# Differentiability of quadratic BSDE generated by continuous martingales and hedging in incomplete markets

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**Abstract**

In this paper we consider a class of BSDE with drivers of quadratic growth, on a stochastic basis generated by continuous local martingales. We first derive the Markov property of a forward-backward system (FBSDE) if the generating martingale is a strong Markov process. Then we establish the differentiability of a FBSDE with respect to the initial value of its forward component. This enables us to obtain the main result of this article which from the perspective of a utility optimization interpretation of the underlying control problem on a financial market takes the following form. The control process of the BSDE steers the system into a random liability depending on a market external uncertainty and this way describes the optimal derivative hedge of the liability by investment in a capital market the dynamics of which is described by the forward component. This delta hedge is described in a key formula in terms of a derivative functional of the solution process and the correlation structure of the internal uncertainty captured by the forward process and the external uncertainty responsible for the market incompleteness. The formula largely extends the scope of validity of the results obtained by several authors in the Brownian setting, designed to give a genuinely stochastic representation of the optimal delta hedge in the context of cross hedging insurance derivatives generalizing the derivative hedge in the Black-Scholes model. Of course, Malliavin's calculus needed in the Brownian setting is not available in the general local martingale framework. We replace it by new tools based on stochastic calculus techniques.

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# 1 Introduction

In recent years Backward Stochastic Differential Equations (BSDE for short) with drivers of quadratic growth have emerged in several fields of application such as in the study of properties of PDEs (see *e.g.* [14, 4]). Closer to the subject of this work, they were employed for providing a genuinely stochastic approach to describe optimal investment strategies in a financial market in problems of hedging derivatives or liabilities of a small trader whose business depends on market external risk such as the one created by weather or energy demand. The latter scenario was addressed for instance in [8, 2, 3, 13]. A small trader such as an energy retailer has a natural source of income deriving from his usual business. For instance, he may have a random position of revenues from heating oil sales at the end of a heating season. To (cross) hedge his risk arising from the partly market external uncertainty present in the temperature process during the heating season, for example via derivatives written on temperature, he decides to invest in the capital market the inherent uncertainty of which is only correlated with this index process. If the agent values his total income at terminal time by exponential utility, or his risk by the entropic risk measure, he may be interested in finding an optimal investment strategy that maximizes his terminal utility resp. minimizes his total risk. The description of such strategies, even under convex constraints for the set of admissible ones, is classical and may be achieved by convex duality methods, and formulated in terms of the analytic Hamilton-Jacobi-Bellman equation. In a genuinely stochastic approach, [8] interpreted the martingale optimality principle by means of BSDE with drivers of quadratic growth, to come up with a solution of this optimal investment problem even under constraints that are not necessarily convex, just closed. The optimal investment strategy is described by the control process in the solution pair of such a BSDE with an explicitly known driver. Using this approach the authors of [3] investigate utility indifference prices and delta hedges for derivatives or liabilities written on non-tradable underlyings such as temperature in incomplete financial market models. A sensitivity analysis of the dependence of the optimal investment strategies on the initial state of the Markovian forward process modeling the external risk process provides an explicit delta hedging formula from the representation of indifference prices in terms of forward-backward systems of stochastic differential equations (FBSDE). In the framework of a Brownian basis, this analysis requires to account for parametric as well as variational differentiability of the solutions of the BSDE part in the sense of Malliavin's calculus (see [2, 3, 4]). Related optimal investment problems have been investigated in situations in which the Gaussian basis is replaced by the one of a continuous martingale ([12] and [13], see also [9]).

In this paper we intend to present the utility indifference based explicit description of a delta hedge extending the one of the Black-Scholes formula just sketched by a sensitivity analysis of related systems of FBSDE on such a more general stochastic basis created by a continuous local martingale. As the backward component of our system, we consider a BSDE of the form (1.1) driven by a continuous local martingale $M$ with

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T f(s, Y_s, Z_s) dC_s - \int_t^T dL_s + \int_t^T d\langle L, L\rangle_s, \quad t \in [0, T], \qquad (1.1)$$

where the *generator* $f$ is assumed to be quadratic in $z$, the *terminal condition* $B$ is bounded,

$C$ is an increasing process and $L$ is a martingale orthogonal to $M$. A solution of (1.1) is given by the triplet $(Y, Z, L)$. The forward component of our system will be of the form

$$X_s = x + \int_0^s \sigma(r, X_r, M_r) dM_r + \int_0^s b(r, X_r, M_r) dC_r, \quad s \in [0, T]. \tag{1.2}$$

We first prove in Theorem 3.4 that the solution processes $Y$ and $Z$ satisfy the Markov property, provided the terminal condition $B$ is a smooth function of the terminal value of the forward process (1.2), and that the local martingale $M$ is a strong Markov process. There is a subtlety in this setting which goes beyond causing a purely technical complication. In fact, only the pair $(X, M)$ is a Markov process (as proved for example in [16, 5, 17]). Just if $M$ has independent increments, it is a stand-alone Markov process. We next show in Theorem 4.6 that the process $Y$ is differentiable with respect to the initial value of the forward component (1.2) and that the derivatives of $Y$ and $Z$ again satisfy a BSDE. The two properties thus proved will combine to allow us to state and prove the main contribution of this paper to the delta hedge representation problem (Theorem 5.2) generalizing the formula obtained in the Gaussian setting and for quadratic generators in [3, Theorem 6.7] and in [10, Corollary 4.1] for the Lipschitz case. More precisely we show that there exists a deterministic function $u$ such that

$$Z_s = \partial_2 u(s, X_s, M_s) \sigma(s, X_s, M_s) + \partial_3 u(s, X_s, M_s), \tag{1.3}$$

where $Y_s = u(s, X_s, M_s), s \in [0, T]$, and $\partial_i$ denotes the partial derivative with respect to the $i$-th variable (see Theorem 5.2). In addition we show that if $M$ has independent increments and the coefficients of the forward process do not depend on $M$, then $Y_s = u(s, X_s)$ and equality (1.3) becomes $Z_s = \partial_2 u(s, X_s)\sigma(s, X_s)$ which coincides with the formula known for the case in which $M$ is the Brownian motion. To the best of our knowledge relation (1.3) was known only in the Brownian setting and the proof used in the literature relies on the representation of the stochastic process $Z$ as the trace of the Malliavin derivative $D$ (*i.e.* $Z_s = D_s Y_s, s \in [0, T]$) relative to the underlying Brownian motion. Since Malliavin's calculus is not available for general continuous local martingales, we propose a new approach based on stochastic calculus techniques, in which directional variational derivatives of Malliavin's calculus are replaced by absolute continuity properties of mixed variation processes of local martingales. Our main result will provide an extension of the probabilistic representation of the optimal delta hedge of [3], obtained there in the Brownian setting, to more general scenarios in which pricing rules are based on general continuous local martingales. Before entering into the technical details leading to a derivation of our main result, let us sketch in some more details this important application of our results in the chosen general setting.

Indeed, using the exponential utility based indifference price approach for pricing and hedging derivatives in incomplete markets sketched above, we interpret relation (1.3) as a delta hedge formula. It is shown in [3] that it can be expressed by the gradient of the indifference price and the correlation. Let us explain, how this structure translates into our local martingale framework, with the more complex Markovian properties. More precisely consider an $n$-dimensional process describing non-tradable risk

$$R_s^{t,r,m} = r + \int_t^s \sigma(u, R_u^{t,r,m}, M_u^{t,m}) dM_u + \int_t^s b(u, R_u^{t,r,m}, M_u^{t,m}) dC_u, \quad s \in [t, T],$$

where $\sigma$ and $b$ are measurable functions. Now an agent aims to price and hedge a derivative of the form $F(R_T^{t,r,m})$, with $F$ being a bounded and measurable function. The hedging instrument is a financial market consisting of $k$ risky assets in units of the numeraire that evolve according to the following SDE

$$dS_s = S_s\big(\beta(s, R_s^{t,r,m}, M_s^{t,m})dM_s + \alpha(s, R_s^{t,r,m}, M_s^{t,m})dC_s\big), \quad s \in [t,T],$$

where $\alpha \in \mathbb{R}^{k\times 1}$ and $\beta \in \mathbb{R}^{k\times d}$ are measurable processes. Note that the price processes of tradable assets $S$ are linked to the risk process via the martingale $M$, its quadratic variation, $\beta$ and $\sigma$. We assume $k \leq d$ in order to exclude arbitrage opportunities. The preferences of the agent are represented through the exponential utility function with risk aversion coefficient $\kappa > 0$, i.e.

$$U(x) = -e^{-\kappa x}, \quad x \in \mathbb{R}.$$

The agent wants to maximize his expected utility from trading in the market. His value function is given by

$$V^F(x,t,r,m) = \sup_{\lambda} \mathbb{E}\left[U(x + \sum_{i=1}^{k}\int_t^T \lambda_s^{(i)} \frac{dS_s^{(i)}}{S_s^{(i)}} + F(R_T^{t,r,m}))\right],$$

where $x$ is his initial capital and $\lambda^{(i)}$ denotes the momentary value of his portfolio fraction invested in the $i$-th asset. The problem of optimization of this value function can be reduced to solving a quadratic BSDE whose generator has been given in [8] for the Brownian case and then extended to our setting in [13]. A way to price and hedge the derivative $F(R_T^{r,t,m})$ is to consider the indifference price $p(t,r,m)$ given by $V^F(x - p(t,r,m), t, r, m) = V^0(x,t,r,m)$. According to [3] the indifference price can be expressed as $p(t,r,m) = Y^{F,t,r,m} - Y^{0,t,r,m}$, where $(Y^{F,t,r,m}, Z^{F,t,r,m}, L^{F,t,r,m})$ is the solution of the BSDE

$$\begin{aligned} Y_s^{F,t,r,m} =& F(R^{t,r,m}) - \int_s^T Z_u^{F,t,r,m} dM_u + \int_s^T f(u, R_u^{t,r,m}, M_u^{t,m}, Z_u^{F,t,r,m})dC_u \\ &- \int_s^T dL_u^{F,t,r,m} + \int_s^T d\langle L^{F,t,r,m}, L^{F,t,r,m}\rangle_u, \quad t \in [0,T], \end{aligned} \tag{1.4}$$

where the generator $f$ is explicitly obtained through the martingale optimal principle in [8, 13]. To implement utility indifference, we next have to describe the optimal strategies $\hat{\lambda}^F$ and $\hat{\lambda}^0$. In [8] it is shown that $\hat{\lambda}^F\beta(\cdot, R^{t,r,m}, M^{t,m})$ (and $\hat{\lambda}^0\beta(\cdot, R^{t,r,m}, M^{t,m})$) are given by a projection of a linear function of $Z^{F,t,r}q^*$ (and respectively $Z^{0,t,r}q^*$) on the constraint set. Since $R^{t,r,m}$ is not tradable directly $\beta$ plays the role of a filter for trading in the market. Due to [3] the optimal strategy to hedge $F(R_T^{t,r,m})$ can be decomposed into a pure trading part $\hat{\lambda}^0$ and the optimal hedge $\Delta$, which is the part of the strategy that replicates the derivative $F(R_T^{t,r,m})$. Using the Markov property given in Theorem 3.4, we see that there exists a deterministic function $u^F$ such that $Y^{F,t,r} = u^F(\cdot, R^{t,r,m}, M^{t,m})$. Moreover, the projection mentioned above can be explicitly expressed and following [3, proof of Theorems 4.2 and 4.4] we have

$$\hat{\lambda}_s^F - \hat{\lambda}_s^0 = (Z_s^{F,t,r,m} - Z_s^{0,t,r,m})q_s^*\beta^*(\beta\beta^*)^{-1}\beta(s, R_s^{t,r,m}, M_s^{t,m}), \quad s \in [t,T].$$

This leads to

$$\Delta(t,r,m) = (\hat{\lambda}^F - \hat{\lambda}^0)\beta^*(\beta\beta^*)^{-1}(t,r,m) = (Z_t^{F,t,r,m} - Z_t^{0,t,r,m})q_t^*\beta^*(\beta\beta^*)^{-1}(t,r,m).$$

Using formula (1.3) we derive

$$\Delta(t,r,m) = [\partial_2 p(t,r,m)\sigma(t,r,m) + \partial_3 p(t,r,m)]\, q_t^*\beta^*(\beta\beta^*)^{-1}(t,r,m). \tag{1.5}$$

We emphasize that, as a consequence of the particular form of the driver $f$ in (1.4), relation (1.5) is replaced by (1.6), if $M$ has independent increments and if the coefficients $\sigma$, $b$, $\beta$ and $\alpha$ do not depend on $M$ (see Remarks 5.3 *ii)* and *iii)*); with

$$\Delta(t,r) = [\partial_2 p(t,r)\sigma(t,r)]\, q_t^*\beta^*(\beta\beta^*)^{-1}(t,r). \tag{1.6}$$

Finally, note that for technical reasons presented later we obtain formulae (1.3) and (1.5) under condition (MRP) (see Section 4.2). However we believe that this condition is not necessary for deriving (1.5) (we refer to the concluding remarks).

We believe that our techniques are not restricted to the setting of continuous local martingales. We will investigate in a forthcoming paper how to obtain a relation of the type (1.3) for BSDE driven by a mixture of continuous and Poisson processes. We finally emphasize that the local martingale $M$ considered in this paper is not assumed to satisfy the martingale representation property.

To present our results, we proceed as follows. In Section 2 we state the main notations and assumptions used in the paper. We discuss the Markov property of an FBSDE in Section 3. Then in Section 4 we give sufficient conditions on the FBSDE to be differentiable in the initial values of its forward component. The final Section 5 is devoted to the representation formula (1.3).

# 2 Preliminaries

### Notations

Let $(M_t)_{t\in[0,T]}$ be a continuous, square-integrable, $d$-dimensional local martingale with $M_0 = 0$ which is defined on a probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where $T$ is a fixed positive real number. We assume that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is continuous and complete and thus every square integrable $\mathbb{P}$-martingale is of the form $Z\cdot M + L$, where $Z$ is a predictable $d$-dimensional process and $L$ a $\mathbb{R}$-valued martingale strongly orthogonal to $M$. From the Kunita-Watanabe inequality it follows that there exists a continuous, $(\mathcal{F}_t)$-adapted, bounded and increasing real-valued process $(C_t)_{t\in[0,T]}$ and a $\mathbb{R}^{d\times d}$-valued predictable process $(q_t)_{t\in[0,T]}$ such that the quadratic variation process $\langle M, M\rangle$ can be written as

$$\langle M, M\rangle_t = \int_0^t q_r q_r^* dC_r, \quad t \in [0,T].$$

By $\mathcal{P}$ we denote the predictable $\sigma$-field of $\Omega \times [0,T]$. Given an arbitrary non-negative and progressively measurable real-valued process $(\alpha_t)_{t\in[0,T]}$ we define $A$ by $A_t := \int_0^t \alpha_s^2 dC_s, 0 \le t \le T$. For any $\beta > 0$ and $p \in [1,\infty)$ we set

- $\mathcal{S}^\infty := \{X|\ X : \Omega \times [0,T] \to \mathbb{R},\ \text{bounded and continuous}\}$,
- $\mathcal{S}^p := \Big\{X|\ X : \Omega \times [0,T] \to \mathbb{R},\ \text{predictable and } \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] < \infty\Big\}$,
- $L^p(d\langle M,M\rangle \otimes d\mathbb{P}) := \left\{Z|\ Z : \Omega \times [0,T] \to \mathbb{R}^{1\times d},\ \text{predictable and } \mathbb{E}\left[\left(\int_0^T |q_s Z_s^*|^2 dC_s\right)^{\frac{p}{2}}\right] < \infty\right\}$.
- $\mathcal{M}^2 := \Big\{X|\ X : \Omega \times [0,T] \to \mathbb{R},\ \text{martingale and } \mathbb{E}\left[\int_0^T |X_s|^2 dC_s\right] < \infty\Big\}$.
- $\mathbb{L}^p := \{\xi|\ \xi : \Omega \to \mathbb{R},\ \mathcal{F}_T - \text{measurable and } \mathbb{E}\left[|\xi|^p\right] < \infty\}$,
- $\mathbb{L}^2_\beta := \big\{\xi|\ \xi : \Omega \to \mathbb{R},\ \mathcal{F}_T - \text{measurable and } \mathbb{E}\left[e^{\beta A_T}|B|^2\right] < \infty\big\}$,
- $\mathbb{L}^\infty := \{\xi|\ \xi : \Omega \to \mathbb{R},\ \mathcal{F}_T - \text{measurable and bounded}\}$,
- $\mathbb{H}^2_\beta(\mathbb{R}^d) := \Big\{\varphi|\ \varphi : \Omega \times [0,T] \to \mathbb{R}^{d\times 1},\ \text{predictable}, \|\varphi\|^2_\beta := \mathbb{E}\left[\int_0^T e^{\beta A_t}|\varphi_t|^2 dC_t\right] < \infty\Big\}$,
- $\mathbb{S}^2_\beta := \Big\{\varphi|\ \varphi : \Omega \times [0,T] \to \mathbb{R}^{d\times 1},\ \text{continuous}, \|\varphi\|^2_\beta := \mathbb{E}\left[\sup_{t\in[0,T]} e^{\beta A_t}|\varphi_t|^2\right] < \infty\Big\}$.

