t}|Y_u-Y_s|^p] &\le C\Big\{|t-s|^p+\mathbb{E}\Big[|t-s|^p\sup_{s\le u\le t}|Z_u|^{2p}+|t-s|^{\frac{p}{2}}\sup_{s\le u\le t}|Z_u|^p\Big]\Big\}\\ &\le C_p\Big\{|t-s|^p+|t-s|^{\frac{p}{2}}\Big\}.\end{aligned}$$

The result follows.

Part ii): Theorem 5.3 states that $Z\in\mathcal{S}^{2p}$. Therefore we are able to write, using Jensen's inequality

$$\mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}}|Z_t-Z_{t_i}|^2dt\Big)^p\Big]\le\Delta^{p-1}\int_{t_i}^{t_{i+1}}\mathbb{E}[\,|Z_t-Z_{t_i}|^{2p}\,]dt. \tag{36}$$

In view of Theorem 5.2 and the subsequent representation formula for Z in terms the Malliavin derivatives of Y (see (16)), we find an alternative way to express the difference $Z_t-Z_{t_i}$ for $t\in[t_i,t_{i+1}]$ by writing

$$Z_t-Z_{t_i}=\nabla Y_t(\nabla X_t)^{-1}\sigma(X_t)-\nabla Y_{t_i}(\nabla X_{t_i})^{-1}\sigma(X_{t_i})=I_1+I_2+I_3, \tag{37}$$

where $I_1=\Big(\nabla Y_t-\nabla Y_{t_i}\Big)(\nabla X_t)^{-1}\sigma(X_t)$, $I_2=\nabla Y_{t_i}\Big((\nabla X_t)^{-1}-(\nabla X_{t_i})^{-1}\Big)\sigma(X_t)$ and $I_3=\nabla Y_{t_i}(\nabla X_{t_i})^{-1}\Big(\sigma(X_t)-\sigma(X_{t_i})\Big)$.

Estimates for I_2 and I_3 are easy to produce since they rely mainly on $\|\nabla Y\|_{\mathcal{S}^{2p}}<\infty$ and the results presented in subsection 2.5. We give details for I_2 and hints how to deal with I_3, remarking that its treatment is very similar. Hölder's inequality combined with the growth condition of σ produce

$$\begin{aligned}\mathbb{E}[\,|I_2|^{2p}] &\le C\,\mathbb{E}\Big[\sup_{0\le u\le T}|\nabla_x Y_u|^{6p}\Big]^{\frac13}\mathbb{E}\Big[\sup_{t_i\le t\le t_{i+1}}|(\nabla X_t)^{-1}-(\nabla X_{t_i})^{-1}|^{6p}\Big]^{\frac13}\mathbb{E}\Big[\sup_{0\le u\le T}|X_u|^{6p}\Big]^{\frac13}\\ &\le C\,\Delta^{3p\frac13}=C\,\Delta^p.\end{aligned} \tag{38}$$

For the last line we use (4), (9) and $\|\nabla Y\|_{\mathcal{S}^{2p}}<\infty$. For I_3, the method is similar: instead of (4) and (9) we have to use (5) and (8).

We next estimate I_1. Using Fubini's Theorem and Hölder's inequality we get

$$\begin{aligned}\int_{t_i}^{t_{i+1}}\mathbb{E}\big[\,|I_1|^{2p}\,\big]\mathrm{d}t &=\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|I_1|^{2p}\mathrm{d}t\,\Big]\\ &\le\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|(\nabla X_t)^{-1}|^{4p}\mathrm{d}t\Big]^{\frac12}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}|\sigma(X_t)|^{4p}\mathrm{d}t\Big]^{\frac12}\mathbb{E}\Big[\sup_{t_i\le t\le t_{i+1}}|\nabla Y_t-\nabla Y_{t_i}|^{2p}\Big].\end{aligned}$$

We can simplify the integral terms by estimating the integrands by their suprema over the intervals. Using the linear growth condition on σ combined with (4), (8), we show in this way that the first two expectations on the right hand side are bounded by $C\Delta^{1/2}$ each. Applying an appropriate version of (36), and using the previous inequalities, we infer

$$\Delta^{p-1}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\big[\,|I_1|^p\,\big]\mathrm{d}t\le C\Delta^p\sum_{i=0}^{N-1}\mathbb{E}\Big[\sup_{t_i\le t\le t_{i+1}}|\nabla Y_t-\nabla Y_{t_i}|^{2p}\Big].$$

