

From bounds on optimal growth towards a theory of good-deal hedging

Dirk Becherer

Abstract. Good-deal bounds have been introduced as a way to obtain valuation bounds for derivative assets which are tighter than the arbitrage bounds. This is achieved by ruling out not only those prices that violate no-arbitrage restrictions but also trading opportunities that are ‘too good’. We study dynamic good-deal valuation bounds that are derived from bounds on optimal expected growth rates. This leads naturally to restrictions on the set of pricing measure which are local in time, thereby inducing good dynamic properties for the good-deal valuation bounds. We study good-deal bounds by duality arguments in a general semimartingale setting. In a Wiener space setting where asset prices evolve as Itô-processes, good-deal bounds are then conveniently described by backward SDEs. We show how the good-deal bounds arise as the value function for an optimal control problem, where a dynamic coherent a priori risk measure is minimized by the choice of a suitable hedging strategy. This demonstrates how the theory of no-good-deal valuations can be associated to an established concept of dynamic hedging in continuous time.

Key words. Good deal bounds, good deal hedging, incomplete markets, backward stochastic differential equations, dynamic coherent risk measures, logarithmic utility, optimal growth

AMS classification. (MSC2000) 60G35, 60G44, 60H30, 91B16, 91B28

1 Introduction

When pricing a contingent claim solely based on no-arbitrage arguments, the range of possible arbitrage free prices can easily be too wide for practical purposes. This can happen generically if the financial market is not taken as complete or, more generally, if the payoff of the contingent claim cannot be perfectly synthesized by dynamical trading. The approach of good-deal bounds has been developed to derive a tighter range of no-good-deal prices, by using only a subset of martingale (pricing) measures to value contingent claims. This subset is chosen such that some notion of too favorable trading opportunities (good deals) is prevented, if the market is extended by additional price processes which are obtained by using those martingale measures for the pricing of contingent claims.

The most cited reference on the topic is probably [12], where no good deals are

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defined by bounds on Sharpe ratios. For comprehensive references on the topic we refer to [9, 18, 7, 20]. Let us just mention here, that in [7] the authors also consider bounds for Sharpe ratios in a continuous time model which permits for asset price jumps, thereby extending [12]. The starting point of another stream of papers is given by a set of acceptable financial positions. From there, no-good-deal price bounds are then defined as the smallest amount which a seller of a claim must charge, respectively the largest amount a buyer can pay, in order to obtain an acceptable position in the market, see e.g. [9, 18]. The route taken in the present article is to look at bounds for expected optimal growth rates, which corresponds to bounds on expected logarithmic utility. In deriving dynamic good-deal bounds in continuous time from bounds on expected utility, it is maybe closest in spirit to and has been motivated by [20], where a rigorous study of such bounds has been presented in a Levy process setting for the case of exponential utility. For important conceptual earlier contributions in this direction, see [10, 11] and also references in [10, 20] on earlier work by S. Hodges. In [20] it has been pointed out that restrictions on the no-good-deal set of martingale (pricing) measures should be local in time, in order to achieve good dynamic properties of the good-deal valuation bounds. However, the authors have concentrated on the case of exponential utility from terminal wealth where additional, essentially technical, assumptions were needed to ensure such local restrictions. Apparently, those do not arise naturally from the good-deal approach in this case. To ensure that their local restrictions imply certain ‘global’ restrictions that have financial meaning in terms of bounds on utility, the authors have therefore imposed additional structural assumptions, like preservation of a Levy structure of the model under all no-good-deal pricing measures.

To arrive at local restrictions for the set of no-good-deal martingale measures in a more endogenous way, it appears natural to start from bounds on expected optimal growth, since this is linked to logarithmic utility which is known to be myopic. By this choice, one also avoids to have good-deal bounds that depend on an exogenously given time horizon, for which expected terminal utility is maximized. This is the starting point for us in Section 3, where we consider the problem in a general semimartingale model, after the model framework has been set in Section 2. To derive more explicit results we specialize in Section 4 to a Wiener space setting with asset prices that are Itô-processes, but need not be Markovian diffusions. In this setting, we obtain a convenient dynamic description of good-deal bounds in terms of standard backward stochastic differential equations (BSDEs) with generators that satisfy a Lipschitz property in Section 5. It is well known that the good-deal approach is in essence an approach to pricing, but that a theory on notions of hedging that can be linked to it still needs to be further developed and, possibly even more so, constructive descriptions for respective hedging strategies are needed. This is emphasized in the concluding remarks in [7], where the authors write “the task of developing such a [good-deal hedging] theory constitutes a highly challenging open problem”. The final Section 5 discusses such links to hedging, showing that a trading strategy, which corresponds to the (upper) good-deal valuation bound, is given by a strategy that minimizes a suitable coherent risk measure ρ (of good-deal type). The hedging strategy can then be defined as the minimizer. In other words, the hedging strategy is the strategy that minimizes (maximizes) the upper (lower) a priori ρ -valuation bounds for the seller (buyer) of a contingent claim, such that the good-deal bounds arise as the tightest ‘a priori’ ρ -valuation bounds for the