Throughout this paper we will make use of the notation $(M^{t,m})_{s\in[t,T]}$ $(t < T,\ m \in \mathbb{R}^{d\times 1})$ which refers to the martingale

$$M^{t,m}_s := m + M_s - M_t$$

defined with respect to the filtration $(\mathcal{F}^t_s)_{s\in[t,T]}$ with $\mathcal{F}^t_s := \sigma(\{M_u - M_t,\ t \le u \le s\})$. Obviously, all the preceding definitions can be introduced with $M^{t,m}$ in place of $M$ and will inherit the superscript $^{t,m}$. For convenience, we write $M^m := M^{0,m}$. Furthermore $M^{(i)}$, $i = 1,\dots,d$, denotes the entries of the vector $M$. The euclidean norm is denoted by $|\cdot|$ and with $\mathcal{E}$ we refer to the stochastic exponential.

### FBSDE driven by continuous martingales

In this subsection we present the main hypotheses needed in this paper.

Let us fix $x \in \mathbb{R}^{n\times 1}$ and $m \in \mathbb{R}^{d\times 1}$. First we consider the process $X^{x,m} := (X^{x,m}_t)_{t\in[0,T]}$ which is defined as a solution of the following stochastic differential equation (SDE)

$$X^{x,m}_t = x + \int_0^t \sigma(s, X^{x,m}_s, M^m_s) dM_s + \int_0^t b(s, X^{x,m}_s, M^m_s) dC_s, \quad t \in [0,T], \tag{2.1}$$

where the coefficients $\sigma : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{n\times d}$ and $b : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{n\times 1}$ are Borel-measurable functions. By [6, Theorem 1] and [16, Theorem 3.1], where independently the same result is proved, this SDE has a unique solution if it satisfies the following hypothesis:

**(HSDE)** The functions $\sigma$ and $b$ are continuous in $(s,x,m)$ and there exists a $K > 0$ such that for all $x_1, x_2 \in \mathbb{R}^{n\times 1}$ and $m \in \mathbb{R}^{d\times 1}$

$$|\sigma(s,x_1,m) - \sigma(s,x_2,m)| + |b(s,x_1,m) - b(s,x_2,m)| \le K|x_1 - x_2|, \quad s \in [0,T].$$

Next we give properties of BSDE which depend on the forward process $X^{x,m}$. More precisely we consider BSDE of the form

$$Y_t^{x,m} = F(X_T^{x,m}) - \int_t^T Z_r^{x,m} dM_r + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*) dC_r - \int_t^T dL_r^{x,m} + \frac{\kappa}{2} \int_t^T d\langle L^{x,m}, L^{x,m} \rangle_r, \tag{2.2}$$

where $F : \mathbb{R}^{n\times 1} \to \mathbb{R}$ and $f : \Omega \times [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$ are $\mathcal{B}(\mathbb{R}^{n\times 1})$-respectively $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{n\times 1}) \otimes \mathcal{B}(\mathbb{R}^{d\times 1}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^{1\times d})$-measurable functions. A solution of the BSDE with *terminal condition* $F(X_T^{x,m})$, a constant $\kappa$ and *generator* $f$ is defined to be a triple of processes $(Y^{x,m}, Z^{x,m}, L^{x,m}) \in \mathcal{S}^\infty \times L^2(d\langle M, M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2$ satisfying (2.2) and such that $\langle L^{x,m}, M^i \rangle = 0$, $i = 1, \dots, d$, and $\mathbb{P}$-a.s. $\int_0^T |f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*)| dC_r < \infty$.

Let $\mathcal{S} := \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \times \mathbb{R} \times \mathbb{R}^{1\times d}$ and assume that (HSDE) holds. Under the following conditions existence and uniqueness of a solution of the backward equation (2.2) was recently discussed in [13, Theorem 2.5]:

**(H1)** The function $F$ is bounded.

**(H2)** The generator $f$ is continuous in $(y, z)$ and there exists a nonnegative predictable process $\eta$ such that $\int_0^T \eta_s dC_s \le a$, where $a$ is a positive constant. Furthermore there exist positive numbers $b$ and $\gamma$, such that $dC_s \otimes d\mathbb{P}$-a.s.

$$|f(s, x, m, y, z)| \le \eta_s + b\eta_s |y| + \frac{\gamma}{2} |z|^2, \quad \text{with } \gamma \ge |\kappa|,\ \gamma \ge b, \quad (x, m, y, z) \in \mathcal{S}.$$

An additional assumption is needed to obtain uniqueness (see [13, Theorem 2.6]):

**(H3)** For every $\beta \ge 1$ we have $\int_0^T |f(s, 0, 0)| dC_s \in L^\beta(\mathbb{P})$. In addition, there exist two constants $\mu$ and $\nu$, a nonnegative predictable process $\theta$ satisfying $\int_0^T |q_s \theta_s|^2 dC_s \le c_\theta$ ($c_\theta \in \mathbb{R}$), such that $dC_s \otimes d\mathbb{P}$-a.s.

$$(y_1 - y_2)(f(s, x, m, y_1, z) - f(s, x, m, y_2, z)) \le \mu |y_1 - y_2|^2, \quad (x, m, y_i, z) \in \mathcal{S}, i = 1, 2, \text{ and}$$

$$|f(s, x, m, y, z_1) - f(s, x, m, y, z_2)| \le \nu(|q_s \theta_s| + |z_1| + |z_2|)|z_1 - z_2|, \quad (x, m, y, z_i) \in \mathcal{S}, i = 1, 2.$$

In this paper we will deal with martingales of bounded mean oscillation, briefly called BMO-martingales. We refer the reader to [11] for a survey. Furthermore we need the following hypothesis:

**(H4)** There exist a $\mathbb{R}$-valued predictable process $A$ bounded below by one and a $\mathbb{R}^{1\times d}$-valued predictable process $B$ such that $A 1^*_{\mathbb{R}^d} \cdot M$ and $B \cdot M$ are BMO martingales satisfying $dC_s \otimes d\mathbb{P}$-a.s.

$$(y_1 - y_2)(f(s, x, m, y_1, z) - f(s, x, m, y_2, z)) \le |q_s A_s 1_{\mathbb{R}^d}| |y_1 - y_2|^2, \quad (x, m, y_i, z) \in \mathcal{S}, i = 1, 2, \text{ and}$$

$$|f(s, x, m, y, z_1) - f(s, x, m, y, z_2)| \le |q_s B_s^*| |z_1 - z_2|, \quad (x, m, y, z_i) \in \mathcal{S}, i = 1, 2.$$

Throughout this paper we also consider another type of BSDE. Again this BSDE is associated to the forward process $X^{x,m}$ solving (2.1), i.e.

$$U_t^{x,m} = F(X_T^{x,m}) - \int_t^T V_s^{x,m} dM_s + \int_t^T f(s, X_s^{x,m}, M_s^m, U_s^{x,m}, V_s^{x,m} q_s^*) dC_s + \int_t^T dN_s^{x,m}, \tag{2.3}$$

$t \in [0,T]$. This type of BSDE has been studied by El Karoui and Huang in [9]. Existence and uniqueness results are derived under the following assumptions on terminal condition $F(X_T^{x,m})$ and generator $f$:

**(L1)** The function $F$ satisfies $F \circ X_T^{x,m} \in \mathbb{L}_\beta^2$.

**(L2)** The generator $f$ satisfies $dC_s \otimes d\mathbb{P}$-a.s.

$$|f(s,x,m,y_1,z_1) - f(s,x,m,y_2,z_2)| \leq r_s|y_1 - y_2| + \theta_s|z_1 - z_2|, \quad (x,m,y_i,z_i) \in \mathcal{S}, i = 1,2,$$

where $r$ and $\theta$ are two non-negative predictable processes. Let $\alpha_s^2 = r_s + \theta_s^2$. We assume $dC_s \otimes d\mathbb{P}$-a.s. that $\alpha_s^2 > 0$ and $\frac{f(\cdot,0,0)}{\alpha} \in \mathbb{H}_\beta^2$.

We conclude this section by presenting assumptions which will be useful in Section 4, where we find sufficient conditions for a FBSDE to be differentiable in its initial values $(x,m) \in \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1}$. To simplify the notation we introduce for each pair $(\sigma, b)$ of forward coefficients the second order differential operator $\mathcal{L} = \sum_{i=1}^n b_i(\cdot)\partial_i + \frac{1}{2}\sum_{i,j=1}^n [(\sigma q)(\sigma q)^*]_{ij}(\cdot)\partial_{ij}$:

**(D1)** The coefficients $\sigma$ and $b$ have locally Lipschitz partial derivatives.

**(D2)** The function $F$ is twice differentiable and such that $\nabla F \cdot \sigma$ and $\mathcal{L}F$ have locally Lipschitz partial derivatives.

**(D3)** The generator $f$ is differentiable in $x, m, y$ and $z$ and there exist a constant $C > 0$ and a nonnegative predictable process $\theta$ satisfying $\int_0^T |q_s\theta_s|^2 dC_s \leq c_\theta$ $(c_\theta \in \mathbb{R})$, such that the partial derivatives satisfy $dC_s \otimes d\mathbb{P}$-a.s.

$$|\partial_i f(s,x,m,y,z)| \leq C(|q_s\theta_s| + |z|), \quad (x,m,y,z) \in \mathcal{S},\ i = 2,\dots,5.$$

**(D4)** The generator $f$ is differentiable in $x, m, y$ and $z$ and there exist a constant $C > 0$ and a nonnegative predictable process $\theta$ satisfying $\int_0^T |q_s\theta_s|^2 dC_s \leq c_\theta$ $(c_\theta \in \mathbb{R})$, such that the partial derivative $\partial_5 f$ is Lipschitz in $(x,m,y,z)$ and for all $i = 2,\dots,4$ the following inequality holds $dC_s \otimes d\mathbb{P}$-a.s.

$$\begin{aligned} &|\partial_i f(s,x_1,m_1,y_1,z_1) - \partial_i f(s,x_2,m_2,y_2,z_2)| \\ &\leq C(|q_s\theta_s| + |z_1| + |z_2|)(|x_1 - x_2| + |m_1 - m_2| + |y_1 - y_2| + |z_1 - z_2|), \end{aligned}$$

for all $(x_j, m_j, y_j, z_j) \in \mathcal{S}, j = 1,2$.

# 3 The Markov property of FBSDE

For a fixed initial time $t \in [0,T)$ and initial values $x \in \mathbb{R}^{n\times 1}$ and $m \in \mathbb{R}^{d\times 1}$ we consider a SDE of the form

$$X_s^{t,x,m} = x + \int_t^s \sigma(u, X_u^{t,x,m}, M_u^{t,m}) dM_u + \int_t^s b(u, X_u^{t,x,m}, M_u^{t,m}) dC_u, \quad s \in [t,T], \tag{3.1}$$

where $M$ is a local martingale as in Section 2 with values in $\mathbb{R}^{d\times 1}$, $\sigma : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{n\times d}$ and $b : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{n\times 1}$. Throughout this chapter the coefficients $\sigma$ and $b$ satisfy (HSDE) and hence (3.1) has a unique solution $X^{t,x,m}$. Before stating and proving the main results of this section we recall the following proposition whose first point is presented in [5, Theorem (8.11)] (see also [17, V.Theorem 35]) and the second part is a direct application of [15, Theorem 5.3].

**Proposition 3.1.** *i) If $M$ is a strong Markov process then $(M_s^{t,m}, X_s^{t,x,m})_{s\in[t,T]}$ is a strong Markov process.*

*ii) If $M$ is a strong Markov process with independent increments and if the coefficients $\sigma$ and $b$ do not depend on $M$, that is if $X$ satisfies*

$$X_s^{t,x} = x + \int_t^s \sigma(u, X_u^{t,x}) dM_u + \int_t^s b(u, X_u^{t,x}) dC_u$$

*in place of (3.1) then the process $(X_s^{t,x})_{s\in[t,T]}$ itself is a strong Markov process.*

Note that in [2, 3, 10] the martingale considered was a standard Brownian motion which is the paradigm of situation *ii)* of the Proposition above. In fact this case presents at least two major advantages. Firstly, the process $X$ is a Markov process itself and secondly, the quadratic variation of $M$ is deterministic.

This section is organized as follows. We first prove in Proposition 3.2 that the solution of a Lipschitz BSDE associated to a forward SDE of the form (3.1) is already determined by the solution $X^{t,x,m}$ of (3.1) and the Markov process $M^{t,m}$. In Theorem 3.4 we then extend this result to quadratic BSDE.

Consider a BSDE of the form

$$\begin{aligned} U_s^{t,x,m} =& F(X_T^{t,x,m}) - \int_s^T V_r^{t,x,m} dM_r + \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{t,x,m}, V_r^{t,x,m} q_r^*) dC_r \\ & - \int_s^T dN_r^{t,x,m}, \quad s \in [t,T]. \end{aligned} \tag{3.2}$$

We suppose that the driver does not depend on $\Omega$ and hence is a deterministic Borel measurable function $f : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \times \mathbb{R} \times \mathbb{R}^{1\times d} \to \mathbb{R}$. If $F$ and $f$ satisfy hypotheses (L1) and (L2) then the BSDE (3.2) admits a unique solution $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m}) \in \mathbb{S}^2_\beta \times \mathbb{H}^2_\beta \times \mathcal{M}^2$ (see [9, Theorem 6.1]). By $\mathcal{B}_e(\mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1})$ we denote the $\sigma$-algebra generated by the family of functions $(x,m) \mapsto \mathbb{E}\left[\int_t^T \phi(s, X_s^{t,x,m}, M_s^{t,m}) dC_s\right]$, where $\phi : \Omega \times [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}$ is predictable, continuous and bounded.