It remains to estimate $\nabla Y_t - \nabla Y_{t_i}$ for $t \in [t_i, t_{i+1}]$ using the BSDE (3). For $p \geq 1$, the inequalities of Doob and Burkholder-Davis-Gundy combine with **HX1** and **HY1** in the same fashion as in part *i)* to yield for $\Theta = (X, Y, Z)$ and $\nabla\Theta = (\nabla X, \nabla Y, \nabla Z)$

$$\begin{aligned}
&\sum_{i=0}^{N-1} \mathbb{E}\Big[\sup_{t_i \leq t \leq t_{i+1}} |\nabla Y_t - \nabla Y_{t_i}|^{2p} \Big] \\
&\quad \leq C \sum_{i=0}^{N-1} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |\langle (\nabla f)(s, \Theta_s), \nabla\Theta_s \rangle| \mathrm{d}s \Big)^{2p} + \Big(\int_{t_i}^{t_{i+1}} |\nabla Z_s|^2 \mathrm{d}s \Big)^{p} \Big] \\
&\quad \leq C\, \mathbb{E}\Big[\Big(\int_{0}^{T} |\langle (\nabla f)(s, \Theta_s), \nabla\Theta_s \rangle| \mathrm{d}s \Big)^{2p} + \Big(\int_{0}^{T} |\nabla Z_s|^2 \mathrm{d}s \Big)^{p} \Big].
\end{aligned}$$

For the last line we interchange summation and expectation, and apply (34) (Here we seem to lose precision. In particular for small increments the inequality is very rough.). We now use the growth condition of **HY1** combined with the fact that $X, Y, Z, \nabla X, \nabla Y \in \mathcal{S}^{2p}$ and $\nabla Z \in \mathcal{H}^{2p}$. Therefore

$$\sum_{i=0}^{N-1} \mathbb{E}\Big[\sup_{t_i \leq t \leq t_{i+1}} |\nabla Y_t - \nabla Y_{t_i}|^{2p} \Big] < \infty,$$

which obviously implies

$$\Delta^{p-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\big[\, |I_1|^p \,\big] \mathrm{d}t \leq C\Delta^p.$$

Finally we inject (38) and the above inequality into (36) (according to (37)), to obtain the second assertion of the Theorem:

$$\begin{aligned}
\sum_{i=0}^{N-1} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} |Z_t - Z_{t_i}|^2 dt \Big)^{p} \Big] &\leq C\Delta^{p-1} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\big[\, |I_1|^{2p} + |I_2|^{2p} + |I_3|^{2p} \big] dt \\
&\leq C\Delta^{p-1}\big(\Delta + 2\Delta^p\big) \leq C\Delta^p.
\end{aligned}$$

Part iii): Theorem 5.2 states that $t \mapsto D_t Y_t$ is a continuous version of Z. Hence we may express for $s, t \in [0, T]$ the difference $Z_t - Z_s$ by Malliavin derivatives of Y, and its moments for $p \geq 2$ by

$$E[\, |Z_t - Z_s|^p] \leq C(E[\, |D_t Y_t - D_s Y_t|^p] + E[\, |D_s Y_t - D_s Y_s|^p]), \quad \text{with } s \leq t.$$

We estimate both expressions on the right hand side separately, with similar arguments as in *Part ii)*.
From Lemma 5.1 we have $\mathbb{E}[\, |D_t Y_t - D_s Y_t|^p\,] \leq C|t-s|^{\frac{p}{2}}$.
For the other term, a simple calculation using BSDE (14) yields

$$D_s Y_t - D_s Y_s = \int_s^t \big[\langle \nabla f(\Theta_u), D_s\Theta_u \rangle\big] \mathrm{d}u - \int_s^t D_s Z_u \mathrm{d}W_u.$$

By Doob's and Burkholder-Davis-Gundy's inequalities we have for $p \geq 2$

$$\begin{aligned}
\mathbb{E}[\sup_{s \leq u \leq t} |D_s Y_u - D_s Y_s|^p] &\leq C\, \mathbb{E}\Big[\Big(\int_s^t \big[(1 + |Y| + |Z|^2)|D_s X_u| + |D_s Y_u| \\
&\qquad + (1 + |Z_u|)\, |D_s Z_u| \,\big] \mathrm{d}u \Big)^p + \Big(\int_s^t |D_s Z_u|^2 \mathrm{d}u \Big)^{\frac{p}{2}} \Big] \\
&\leq C\, \big\{ |t-s|^p + |t-s|^{\frac{p}{2}} \big\}.
\end{aligned}$$