residual risk that can be obtained by optimal hedging in the market. For our mathematical analysis, BSDE theory provides convenient methods to solve the optimization problems involved and to describe both the valuation bounds and the optimal hedging strategies. To our best knowledge, the connection between growth-optimal dynamic good-deal bounds and a corresponding notion of hedging with respect to dynamic coherent risk measures has not been elaborated so far. Mathematically, this connection fits well with the general theory for inf-convolutions of risk measures, see [4] for an excellent exposition and references.

The contributions of this article are threefold. First, we show that restrictions on optimal expected growth rates lead to good-deal valuation bounds with good dynamic properties in a general model. Secondly, we describe the dynamics of the valuation bounds and of a suitable hedging strategy in a Wiener space setting by BSDE solutions where, finally, the corresponding hedging strategy is obtained as the minimizer of a suitable a-priori risk measure of good-deal type.

2 General framework and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis with fixed time horizon $\bar{T} < \infty$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \bar{T}]}$ satisfying the usual conditions of right-continuity and completeness. All semimartingales are taken to have right-continuous paths with left limits (RCLL paths). For simplicity let \mathcal{F}_0 be trivial and $\mathcal{F} = \mathcal{F}_{\bar{T}}$. Conditional expectations with respect to \mathcal{F}_t under P are denoted by $E_t[\cdot] = E_t^P[\cdot]$. For random variables X we define $E_t[X]$ as $E_t[X^+] - E_t[X^-]$ where the latter is well defined, and as $-\infty$ elsewhere. Inequalities between random variables (processes) are meant to hold almost surely (respectively $P \times dt$ almost everywhere). Equality between processes with RCLL paths then means indistinguishability. By convention, $\frac{0}{0}$ is defined to be 0.

All prices of assets in our model for the financial market are expressed in terms of a numeraire asset, whose (discounted) price is thus constant at one. A common choice for the numeraire is a cash account where money can be deposited and lent from at the same rate of interest; for such a choice, asset prices are expressed in units of one Euro put into the cash account at time 0. Alternatively, if the numeraire is chosen to be the zero coupon bond with maturity \bar{T} , then asset prices are expressed in \bar{T} -forward units, i.e. in terms of Euros at time \bar{T} . We consider a financial market model with $d+1$ tradable assets, which comprise the numeraire asset and furthermore d risky assets, whose price processes $S = (S^i)_{i=1, \dots, d}$ are semimartingales. All S^i are assumed to be locally bounded from below. For example, a natural (globally uniform) lower bound for stocks would be zero.

As usual, a (self-financing) portfolio strategy is defined by some initial capital V_0 and a predictable and S -integrable \mathbb{R}^d -valued process ϑ (i.e. $\vartheta \in L(S)$), describing the numbers of risky assets to be held dynamically over time. The wealth process of such strategy is given by $V = V_0 + \vartheta \cdot S = V_0 + \int \vartheta dS$. Since we will be interested in the returns V_t/V_s ($s \leq t \leq \bar{T}$) of wealth processes that are strictly positive, it is convenient

to normalize initial wealth to one by simple scaling. To this end, we define

$$\mathcal{N} := \mathcal{N}(S) := \{N \mid N = 1 + \theta \cdot S, \theta \in L(S), N > 0\} \quad (2.1)$$

as the family of all strictly positive wealth processes starting at one. These processes are called tradable numeraires (with respect to S).

We denote by $\mathcal{M}^e(\mathcal{N})$ the set of all probability measures equivalent to P ($Q \sim P$) such that any $N \in \mathcal{N}$ is a local Q -martingale. It is easy to see that $Q \sim P$ is a local martingale measure for S if and only if it is a local martingale measure for any $N \in \mathcal{N}$. Indeed, one inclusion of the identity $\mathcal{M}^e(S) = \mathcal{M}^e(\mathcal{N})$ follows from S being locally bounded from below by adding a suitable constant and a simple scaling argument, while the other inclusion follows from [1], Corollary 3.5.

Throughout the sequel, we assume that the market is free of arbitrage in the sense that

$$\mathcal{M}^e := \mathcal{M}^e(\mathcal{N}(S)) \neq \emptyset. \quad (2.2)$$

More precisely, condition (2.2) is equivalent to the property of ‘no free lunch with vanishing risk’ introduced in [14], cf. Proposition 2.3 in [5].