**Proposition 3.2.** *Assume that $M$ is a strong Markov process and that (L1) and (L2) are in force. Then there exist deterministic functions $u : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}$, $\mathcal{B}([0,T]) \otimes \mathcal{B}_e(\mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1})$-measurable and $v : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{1\times d}$, $\mathcal{B}([0,T]) \otimes \mathcal{B}_e(\mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1})$-measurable such that*

$$U_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad V_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}), \quad s \in [t,T]. \tag{3.3}$$

**Remark 3.3.** *Before turning to the proof of Proposition 3.2 we stress the following point. Assume $M$ and $X$ are as in Proposition 3.1 ii) and that the driver $f$ in (3.2) does not depend on $M$, then Proposition 3.2 boils down to: there exist deterministic functions $u : [0,T]\times\mathbb{R}^{n\times 1} \to \mathbb{R}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1})$-measurable and $v : [0,T]\times\mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times d}$, $\mathcal{B}([0,T])\otimes \mathcal{B}_e(\mathbb{R}^{n\times 1})$-measurable such that*

$$U_s^{t,x} = u(s, X_s^{t,x}), \quad V_s^{t,x} = v(s, X_s^{t,x}), \quad s \in [t,T].$$

*Proof of Proposition 3.2.* Consider the following sequence $(U^{k,t,x,m}, V^{k,t,x,m}, N^{t,x,m})_{k\geq 0}$ of BSDE

$$\begin{aligned} U^{0,t,x} =& V^{0,t,x} = 0, \\ U_s^{k+1,t,x} =& F(X_T^{t,x,m}) + \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{k,t,x,m}, V_r^{k,t,x,m} q_r^*) dC_r \\ & - \int_s^T V_r^{k+1,t,x,m} dM_r - \int_s^T dN_r^{t,x,m}. \end{aligned} \tag{3.4}$$

We recall an estimate obtained in [9, p. 35]. Let $\beta > 0$ and let $\alpha$ be as in Section 2. Then

$$\begin{aligned} &\|\alpha(U^{k+1,t,x,m} - U^{k,t,x,m})\|_\beta^2 + \|q(V^{k+1,t,x,m} - V^{k,t,x,m})^*\|_\beta^2 \\ &\leq \varepsilon \|\alpha(U^{k,t,x,m} - U^{k-1,t,x,m})\|_\beta^2 + \|q(V^{k,t,x,m} - V^{k-1,t,x,m})^*\|_\beta^2, \end{aligned}$$

where $\varepsilon$ is a constant depending on $\beta$ which can be chosen such that $\varepsilon < 1$. Applying the result recursively we obtain

$$\begin{aligned} &\|\alpha(U^{k+1,t,x,m} - U^{k,t,x,m})\|_\beta^2 + \|q(V^{k+1,t,x,m} - V^{k,t,x,m})^*\|_\beta^2 \\ &\leq \varepsilon^k \left( \|\alpha(U^{1,t,x,m} - U^{0,t,x,m})\|_\beta^2 + \|q(V^{1,t,x,m} - V^{0,t,x,m})^*\|_\beta^2 \right). \end{aligned}$$

Since $\sum_{k=0}^\infty \|\alpha(U^{k+1,t,x,m} - U^{k,t,x,m})\|_\beta^2 + \|q(V^{k+1,t,x,m} - V^{k,t,x,m})^*\|_\beta^2 < \infty$ the sequence $(U^{k,t,x,m}, V^{k,t,x,m}, N^{t,x,m})_k$ converges $dC_s \otimes d\mathbb{P}$-almost surely to $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$ as $k$ tends to infinity.

We show by induction on $k \geq 1$ the following property *($\mathcal{P}_k$)*:

*($\mathcal{P}_k$)* there exist deterministic functions $\Phi^k : [0,T]\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1} \to \mathbb{R}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1}\times \mathbb{R}^{d\times 1})$-measurable and $\Psi^k : [0,T]\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1} \to \mathbb{R}^{1\times d}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1})$-measurable such that $U_s^{k,t,x,m} = \Phi^k(s, X_s^{t,x,m}, M_s^{t,m})$ and $V_s^{k,t,x,m} = \Psi^k(s, X_s^{t,x,m}, M_s^{t,m})$, for $t \leq s \leq T, k \in \mathbb{N}$.

Proof of *($\mathcal{P}_1$)*: From the definition of $U^{1,t,x,m}$ and since $N^{t,x,m}$ is a martingale we have for $s \in [t,T]$

$$U_s^{1,t,x,m} = \mathbb{E}\left[U_s^{1,t,x,m}|\mathcal{F}_s^t\right] = \mathbb{E}\left[F(X_T^{t,x,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0)dC_r\big|\mathcal{F}_s^t\right]. \quad (3.5)$$

The Markov property and Doob-Dynkin's Lemma give

$$\begin{aligned} U_s^{1,t,x,m} &= \mathbb{E}\left[F(X_T^{t,x,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0)dC_r\big|\mathcal{F}_s^t\right] \\ &= \mathbb{E}\left[F(X_T^{t,x,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0)dC_r\big|(X_s^{t,x,m}, M_s^{t,m})\right] \\ &= \Phi^1(s, X_s^{t,x,m}, M_s^{t,m}) \end{aligned}$$

where $\Phi^1 : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}$. Now let

$$R_s^{1,t,x,m} = U_s^{1,t,x,m} + \int_t^s f(r, X_r^{t,x,m}, M_r^{t,m}, 0, 0)dC_r - N_s^{t,x,m} + N_t^{t,x,m}, \quad s \in [t,T].$$

Then for $s \in [t,T]$

$$R_s^{1,t,x,m} = \int_t^s V_r^{1,t,x,m} dM_r \quad (3.6)$$

and hence $R^{1,t,x,m}$ is a strongly additive (in the sense of [5, p. 169]) square integrable martingale. By [5, Theorem (6.27)] there exists a function $\Psi^1 : [0,T] \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \to \mathbb{R}^{1\times d}$ such that

$$\langle R^{1,t,x,m}, R^{1,t,x,m}\rangle_s = \int_t^s \Psi^1(r, X_r^{t,x,m}, M_r^{t,m}) d\langle M, M\rangle_r \Psi^1(r, X_r^{t,x,m}, M_r^{t,m})^*, \quad (3.7)$$

and from the definition of $R^{1,t,x,m}$ we know

$$\langle R^{1,t,x,m}, R^{1,t,x,m}\rangle_s = \int_t^s V_r^{1,t,x,m} d\langle M, M\rangle_r (V_r^{1,t,x,m})^*, \quad s \in [t,T]. \quad (3.8)$$

But (3.7) and (3.8) imply

$$V_s^{1,t,x,m} = \Psi^1(s, X_s^{t,x,m}, M_s^{t,m}), \quad dC_s \otimes d\mathbb{P} - a.e.$$

Let $k \geq 1$. We prove *($\mathcal{P}_k$) $\Longrightarrow$ ($\mathcal{P}_{k+1}$)*: We have for $s \in [t,T]$

$$\begin{aligned} U_s^{k+1,t,x,m} &= \mathbb{E}\left[U_s^{k+1,t,x,m}|\mathcal{F}_s^t\right] \\ &= \mathbb{E}\left[F(X_T^{t,x,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{k,t,x,m}, V_r^{k,t,x,m} q_r^*)dC_r\big|\mathcal{F}_s^t\right] \\ &= \mathbb{E}\left[F(X_T^{t,x,m}) - \int_s^T f(r, X_r^{t,x,m}, M_r^{t,m}, \Phi^k(r, X_r^{t,x,m}, M_r^{t,m}), \Psi^k(r, X_r^{t,x,m}, M_r^{t,m}) q_r^*)dC_r\big|\mathcal{F}_s^t\right] \\ &= \mathbb{E}\left[F(X_T^{t,x,m}) - \int_s^T f^k(r, X_r^{t,x,m}, M_r^{t,m})dC_r\big|\mathcal{F}_s^t\right], \end{aligned}$$

where $f^k(r,y,z) := f(r,y,\Phi^k(r,y,z),\Psi^k(r,y,z)q_r^*)$. Using the same argument as in step $k=1$ we deduce that there exists a function $\Phi^{k+1}:[0,T]\times\mathbb{R}^{m\times 1}\times\mathbb{R}^{d\times 1}\to\mathbb{R}$ such that

$$U_s^{k+1,t,x,m} = \Phi^{k+1}(s, X_s^{t,x,m}, M_s^{t,m}).$$

For $s\in[t,T]$ let

$$R_s^{k+1,t,x,m} = U_s^{k+1,t,x,m} + \int_t^s f^k(r, X_r^{t,x,m}, M_r^{t,m})dC_r - N_s^{t,x,m} + N_t^{t,x,m}.$$

Following the same procedure as in step $k=1$, and omitting the details, we deduce that there exists a function $\Psi^{k+1}:[0,T]\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1}\to\mathbb{R}^{1\times d}$ such that

$$V_s^{k+1,t,x,m} = \Psi^{k+1}(s, X_s^{t,x,m}, M_s^{t,m}).$$

Let

$$u(r,y,z) := \limsup_{k\to\infty}\Phi^k(r,y,z), \quad v(r,y,z) := \limsup_{k\to\infty}\Psi^k(r,y,z).$$

Since the sequence $(U^{k,t,x}, V^{k,t,x}, N^{t,x,m})_k$ converges $dC_s\otimes d\mathbb{P}$-almost surely to $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$ as $k$ tends to infinity we have for $s\in[t,T]$

$$\begin{aligned} u(s, X_s^{t,x,m}, M_s^{t,m}) =& (\limsup_{k\to\infty}\Phi^k)(s, X_s^{t,x,m}, M_s^{t,m}) = \limsup_{k\to\infty}(\Phi^k(s, X_s^{t,x,m}, M_s^{t,m})) \\ =& \limsup_{k\to\infty} U_s^{k,t,x,m} = U_s^{t,x,m}. \end{aligned}$$

Similarly we obtain

$$v(s, X_s^{t,x,m}, M_s^{t,m}) = V_s^{t,x,m}.$$

□

We conclude this section by extending Proposition 3.2 to a quadratic FBSDE. More precisely we consider the following BSDE

$$\begin{aligned} Y_s^{t,x,m} =& F(X_T^{t,x,m}) - \int_s^T Z_u^{t,x,m}dM_u + \int_s^T f(u, X_u^{t,x,m}, M_u^{t,m}, Y_u^{t,x,m}, Z_u^{t,x,m}q_u^*)dC_u \\ & - \int_s^T dL_u^{t,x,m} + \frac{\kappa}{2}d\langle L^{t,x,m}, L^{t,x,m}\rangle_u, \quad s\in[t,T], \end{aligned} \tag{3.9}$$

where the forward process $X^{t,x,m}$ is a solution of (3.1). Again we suppose that the driver $f$ does not depend on $\Omega$ and hence is a deterministic Borel measurable function $f:[0,T]\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1}\times\mathbb{R}\times\mathbb{R}^{1\times d}\to\mathbb{R}$. If $F$ satisfies (H1) and $f$ hypotheses (H2) and (H3) then the BSDE (3.9) admits a unique solution $(Y^{t,x,m}, Z^{t,x,m}, L^{t,x,m})\in\mathbb{S}^\infty\times L^2(d\langle M,M\rangle\otimes d\mathbb{P})\times\mathcal{M}^2$ (see [13, Theorem 2.5]).

**Theorem 3.4.** *For $t\in[0,T]$ we assume that $M$ is a strong Markov process and that (H1)-(H3) hold. Then there exist deterministic functions $u:[0,T]\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1}\to\mathbb{R}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1})$-measurable and $v:[0,T]\times\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1}\to\mathbb{R}^{1\times d}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1}\times\mathbb{R}^{d\times 1})$-measurable such that*

$$Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m}), \quad Z_s^{t,x,m} = v(s, X_s^{t,x,m}, M_s^{t,m}), \quad s\in[t,T]. \tag{3.10}$$

**Remark 3.5.** *As mentioned in Remark 3.3, in the framework of Proposition 3.1 ii) and if the driver $f$ in* (3.9) *does not depend on $M$, Theorem 3.4 simplifies to: there exist deterministic functions $u : [0,T]\times\mathbb{R}^{n\times 1} \to \mathbb{R}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1})$-measurable and $v : [0,T]\times\mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times d}$, $\mathcal{B}([0,T])\otimes\mathcal{B}_e(\mathbb{R}^{n\times 1})$-measurable such that*

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad Z_s^{t,x} = v(s, X_s^{t,x}), \quad s \in [t,T].$$

*Proof of Theorem 3.4.* Existence and uniqueness of the solution of (3.9) under hypotheses (H1)-(H3) have been obtained in [13, Theorems 2.5 and 2.6]. More precisely it is shown in the proof of [13, Theorem 2.5] that the solution of a quadratic BSDE can be derived as the limit of solutions of a sequence of BSDE with Lipschitz generators. We follow this proof and begin by relaxing condition (H2). Indeed consider the following assumption (H2').

(H2') The generator $f$ is continuous in $(y,z)$ and there exists a predictable process $\eta$ such that $\eta \geq 0$ and $\int_0^T \eta_s dC_s \leq a$, where $a$ is a positive constant. Furthermore there exists a constant $\gamma > 0$ such that $dC_s \otimes d\mathbb{P}$-a.s.

$$|f(s,x,m,y,z)| \leq \eta_s + \frac{\gamma}{2}|z|^2, \quad \text{with } \gamma \geq |\kappa|, \quad (x,m,y,z) \in \mathcal{S}.$$

Assume that one can prove existence of a solution of (3.9) if $f$ satisfies (H2') instead of (H2). Let $f_K$ be the generator $f$ truncated in $Y$ at a well-chosen level $K$ (according to [13, Lemma 3.1]). More precisely, set $f_K(s,x,m,y,z) := f(s,x,m,\rho(y)_K,z)$ with

$$\rho_K(y) := \begin{cases} -K, & \text{if } y < -K, \\ y, & \text{if } |y| \leq K, \\ K, & \text{if } y > K. \end{cases}$$

It is shown in [13, proof of Theorem 2.5, Step 1] that $f_K$ satisfies (H2'). Hence by hypothesis there exists a triple of stochastic processes $(Y_K^{t,x,m}, Z_K^{t,x,m}, L_K^{t,x,m})$ which solves (3.9) with generator $f_K$. Due to a comparison argument and since $f_K$ and $f$ coincide along the sample paths of the solution $(Y_K^{t,x,m}, Z_K^{t,x,m}, L_K^{t,x,m})$, it also solves (3.9) with $f$ satisfying (H2). As a consequence our proof is finished if we show that (3.10) holds for a generator $f$ satisfying (H2').

The next step is to consider a BSDE which is equivalent to the BSDE (3.9) and is obtained via an exponential coordinate change. We only give a brief survey and refer to [13, proof of Theorem 2.5, Step 2] for a complete treatment. Setting $U^{t,x,m} := e^{\kappa Y^{t,x,m}}$ transforms (3.9) into the following BSDE

$$\begin{aligned} U_s^{t,x,m} =& e^{\kappa F(X_T^{t,x,m})} - \int_s^T V_r^{t,x,m} dM_r + \int_s^T g(r, X_r^{t,x,m}, M_r^{t,m}, U_r^{t,x,m}, V_r^{t,x,m} q_r^*) dC_r \\ & - \int_s^T dN_r^{t,x,m}, \quad s \in [t,T]. \end{aligned} \tag{3.11}$$

We refer to a solution of this BSDE as $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$. The new generator

$$g(s,x,m,u,v) := \left(\kappa u f\left(s,x,m,\frac{\ln u}{\kappa}, \frac{v}{\kappa u}\right) - \frac{1}{2u}|v|^2\right) 1_{]0,\infty[}(u),$$

$(x, m, u, v) \in \mathcal{S}$, satisfies (H2') and the triple $(Y^{t,x,m}, Z^{t,x,m}, L^{t,x,m})$ with

$$Y^{t,x,m} := \frac{\log(U^{t,x,m})}{\kappa}, \quad Z^{t,x,m} := \frac{V^{t,x,m}}{\kappa U^{t,x,m}}, \quad L^{t,x,m} := \frac{1}{\kappa U^{t,x,m}} \cdot N^{t,x,m} \tag{3.12}$$

is well defined and is solution to (3.9) with generator $f$ satisfying (H2).