This last line follows, because all the integrand processes belong to $\mathcal{S}^p$ for all $p \geq 2$ (see Theorem 5.3). Combining the two above estimates we have $E[\,|Z_t - Z_s|^p] \leq C\,|t-s|^{\frac{p}{2}}$ as intended. Kolmogorov's continuity criterion yields the continuity statement. □

5.3 Zhang's path regularity Theorem

Now let π be an equidistant partition of $[0,T]$ with N points and constant mesh size $\Delta^\pi = h$. We define a set of random variables

$$\bar{Z}^\pi_{t_i} = \frac{1}{h}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} Z_s \mathrm{d}s \big| \mathcal{F}_{t_i}\Big], \quad \text{for all partition points} \quad t_i, 0 \leq i \leq N-1, \tag{39}$$

where Z is the control process in the solution of FBSDE (2)-(3) under **HX0** and **HY0**. It is not difficult to show that $\bar{Z}^\pi_{t_i}$ is the best $\mathcal{F}_{t_i}$-adapted $\mathcal{H}^2([t_i, t_{i+1}])$ approximation of Z, i.e.

$$\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}^\pi_{t_i}|^2 \mathrm{d}s\Big] = \inf_{Z_i \in L^2(\Omega, \mathcal{F}_{t_i})} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - Z_i|^2 \mathrm{d}s\Big].$$

Let now $\bar{Z}^\pi_t = \bar{Z}^\pi_{t_i}$ for $t \in [t_i, t_{i+1}[, 0 \leq i \leq N-1$. It is equally easy to see that $\bar{Z}^\pi_t \to Z_t$ in $\mathcal{H}^2$ as $h \to 0$: since Z is adapted there exists an adapted family of processes Z^π indexed by our equidistant partitions such that $Z^\pi_t = Z_{t_i}$ for $t \in [t_i, t_{i+1})$ and that $Z^\pi \to Z$ in $\mathcal{H}^2$ as $h \to 0$. Since $\{\bar{Z}^\pi\}$ is the best $\mathcal{H}^2$– approximation of Z, we obtain

$$\|Z - \bar{Z}^\pi\|_{\mathcal{H}^2} \leq \|Z - Z^\pi\|_{\mathcal{H}^2} \to 0$$

as $h \to 0$. As an immediate corollary of *ii)* in the previous Theorem we get the extension to the setting of drivers with quadratic growth of the famous Theorem 3.4.3 in [Zha01]. Let $p = 1$ in Theorem 5.5. Then

Theorem 5.6. *Under* **HX1** *and* **HY1** *and for each* $0 \leq i \leq N-1$ *we have*

$$\max_{0 \leq i \leq N-1} \Big\{ \sup_{t \in [t_i, t_{i+1})} \mathbb{E}\Big[\,|Y_t - Y_{t_i}|^2\Big] \Big\} + \sum_{i=0}^{N-1} \mathbb{E}\Big[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}^\pi_{t_i}|^2 \mathrm{d}s\Big] \leq C|\pi|,$$

where C *is a positive constant independent of* π.

6 Numerics for qgBSDE - a truncation procedure

A common method to deal with non-linearities or unbounded functions consists in truncating them. In our BSDE 3, the driver has a quadratic nonlinearity in z. Our truncation of the nonlinear driver will relate BSDE with drivers of quadratic growth BSDE with globally Lipschitz drivers. For this type of BSDE numerical schemes are readily available (see [BT04], [Eli06] and references therein), and the error committed in the numerical approximation for BSDE with Lipschitz driver is well known. So to fully analyze the error related to successive approximations in the case of BSDE with drivers of quadratic growth, it only remains to provide an estimate for the error arising from truncation. This is what we propose to do in this section.

For our qgFBSDE system (2)-(3) we assume that **HX1** and **HY1** hold. In this section the diffusion process X appearing in the BSDE's terminal condition and driver plays a secondary role, especially in the calculations we will be presenting.

To truncate the driver of quadratic growth, we use the already familiar sequence $\{h_n\}_{n \in \mathbb{N}}$ defined in (25). We will have to justify that this sequence indeed does the job: we will need

(32) to make our calculations work; we will have to justify that the truncated FBSDE obtained in this way satisfy **HX1** and **HY1**.