Let us recall some properties about the set of numeraires $\mathcal{N}(S)$.

Lemma 2.0.1 *I. \mathcal{N} is convex.*

2. For $N^1, N^2 \in \mathcal{N}$ and for stopping times T, τ with $0 \leq T \leq \tau \leq \bar{T}$, $A \in \mathcal{F}_T$, the process

$$N := I_{[0, T]} N^1 + I_{]T, \tau]} \left(I_A \frac{N_T^1}{N_T^2} N^2 + I_{A^c} N^1 \right) + I_{] \tau, \bar{T}]} \left(I_A \frac{N_T^1}{N_T^2} \frac{N_\tau^2}{N_\tau^1} + I_{A^c} \right) N^1$$

is an element of \mathcal{N} .

3. Every $N \in \mathcal{N}$ can be written as a stochastic exponential, and the set

$$\mathcal{L} := \left\{ L \mid L \text{ semimartingale, } L_0 = 0, \mathcal{E}(L) \in \mathcal{N} \right\}$$

contains 0 and is predictably convex, i.e., for any predictable $[0, 1]$ -valued process H and $L^1, L^2 \in \mathcal{L}$, the process $L := H \cdot L^1 + (1 - H) \cdot L^2$ is an element of \mathcal{L} .

4. Any $N \in \mathcal{N}$ is bounded away from 0 uniformly in t : $P[\inf_{t \leq \bar{T}} N_t > 0] = 1$, and is a strictly positive supermartingale under Q for any $Q \in \mathcal{M}^e$.

Proof: Part 1 is immediate. Part 3 can be shown using that any $N \in \mathcal{N}$, being a strictly positive Q -martingale for $Q \in \mathcal{M}^e(\mathcal{N})$, can be written as a stochastic exponential $N = \mathcal{E}(L) \equiv 1 + N_- \cdot L$ for $L := (1/N_-) \cdot N$. Letting $L := H \cdot L^1 + (1 - H) \cdot L^2$ for $N^i = \mathcal{E}(L^i) \in \mathcal{N}$, $i = 1, 2$, it is straightforward to show that N is in \mathcal{N} . For part 2, let $H := I_{[0, T]} + I_{]T, \tau]} I_{A^c} + I_{] \tau, \bar{T}]}$ and $L := H \cdot L^1 + (1 - H) \cdot L^2$. Then one can check that $N = \mathcal{E}(L)$ and the claim follows by part 3. Details on parts 2,3 are given in [5], Lemma 2.5. For part 4, the second claim holds since any positive local martingale is a supermartingale by Fatou’s lemma. The first claim then follows by the minimum principle for positive supermartingales, which is an application of the optional stopping theorem, see [23] II.3. □

Dynamic good-deal bounds: Preliminaries

For equivalent martingale measures $Q \in \mathcal{M}^e$ we will for convenience often identify the measure Q with its density process $Z_t = Z_t^Q = E_t[dQ/dP]$, omitting indices like Q when there is no ambiguity. To limit notations, we write compactly $Z \in \mathcal{M}^e$ for Z^Q of some $Q \in \mathcal{M}^e$.

We denote by \mathcal{Q} the set

$$\mathcal{Q} := \mathcal{Q}(S) := \{Q \in \mathcal{M}^e(S) \mid E[-\log Z_{\bar{T}}] < \infty\}. \quad (2.3)$$

The integrability condition in (2.3) requires that the relative entropy $H(P|Q)$ of P with respect to Q , that is

$$H(P|Q) := E[-\log Z_{\bar{T}}] = E^Q \left[\frac{1}{Z_{\bar{T}}} \log \frac{1}{Z_{\bar{T}}} \right], \quad (2.4)$$

is finite. For an equivalent martingale measure Q , it is usual to call $H(P|Q)$ the reverse relative entropy of Q , whereas $H(Q|P)$ is called the relative entropy of Q .

Let

$$h = (h_t) \geq 0$$

be a predictable process that is bounded, measurable and positive. For given h , let $\mathcal{Q}^{\text{ngd}} := \mathcal{Q}^{\text{ngd}}(S)$ denote the subset of those measures $Q \in \mathcal{M}^e(S)$ which satisfy that

$$E_T \left[-\log \frac{Z_\tau}{Z_T} \right] \leq \frac{1}{2} E_T \left[\int_T^\tau h_u^2 du \right] \quad \text{for all } T \leq \tau \leq \bar{T}, \quad (2.5)$$

where T, τ are stopping times. Let us note here, that we have not included the factor $1/2$ and the square into the function h to simplify formulae in the later sections, e.g. 5.