To derive the existence of a solution of (3.11) an approximating sequence of BSDE with Lipschitz generator $g^p$ and terminal condition $e^{(\kappa F(X_T^{t,x,m}))}$ is introduced in such a way that $g^p$ converges $dC_s \otimes d\mathbb{P}$-almost surely to $g$ as $p$ tends to infinity. We do not specify the explicit expression for $g^p$, since it does not matter for our purposes, and besides the aforementioned properties we only need that the sequence is increasing in $y$, implying the same property for the solution component $(U^{p,t,x,m})_{p\in\mathbb{N}}$. For more details we refer to [13, proof of Theorem 2.5, step 3].

Let $p \geq 1$. We consider the BSDE (3.11) with generator $g^p$ and terminal condition $e^{(\kappa F(X_T^{t,x,m}))}$. Since $g^p$ is Lipschitz continuous we know from [9, Theorem 6.1] that a unique solution $(U^{p,t,x,m}, V^{p,t,x,m}, N^{p,t,x,m})$ exists. Now we can apply Proposition 3.2 which provides deterministic functions $a^p$ and $b^p$ such that

$$U_s^{p,t,x,m} = a^p(s, X_s^{t,x,m}, M_s^{t,m}) \text{ and } V_s^{p,t,x,m} = b^p(s, X_s^{t,x,m}, M_s^{t,m}), \quad s \in [t, T].$$

A subsequence, for convenience again denoted by $(U^{p,t,x,m}, V^{p,t,x,m}, N^{p,t,x,m})_{p\in\mathbb{N}}$, converges almost surely (with respect to $dC_s \otimes d\mathbb{P}$) to the solution $(U^{t,x,m}, V^{t,x,m}, N^{t,x,m})$ of (3.11). Letting

$$a(s, y, m) := \liminf_{p\to\infty} a^p(s, y, m), \quad b(s, y, m) := \liminf_{p\to\infty} b^p(s, y, m),$$

$(s, y, m) \in [0, T] \times \mathbb{R}^{d\times 1} \times \mathbb{R}^{n\times 1}$, we conclude that $U_s^{t,x,m} = a(s, X_s^{t,x,m}, M_s^{t,m})$ and $V_s^{t,x,m} = b(s, X_s^{t,x,m}, M_s^{t,m}), s \in [t, T]$. Since $(U^{p,t,x,m})_{p\in\mathbb{N}}$ is increasing, we may set

$$u := \frac{\ln a}{\kappa}, \quad v := \frac{b}{\kappa a}.$$

Hence the result follows by (3.12). □

# 4 Differentiability of FBSDE

In this section we derive differentiability of the FBSDE of (2.1)-(2.2) with respect to the initial data $x$ and $m$. The presence of the quantity $\langle L, L\rangle$ in the equation, where we recall that $L$ is part of the solution of (2.2), prevents us from extending directly the usual techniques presented for example in [2, 3, 4]. Under an additional assumption (MRP) defined in Section 4.2 we deduce the differentiability of (2.2) from the one of the auxiliary BSDE (4.1).

## 4.1 Differentiability of an auxiliary FBSDE

As mentioned above we first prove the differentiability of an auxiliary BSDE which will allow us to deduce the result for (2.2) in Section 4.2.

For every $(x,m) \in \mathbb{R}^{(n+d)\times 1}$ let us consider the following forward backward system of equations

$$\begin{aligned} X_t^{x,m} &= x + \int_0^t \sigma(r, X_r^{x,m}, M_r^m) dM_r + \int_0^t b(r, X_r^{x,m}, M_r^m) dC_r, \\ Y_t^{x,m} &= F(X_T^{x,m}) - \int_t^T Z_r^{x,m} dM_r + \int_t^T f(r, X_r^{x,m}, M_r^m, Y_r^{x,m}, Z_r^{x,m} q_r^*) dC_r, \end{aligned} \tag{4.1}$$

where $M$ is a continuous local martingale in $\mathbb{R}^{d\times 1}$ satisfying the martingale representation property and $C, q, \sigma, b, F, f$ are as described in Section 2. A solution of this system is given by the triple $(X^{x,m}, Y^{x,m}, Z^{x,m}) \in \mathcal{S}^p \times \mathcal{S}^\infty \times L^2(d\langle M, M\rangle \otimes d\mathbb{P})$ of stochastic processes. Note that the system (4.1) has a unique solution if the coefficients $\sigma$ and $b$ of the forward component satisfy (HSDE) and the terminal condition $F$ and the generator $f$ of the backward part satisfy (H1)-(H3).

In this section we will give sufficient conditions for the system (4.1) to be differentiable in $(x,m) \in \mathbb{R}^{(n+d)\times 1}$. Before turning to the backward SDE of the system we provide some material about the differentiability of the forward component obtained in [17, V.7].

**Proposition 4.1.** *Assume that $\sigma$ and $b$ satisfy (D1). Then for almost all $\omega \in \Omega$ there exists a solution $X^{x,m}(\omega)$ of (4.1) which is continuously differentiable in $x$ and $m$. In addition the derivatives $D^x_{ik} := \frac{\partial}{\partial x_k} X^{(i)x,m}$, $i,k = 1,\dots n$, and $D^m_{ik} := \frac{\partial}{\partial m_k} X^{(i)x,m}$, $i = 1,\dots n$, $k = 1,\dots d$, satisfy the following SDE for $t \in [0,T]$*

$$\begin{aligned} D^x_{ikt} =& \delta_{ik} + \sum_{\alpha=1}^d \sum_{j=1}^n \int_0^t \partial_{1+j}\sigma_{i\alpha}(s, X_s^{x,m}, M_s^m) D^x_{jks} dM_s^{(\alpha)} \\ &+ \sum_{j=1}^n \int_0^t \partial_{1+j} b^{(i)}(s, X_s^{x,m}, M_s^m) D^x_{jks} dC_s, \end{aligned} \tag{4.2}$$

$$\begin{aligned} D^m_{ikt} =& \sum_{\alpha=1}^d \sum_{j=1}^n \int_0^t \partial_{1+j}\sigma_{i\alpha}(s, X_s^{x,m}, M_s^m) D^m_{jks} dM_s^{(\alpha)} + \sum_{j=1}^n \int_0^t \partial_{1+j} b^{(i)}(s, X_s^{x,m}, M_s^m) D^m_{jks} dC_s \\ &+ \sum_{\alpha=1}^d \int_0^t \partial_{1+n+k}\sigma_{i\alpha}(s, X_s^{x,m}, M_s^m) dM_s^{(\alpha)} + \int_0^t \partial_{1+n+k} b^{(i)}(s, X_s^{x,m}, M_s^m) dC_s \end{aligned} \tag{4.3}$$

*and $\frac{\partial}{\partial m_k} M^{(j)m} = \delta_{kj}$, $k,j = 1,\dots,d$. Furthermore for all $p > 1$ there exists a positive constant $C$ such that the following estimate holds*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^{x,m} - X_t^{x',m'}|^{2p}\right] \le C(|x-x'|^2 + |m-m'|^2)^p. \tag{4.4}$$

*Proof.* Let $\left(\tilde{X}_t^{x,m}\right)_{t\in[0,T]}$ be the stochastic process with values in $\mathbb{R}^{(1+n+d)\times 1}$ defined as

$$\tilde{X}_t^{x,m} = \begin{pmatrix} t \\ X_t^{x,m} \\ M_t^m \end{pmatrix}.$$

This process is the solution of the SDE

$$d\tilde{X}_t^{x,m} = \tilde{\sigma}(\tilde{X}_t^{x,m})d\tilde{M}_t,\ \tilde{X}_0^{x,m} = (0,x,m)$$

with

$$\tilde{\sigma}(\tilde{X}_t^{x,m}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma(\tilde{X}_t^{x,m}) & b(\tilde{X}_t^{x,m}) \\ 0 & I_d & 0 \end{pmatrix},\ \tilde{M}_t = \begin{pmatrix} t \\ M_t \\ C_t \end{pmatrix}.$$

According to [17, Theorem V.39] the derivatives $D^x$, $D^m$ and $\frac{\partial}{\partial m_k} M^{(j)m}$, $k,j = 1,\dots,d$, exist and are continuous in $x$ and $m$. In addition, formula [17, (D) p. 312] leads to (4.2) and (4.3). The estimate (4.4) follows immediately from [17, $(***)$, p. 309]. □

We now focus on the BSDE of system (4.1). Let $\tilde{x} := (x,m) \in \mathbb{R}^{(n+d)\times 1}$ and $e_i$, $i = 1,...,n+d$, the unit vectors in $\mathbb{R}^{(n+d)\times 1}$. For all $\tilde{x}$, $h \neq 0$ and $i \in \{1,\dots,n+d\}$ let $\xi^{\tilde{x},h,i} = \frac{1}{h}(F(X_T^{\tilde{x}+he_i}) - F(X_T^{\tilde{x}}))$. The following Lemma will be needed later in order to prove the differentiability of the backward component. To simplify the notation we suppress the superscript $i$ from now on.

**Lemma 4.2.** *Let (D1) and (D2) be satisfied. Then for every $p > 1$ there exists a constant $C > 0$, such that for all $\tilde{x}, \tilde{x}' \in \mathbb{R}^{(n+d)\times 1}$, $h, h' \neq 0$*

$$\mathbb{E}\left[|\xi^{\tilde{x},h} - \xi^{\tilde{x}',h'}|^{2p}\right] \leq C(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2)^p \tag{4.5}$$

*Proof.* We know from Itô's formula

$$\begin{aligned} F(X_t^{\tilde{x}}) =& F(X_0^{\tilde{x}}) + \int_0^t \nabla F(X_s^{\tilde{x}})\sigma(s, X_s^{\tilde{x}}, M_s^m)dM_s \\ &+ \int_0^t \left( \sum_{i=1}^n \partial_i F(X_s^{\tilde{x}}) b_i(s, X_s^{\tilde{x}}, M_s^m) \right. \\ &\left. + \frac{1}{2}\sum_{i,j=1}^n \partial_{ij} F(X_s^{\tilde{x}})[(\sigma q)\cdot(\sigma q)^*]_{ij}(s, X_s^{\tilde{x}}, M_s^m) \right) dC_s. \end{aligned}$$

Hence $F(X_t^{\tilde{x}})$ is a SDE with coefficients $\hat{\sigma}(s,x,m) = \nabla F(x)\sigma(s,x,m)$ and $\hat{b}(s,x,m) = \sum_{i=1}^n \partial_i F(x) b_i(s,x,m) + \frac{1}{2}\sum_{i,j=1}^n \partial_{ij} F(x)[(\sigma q)\cdot(\sigma q)^*]_{ij}(s,x,m)$. By (D2) the coefficients $\hat{\sigma}$ and $\hat{b}$ have locally Lipschitz partial derivatives and thus with (4.4) the estimate (4.5) follows. □

The next Lemma shows that we can choose the family $(Y^{\tilde{x}})$ to be continuous in $\tilde{x} \in \mathbb{R}^{(n+d)\times 1}$.

**Lemma 4.3.** *Let (H1)-(H3) and (D1)-(D3) be satisfied. Then for all $p > 1$ there exists a constant $C > 0$, such that for all $\tilde{x}, \tilde{x}' \in \mathbb{R}^{(n+d)\times 1}$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{\tilde{x}} - Y_t^{\tilde{x}'}|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T |q_t(Z_t^{\tilde{x}} - Z_t^{\tilde{x}'})^*|^2 dC_t\right)^p\right] \leq C|\tilde{x}-\tilde{x}'|^{2p}. \tag{4.6}$$

*Furthermore for almost all $\omega \in \Omega$ there exists a solution $Y^{\tilde{x}}(\omega)$ of (4.1) which is continuous in $\tilde{x} \in \mathbb{R}^{(n+d)\times 1}$.*

*Proof.* Let $\delta Y := Y^{\tilde{x}} - Y^{\tilde{x}'}$, $\delta Z := Z^{\tilde{x}} - Z^{\tilde{x}'}$, $\delta M := M^m - M^{m'}$ and $\delta X := X^{\tilde{x}} - X^{\tilde{x}'}$. We also set for $s \in [0,T]$

$$\begin{aligned}
A_r^Z &:= \int_0^1 \partial_5 f(r, X_r^{\tilde{x}}, M_r^m, Y_r^{\tilde{x}}, Z_r^{\tilde{x}'} q_r^* + \zeta(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'}) q_r^*) d\zeta \\
A_r^Y &:= \int_0^1 \partial_4 f(r, X_r^{\tilde{x}}, M_r^m, Y_r^{\tilde{x}'} + \zeta(Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}), Z_r^{\tilde{x}'} q_r^*) d\zeta \\
A_r^M &:= \int_0^1 \partial_3 f(r, X_r^{\tilde{x}}, M_r^{m'} + \zeta(M_r^m - M_r^{m'}), Y_r^{\tilde{x}'}, Z_r^{\tilde{x}'} q_r^*) d\zeta \\
A_r^X &:= \int_0^1 \partial_2 f(r, X_r^{\tilde{x}'} + \zeta(X_r^{\tilde{x}} - X_r^{\tilde{x}'}), M_r^{m'}, Y_r^{\tilde{x}'}, Z_r^{\tilde{x}'} q_r^*) d\zeta.
\end{aligned}$$

Considering the difference $\delta Y$ of the backward component in (4.1) we see that for $t \in [0,T]$

$$\begin{aligned}
\delta Y_t =& F(X_T^{\tilde{x}}) - F(X_T^{\tilde{x}'}) - \int_t^T \delta Z_r dM_r \\
&+ \int_t^T [f(r, X_r^{\tilde{x}}, M_r^m, Y_r^{\tilde{x}}, Z_r^{\tilde{x}} q_r^*) - f(r, X_r^{\tilde{x}'}, M_r^{m'}, Y_r^{\tilde{x}'}, Z_r^{\tilde{x}'} q_r^*)] dC_r \\
=& F(X_T^{\tilde{x}}) - F(X_T^{\tilde{x}'}) - \int_t^T \delta Z_r dM_r + \int_t^T \underbrace{(\delta Z_r q_r^* A_r^Z + \delta Y_r A_r^Y + \delta M_r^* A_r^M + \delta X_r^* A_r^X)}_{=: g(r, \delta Y_r, \delta Z_r q_r^*)} dC_r
\end{aligned}$$

holds. Note that $(\delta Y, \delta Z)$ can be seen as a BSDE whose generator $g$ satisfies (H4) and whose terminal condition $F(X_T^{\tilde{x}}) - F(X_T^{\tilde{x}'})$ is bounded (see (H1)). More precisely we derive with (D3) and [13, Lemma 3.1]the existence of a constant $C$ such that for all $y, y_1, y_2 \in \mathbb{R}$ and $z, z_1, z_2 \in \mathbb{R}^{1\times d}$ $dC_r \otimes d\mathbb{P}$-a.s.

$$\begin{aligned}
|g(r,y,z_1) - g(r,y,z_2)| \leq& |A_r^Z||z_1 - z_2| \\
\leq& C(|q_r \theta_r| + |Z_r^{\tilde{x}'} q_r^*| + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'}) q_r^*|)|z_1 - z_2| \quad \text{and} \\
|g(r,y_1,z) - g(r,y_2,z)| \leq& |A_r^M||y_1 - y_2| \\
\leq& C(|q_r \theta_r| + |Z_r^{\tilde{x}'} q_r^*|)|y_1 - y_2|.
\end{aligned}$$