Recalling the driver of BSDE (3), we define a family of functions $f_n : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, (\omega, t, x, y, z) \mapsto f(\omega, t, x, y, h_n(z))$ and with it, the family of truncated FBSDE

$$Y_t^n = \xi + \int_t^T f(s, X_s, Y_s^n, h_n(Z_s^n))\mathrm{d}s - \int_t^T Z_s^n \mathrm{d}W_s, \tag{40}$$

with $\xi = g(X_T)$. The solution process of (3) is denoted by (Y, Z) and the solution process of its truncated counterpart (40) by (Y^n, Z^n). Furthermore, we recall that in (25) $(h_n)_{n\in\mathbb{N}}$ was defined as a sequence of C^1 functions, and that by Theorem 2.6 we have

$$\sup_{n\in\mathbb{N}} \|Z^n * W\|_{BMO} \leq \|Z * W\|_{BMO} \leq C < \infty.$$

This means that the martingales $Z^n * W, n \in \mathbb{N}$, satisfy the inverse Hölder inequality with the uniform exponent $\bar{r}$ (see subsection 2.3, and also Remark 4.3).

Remark 6.1. If (3) satisfies **HX1** and **HY1**, by inspection of the hypotheses it is easy to see that family (40) satisfies **HX1** and **HY1** as well. This means that the results on differentiability in subsection 2.6 and on continuity and bounds in section 5.1 are available for the truncated BSDE (40) as well.

Our result on the truncation error contains a parameter which arises from the inverse Hölder inequality.

Theorem 6.2. *Assume that* **HX1** *and* **HY1** *are satisfied, and let* (Y, Z) *and* (Y^n, Z^n) *be solutions of (3) and (40) respectively. Then for all* $p \geq 1$ *there exists a positive constant* C_p *such that for all* $n \in \mathbb{N}$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p}\Big] + \mathbb{E}\Big[\Big(\int_0^T |Z_s^n - Z_s|^2 \mathrm{d}s\Big)^p\Big] \leq C_p n^{-\frac{1}{2\bar{q}^2}}.$$

The constant $\bar{q}$ *is the Hölder conjugate of* $\bar{r} \in (1, \infty)$ *which is related to the BMO martingale* $Z * W$ *as explained in Remark 2.7.*

Proof. As usual we have to rely on an a priori estimate for the difference of original and truncated BSDE. To this end we use the notation, methods and arguments of the proofs of Lemmas 3.1 and 3.2 (see Section 3), without repeating all the details. For

$$b_t^n = \frac{f_n(\cdot, t, X_t, Y_t^n, Z_t^n) - f(\cdot, t, X_t, Y_t^n, Z_t)}{|Z_t^n - Z_t|^2}(Z_t^n - Z_t)\mathbf{1}_{\{Z_t^n \neq Z_t\}}, \quad t \in [0,T], n \in \mathbb{N},$$

the associated equivalent measure obtained after measure change by subtracting the drift related to b^n will be denoted by $\mathbb{Q}^{b_n}$. The supersript will be omitted for convenience. Following the

proof of Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\sup_{0\le t\le T}|Y_t-Y_t^n|^{2p}\Big]+\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|Z_s-Z_s^n|^2\mathrm{d}s\Big)^p\Big]\\
&\quad\le C\,\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|(f(s,Y_s^n,Z_s^n)-f(s,Y_s^n,h_n(Z_s^n))|\mathrm{d}s\Big)^{2p}\Big]\\
&\quad\le C\,\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|M(1+|Z_s^n|+|h_n(Z_s^n)|)|Z_s^n-h_n(Z_s^n)|\mathrm{d}s\Big)^{2p}\Big]\\
&\quad\le C\,\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|M(1+|Z_s^n|+|h_n(Z_s^n)|)^2\mathrm{d}s\Big)^{2p}\Big]^{\frac12}\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|Z_s^n-h_n(Z_s^n)|^2\mathrm{d}s\Big)^{2p}\Big]^{\frac12}\\
&\quad\le C\,\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|Z_s^n-h_n(Z_s^n)|^2\mathrm{d}s\Big)^{2p}\Big]^{\frac12}.
\end{aligned}$$

Here we made use of the growth assumption on f stated in **HY1**, Hölder's inequality and (22). A closer look at the properties of h_n reveals that for $n\in\mathbb{N}, s\in[0,T]$ we have

$$|Z_s^n-h_n(Z_s^n)|^2\le 4|Z_s^n|^2\mathbf{1}_{\{|Z_s^n|>n\}}.$$

The validity of **HX1** and **HY1** for the family of drivers used entitles us to employ the crucial (32) of Theorem 5.3 in the following estimates. Besides, we use Jensen's inequality, Fubini's Theorem and Chebyshev's inequality, to continue the estimation of the expression in the last line of our a priori estimate by