We shall write $h(t)$ instead of h_t ($0 \leq t \leq \bar{T}$) if h is deterministic, depending only on time but not on $\omega \in \Omega$. The simplest case is to take $h(t) = \bar{h} \geq 0$ to be constant. Permitting h to depend on time t and ω increases generality. In later sections, \mathcal{Q}^{ngd} will serve as a set of pricing measures which are taken into consideration to determine a range of no-good-deal valuations. If, for instance, market prices of risk in the distant future are considered to be of higher variability this could be captured by $t \mapsto h(t)$ being increasing.

Clearly, we have $\mathcal{Q}^{\text{ngd}} \subset \mathcal{Q} \subset \mathcal{M}^e$. Going beyond condition (2.2), we will subsequently assume that \mathcal{Q}^{ngd} contains at least one element \hat{Q} , i.e. it is non-empty

$$\mathcal{Q}^{\text{ngd}} \neq \emptyset. \quad (2.6)$$

The next result collects properties of the density process Z for measures Q in \mathcal{Q} .

Proposition 2.0.2 *For any $Q \sim P$ with $-\log Z_{\bar{T}} \in L^1(P)$ the following holds.*

1. *The process $(-\log Z_t)_{t \leq \bar{T}}$ is a submartingale of class (D), that is the family $\{-\log Z_T\}$ is uniformly P -integrable, with T ranging over the set of all stopping times $T \leq \bar{T}$.*

2. The process $-\log Z$ has a unique Doob-Meyer decomposition

$$-\log Z_t = M_t + A_t, \quad t \leq \bar{T}, \quad (2.7)$$

with M being a uniformly integrable martingale and A being an integrable increasing predictable process with $A_0 = 0$.

3. For any stopping times $T \leq \tau \leq \bar{T}$, it holds

$$E_T \left[-\log \frac{Z_\tau}{Z_T} \right] = E_T[A_\tau - A_T]. \quad (2.8)$$

Proof:

1. By assumption we have $-\log Z_{\bar{T}} \in L^1(P)$. For any stopping time $T \leq \bar{T}$,

$$-E_T[Z_{\bar{T}}] + 1 \leq -Z_T + 1 \leq -\log Z_T \leq E_T[-\log Z_{\bar{T}}]$$

holds by Jensen's inequality. Since the upper and lower bounds are uniformly integrable martingales stopped at T , the family $\{-\log Z_T\}$ is of class (D). Using again Jensen's inequality, we obtain that $-\log Z$ is a supermartingale.

2. This follows by a version of the Doob-Meyer decomposition, see Theorem 8 in ch.III of [22]. That A is integrable, means that $E[A_{\bar{T}}] < \infty$.

3. This follows from part 2 since martingale increments vanish in expectation. \square

For any Q as in Proposition 2.0.2 with density process Z and corresponding A from (2.7), we can define a finite measure $\mu = \mu^Q$ on the predictable σ -field \mathcal{P} by

$$\mu(B) := E \left[\int 1_B dA_t \right], \quad B \in \mathcal{P}. \quad (2.9)$$

The next result shows that μ corresponding $Q \in \mathcal{Q}$ is dominated by the measure

$$\nu(B) := \frac{1}{2} E \left[\int 1_B h_t^2 dt \right], \quad B \in \mathcal{P}, \quad (2.10)$$

and thereby provides simpler criteria (2.11), (2.13) for Q being in \mathcal{Q}^{ngd} , which are formulated with respect to deterministic times only instead of stopping times as (2.5).

Proposition 2.0.3 *If the measure $Q \sim P$ satisfies*

$$E_s \left[-\log \frac{Z_t}{Z_s} \right] \leq \frac{1}{2} E_s \left[\int_s^t h_u^2 du \right] \quad \text{for all } s \leq t \leq \bar{T} \quad (2.11)$$

with s, t ranging over the set of all deterministic times, then

$$\mu(B) \leq \nu(B) \quad (2.12)$$

holds for all predictable sets $B \in \mathcal{P}$. In particular, the conditions (2.11) and (2.5) are equivalent, and for $Q \in \mathcal{M}^e(S)$ condition (2.11) implies that Q is in \mathcal{Q}^{ngd} .

In the case where $h_t = h(t)$ is deterministic, (2.11) simplifies to

$$E_s \left[-\log \frac{Z_t}{Z_s} \right] \leq \frac{1}{2} \int_s^t h^2(u) du \quad \text{for all } s \leq t \leq \bar{T}. \quad (2.13)$$