Hence we can apply the apriori estimates of Lemma A.1 and hence we know that for every $p > 1$ there exist constants $q > 1$ and $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |\delta Y_t|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T |q_t \delta Z_t^*|^2 dC_t\right)^p\right] \\
&\leq C\mathbb{E}\left[|F(X_T^{\tilde{x}}) - F(X_T^{\tilde{x}'})|^{2pq} + \left(\int_0^T |\delta M_r^* A_r^M + \delta X_r^* A_r^X| dC_r\right)^{2pq}\right]^{\frac{1}{q}}.
\end{aligned} \tag{4.7}$$

By condition (D3) and Hölder's inequality we get

$$\mathbb{E}\left[\left(\int_0^T |\delta M_r^* A_r^M + \delta X_r^* A_r^X| dC_r\right)^{2pq}\right]$$

$$\leq C\mathbb{E}\left[\left(\int_0^T |\delta M_r|^2 dC_r\right)^{2pq}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T (|q_r\theta_r| + |Z_r^{\tilde{x}'} q_r^*|)^2 dC_r\right)^{2pq}\right]^{\frac{1}{2}}$$
$$+ C\mathbb{E}\left[\left(\int_0^T |\delta X_r|^2 dC_r\right)^{2pq}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T (|q_r\theta_r| + |Z_r^{\tilde{x}'} q_r^*|)^2 dC_r\right)^{2pq}\right]^{\frac{1}{2}}$$

Note that $\mathbb{E}\left[\left(\int_0^T |q_r\theta_r|^2 dC_r\right)^{2pq}\right]$ is bounded by (D3). Furthermore $\mathbb{E}\left[\left(\int_0^T |Z_r^{\tilde{x}'} q_r^*|^2 dC_r\right)^{2pq}\right]$ is bounded, as is seen by applying Lemma A.1. Hence

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T |\delta M_r^* A_r^M + \delta X_r^* A_r^X| dC_r\right)^{2pq}\right] \leq & C|m-m'|^{2pq} + C\mathbb{E}\left[\left(\sup_{t\in[0,T]} |\delta X_t|^2 C_T\right)^{2pq}\right]^{\frac{1}{2}} \\ \leq & C\left(|m-m'|^{2pq} + |\tilde{x}-\tilde{x}'|^{2pq}\right),\end{aligned}$$

where the last inequality is due to (4.4). Combining (4.7), condition (D2) and the last inequality we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\delta Y_t|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T |q_s \delta Z_s^*|^2\right)^p\right] \leq C|\tilde{x}-\tilde{x}'|^{2p}.$$

Now Kolmogorov's Lemma (see Theorem 73, Chapter IV in [17]) implies that there exists a version of $(Y^{\tilde{x}})$ which is continuous in $\tilde{x}$ for almost all $\omega \in \Omega$. □

For all $h \neq 0, \tilde{x} \in \mathbb{R}^{(n+d)\times 1}, t \in [0,T]$ let $U_t^{\tilde{x},h} = \frac{1}{h}(Y_t^{\tilde{x}+he_i} - Y_t^{\tilde{x}})$, $V_t^{\tilde{x},h} = \frac{1}{h}(Z_t^{\tilde{x}+he_i} - Z_t^{\tilde{x}})$, $\Delta_t^{\tilde{x},h} = \frac{1}{h}(X_t^{\tilde{x}+he_i} - X_t^{\tilde{x}})$, $\varpi_t^{\tilde{x},h} = \frac{1}{h}(M_t^{\tilde{x}+he_i} - M_t^{\tilde{x}})$ (where it is implicit that $M^{\tilde{x}}$ depends only on the component $m$ of $\tilde{x} = (x,m)$), and $\xi^{\tilde{x},h} = \frac{1}{h}(F(X_T^{\tilde{x}+he_i}) - F(X_T^{\tilde{x}}))$. We define $\delta U$ by $\delta U = U^{\tilde{x},h} - U^{\tilde{x}',h'}$ and the processes $\delta V$, $\delta\Delta$, $\delta\varpi$, and $\delta\xi$ in an analogous way. We give estimates on the differences of difference quotients of the family $(Y^{\tilde{x}})$.

**Lemma 4.4.** *Let (H1)-(H3) and (D1)-(D3) be satisfied. Then for each $p > 1$ there exists a constant $C > 0$ such that for any $\tilde{x}, \tilde{x}' \in \mathbb{R}^{(n+d)\times 1}$ and $h, h' \neq 0$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |U_t^{\tilde{x},h} - U_t^{\tilde{x}',h'}|^{2p}\right] \leq C(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2)^p. \tag{4.8}$$

*Proof.* This proof is similar to the one of Lemma 4.3. By definition of $U^{\tilde{x},h}$ and of $U^{\tilde{x}',h'}$ we have

$$\begin{aligned}U_t^{\tilde{x},h} = & \xi^{\tilde{x},h} - \int_t^T V_r^{\tilde{x},h} dM_r \\ & + \int_t^T \frac{1}{h}\left[f(r, X_r^{\tilde{x}+he_i}, M_r^{\tilde{x}+he_i}, Y_r^{\tilde{x}+hei}, Z_r^{\tilde{x}+he_i} q_r^*) - f(r, X_r^{\tilde{x}}, M_r^{\tilde{x}}, Y_r^{\tilde{x}}, Z_r^{\tilde{x}} q_r^*)\right] dC_r.\end{aligned} \tag{4.9}$$

As in the proof of Lemma 4.3 we decompose the integrand in the last term of the RHS of the equality above by writing

$$\begin{aligned}&\frac{1}{h}(f(r,X_r^{\tilde{x}+he_i},M_r^{\tilde{x}+he_i},Y_r^{\tilde{x}+hei},Z_r^{\tilde{x}+he_i}q_r^*)-f(r,X_r^{\tilde{x}},M_r^{\tilde{x}},Y_r^{\tilde{x}},Z_r^{\tilde{x}}q_r^*))\\ =&V_t^{\tilde{x},h}q_r^*(A^Z)_r^{\tilde{x},h}+U_t^{\tilde{x},h}(A^Y)_r^{\tilde{x},h}+\varpi_r^{\tilde{x},h*}(A^M)_r^{\tilde{x},h}+\Delta^{\tilde{x},h}(A^X)_r^{\tilde{x},h},\end{aligned}$$

where $A^Z, A^Y, A^M, A^X$ are defined as in the proof of Lemma 4.3, for instance

$$(A^Z)_r^{\tilde{x},h}:=\int_0^1\partial_5 f(r,X_r^{\tilde{x}+he_i},M_r^{\tilde{x}+he_i},Y_r^{\tilde{x}+he_i},Z_r^{\tilde{x}}q_r^*+\theta(Z_r^{\tilde{x}+he_i}-Z_r^{\tilde{x}})q_r^*)d\theta.$$

Taking the difference of two equations of the form (4.9) we obtain that $(\delta U,\delta V)$ satisfies the BSDE

$$\begin{aligned}\delta U_t=&\delta\xi-\int_0^T\delta V_r dM_t+\int_t^T\delta V_r q_r^*(A^Z)_r^{\tilde{x},h}+\delta U_r(A^Y)_r^{\tilde{x},h}+\Big[q_r^*((A^Z)_r^{\tilde{x},h}-(A^Z)_r^{\tilde{x}',h'})V_r^{\tilde{x}',h'}\\&+U_r^{\tilde{x}',h'}((A^Y)_r^{\tilde{x},h}-(A^Y)_r^{\tilde{x}',h'})+\varpi_r^{\tilde{x},h*}(A^M)_r^{\tilde{x},h}-\varpi_r^{\tilde{x}',h'*}(A^M)_r^{\tilde{x}',h'}\\&+\Delta^{\tilde{x},h*}(A^X)_r^{\tilde{x},h}-\Delta^{\tilde{x}',h'*}(A^X)_r^{\tilde{x}',h'}\Big]dC_r.\end{aligned}\tag{4.10}$$

The generator of this BSDE satisfies condition (H4) due to assumption (D3) (details are similar to those of the proof of Lemma 4.3 and are left to the reader). By Lemma A.1 for every $p>1$ there exist constants $q>1$ and $C>0$ such that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}|\delta U_t|^{2p}+\left(\int_0^T|q_s\delta V_s^*|^2dC_s\right)^p\right]\\&\le C\mathbb{E}\Big[|\delta\xi|^{2pq}+\left(\int_0^T|q_r^*((A^Z)_r^{\tilde{x},h}-(A^Z)_r^{\tilde{x}',h'})||V_r^{\tilde{x}',h'}|+|U_r^{\tilde{x}',h'}||((A^Y)_r^{x,h}-(A^Y)_r^{\tilde{x}',h'})|\right.\\&\quad\left.+|\varpi_r^{\tilde{x},h*}(A^M)_r^{\tilde{x},h}-\varpi_r^{\tilde{x}',h'*}(A^M)_r^{\tilde{x}',h'}|+|\Delta^{\tilde{x},h*}(A^X)_r^{\tilde{x},h}-\Delta^{\tilde{x}',h'*}(A^X)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\Big]^{1/q}.\end{aligned}$$

We estimate separately each part of the right hand side of the inequality presented. First, by Cauchy-Schwarz' inequality we have

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^T|q_r^*((A^Z)_r^{\tilde{x},h}-(A^Z)_r^{\tilde{x}',h'})||V_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right]\\&\le\mathbb{E}\left[\left(\int_0^T|q_r^*((A^Z)_r^{\tilde{x},h}-(A^Z)_r^{\tilde{x}',h'})|^2dC_r\right)^{2pq}\right]^{1/2}\mathbb{E}\left[\left(\int_0^T|V_r^{\tilde{x}',h'}|^2dC_r\right)^{2pq}\right]^{1/2}\\&\le C\mathbb{E}\left[\left(\int_0^T|q_r^*((A^Z)_r^{\tilde{x},h}-(A^Z)_r^{\tilde{x}',h'})|^2dC_r\right)^{2pq}\right]^{1/2}\end{aligned}$$

since $\mathbb{E}\left[\left(\int_0^T |V_r^{\tilde{x}',h'}|^2 dC_r\right)^{2pq}\right]$ is bounded by Lemma A.1. Then hypothesis (D4) and a combination of Lemma 4.3 and (4.4) lead to the following estimate, in which the constant $C$ may change from line to line

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |q_r^*((A^Z)_r^{\tilde{x},h} - (A^Z)_r^{\tilde{x}',h'})||V_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right] \\
&\leq C\mathbb{E}\Bigg[\Bigg(\int_0^T |q_r^*(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'})|^2 + |X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}|^2 + |\underbrace{M_r^{\tilde{x}+he_i} - M_r^{\tilde{x}'+h'e_i}}_{=\tilde{x}+he_i-\tilde{x}'-h'e_i}|^2 \\
&\quad + |Y_r^{\tilde{x}+he_i} - Y_r^{\tilde{x}'+h'e_i}|^2 + |q_r^*(Z_r^{\tilde{x}+he_i} - Z_r^{\tilde{x}'+h'e_i})|^2 dC_r\Bigg)^{2pq}\Bigg]^{1/2} \\
&\leq C\left(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2\right)^{pq}.
\end{aligned}$$

Similarly we derive the estimate

$$\mathbb{E}\left[\left(\int_0^T |U_r^{\tilde{x}',h'}||(A^Y)_r^{x,h} - (A^Y)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right] \leq C\left(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2\right)^{pq}.$$

We next estimate, with constants $C$ allowed to change from line to line

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |\varpi_r^{\tilde{x},h*}(A^M)_r^{\tilde{x},h} - \varpi_r^{\tilde{x}',h'*}(A^M)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^T |\underbrace{\varpi_r^{\tilde{x},h} - \varpi_r^{\tilde{x}',h'}}_{=0}||(A^M)_r^{\tilde{x},h}|dC_r\right)^{2pq}\right] + C\mathbb{E}\left[\left(\int_0^T |\underbrace{\varpi_r^{\tilde{x}',h'}}_{=e_i}||(A^M)_r^{\tilde{x},h} - (A^M)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^T |(A^M)_r^{\tilde{x},h} - (A^M)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^T (|q_r\theta_r| + |Z_r^{\tilde{x}}q_s^*| + |Z_r^{\tilde{x}'}q_s^*|)^2 dC_r\right)^{2pq}\right]^{1/2} \mathbb{E}\Bigg[\Bigg(\int_0^T (|X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}| \\
&\quad + |Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}| + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'})q_r^*| + |M_r^{\tilde{x}} - M_r^{\tilde{x}'}| + |M_r^{\tilde{x}+he_i} - M_r^{\tilde{x}'+h'e_i}|)^2 dC_r\Bigg)^{2pq}\Bigg]^{1/2},
\end{aligned}$$

where the last inequality is due to hypotheses (D4) and Hölder's inequality. An application of the a priori estimates from Lemma A.1 implies that $\mathbb{E}\left[\left(\int_0^T (|q_r\theta_r| + |Z_r^{\tilde{x}}q_r^*| + |Z_r^{\tilde{x}'}q_r^*|)^2 dC_r\right)^{2pq}\right]$ is bounded. Then with the estimates (4.4) and (4.6) we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |\varpi_r^{\tilde{x},h*}(A^M)_r^{\tilde{x},h} - \varpi_r^{\tilde{x}',h'*}(A^M)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right] \\
&\leq C\mathbb{E}\Bigg[\Bigg(\int_0^T |X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}|^2 + |Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}|^2 + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'})q_r^*|^2
\end{aligned}$$

$$+|M_r^{\tilde{x}} - M_r^{\tilde{x}'}|^2 + |M_r^{\tilde{x}+he_i} - M_r^{\tilde{x}'+h'e_i}|^2 dC_r\Big)^{2pq}\Big]^{1/2}$$
$$\leq C(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2)^{pq}.$$

We now consider the last term whose treatment is quite similar to the one of the term just discussed. We therefore give the main computations without providing detailed arguments. We have, with constants $C$ the value of which may change from line to line

$$\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h*}(A^X)_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'*}(A^X)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right]$$
$$\leq C\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'}|^2 dC_r\right)^{2pq}\right]^{1/2} \mathbb{E}\left[\left(\int_0^T |(A^X)_r^{\tilde{x},h}|^2 dC_r\right)^{2pq}\right]^{1/2}$$
$$+ C\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x}',h'}||(A^X)_r^{\tilde{x},h} - (A^X)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right].$$

Using (D3) and Lemma A.1 we deduce that $\mathbb{E}\left[\left(\int_0^T |(A^X)_r^{\tilde{x},h}|^2 dC_r\right)^{2pq}\right]$ is bounded. Using hypothesis (D4) and estimate (4.4) again we obtain

$$\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h*}(A^X)_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'*}(A^X)_r^{\tilde{x}',h'}|dC_r\right)^{2pq}\right]$$
$$\leq C\mathbb{E}\left[\left(\int_0^T |\Delta_r^{\tilde{x},h} - \Delta_r^{\tilde{x}',h'}|^2 dC_r\right)^{2pq}\right]^{1/2}$$
$$+ C\mathbb{E}\left[\left(\int_0^T \left(|q_r\theta_r| + |Z_r^{\tilde{x}'}q_r^*| + |Z_r^{\tilde{x}}q_r^*|\right)\left(|X_r^{\tilde{x}} - X_r^{\tilde{x}'}| + |X_r^{\tilde{x}+he_i} - X_r^{\tilde{x}'+h'e_i}| + |M_r^{\tilde{x}} - M_r^{\tilde{x}'}|\right.\right.\right.$$
$$\left.\left.\left. + |Y_r^{\tilde{x}} - Y_r^{\tilde{x}'}| + |(Z_r^{\tilde{x}} - Z_r^{\tilde{x}'})q_r^*|\right) dC_r\right)^{2pq}\right]$$
$$\leq C(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2)^{pq}.$$