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|Z_s^n-h_n(Z_s^n)|^2\mathrm{d}s\Big)^{2p}\Big]\\
&\quad\le C\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|Z_s^n|^4\mathrm{d}s\Big)^{2p}\Big]^{\frac12}\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T\mathbf{1}_{\{|Z_s^n|>n\}}\mathrm{d}s\Big)^{4p}\Big]^{\frac12}\le C\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T\mathbf{1}_{\{|Z_s^n|>n\}}\mathrm{d}s\Big)^{4p}\Big]^{\frac12}\\
&\quad\le CT^{\frac{2k-1}{2}}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T\mathbf{1}_{\{|Z_s^n|>n\}}\mathrm{d}s\Big]^{\frac12}\le C_{p,T}\Big(\int_0^T\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{|Z_s^n|>n\}}]\mathrm{d}s\Big)^{\frac12}\\
&\quad= C_{p,T}\Big(\int_0^T\mathbb{Q}\big[\{|Z_s^n|>n\}\big]\mathrm{d}s\Big)^{\frac12}\le C_{p,T}\Big(\int_0^T\frac{1}{n^2}\mathbb{E}^{\mathbb{Q}}[\,|Z_s^n|^2]\mathrm{d}s\Big)^{\frac12}\\
&\quad\le\frac{C_{p,T}}{n}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|Z_s^n|^2\mathrm{d}s\Big]^{\frac12}\le C_{p,T}\frac{1}{n}.
\end{aligned}$$

We emphasize that the constants $C_{p,T}$ which may vary from line to line may depend on p and T, but not on n. Returning to the inequality we started with we have proved so far

$$\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|Y_t-Y_t^n|^{2p}+\Big(\int_0^T|Z_s-Z_s^n|^2\mathrm{d}s\Big)^p\Big]\le C_{p,T}n^{-\frac12}.$$

Finally using the same arguments as in the terminal part of Lemma 3.1, we are able to get rid of the dependence on $\mathbb{Q}$, and derive an inequality with respect to the original measure $\mathbb{P}$ by writing

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t-Y_t^n|^{2p}+\Big(\int_0^T|Z_s-Z_s^n|^2\mathrm{d}s\Big)^p\Big]\le C_pn^{-\frac{1}{2\bar q^2}},$$

with C_p a positive constant independent of n. (This argument is a little short. In particular, why do you write $\bar q$ instead of $\bar q^2$?) □

A Appendix

(I think in this appendix we have to give the condition $v \leq u$ some attention. If it is needed, we should let it appear somehow in the hypotheses already.) In this appendix we give the technical details left out in section 4 in the proof of second order Malliavin differentiability of the solution processes of a BSDE the driver of which satisfies Lipschitz conditions.

The techniques we will use are not new. They are based on a Picard iteration argument. It does not only give existence and uniqueness of solutions. It also allows to establish Malliavin differentiability in each step for the respective approximation of the solution. By means of a contraction argument in a suitable Sobolev norm, Malliavin smoothness is carried over to the solution in the limit. In contrast to previous applications, here the scheme deals with an equation that already has a Malliavin derivative as its solution.

We start with canonical coefficients that are given by an $\mathcal{F}_T$-measurable random variable ξ and a measurable function $f : \Omega \times [0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, such that

$$f(\cdot, s, u, y, z) = a_{u,s} + b_s\, y + \langle c_s, z\rangle.$$

The coefficient functions defining this driver will be supposed to satisfy the following assumptions.

(A1) $b : \Omega \times [0,T] \to \mathbb{R}$ and $c : \Omega \times [0,T] \to \mathbb{R}^d$ are measurable $(\mathcal{F}_t)$-adapted processes bounded by a constant $M > 0$.

$a : \Omega \times [0,T] \times [0,T] \to \mathbb{R}$ satisfies $\sup_{0\leq u\leq T}\|a_{u,\cdot}\|_{\mathcal{S}^{2p}} < \infty$ for all $p \geq 1$, and for each $u \in [0,T]$ $a_{u,\cdot}$ is $(\mathcal{F}_t)$-adapted.