Proof: By assumption (2.11), inequality (2.12) holds for predictable sets of the form $B = A_s \times (s, t]$ with $s < t \leq \bar{T}$ and $A_s \in \mathcal{F}_s$. The class of all such sets is a semi-ring that generates the predictable σ -field \mathcal{P} . Since both measures are finite and (2.12) holds on the semi-ring, it follows that (2.12) also holds on the predictable σ -field \mathcal{P} generated by it. Indeed, the inequality extends directly from the semi-ring to \mathcal{P} for the outer measures that are generated from the restrictions of μ, ν onto the semi-ring. Since these outer measures coincide with μ respectively ν on \mathcal{P} by the extension theorems of measure theory, (2.12) holds for $B \in \mathcal{P}$. Hence, condition (2.11) implies (2.5) since sets of the form $B = A_T \times]T, \tau]$ are in \mathcal{P} . Necessity is obvious. The last claim follows from the definition of \mathcal{Q} . \square

Definition 2.0.4 A set \mathcal{S} of probability measures, all elements of which are equivalent to P , is called *m-stable* (m-stable) if for all elements $Q^1, Q^2 \in \mathcal{S}$ with density processes Z^1, Z^2 and for all stopping times $T \leq \bar{T}$, it holds that $Z_T^1 := Z_T^1 \frac{Z_T^2}{Z_T^2}$ is the density of some $Q \in \mathcal{S}$.

This definition follows [13], where m-stable sets of measures are studied in a general framework. Examples for m-stable sets that play a role in the sequel are given by

Proposition 2.0.5 The sets \mathcal{M}^e , \mathcal{Q} and \mathcal{Q}^{ngd} are m-stable.

Proof: The arguments for Proposition 5 in [13] show that \mathcal{M}^e is m-stable. This holds also without S being locally bounded. The m-stability \mathcal{Q} follows then by part 3 of Proposition 2.0.2, which implies

$$E \left[-\log \left(Z_T^1 \frac{Z_T^2}{Z_T^2} \right) \right] \leq E[A_T^1 + (A_T^2 - A_T^2)] < \infty.$$

To show that the property (2.5) defining \mathcal{Q}^{ngd} is consistent with m-stability of \mathcal{Q}^{ngd} , let $\tau' \leq \tau$ be stopping times. Letting $T' = (T \vee \tau') \wedge \tau$ for T and Z as in Definition 2.0.4, we obtain, using again part 3 of Proposition 2.0.2, that

$$\begin{aligned} E_{\tau'} \left[-\log \frac{Z_{\tau'}}{Z_{\tau'}} \right] &= E_{\tau'} [A_{\tau'}^2 - A_{T'}^2 + A_{T'}^1 - A_{\tau'}^1] \\ &\leq \frac{1}{2} E_{\tau'} \left[\int_{\tau'}^{\tau} h_u^2 du \right]. \end{aligned}$$

\square

With respect to a given m-stable set \mathcal{S} of equivalent measures, we define for $X \in L^\infty$ the upper and lower valuation bounds by

$$\pi_t^u(X; \mathcal{S}) = \text{ess sup}_{Q \in \mathcal{S}} E_t^Q[X], \quad (2.14)$$

$$\pi_t^\ell(X; \mathcal{S}) = \text{ess inf}_{Q \in \mathcal{S}} E_t^Q[X]. \quad (2.15)$$

Clearly, $\pi^\ell(X; \mathcal{S}) = -\pi^u(-X; \mathcal{S})$ holds. Therefore, we can restrict the analysis to the upper bounds π^u in the sequel, without loss of generality.

The next results recalls that the family $\{\pi_t^u(X)\}_{t \leq \bar{T}}$ satisfies the properties of a dynamic coherent risk measure (or a dynamic monetary coherent utility functional), and can be represented by a process with good path properties. We will always choose such a version in the sequel without further notice. Moreover, it shows that the family of mappings $X \mapsto \pi_T^u(X)$ where T ranges over all stopping times $T \leq \bar{T}$, exhibits good dynamic consistency properties.

Proposition 2.0.6 *Assume $\mathcal{S} \neq \emptyset$ (e.g. $\mathcal{S} = \mathcal{Q}^{\text{ngd}}$). As mappings from L^∞ to $L^\infty(\mathcal{F}_t)$ the family $X \mapsto \pi_t^u(X; \mathcal{S})$ ($t \leq \bar{T}$) has the following properties.*

1. (Path properties) *For any $X \in L^\infty$, there is a version of $(\pi_t^u(X))_{t \leq \bar{T}}$ having RCLL paths and such that*

$$\pi_T^u(X) = \operatorname{ess\,sup}_{Q \in \mathcal{S}} E_T^Q[X] \quad \text{for all stopping times } T \leq \bar{T}.$$