We derive the estimate

$$\mathbb{E}\left[|\delta\xi|^{2pq}\right] \leq C(|\tilde{x}-\tilde{x}'|^2 + |h-h'|^2)^{pq}$$

immediately from (4.5). This completes the proof of (4.8). □

**Proposition 4.5.** *Let (H1)-(H3) and (D1)-(D4) be satisfied. Then there exists a solution $(X^{\tilde{x}}, Y^{\tilde{x}}, Z^{\tilde{x}})$ of (4.1), such that $X^{\tilde{x}}(\omega)$ and $Y^{\tilde{x}}(\omega)$ are continuously differentiable in $\tilde{x} \in \mathbb{R}^{(n+d)\times 1}$ for almost all $\omega \in \Omega$. Furthermore there exist processes $\frac{\partial}{\partial x}Z^{x,m}, \frac{\partial}{\partial m}Z^{x,m} \in L^2(d\langle M,M\rangle \otimes d\mathbb{P})$ such that the derivatives $(U_k^x, V_{ik}^x) := (\frac{\partial}{\partial x_k}Y^{x,m}, \frac{\partial}{\partial x_k}Z^{(i),x,m})$, $i = 1,\dots,d$, $k = 1,\dots,n$, and $(U_k^m, V_{ik}^m) := (\frac{\partial}{\partial m_k}Y^{x,m}, \frac{\partial}{\partial m_k}Z^{(i),x,m})$, $i,k = 1,\dots,d$, solve the following*

*BSDE for* $t \in [0,T]$

$$
\begin{aligned}
U^x_{kt} = &\sum_{j=1}^{n} \partial_j F(X^{x,m}_T) D^x_{jkT} - \sum_{\alpha=1}^{d} \int_t^T V^x_{i\alpha s} dM^{(\alpha)}_s \\
&+ \sum_{j=1}^{n} \int_t^T \partial_{1+j} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s q^*_s) D^x_{jks} dC_s \\
&+ \int_t^T \partial_{1+n+d+1} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s q^*_s) U^x_{ks} dC_s \\
&+ \sum_{j=1}^{n} \int_t^T \partial_{1+n+d+1+j} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s q^*_s) \frac{\partial}{\partial x_k} q_{jks} Z^{(j),x,m} dC_s,
\end{aligned}
$$

$$
\begin{aligned}
U^m_{kt} = &\sum_{j=1}^{d} \partial_{n+j} F(X^{x,m}_T) D^m_{jkT} - \sum_{\alpha=1}^{d} \int_t^T V^m_{i\alpha s} dM^{(\alpha)}_s \\
&+ \sum_{j=1}^{n} \int_t^T \partial_{1+j} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s) D^m_{jks} dC_s \\
&+ \int_t^T \partial_{1+n+k} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s q^*_s) dC_s \\
&+ \int_t^T \partial_{1+n+d+1} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s q^*_s) U^m_{ks} dC_s \\
&+ \sum_{j=1}^{n} \int_t^T \partial_{1+n+d+1+j} f(s, X^{x,m}_s, M^m_s, Y^{x,m}_s, Z^{x,m}_s q^*_s) q_{jks} V^m_{jks} dC_s.
\end{aligned}
$$

*Proof.* From 4.4 and Kolmogorov's Lemma (see Theorem 73, Chapter IV in [17]) we deduce that there exists a family of solutions $(Y^{\tilde{x}})$ of (4.1) which is continuously differentiable in $\tilde{x}$ for almost all $\omega \in \Omega$. Finally from equation (4.10) taking $h \to 0$ the BSDE follows. □

## 4.2 Differentiability of FBSDE

Now we come back to the initial FBSDE (2.1)-(2.2). In order to obtain the differentiability of this system we require the following additional assumption:

**(MRP)** There exists a continuous $L^2$-martingale $N := (N_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ strongly orthogonal to $M$ ($\langle M^i, N\rangle = 0$, $i = 1, \dots, d$) such that the pair $(M, N)$ has the martingale representation property.

The presence of the additional bracket $\langle L, L\rangle$ in the BSDE prevents us from applying the usual techniques for differentiability as for example in [2, 3, 4]. More precisely, under (MRP) we can show that the BSDE in (2.2) can be written as

$$
Y^{x,m}_t = F(X^{x,m}_T) - \int_t^T Z^{x,m}_r dM_r - \int_t^T U^{x,m}_r dN_r + \int_t^T h(r, X^{x,m}_r, M^m_r, Y^{x,m}_r, Z^{x,m}_r q^*_r, U^{x,m}_r) d\tilde{C}_r, \tag{4.11}
$$

$t \in [0,T]$, where $\tilde{C}$ and $h$ are defined as in section A.1. Due to hypotheses (MRP) and the orthogonality of the martingales $L$ and $M$, the representation of $L$ as $L = U \cdot N$ where $U$ is a predictable square integrable stochastic process is obtained. So the solution $(Y,Z,L)$ of the backward part (2.2) becomes $(Y,Z,U)$ in (4.11). The bracket $\langle L,L\rangle$ is then a component of the new generator $h$, which is quadratic in $U$. We refer to the beginning of Section A.1, where a discussion of the technical aspects of this claim is given. Now we can write the system (2.1)-(2.2) as

$$X_t^{x,m} = x + \int_0^t \tilde{\sigma}(s, X_s^{t,x,m}, M_s^{t,m}) d\tilde{M}_s + \int_0^t \tilde{b}(s, X_s^{x,m}, M_s^m) d\tilde{C}_s,$$
$$Y_t^{x,m} = F(X_T^{x,m}) - \int_t^T \tilde{Z}_s^{x,m} d\tilde{M}_s + \int_t^T h(s, X_s^{x,m}, M_s^m, Y_s^{x,m}, \tilde{Z}_s^{x,m} \tilde{q}_s^*) d\tilde{C}_s,$$

$t \in [0,T]$, where $\tilde{M}$, $\tilde{q}$, $\tilde{Z}$, are defined as in Section A.1 and $\tilde{\sigma} := \begin{pmatrix} \sigma & 0 \end{pmatrix}$, $\tilde{b} := b \times \frac{1+\left(\sum_{i=1}^d \langle M^i, M^i\rangle_s + \langle N,N\rangle_s\right)^2}{1+\left(\sum_{i=1}^d \langle M^i, M^i\rangle_s\right)^2}$. A solution $(X^{x,m}, Y^{x,m}, \tilde{Z}^{x,m}) \in \mathcal{S}^p \times \mathcal{S}^\infty \times L^2(d\langle \tilde{M}, \tilde{M}\rangle \otimes d\mathbb{P})$ of this system exists for $\sigma$, $b$ satisfying (HSDE) and $F$, $h$ satisfying (H1)-(H3). Therefore we obtain the following result.

**Theorem 4.6.** *Let (H1)-(H3) and (D1)-(D4) be satisfied. Then there exists a solution $(X^{\tilde{x}}, Y^{\tilde{x}}, \tilde{Z}^{\tilde{x}})$ of (2.1)-(2.2), such that $X^{\tilde{x}}(\omega)$ and $Y^{\tilde{x}}(\omega)$ are continuously differentiable in $\tilde{x} \in \mathbb{R}^{(n+d)\times 1}$ for almost all $\omega \in \Omega$.*

*Proof.* The Theorem follows from Proposition 4.5. □

# 5 Representation formula

In this Section we provide the representation formula (1.3) which generalizes the one obtained in [2, 3], where $M$ is a Brownian motion. We recall that in the Gaussian setting the proof of this formula is based on the representation of the stochastic process $Z$ as the trace of the Malliavin derivative of $Y$. In the general martingale setting of this paper, however, Malliavin's calculus is not available. Therefore we propose a new proof based on stochastic calculus techniques. We also stress that the last term in formula (1.3) vanishes if we assume that $M$ has independent increments, $\sigma$ and $b$ do not depend on $M$ in (2.1) and that the driver $f$ in (2.2) is free of $M$.

**Proposition 5.1.** *Let $u$ be as in (3.10), that is, $Y_s^{t,x,m} = u(s, X_s^{t,x,m}, M_s^{t,m})$, $s \in [t,T]$. Under the hypotheses of Theorem 3.4 and Theorem 4.6 we have*

*i)* $x \mapsto u(t,x,m) \in \mathscr{C}^1(\mathbb{R}^{n\times 1})$, $(t,m) \in [0,T] \times \mathbb{R}^{d\times 1}$,

*ii)* $m \mapsto u(t,x,m) \in \mathscr{C}^1(\mathbb{R}^{d\times 1})$, $(t,x) \in [0,T] \times \mathbb{R}^{n\times 1}$.

*Proof.*

i) Fix $(t,m)$ in $[0,T]\times\mathbb{R}^{d\times 1}$. As already mentioned, $Y_t^{t,x,m}$ is deterministic and $u(t,x,m) = Y_t^{t,x,m}$. By differentiability of $Y^{t,x,m}$ with respect to $x$ (Theorem 4.6), we obtain that $x \mapsto u(t,x,m)$ belongs to $\mathscr{C}^1(\mathbb{R}^{n\times 1})$.

ii) The proof is similar to *i)*.

□

We present the main result of this paper. We introduce the following assumption (ACL) which will be used in the proof of Theorem 5.2.

**(ACL)** For every $t \in [0,T]$ and $h > 0$ the law of the vector $(M_t, X_t, M_{t+h} - M_t, X_{t+h} - X_t)$ has an absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}^{d\times 1} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \times \mathbb{R}^{n\times 1}$.

**Theorem 5.2.** *Let the assumptions of Theorem 3.4 and Theorem 4.6 be in force. Assume in addition that hypothesis (ACL) is satisfied. Then for all $s \in [t,T]$ we have $dC_s \otimes d\mathbb{P} - a.e$*

$$Z_s^{t,x,m} = \partial_2 u(s, X_s^{t,x,m}, M_s^{t,m})\sigma(s, X_s^{t,x,m}, M_s^{t,m}) + \partial_3 u(s, X_s^{t,x,m}, M_s^{t,m}), \tag{5.1}$$

*where* $\partial_2 u(s, X_s^{t,x,m}, M_s^{t,m}) := \left(\partial_{1+j} u(s, X_s^{t,x,m}, M_s^{t,m})\right)_{1\le j\le n}$ *and* $\partial_3 u(s, X_s^{t,x,m}, M_s^{t,m}) := \left(\partial_{1+n+j} u(s, X_s^{t,x,m}, M_s^{t,m})\right)_{1\le j\le d}$.

**Remark 5.3.** *i) The technique used in the following proof is independent on the assumptions made in Section 3 and 4. More precisely only the validity of Proposition 5.1 is required to prove Theorem 5.2.*

*ii) An interesting particular case of Theorem 5.2 is given when $X$ and $M$ are like in Proposition 3.1 ii) and when $f$ in* (3.9) *does not depend on $M$. In this situation equation* (5.1) *becomes:*

$$Z_s^{t,x} = \partial_2 u(s, X_s^{t,x})\sigma(s, X_s^{t,x}), \quad dC_s \otimes d\mathbb{P} - a.e$$

*which coincides with the representation formula derived in [2, 3] when $M$ is a standard Brownian motion.*

*iii) In contradistinction, one can be interested in knowing when $u$ in Theorem 5.2 does not depend trivially on $M$ in other words when the third term in* (5.1) *does not vanish. This is related to the Markov property given for $Y$ and we exhibit in Section A.3 an explicit example where $u$ depends non-trivially on $M$.*

*iv) We think that the assumption (ACL) is not necessary but we use it in order to avoid some technicalities and to keep the length of the present paper within limit.*

*Proof of Theorem 5.2.* Fix $s$ in $[t,T]$. For simplicity of notations we will drop the superscript $(t,x,m)$. Assume for a moment that the following equality holds

$$\langle u(\cdot, X_\cdot, M_\cdot), M_\cdot\rangle_s = \int_t^s [\partial_2 u(r, X_r, M_r)\sigma(r, X_r, M_r) + \partial_3 u(r, X_r, M_r)] d\langle M, M\rangle_r, \tag{5.2}$$

and denote by $\langle u(\cdot, X_s, M_s), M_\cdot\rangle_s$ the vector $\Big(\langle u(\cdot, X_\cdot, M_\cdot), M_\cdot^{(1)}\rangle_s, \ldots, \langle u(\cdot, X_\cdot, M_\cdot), M_\cdot^{(d)}\rangle_s\Big)$. Then since $(Y, Z)$ is solution of (3.9) we have that

$$\langle Y, M\rangle_s = \int_t^s Z_r d\langle M, M\rangle_r, \quad s \in [t, T]. \tag{5.3}$$

The conclusion of the theorem then follows from the fact that $Y_s = u(s, X_s, M_s)$, $s \in [t, T]$, and from relations (5.2) and (5.3). The rest of the proof is therefore devoted to show relation (5.2).

Let $r \geq 1$ and $\pi^{(r)} := \{t_j^{(r)},\ j = 1, \ldots, r+1\}$ be a deterministic subdivision of $[t, s]$ whose mesh $|\pi^{(r)}|$ tends to zero as $r$ goes to infinity with $t_0^{(r)} = t$ and $t_{r+1}^{(r)} = s$. For simplicity of notation the superscript $(r)$ will be abandoned. In addition we denote the increments of the stochastic process $M$ on $[t_j, t_{j+1}]$ as $\Delta_j M$. Let $i \in \{1, \ldots, d\}$. We have that

$$\begin{aligned}
&\langle u(\cdot, X_\cdot, M_\cdot), M_\cdot^{(i)}\rangle_s \\
&\overset{\mathbb{P}}{=} \lim_{r\to\infty} \sum_{j=0}^{r} \big(u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_j}, M_{t_j})\big)\, \Delta_j M^{(i)} \\
&\overset{\mathbb{P}}{=} \lim_{r\to\infty} \Bigg[ \sum_{j=0}^{r} \big(u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_{j+1}}, M_{t_{j+1}})\big)\, \Delta_j M^{(i)} \\
&+ \sum_{j=0}^{r} \big(u(t_j, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_j}, M_{t_j})\big)\, \Delta_j M^{(i)} \Bigg] \\
&\overset{\mathbb{P}}{=:} \lim_{r\to\infty} \Big[ S_{r,1}^{(i)} + S_{r,2}^{(i)} \Big].
\end{aligned} \tag{5.4}$$

We treat the two previous parts separately. Assume for sake of clarity that the second term converges and more precisely that the relation (5.5) below holds:

$$\begin{aligned}
\lim_{r\to\infty} S_{r,2}^{(i)} &= \lim_{r\to\infty} \sum_{j=0}^{r} \big(u(t_j, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_j}, M_{t_j})\big)\, \Delta_j M^{(i)} \\
&\overset{\mathbb{P}}{=} \left( \int_t^s [\partial_2 u(r, X_r, M_r)\sigma(r, X_r, M_r) + \partial_3 u(r, X_r, M_r)] d\langle M, M\rangle_r \right)^{(i)}.
\end{aligned} \tag{5.5}$$

Then if the convergence (5.5) holds it follows by relations (5.3) and (5.4) that the limit

$$\lim_{r\to\infty} S_{r,1}^{(i)} = \lim_{r\to\infty} \sum_{j=0}^{r} \big(u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_{j+1}}, M_{t_{j+1}})\big)\, \Delta_j M^{(i)}$$

exists in probability and is equal to $\big(\int_t^s H_r d\langle M, M\rangle_r\big)^{(i)}$ where $H$ is defined as

$$H_r := Z_r - \partial_2 u(r, X_r, M_r)\sigma(r, X_r, M_r) + \partial_3 u(r, X_r, M_r), \quad r \in [t, s].$$