(A2) ξ is a Malliavin differentiable, $\mathcal{F}_T$-measurable, bounded random variable with Malliavin derivative given by $D\xi$ satisfying

$$\sup_{0\leq u\leq T}\|D_u\xi\|_{L^p} < \infty, \text{ for all } p \geq 1.$$

(A3) $a_{u,t}, b_t, c_t$ are Malliavin differentiable for all $u \in [0,T]$ and $t \in [0,T]$. Measurable versions of their Malliavin derivatives are respectively given by $D_v a_{u,t}$, $D_v b_t$ and $D_v c_t$ for $v \in [0,T]$ such that for all $p \geq 1$

$$\sup_{0\leq v,u\leq T}\mathbb{E}\Big[\Big(\int_0^T \big[\,|D_v a_{u,s}|^2 + |D_v b_s|^2 + |D_v c_s|^2\big]\mathrm{d}s\Big)^p\Big] < \infty.$$

(A4) For all $u \in [0,T]$ $D_u\xi$ is Malliavin differentiable, with second derivative given by $D_vD_u\xi, v, u \in [0,T]$, and satisfying $\sup_{0\leq u,v\leq T}\|D_vD_u\xi\|_{L^p} < \infty$ for all $p \geq 1$.

Under these assumptions we consider the following backward stochastic differential equation

$$\begin{aligned} &U_{u,t} = 0, \qquad V_{u,t} = 0, \qquad t \in [0,u), \\ &U_{u,t} = D_u\xi - \int_t^T V_{u,s}\mathrm{d}W_s + \int_t^T f(s,u,U_{u,s},V_{u,s})\mathrm{d}s, \qquad t \in [u,T]. \end{aligned} \tag{41}$$

Theorem A.1. *Under (A1) and (A2), the BSDE (41) has a unique solution (U,V) in $\mathcal{H}^{2p}\times\mathcal{H}^{2p}$ for $p \geq 1$. Furthermore, if (A3) and (A4) hold, then (U,V) is Malliavin differentiable and a*

version of $\{(D_vU_{u,t}, D_vV_{u,t}); 0 \le v \le u \le t \le T\}$ *satisfies for any* $0 \le v \le u \le t \le T$

$$D_vU_{u,t} = D_vD_u\xi - \int_t^T D_vV_{u,s}\mathrm{d}W_s + \int_t^T \Big[(D_vf)(s,u,U_{u,s},U_{u,s}) + \big\langle \nabla f(s,u,U_{u,s},U_{u,s}), (D_vU_{u,s}, D_vU_{u,s})\big\rangle\Big]\mathrm{d}s. \quad (42)$$

A version of $\{V_{u,t}; 0 \le u \le t \le T\}$ *is given by* $\{D_tU_{u,t}; 0 \le u \le t \le T\}$.

Proof. For the sake of notational simplicity and clarity, we provide a proof for the case $d = 1$. This proof splits into two steps. In the first one, we are concerned with existence and uniqueness of solutions for (41) using a Picard iteration. In the second step we prove Malliavin differentiability. To this end, we show that the sequence arising in the Picard scheme is in fact Malliavin differentiable, and by contraction that its limit must be the Malliavin derivative of the solution process constructed in the first part.

Part i): To simplify notation we refer to $D_u\xi$ as ξ_u if there is no ambiguity.

We have a "standard" BSDE with Lipschitz continuous driver and a smooth terminal condition. The usual arguments for existence and uniqueness in this setting are well known. We recall the Picard iteration argument of the proof of Proposition 5.3 in [EKPQ97]. Let $(U^0_{u,t}, V^0_{u,t}) = (0,0)$ and for $k \ge 0$ define recursively the pair $(U^{k+1}_{u,t}, V^{k+1}_{u,t})$ as the solution of

$$U^{k+1}_{u,t} = D_u\xi - \int_t^T V^{k+1}_{u,s}\mathrm{d}W_s + \int_t^T f(s,u,U^k_{u,s},V^k_{u,s})\mathrm{d}s, \quad \text{for } 0 \le u \le t \le T.$$

Under (A1) and (A2) the iteration scheme is well defined and the following moment estimates hold for all $k \in \mathbb{N}$ and $p \ge 2$ (see Proposition 2.1 in [EKPQ97]):

$$\sup_{0\le u\le T}\Big\{\|U^k_u\|^p_{\mathcal{S}^p} + \|V^k_u\|^p_{\mathcal{H}^p}\Big\} \le C \sup_{0\le u\le T}\Big\{\|D_u\xi\|^p_{L^p} + \|a_{u,\cdot}\|^p_{\mathcal{H}^p}\Big\} < \infty. \quad (43)$$