2. (Recursiveness) *For any stopping times $T \leq \tau \leq \bar{T}$, it holds that*

$$\pi_T^u(X) = \pi_T^u(\pi_\tau^u(X)).$$

3. (Stopping-time consistency) *For stopping times $T \leq \tau \leq \bar{T}$, the inequality $\pi_\tau^u(X^1) \geq \pi_\tau^u(X^2)$ implies $\pi_T^u(X^1) \geq \pi_T^u(X^2)$.*
4. (Supermartingale property) *$(\pi_t^u(X))$ is a supermartingale under any $Q \in \mathcal{S}$.*
5. (Coherent risk measure) *For any stopping time $T \leq \bar{T}$ and $m_T, \alpha_T, \lambda_T \in L^\infty(\mathcal{F}_T)$ with $0 \leq \alpha_T \leq 1$, $\lambda_T \geq 0$, the mapping $X \mapsto \pi_T^u(X)$ satisfies the properties:*

- *monotonicity: $X^1 \geq X^2$ implies $\pi_T^u(X^1) \geq \pi_T^u(X^2)$*
- *translation invariance: $\pi_T^u(X + m_T) = \pi_T^u(X) + m_T$*
- *convexity: $\pi_T^u(\alpha_T X^1 + (1 - \alpha_T) X^2) \leq \alpha_T \pi_T^u(X^1) + (1 - \alpha_T) \pi_T^u(X^2)$*
- *positive homogeneity: $\pi_T^u(\lambda_T X) = \lambda_T \pi_T^u(X)$*

6. *No arbitrage consistency: If moreover $\mathcal{S} \subset \mathcal{M}^e$, then $\pi_T^u(X) = x + \vartheta \cdot S_T$ holds for any $X = x + \vartheta \cdot S_{\bar{T}}$ with $((\vartheta \cdot S_t)_{t \leq \bar{T}})$ being uniformly bounded.*

Proposition 2.0.6 is analogous to Theorem 2.7 in [20] and a direct consequence of results by Delbaen [13]. As the proof shows, it follows essentially from m -stability of \mathcal{S} .

Remark 2.0.7 In the literature on risk measures, those are often applied to the net value Y of a position. Under such convention, one would rather call $Y \mapsto \pi^u(-Y) = -\pi^\ell(Y)$ a dynamic coherent risk measure and call $Y \mapsto \pi^\ell(Y)$ a monetary coherent utility functional. Clearly, the difference is only a matter of sign conventions, where we take $X = -Y$ as a liability instead of a net value. This should not cause confusion.

Proof of Proposition 2.0.6: Since \mathcal{S} is m-stable by assumption, part 1 follows by Lemmata 22 and 23 from [13], and using his Theorem 12 yields the claims 2-4. Finally, part 5 is immediate from part 1. Also part 6 follows from 1, since bounded local martingales are uniformly integrable. \square

In the sequel, the aim is to study good-deal bounds with respect to optimal growth. In the next section, those bounds are shown to be valuations bounds as above for $\mathcal{S} = \mathcal{Q}^{\text{ngd}}(S)$. This explains the terminology of the next definition.

Definition 2.0.8 *The upper and lower good-deal bounds for $X \in L^\infty$ are defined by (2.14) respectively (2.15) for $\mathcal{S} = \mathcal{Q}^{\text{ngd}}(S)$. For brevity of notation, they are denoted in the sequel by*

$$\pi_t^u(X) := \pi_t^u(X; \mathcal{Q}^{\text{ngd}}(S)) \quad \text{and} \quad \pi_t^\ell(X) := \pi_t^\ell(X; \mathcal{Q}^{\text{ngd}}(S)). \quad (2.16)$$

3 Good-deal bounds for optimal growth rates: Duality results

To motivate the subsequent results, let us suppose that the market is to be extended by adding further tradable risky assets. From the general theory of no-arbitrage pricing, it is known that each arbitrage-free price process of contingent claims with (discounted) payoffs $X \in L^0(\mathcal{F}_{\bar{T}}, \mathbb{R}^{d'})$ should in principle be of the form

$$S'_t = S'_t(X, Q) = E_t^Q[X], \quad t \leq \bar{T}, \quad (3.1)$$

for some $Q \in \mathcal{M}^e$. Indeed, this ensures ‘no arbitrage’ for the extended market with risky asset price processes

$$\bar{S} = (S, S') = (S, S'(X, Q)) \quad (3.2)$$

evolving in $\mathbb{R}^{d+d'}$, in the sense that $Q \in \mathcal{M}^e(\bar{S}) \neq \emptyset$, subject to suitable integrability assumptions. A sufficient condition to ensure that \bar{S} meets the same assumptions that our general setting imposes on S is obviously that $X \in L^\infty$. A more general condition is that all coordinates of $X = (X^i)_{1 \leq i \leq d'}$ are bounded from below and integrable with respect to the $Q \in \mathcal{M}^e$ chosen in (3.1), i.e.

$$(X^i)^- \in L^\infty \quad \text{and} \quad X^i \in L^1(Q) \quad \text{for all } i = 1, \dots, d', \quad (3.3)$$

Then $S'(X, Q)$ is finite and bounded from below in each coordinate.