Considering $\left(\int_{t_j}^{t_{j+1}} H_r d\langle M, M\rangle_r\right)^{(i)}$ for $j$ and $r$ as above, we have that

$$\left(u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_{j+1}}, M_{t_{j+1}})\right) \Delta_j M^{(i)} = \Delta_j M^{(i)} \sum_{k=1}^{d} H_{t_j}^{(k)} \Delta_j M^{(k)} + \mathcal{R}_{j,r} \quad (5.6)$$

where $\mathcal{R}_{j,r}$ is a remaining such that $\lim_{r\to\infty} \sum_{j=0}^{r} \mathcal{R}_{j,r} = 0$. From the representation of $Z$ obtained in Theorem 3.4, there exists a deterministic function $\alpha^{(k)}$ such that $H_{t_j}^{(k)} = \alpha^{(k)}(t_j, X_{t_j}, M_{t_j})$. Let us consider the part of the left hand side of equality (5.6) which actually contributes to the density $H$. That is there exists a deterministic function $\beta$ continuous in its two last variables such that

$$\beta(t_j, t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) \Delta_j M^{(i)} = \Delta_j M^{(i)} \sum_{k=1}^{d} \alpha^{(k)}(t_j, X_{t_j}, M_{t_j}) \Delta_j M^{(k)} \quad (5.7)$$

and

$$\left(u(t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_{j+1}}, M_{t_{j+1}}) - \beta(t_j, t_{j+1}, X_{t_{j+1}}, M_{t_{j+1}})\right) \Delta_j M^{(i)} = R_{j,r}.$$

Since for fixed $j$ and $r$ the random vector $(M_{t_j}, \Delta_j M, X_{t_j}, \Delta_j X)$ has absolutely continuous law with respect to the Lebesgue measure (by assumption (ACL)), relation (5.7) shows that

$$\beta(t_j, t_{j+1}, x + \xi_1, m + \xi_2) = \sum_{k=1}^{d} \alpha^{(k)}(t_j, x, m) \xi_2^{(k)} \quad (5.8)$$

for almost all $(x, m, \xi_1, \xi_2)$ in $\mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{d\times 1}$. Let $(\xi_{2,p})_{p\geq 1}$ be a sequence in $\mathbb{R}^{d\times 1}$ such that $\lim_{p\to\infty} \xi_{2,p} = 0$. Replacing $\xi_2$ by $(\xi_{2,p})_{p\geq 1}$ in (5.8) shows that the deterministic functions $\alpha$ and $\beta$ vanish. As a consequence, $H_{t_j}$ in equation (5.6) is equal to zero and

$$\lim_{r\to\infty} S_{r,1}^{(i)} \overset{\mathbb{P}}{=} 0.$$

It remains to show that relation (5.5) holds.
*Proof of* (5.5)*:*

$$\begin{aligned}
\lim_{r\to\infty} S_{r,2}^{(i)} &\overset{\mathbb{P}}{=} \lim_{r\to\infty} \sum_{j=0}^{r} \left(u(t_j, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_j}, M_{t_j})\right) \Delta_j M^{(i)} \\
&\overset{\mathbb{P}}{=} \lim_{r\to\infty} \Bigg[ \sum_{j=0}^{r} \left(u(t_j, X_{t_{j+1}}, M_{t_j}) - u(t_j, X_{t_j}, M_{t_j})\right) \Delta_j M^{(i)} \\
&\quad + \sum_{j=0}^{r} \left(u(t_j, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_{j+1}}, M_{t_j})\right) \Delta_j M^{(i)} \Bigg]. \quad (5.9)
\end{aligned}$$

In addition we can write

$$u(t_j, X_{t_{j+1}}, M_{t_j}) - u(t_j, X_{t_j}, M_{t_j})$$

$$= \sum_{k=1}^{n} \left( u(t_j, X_{t_j}^{(1)}, \ldots, X_{t_j}^{(k-1)}, X_{t_{j+1}}^{(k)}, \ldots, X_{t_{j+1}}^{(n)}, M_{t_j}) - u(t_j, X_{t_j}^{(1)}, \ldots, X_{t_j}^{(k-1)}, X_{t_j}^{(k)}, \ldots, X_{t_{j+1}}^{(n)}, M_{t_j}) \right).$$

Each term of this sum can be written as

$$\begin{aligned} &u(t_j, X_{t_j}^{(1)}, \ldots, X_{t_j}^{(k-1)}, X_{t_{j+1}}^{(k)}, \ldots, X_{t_{j+1}}^{(n)}, M_{t_j}) - u(t_j, X_{t_j}^{(1)}, \ldots, X_{t_j}^{(k-1)}, X_{t_j}^{(k)}, \ldots, X_{t_{j+1}}^{(n)}, M_{t_j}) \\ &= (\bar{\partial_2} u)(\Delta_j X^{(1)}, \ldots, \Delta_j X^{(n)})^* \end{aligned} \tag{5.10}$$

where $\bar{\partial_2} u := \left( \partial_{1+k} u(t_j, X_{t_j}^{(1)}, \ldots, X_{t_j}^{(k-1)}, \bar{X}_{t_j}^{(k)}, X_{t_{j+1}}^{(k+1)}, \ldots, X_{t_{j+1}}^{(n)}, M_{t_j}) \right)_{1 \le k \le n}$ and $\bar{X}_{t_j}^{(k)}$ is a suitable random point in the interval $[X_{t_j}^{(k)} \wedge X_{t_{j+1}}^{(k)}, X_{t_j}^{(k)} \vee X_{t_{j+1}}^{(k)}]$. Similarly we obtain

$$u(t_j, X_{t_{j+1}}, M_{t_{j+1}}) - u(t_j, X_{t_{j+1}}, M_{t_j}) = (\bar{\partial_3} u)(\Delta_j M^{(1)}, \ldots, \Delta_j M^{(d)})^* \tag{5.11}$$

with $\bar{\partial_3} u := \left( \partial_{1+n+k} u(t_j, X_{t_{j+1}}, M_{t_j}^{(1)}, \ldots, M_{t_j}^{(k-1)}, \bar{M}_{t_j}^{(k)}, M_{t_{j+1}}^{(k+1)}, \ldots, M_{t_{j+1}}^{(d)}) \right)_{1 \le k \le d}$. Combining relations (5.9), (5.10) and (5.11) we conclude

$$\begin{aligned} \lim_{r \to \infty} S_{r,2}^{(i)} &\overset{\mathbb{P}}{=} \lim_{r \to \infty} \sum_{j=0}^{r} [(\bar{\partial_2} u)(\Delta_j X^{(1)}, \ldots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} + (\bar{\partial_3} u)(\Delta_j M^{(1)}, \ldots, \Delta_j M^{(d)})^* \Delta_j M^{(i)}] \\ &\overset{\mathbb{P}}{=} \lim_{r \to \infty} \sum_{j=0}^{r} \Big[ \partial_2 u(t_j, X_{t_j}, M_{t_j})(\Delta_j X^{(1)}, \ldots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} \\ &\quad + \partial_3 u(t_j, X_{t_j}, M_{t_j})(\Delta_j M^{(1)}, \ldots, \Delta_j M^{(d)})^* \Delta_j M^{(i)} + R(i,j,r) \Big], \end{aligned} \tag{5.12}$$

where $R(i,j,r)$ is defined as

$$\begin{aligned} R(i,j,r) = &\left( (\bar{\partial_2} u) - \partial_2 u(t_j, X_{t_j}, M_{t_j}) \right) (\Delta_j X^{(1)}, \ldots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} \\ &+ \left( (\bar{\partial_3} u) - \partial_3 u(t_j, X_{t_j}, M_{t_j}) \right) (\Delta_j M^{(1)}, \ldots, \Delta_j M^{(d)})^* \Delta_j M^{(i)}. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{r \to \infty} \sum_{j=0}^{r} [\partial_2 u(t_j, X_{t_j}, M_{t_j})(\Delta_j X^{(1)}, \ldots, \Delta_j X^{(n)})^* \Delta_j M^{(i)} + (\bar{\partial_3} u)(\Delta_j M^{(1)}, \ldots, \Delta_j M^{(d)})^* \Delta_j M^{(i)}] \\ &\overset{\mathbb{P}}{=} \left( \int_t^s \left[ \partial_2 u(r, X_r, M_r) \sigma(r, X_r, M_r) + \partial_3 u(r, X_{t_j}, M_{t_j}) \right] d\langle M, M \rangle_r \right)^{(i)} \end{aligned} \tag{5.13}$$

relation (5.2) follows from equations (5.12) and (5.13) provided the following equation (5.14) holds:

$$\lim_{r \to \infty} \left| \sum_{j=0}^{r} R(i,j,r) \right| \overset{\mathbb{P}}{=} 0. \tag{5.14}$$

We conclude the proof by showing relation (5.14). Let

$$A^{(r)} := \sup_{z \in [t,s], |a-b| \le |\pi^{(r)}|, k=1,\ldots,n} \{ |\partial_{1+k} u(z, X_a, M_z) - \partial_{1+k} u(z, X_b, M_z)| \} \quad \text{and}$$

$$B^{(r)} := \sup_{z\in[t,s],|a-b|\leq|\pi^{(r)}|,\ |c-d|\leq|\pi^{(r)}|,k=1\ldots,d} \{|\partial_{1+n+k}u(z,X_c,M_a)-\partial_{1+n+k}u(z,X_d,M_b)|\}.$$

For $1\leq i\leq d$, $r\in\mathbb{N}$ we have

$$\begin{aligned}
&\sum_{j=0}^{r}|R(i,j,r)|\\
&\leq CA^{(r)}\sum_{j=0}^{r}\sum_{k=1}^{n}|\Delta_j X^{(k)}\Delta_j M^{(i)}|+CB^{(r)}\sum_{j=0}^{r}\sum_{k=1}^{d}|\Delta_j M^{(k)}\Delta_j M^{(i)}|\\
&\leq \frac{C}{2}A^{(r)}\sum_{j=0}^{r}\sum_{k=1}^{n}\left[|\Delta_j X^{(k)}|^2+|\Delta_j M^{(i)}|^2\right]+\frac{C}{2}B^{(r)}\sum_{j=0}^{r}\sum_{k=1}^{d}\left[|\Delta_j M^{(k)}|^2+|\Delta_j M^{(i)}|^2\right]\\
&\leq \frac{C}{2}A^{(r)}\sum_{k=1}^{n}\left[\sum_{j_1=1}^{r}|\Delta_{j_1}X^{(k)}|^2+\sum_{j_2=1}^{r}|\Delta_{j_2}M^{(i)}|^2\right]+\frac{C}{2}B^{(r)}\sum_{k=1}^{d}\left[\sum_{j_1=1}^{r}|\Delta_{j_1}M^{(k)}|^2+\sum_{j_2=1}^{r}|\Delta_{j_2}M^{(i)}|^2\right].
\end{aligned}$$

According to Proposition 5.1 we have that

$$\lim_{r\to\infty}A^{(r)}=\lim_{r\to\infty}B^{(r)}=0,\quad \mathbb{P}-a.s.$$

On the other hand

$$\begin{cases}\lim_{r\to\infty}\sum_{j=0}^{r}|\Delta_j X^{(k)}|^2\overset{\mathbb{P}}{=}\langle X^{(k)},X^{(k)}\rangle_z,\\ \lim_{r\to\infty}\sum_{j=0}^{r}|\Delta_j M^{(k)}|^2\overset{\mathbb{P}}{=}\langle M^{(k)},X^{(k)}\rangle_z,\end{cases}$$

which concludes the proof. □

## Concluding remarks

In this paper we prove the representation formula (1.3) for the control process of a quadratic growth BSDE driven by a continuous local martingale. This can be used for giving an explicit representation of the delta hedge in utility indifference based hedging of insurance derivatives with exponential preferences. We also provide the Markov property and differentiability of the FBSDE (2.1)-(2.2) in the initial state parameter of its forward part. This last property is obtained under an additional assumption (MRP). As explained at the beginning of Section 4, this hypothesis allows us to deduce the differentiability of (2.2) from the one of the BSDE (4.1) which differs from the former by the absence of the quantity $\langle L,L\rangle$ that prevents us from employing existing proofs of differentiability. However, we think that differentiability should hold without this assumption, and that different techniques have to be developed for achieving this goal.

Additionally, as already mentioned in this paper Malliavin's calculus has been used by several authors to recover formula (5.1) in the Brownian framework. Our alternative method is valid in this setting and it seems to us that it could present advantages in some practical

situations. Actually, Malliavin's calculus is known for its efficiency in several topics however it also requires usually more regularity than the problems requires intrinsically. In [1] the authors study the quadratic hedging problem of contingent claims with basis risk when the hedging instrument and the underlying of the contingent are related via a random correlation process. As exposed in [1] the hedging strategy is described via a representation formula of the form (5.1) for the control process of the backward part of a FBSDE driven by a Brownian motion. In their case the coefficient of the forward process depends on a correlation process $\rho$ which is itself solution of a Brownian SDE. As explained in a comment in [1, Section 3.4] the use of Malliavin's calculus imposes to assume that the derivatives of the coefficients of the SDE defining $\rho$ have bounded derivatives. This additional regularity is not necessary in our approach and this would allow one to consider other examples of correlation processes with only locally Lipschitz bounded derivatives.


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## A Appendix

In the first section of this appendix we provide the transformation of a BSDE of the form (2.2) which is needed in section 4, and give a priori estimates on the solution of the transformed BSDE with respect to its terminal condition and its generator. Then in Section A.3 we present an explicit example of situation described in Proposition 3.1 *ii)*.

### A.1 Transformation of the BSDE (2.2) under (MRP)

We start giving a justification that under (MRP) the BSDE of the form

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T f(s, Y_s, Z_s q_s^*) dC_s - \int_t^T dL_s + \frac{\kappa}{2} \int_t^T d\langle L, L\rangle_s \tag{A.1}$$

can be transformed into a BSDE of the form

$$Y_t = B - \int_t^T \tilde{Z}_s d\tilde{M}_s + \int_t^T h(s, Y_s, \tilde{Z}_s \tilde{q}_s^*) d\tilde{C}_s, \tag{A.2}$$

where for all $s \in [0, T]$

$$\tilde{M}_s := \begin{pmatrix} M_s \\ N_s \end{pmatrix}, \quad \tilde{q}_s := \begin{pmatrix} q_s & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{C}_s := \arctan\left( \sum_{i=1}^{d} \langle M^{(i)}, M^{(i)} \rangle_s + \langle N, N \rangle_s \right),$$

$\tilde{Z}_s := \begin{pmatrix} Z_s & U_s \end{pmatrix}$. The precise definition of $h$ will follow from the subsequent calculations. Recall that

$$dC_s = \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)}\rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s\right)^2}.$$

We have for $t \in [0,T]$

$$\begin{aligned}
&\int_t^T f(s, Y_s, Z_s q_s^*) dC_s + \frac{\kappa}{2}\int_t^T d\langle L, L\rangle_s \\
&= \int_t^T f(s, Y_s, Z_s q_s^*) dC_s + \frac{\kappa}{2}\int_t^T U_s^2 d\langle N, N\rangle_s \\
&= \int_t^T \tilde{f}(s, Y_s, Z_s q_s^*) \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)}\rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s + \langle N, N\rangle_s\right)^2} \\
&\quad + \int_t^T g(U_s) \frac{d\langle N, N\rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s + \langle N, N\rangle_s\right)^2},
\end{aligned}$$

where for $s \in [t,T]$

$$\tilde{f}(\cdot, s, Y_s, Z_s q_s^*) := f(\cdot, s, Y_s, Z_s q_s^*) \times \frac{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s + \langle N, N\rangle_s\right)^2}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s\right)^2}$$

and

$$g(\cdot, U_s) := \frac{\kappa}{2} U_s^2 \left(1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s + \langle N, N\rangle_s\right)^2\right).$$

Hence

$$\begin{aligned}
&\int_t^T f(s, Y_s, Z_s q_s^*) dC_s + \frac{\kappa}{2}\int_t^T d\langle L, L\rangle_s \\
&= \int_t^T \left(\tilde{f}(s, Y_s, Z_s q_s^*)\varphi_1(s) + g(U_s)\varphi_2(s)\right) \frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)}\rangle_s + d\langle N, N\rangle_s}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_s + \langle N, N\rangle_s\right)^2} \\
&= \int_t^T \left(\tilde{f}(s, Y_s, Z_s q_s^*)\varphi_1(s) + g(U_s)\varphi_2(s)\right) d\tilde{C}_s,
\end{aligned}$$

where $\varphi_1$ and $\varphi_2$ are two predictable, with respect to the random measure $\frac{\sum_{i=1}^d d\langle M^{(i)}, M^{(i)}\rangle_\cdot + d\langle N, N\rangle_\cdot}{1 + \left(\sum_{i=1}^d \langle M^{(i)}, M^{(i)}\rangle_\cdot + \langle N, N\rangle_\cdot\right)^2}$ square integrable functions obtained from Riesz's representation theorem (see *e.g.* [7, p. 176]). As a consequence, letting $h(\cdot, s, Y_s, Z_s q_s^*, U_s) := \tilde{f}(\cdot, s, Y_s, Z_s q_s^*)\varphi_1(s) + g(\cdot, U_s)\varphi_2(s)$ we obtain that (A.1) can be written as

$$Y_t = B - \int_t^T Z_s dM_s - \int_t^T U_s dN_s + \int_t^T h(s, Y_s, Z_s q_s^*, U_s) d\tilde{C}_s$$

and the generator $h$ will satisfy hypotheses (H4). In particular $h$ preserves the growth in the variables $y, z, u$. Hence we arrive at BSDE (A.2).