Along classical lines of argument[4] of Corollary 2.1 in [EKPQ97] we obtain: $(U^k, V^k) \to (U, V)$ $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t \otimes \mathrm{d}u$ a.e. as well as

$$\lim_{k\to\infty} \sup_{0\le u\le T}\Big\{\|U^{k+1}_u - U_u\|_{\mathcal{S}^{2p}} + \|V^{k+1}_u - V_u\|_{\mathcal{H}^{2p}}\Big\} = 0, \quad \text{for all } p \ge 1.$$

Part ii): In what follows we prove that the sequence $(U^k, V^k)_{k\in\mathbb{N}}$ is Malliavin differentiable and its Malliavin derivatives converge to a version of the Malliavin derivative of (U, V). This is done recursively, starting with the initial step. From (A3) and (A4), the differentiability of ξ_u and $a_{u,t}$ implies that for all $0 \le u \le t \le T$ the process $\mathbb{E}\big[\xi_u + \int_t^T f(s,u,0,0)\mathrm{d}s|\mathcal{F}_t\big] \in \mathbb{D}^{1,2}$ and hence

$$\mathbb{E}\big[\xi_u + \int_t^T f(s,u,0,0)\mathrm{d}s|\mathcal{F}_t\big] = U^1_{u,t} \in \mathbb{D}^{1,2}.$$

Since $\xi + \int_t^T f(s,u,0,0)\mathrm{d}s - U^1_{u,t} = \int_t^T V^1_{u,s}\mathrm{d}W_s$, Lemma 5.1 of [EKPQ97] implies $V^1_{u,t} \in \mathbb{D}^{1,2}$. For the recursive step, we next show that if $(U^k_{u,t}, V^k_{u,t}) \in \mathbb{D}^{1,2}$, then also $(U^{k+1}_{u,t}, V^{k+1}_{u,t}) \in \mathbb{D}^{1,2}$. Assume

[4] Applying the Itô formula to $e^{\beta t}(U^{k+1}_{u,t} - U^k_{u,t})^2, t \in [0,T]$, one proves in the usual fashion norm contraction of the sequence through the a priori estimates. As this argument is well known we omit it.

that $(U^k_{u,t}, V^k_{u,t}) \in \mathbb{D}^{1,2}$. Since $b, c \in \mathbb{D}^{1,2}$, by the rules of Malliavin calculus we have $\mathbb{E}\big[\xi_u + \int_t^T f(s,u,U^k_{u,s},V^k_{u,s})\mathrm{d}s|\mathcal{F}_t\big] \in \mathbb{D}^{1,2}$ and hence $U^{k+1}_{u,t} \in \mathbb{D}^{1,2}$. Consequently for $\int_t^T V^{k+1}_{u,s}\mathrm{d}W_s = \xi_u + \int_t^T f(s,u,U^k_{u,s},V^k_{u,s})\mathrm{d}s - U^{k+1}_{u,t}$ again Lemma 5.1 in [EKPQ97] yields $V^{k+1}_{u,t} \in \mathbb{D}^{1,2}$. Given these properties we have for $0 \le v \le u \le t \le T$

$$\begin{aligned} D_v U^{k+1}_{u,t} &= D_v\xi_u - \int_t^T D_v V^{k+1}_{u,s}\mathrm{d}W_s \\ &\quad + \int_t^T \Big[(D_v f)(s,u,U^k_{u,s},V^k_{u,s}) + \big\langle(\nabla f)(s,u,U^k_{u,s},V^k_{u,s}),(D_vU^k_{u,s},D_vV^k_{u,s})\big\rangle\Big]\mathrm{d}s. \end{aligned}$$

We continue by showing that the sequence $(D_vU^k_{u,t}, D_vV^k_{u,t})$ converges and identify its limit as $(D_vU_{u,t}, D_vV_{u,t})$ which in addition is a solution of (42).