However, in incomplete markets the set \mathcal{M}^e generically contains not one unique martingale measure but infinitely many. The price range, which would be obtained from (3.1) by letting Q range over all \mathcal{M}^e is typically too wide for practical purposes. In other words, the restrictions on prices that result from no-arbitrage arguments alone will often lead to price bounds that are too wide. This leads naturally to the idea to let

Q range over a suitable smaller set (say \mathcal{Q}^{ngd}) to obtain tighter price bounds. To give financial meaning to the desired tighter bounds, the general idea of good-deal bounds is to let Q range over a subset of \mathcal{M}^e which is taken such that price processes $S'(X, Q)$ do not permit trading opportunities that are ‘too good’ to be realistic. Prices within the resulting range are then considered to be consistent with ‘no good deals’ while prices outside the range might be interpreted as ‘good deals’ for either the seller or buyer, depending whether they are above π^u or below π^ℓ .

There are different possibilities for defining good deals, which can result in different subsets of \mathcal{M}^e . We are going to show that our definition of \mathcal{Q}^{ngd} is such that *any* market extension $\bar{S} = (S, S')$ by price processes of the form (3.1) for $Q \in \mathcal{Q}^{\text{ngd}}$ does only permit for (conditional) expected optimal growth of returns at a rate not exceeding the bound $h_t^2/2$ specified via h in the definition (2.5) of \mathcal{Q}^{ngd} .

To show this, we are going to apply convex duality arguments. Recall that the conjugate function of $U(x) = \log x$ with $x > 0$ is

$$V(y) := \sup_{x>0} U(x) - xy = -\log y - 1, \quad y > 0. \quad (3.4)$$

It follows that

$$U(x) \leq V(y) + xy \quad \text{for all } x, y > 0, \quad (3.5)$$

with equality holding for $x = 1/y$.

Any local martingale $\bar{N} > 0$ with respect to some $Q \in \mathcal{M}^e$ is a Q -supermartingale. Hence, $\bar{N}Z$ is a P -supermartingale. Letting $x = \bar{N}_\tau/\bar{N}_T$ and $y = Z_\tau/Z_T$ in (3.5) for stopping times $T \leq \tau \leq \bar{T}$, we obtain by taking conditional expectations that

$$\begin{aligned} E_T \left[\log \frac{\bar{N}_\tau}{\bar{N}_T} \right] &\leq E_T \left[-\log \frac{Z_\tau}{Z_T} \right] + E_T \left[\frac{Z_\tau \bar{N}_\tau}{Z_T \bar{N}_T} \right] - 1 \\ &\leq E_T \left[-\log \frac{Z_\tau}{Z_T} \right] = E_T^Q \left[\frac{Z_T}{Z_\tau} \log \frac{Z_T}{Z_\tau} \right], \end{aligned} \quad (3.6)$$

and these inequalities become equalities for $\bar{N} = 1/Z$.

Theorem 3.0.9 *1. For any $Q \in \mathcal{Q}^{\text{ngd}}$ with density process Z and any X satisfying (3.3), it holds that any tradable numeraire $\bar{N} = 1 + \bar{\vartheta} \cdot \bar{S} \in \mathcal{N}(\bar{S})$ in the extended market (3.2) satisfies*

$$E_T \left[\log \frac{\bar{N}_\tau}{\bar{N}_T} \right] \leq E_T \left[-\log \frac{Z_\tau}{Z_T} \right] \leq \frac{1}{2} E_T \left[\int_T^\tau h_u^2 du \right] \quad (3.7)$$

for all stopping times $T \leq \tau \leq \bar{T}$.

2. The former inequality in (3.7) is sharp: For any $Q \in \mathcal{Q}^{\text{ngd}}$ with density process Z , there exists a real-valued X satisfying (3.3) (with $d' = 1$) such that $\bar{N} :=$

$S'(X, Q)/S'_0(X, Q)$ in $\mathcal{N}(\bar{S})$ is a tradable numeraire in the extended market (3.2), satisfying

$$E_T \left[\log \frac{\bar{N}_\tau}{\bar{N}_T} \right] = E_T \left[-\log \frac{Z_\tau}{Z_T} \right] \quad (3.8)$$

for all stopping times $T \leq \tau \leq \bar{T}$.