## A.2 Apriori estimates

Now we assume that $M$ itself satisfies the martingale representation theorem. We consider the following BSDE

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T f(s, Y_s, Z_s q_s^*) dC_s, \tag{A.3}$$

where $M, q, C$ are defined as in Section 2. Suppose that the terminal condition $B$ is a bounded real-valued random variable and the generator $f$ satisfies assumption (H4). The following a priori inequality is crucial for our differentiability and representation results.

**Lemma A.1.** *We assume that for every $\beta \geq 1$ we have $\int_0^T |f(s,0,0)| dC_s \in L^\beta(\mathbb{P})$. Let $p > 1$. Then there exist constants $q > 1$, $C > 0$ depending only on $T$, $p$ and on the BMO-norm of $H \cdot M$ ($H$ being defined below) such that*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T |q_s Z_s^*|^2 dC_s\right)^p\right] \\ &\leq C\mathbb{E}\left[|B|^{2pq} + \left(\int_0^T |f(s,0,0)| dC_s\right)^{2pq}\right]^{\frac{1}{q}}.\end{aligned}$$

*Proof.* In the following, constants appearing in inequalities will be denoted by $C$ and may change from line to line. We follow [4, Lemmata 7-8 and Corollary 9] (see also [3, Lemma 6.1]) which have been designed for the Brownian setting. However, as we will show below their arguments can be extended to the case of continuous local martingales. We proceed in two steps.
In Step 1, we provide the estimate on $Y$. Let

$$J_s = \begin{cases} \frac{f(s,Y_s,Z_s q_s^*) - f(s,0,Z_s q_s^*)}{Y_s}, & \text{if } Y_s \neq 0, \\ 0, & \text{otherwise, and} \end{cases}$$

$$H_s = \begin{cases} \frac{f(s,0,Z_s q_s^*) - f(t,0,0)}{|q_s Z_s^*|^2} Z_s, & \text{if } |q_s Z_s^*|^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then BSDE (A.3) has the form

$$Y_t = B - \int_t^T Z_s dM_s + \int_t^T \left(J_s Y_s + (q_s H_s^*)(q_s Z_s^*)^* + f(s,0,0)\right) dC_s, \quad t \in [0,T]. \tag{A.4}$$

Due to (H4) we have $|qH^*| \leq |qB^*|$ and it follows that $H \cdot M$ is a BMO($\mathbb{P}$) martingale. By [11, Theorem 2.3] the measure $\mathbb{Q}$ defined by $d\mathbb{Q} = \mathcal{E}(H \cdot M)_T d\mathbb{P}$ is a probability measure. Girsanov's theorem implies that

$$K = Z \cdot M - \int_0^\cdot (q_s H_s^*)(q_s Z_s^*)^* dC_s$$

is a local $\mathbb{Q}$-martingale. This means that there exists an increasing sequence of stopping times $(\tau^n)_{n\in\mathbb{N}}$ such that $K_{\cdot\wedge\tau_n}$ is a $\mathbb{Q}$-martingale for any $n \in \mathbb{N}$. Letting $e = e^{\int_0^\cdot J_s dC_s}$ we

have with Itô's formula for $t \in [0, \tau^n]$

$$e_t Y_t = e_T B - \int_t^{\tau^n} e_s dK_s + \int_t^{\tau^n} e_s f(s,0,0) dC_s.$$

Taking the conditional expectation on both sides gives

$$e_t Y_t = \mathbb{E}^{\mathbb{Q}} \left[ e_T B + \int_t^{\tau^n} e_s f(s,0,0) dC_s | \mathcal{F}_t \right].$$

By [11, Theorem 3.1] there exists a $r > 1$ such that $\mathcal{E}(H \cdot M)_T \in L^{\frac{r}{r-1}}(\mathbb{P})$ and the reverse Hölder inequality holds, *i.e.* there exists a constant $C$ (depending on $r$) such that

$$\mathcal{E}(H \cdot M)_t^{-1} \mathbb{E} \left[ \mathcal{E}(H \cdot M)_T^{\frac{r}{r-1}} | \mathcal{F}_t \right]^{\frac{r-1}{r}} \leq C.$$

Since $J_s \leq A_s$ and hence $e_s / e_t \leq \int_0^T |q_r A_r \mathbf{1}_{\mathbb{R}^d}|^2 dC_r$ for all $s \geq t$, these two facts lead to

$$\begin{aligned}
|Y_t| &\leq \mathcal{E}(H \cdot M)_t^{-1} \mathbb{E} \left[ \mathcal{E}(H \cdot M)_T (\frac{e_T}{e_t} |B| + \int_0^T \frac{e_s}{e_t} |f(s,0,0)| dC_s) | \mathcal{F}_t \right] \\
&\leq \mathcal{E}(H \cdot M)_t^{-1} \mathbb{E} \left[ \mathcal{E}(H \cdot M)_T \int_0^T |q_s A_s \mathbf{1}_{\mathbb{R}^d}|^2 dC_s \left( |B| + \int_0^T |f(s,0,0)| dC_s \right) | \mathcal{F}_t \right] \\
&\leq \mathcal{E}(H \cdot M)_t^{-1} \mathbb{E} \left[ \mathcal{E}(H \cdot M)_T^{\frac{r}{r-1}} | \mathcal{F}_t \right]^{\frac{r-1}{r}} \mathbb{E} \left[ \left( \int_0^T |q_s A_s \mathbf{1}_{\mathbb{R}^d}|^2 dC_s \right)^r \left( |B| + \int_0^T |f(s,0,0)| dC_s \right)^r | \mathcal{F}_t \right]^{\frac{1}{r}} \\
&\leq C \mathbb{E} \left[ \left( \int_0^T |q_s A_s \mathbf{1}_{\mathbb{R}^d}|^2 dC_s \right)^r \left( |B| + \int_0^T |f(s,0,0)| dC_s \right)^r | \mathcal{F}_t \right]^{\frac{1}{r}}.
\end{aligned}$$

Let $p > 1$. Then

$$\sup_{t \in [0,T]} |Y_t|^{2p} \leq C \sup_{t \in [0,T]} \left( \mathbb{E} \left[ \left( \int_0^T |q_s A_s \mathbf{1}_{\mathbb{R}^d}|^2 dC_s \right)^r \left( |B| + \int_0^T |f(s,0,0)| dC_s \right)^r | \mathcal{F}_t \right] \right)^{\frac{2p}{r}}.$$

Now apply Doob's $L^p$-inequality to obtain

$$\begin{aligned}
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^{2p} \right] &\leq C \mathbb{E} \left[ \left( \mathbb{E} \left[ \left( \int_0^T |q_s A_s \mathbf{1}_{\mathbb{R}^d}|^2 dC_s \right)^r \left( |B| + \int_0^T |f(s,0,0)| dC_s \right)^r | \mathcal{F}_T \right] \right)^{\frac{2p}{r}} \right] \\
&\leq C \mathbb{E} \left[ \left( \int_0^T |q_s A_s \mathbf{1}_{\mathbb{R}^d}|^2 dC_s \right)^{2p} \left( |B| + \int_0^T |f(s,0,0)| dC_s \right)^{2p} \right].
\end{aligned}$$

As above there exists a $q > 1$ such that $\mathcal{E}(A \mathbf{1}_{\mathbb{R}^d} \cdot M)_T \in L^{\frac{q}{q-1}}(\mathbb{P})$ and Hölder inequality entails

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t|^{2p} \right] \leq C \mathbb{E} \left[ \left( |B| + \int_0^T |f(s,0,0)| dC_s \right)^{2pq} \right]^{\frac{1}{q}}.$$

In Step 2, we give an estimate on $Z$. Assumption (H4) and Young's inequality imply for $s \in [0,T], y \in \mathbb{R}, z \in \mathbb{R}^{1\times d}$

$$2yf(s,y,q_sz^*) \le 2|y|f(s,0,0) + 2|q_sA_s\mathbf{1}_{\mathbb{R}^d}|y^2 + 2|q_sB_s^*|^2|y|^2 + \frac{1}{2}|q_sz^*|^2.$$

Let us define the following sequence of stopping times

$$\tau^r = \inf\{t \in [0,T] : \int_0^t |q_sZ_s^*|^2 dC_s \ge r\} \wedge T, \quad r \in \mathbb{N}.$$

Now we apply Itô's formula to $Y^2$ with the result

$$\begin{aligned}|Y_0|^2 \le& |Y_{\tau^r}|^2 - 2\int_0^{\tau^r} Y_sZ_s dM_s \\ &+ \int_0^{\tau^r} (2|Y_s|f(s,0,0) + 2|q_sA_s\mathbf{1}_{\mathbb{R}^d}|Y_s^2 + 2|q_sB_s^*|^2|Y_s|^2 + \frac{1}{2}|q_sZ_s^*|^2)dC_s - \int_0^{\tau^r} |q_sZ_s^*|^2 dC_s.\end{aligned}$$

Since $\tau^r \le T$, we deduce

$$\begin{aligned}\frac{1}{2}\int_0^{\tau^r} |q_sZ_s^*|^2 dC_s \le& |Y_{\tau^r}|^2 + 2\left|\int_0^{\tau^r} Y_sZ_s dM_s\right| \\ &+ \int_0^T (2|Y_s||f(s,0,0)| + 2|q_sA_s\mathbf{1}_{\mathbb{R}^d}||Y_s|^2 + 2|q_sB_s^*|^2|Y_s|^2)dC_s \\ \le& \sup_{t\in[0,T]} |Y_t|^2 + 2\left|\int_0^{\tau^r} Y_sZ_s dM_s\right| \\ &+ 2\sup_{t\in[0,T]} |Y_t| \int_0^T |f(s,0,0)|dC_s + 2\sup_{t\in[0,T]} |Y_t|^2 \int_0^T (|q_sA_s\mathbf{1}_{\mathbb{R}^d}|^2 + |q_sB_s^*|^2)dC_s.\end{aligned}$$

Then the inequalities of Burkholder-Davis-Gundy and Young give for $p > 1$

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^{\tau^r} |q_sZ_s^*|^2 dC_s\right)^p\right] \\ &\le C\Bigg(\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T Y_s^2|q_sZ_s^*|^2 dC_s\right)^{p/2}\right] + \mathbb{E}\left[\left(\int_0^T |f(s,0,0)|dC_s\right)^{2p}\right] \\ &\quad + \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\left\{1 + \left(\int_0^T (|q_sA_s\mathbf{1}_{\mathbb{R}^d}|^2 + |q_sB_s^*|^2)dC_s\right)^p\right\}\right]\Bigg) \\ &\le C\Bigg((C+1)\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right] + \mathbb{E}\left[\frac{1}{2C}\left(\int_0^T |q_sZ_s^*|^2 dC_s\right)^p\right] \\ &\quad + \mathbb{E}\left[\left(\int_0^T |f(s,0,0)|dC_s\right)^{2p}\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right] C\Bigg).\end{aligned}$$

In the last inequality we have applied the inequalities of Young and Hölder and used that $A\mathbf{1}_{\mathbb{R}^d} \cdot M$ and $B \cdot M$ are BMO-martingales. Now Fatou's Lemma gives

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T |q_s Z_s^*|^2 dC_s\right)^p\right] \leq & C\left(\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right] + \mathbb{E}\left[\left(\int_0^T |f(s,0,0)|dC_s\right)^{2p}\right]\right) \\ \leq & C\mathbb{E}\left[\left(|B| + \int_0^T |f(s,0,0)|dC_s\right)^{2pq}\right]^{1/q},\end{aligned}$$

where we have also used Hölder's inequality to state

$$\mathbb{E}\left[\left(\int_0^T |f(s,0,0)|dC_s\right)^{2p}\right] \leq \mathbb{E}\left[\left(\int_0^T |f(s,0,0)|dC_s\right)^{2pq}\right]^{1/q}.$$

□

### A.3 Additional material on Markov processes

We provide an example where the function $u$ in Theorem 3.4 does not depend trivially on $M$.

Let $M := (M_t)_{t\in[0,T]}$ be a continuous martingale with non-independent increments which is a Markov process with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let $X := (X_t)_{t\in[0,T]}$ be the solution of the SDE

$$dX_t = \int_0^t \sigma(a, X_a) dM_a, \quad t \in [0,T] \text{ and } X_0 = 0,$$

with

$$\sigma(a,x) = \begin{cases} 1+x, & \text{if } a \geq \frac{T}{2} \\ 0, & \text{if } a < \frac{T}{2}. \end{cases}$$

Note that the coefficient $\sigma$ is Lipschitz in $x$ for every $a$ and that it is right continuous with left limits in $a$ for every $x$; as a consequence $X$ admits an unique solution by [17, V. Theorem 35]. Consider a simple BSDE of the form (3.9) with $f \equiv 0$, $\kappa = 0$ and $F(x) := \log(1+x)$. Note that $F$ is not bounded but in this special case the existence of a solution to the BSDE is assured. Our aim is to show that $\mathbb{E}[F(X_T)|\mathcal{F}_t]$ is not a trivial function of $M$ for $t \in [0,T]$. By Itô's formula we have

$$F(X_T) = \log(1+X_t) + \int_t^T (1+X_s)^{-1} dX_s - \frac{1}{2}(\langle M, M\rangle_T - \langle M, M\rangle_t),$$

and hence

$$\begin{aligned}\mathbb{E}[F(X_T)|\mathcal{F}_t] = & \log(1+X_t) - \frac{1}{2}\mathbb{E}[\langle M, M\rangle_T - \langle M, M\rangle_t|\mathcal{F}_t] \\ = & \log(1+X_t) - \frac{1}{2}\mathbb{E}[M_T^2 - M_t^2|\mathcal{F}_t]\end{aligned}$$

$$= \log(1 + X_t) - \frac{1}{2}\mathbb{E}[M_T^2 - M_t^2 | M_t]$$

since $M$ is a Markov process. Choose $0 < t < \frac{T}{2}$. By definition of $X$, the last term on the right hand side above cannot be expressed as a non-trivial deterministic function of $(t, X_t)$ since $M_s$ cannot be deduced from $X_s$ for $s < \frac{T}{2}$. However this term is deterministic and only depends on $t$ if $M$ has independent increments. This gives an example of a situation where the function $u$ in Theorem 3.4 does not depend trivially on $M$.

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