If we assume that equation (42) has a solution (D_vU_u, D_vV_u) then the usual moment estimation techniques combined with the current assumptions produce

$$\sup_{0\le v,u\le T}\Big\{\|D_vU_u\|_{\mathcal{S}^{2p}} + \|D_vV_u\|_{\mathcal{H}^{2p}}\Big\} < \infty, \qquad p \ge 1. \tag{44}$$

Fix $N \in \mathbb{N}$ to be chosen later, fix $0 \le u \le v \le T$, set $\delta = T/N$ and define a partition $\tau_i = i\delta$ for $i \in \{1,\dots,N\}$. Then a priori estimates yield for $0 \le i \le N-1$

$$\begin{aligned} A^{k+1}_{u,v,i} &= \|D_vU^{k+1}_u - D_vU_u\|^2_{\mathcal{S}^2([\tau_i,\tau_{i+1}])} + \|D_vV^{k+1}_u - D_vV_u\|^2_{\mathcal{H}^2([\tau_i,\tau_{i+1}])} \\ &\le C\Big\{\mathbb{E}\Big[|D_vU^{k+1}_{u,\tau_{i+1}} - D_vU_{u,\tau_{i+1}}|^2\Big] + B^k_{u,v,i} + C^k_{u,v,i}\Big\} \end{aligned} \tag{45}$$

with

$$\begin{aligned} B^k_{u,v,i} &= \big\| \,|D_vb|\,|U^k_u - U_u| + |D_vc|\,|V^k_u - V_u|\,\big\|^2_{\mathcal{H}^2([\tau_i,\tau_{i+1}])}, \\ C^k_{u,v,i} &= \mathbb{E}\Big[\Big(\int_{\tau_i}^{\tau_{i+1}}\Big[\,|b_s|\,|D_vU^k_{u,s} - D_vU_{u,s}| + |c_s|\,|D_vV^k_{u,s} - D_vV_{u,s}|\,\Big]\mathrm{d}s\Big)^2\Big]. \end{aligned}$$

Since both b and c are bounded, Jensen's inequality yields

$$C^k_{u,v,i} \le C\delta A^k_{u,v,i}$$

and hence, an induction argument combined with (44), (45) and the assumptions provides

$$\sup_{0\le u,v\le T} A^k_{u,v,i} < \infty, \quad \text{for all } k \ge 0. \tag{46}$$

To estimate $B^k_{u,v,i}$, note that according to (A3), $\sup_{0\le v\le T}\big\{\|D_vb\|_{H^2} + \|D_vc\|_{H^2}\big\} < \infty$ and that according to the first part of the proof $(U^k - U, V^k - V) \to 0$ in $\mathcal{S}^p \times \mathcal{H}^p$. Now choose N large enough to guarantee $\alpha = C\delta < 1$. Therefore for any $\eta > 0$ one finds a $K^* \ge 0$, independent of u, v for which

$$A^{k+1}_{u,v,i} \le C\mathbb{E}\Big[|D_vU^{k+1}_{u,\tau_{i+1}} - D_vU_{u,\tau_{i+1}}|^2\Big] + \eta + \alpha A^k_{u,v,i}, \qquad \text{for } k \ge K^*.$$

The equation $D_vU_{u,T} = D_vU^k_{u,T}$ allows us to write for $i = N-1$ and $k \ge K^*$

$$\sup_{0\le u,v\le T} A^k_{u,v,N-1} \le \eta + \alpha^{k-K^*}\sup_{0\le u,v\le T} A^{K^*}_{u,v,N-1}.$$

As a consequence, (46) implies that $\sup_{0\le u,v\le T} A^k_{u,v,N-1} \to 0$ as $k \to \infty$. One can expand the argument and show recursively that for all $0 \le i \le N-1$ one has $\sup_{0\le u,v\le T} A^k_{u,v,i} \to 0$. Summing over i, one arrives at

$$\sup_{0\le u,v\le T} \left\{ \|D_v U^{k+1}_u - D_v U_u\|^2_{\mathcal{S}^2([0,T])} + \|D_v V^{k+1}_u - D_v V_u\|^2_{\mathcal{H}^2([0,T])} \right\} \overset{k\to\infty}{\longrightarrow} 0.$$

The conclusion is that (U_u, V_u) are indeed Malliavin differentiable and a version of its Malliavin derivatives is given by the limit of $(D_v U^k_u, D_v V^k_u)$.

The last statement of our theorem follows from Lemma 5.1 in [EKPQ97]. We write our BSDE (41) for terminal time t, apply the Malliavin derivative operator, and obtain by the quoted Lemma

$$\begin{aligned} D_v U_{u,t} = V_{u,t} &- \int_v^t D_v V_{u,s} \mathrm{d}W_s \\ &+ \int_v^t \Big[(D_v f)(s,u,U_{u,s},U_{u,s}) + \big\langle \nabla f(s,u,U_{u,s},U_{u,s}), (D_v U_{u,s}, D_v U_{u,s}) \big\rangle \Big] \mathrm{d}s. \end{aligned}$$

Choosing $v = t$ leads to the desired representation. □

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