Proof: Equation (3.6) and the remark following it imply the claim. \square

The first part of the theorem shows that \mathcal{Q}^{ngd} is defined such that all market extensions (3.2) based on $Q \in \mathcal{Q}^{\text{ngd}}$ will respect the good-deal bounds on expected growth rates specified via h . The second part shows that the duality relation is sharp, in that the duality bound (3.6) is attained by a suitable market extension with respect to a given Q . Whether the second inequality in (3.7) is sharp and might be attained for some Q with equality, in general depends on the market model under consideration. For instance, if \mathcal{M}^e were a singleton $\{\hat{Q}\}$ with $\hat{Q} \in \mathcal{Q}^{\text{ngd}}$, the second inequality in (3.7) easily becomes strict if h is taken somewhat larger.

4 A Financial Market Model with Itô-Processes

In the next sections, we are going to obtain more explicit results on good-deal valuation bounds arising from restrictions on expected growth rates, in comparison to the convex duality results for general semimartingale models in the previous section. To this end, we study a model where the dynamics of financial market prices S are described by Itô processes on a Wiener space. This permits to use in Section 5 the well-developed theory of backward stochastic differential equations (BSDEs) to describe the dynamics of the good-deal valuation bounds and a corresponding notion of hedging more explicitly.

An Itô process model

From the remainder of the paper, we strengthen the assumptions from Section 2 on the underlying model, by assuming that the filtration \mathbb{F} of our stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is the filtration generated by an n -dimensional Brownian motion (Wiener process) $(W_t)_{t \in [0, \bar{T}]}$, completed by nullsets. It is well known that \mathbb{F} is then right-continuous. Furthermore, we assume there are $d \leq n$ risky assets whose price processes $S = (S_t^i)_{1 \leq i \leq d}$ are described by the unique solution to the stochastic differential equation (SDE)

$$dS_t^i = S_t^i dR_t^i, \quad t \leq \bar{T}, \quad 1 \leq i \leq d, \quad (4.1)$$

with $S_0^i > 0$, where the return process $R = (R_t^i)_{1 \leq i \leq d}$ is given by the solution to the SDE

$$dR_t = \gamma_t dt + \sigma_t dW_t. \quad (4.2)$$

We assume that γ and σ are predictable processes taking values in \mathbb{R}^d and $\mathbb{R}^{d \times n}$. The volatility process σ is taken to have full rank d , in the sense that

$$\det \sigma \sigma^{\text{tr}} \neq 0 \quad (P \times dt - a.e.). \quad (4.3)$$

Denoting the market price of risk process by

$$\xi := \sigma^{\text{tr}}(\sigma\sigma^{\text{tr}})^{-1}\gamma \quad (4.4)$$

one can write the SDE describing R compactly as

$$R_0 = 0, \quad dR_t = \sigma_t(\xi_t dt + dW_t) =: \sigma_t d\widehat{W}_t, \quad t \leq \bar{T}. \quad (4.5)$$

We assume that there exists some $\varepsilon \in (0, \infty)$ such that, for h from (2.5), it holds

$$h - |\xi| > \varepsilon \quad (dP \times dt - a.e.). \quad (4.6)$$

By boundedness of h , this implies in particular that

$$\text{the market price of risk process } \xi \text{ is bounded.} \quad (4.7)$$

Each $S^i = S_0^i \mathcal{E}(R^i)$ is a stochastic exponential and can be written explicitly as

$$S_t^i = S_0^i \exp \left(\int_0^t (\sigma_u)^i dW_u + \int_0^t (\sigma_u \xi_u)^i - \frac{1}{2} (\sigma_u \sigma_u^{\text{tr}})^{ii} du \right).$$

For subsequent analysis, it turns out to be more convenient to describe trading strategies not in terms of numbers $\vartheta = (\vartheta_t^i)$ of risky assets held, but instead by amounts of wealth $\varphi = (\varphi_t^i)$ invested in each of the risky assets. To this end, we define the set Φ_φ of permitted trading strategies to consist of those predictable processes $\varphi = (\varphi_t^i)$ which satisfy

$$E \left[\int_0^{\bar{T}} |\varphi_t^{\text{tr}} \sigma_t|^2 dt \right] < \infty. \quad (4.8)$$

The wealth process V that is obtained from initial wealth V_0 by investing according to $\varphi \in \Phi$ is given by the solution to the SDE

$$dV_t = \varphi_t^{\text{tr}} dR_t = \varphi_t^{\text{tr}} \sigma_t (\xi_t dt + dW_t) = \varphi_t^{\text{tr}} \sigma_t d\widehat{W}_t, \quad (4.9)$$

where all occurring integral terms are being well defined thanks to

$$\int_0^{\bar{T}} |\xi_t^{\text{tr}} \sigma_t^{\text{tr}} \varphi_t| dt \leq \left(\int_0^{\bar{T}} |\sigma_t^{\text{tr}} \varphi_t|^2 dt \right)^{\frac{1}{2}} \left(\int_0^{\bar{T}} |\xi_t|^2 dt \right)^{\frac{1}{2}} < \infty.$$

Remark 4.0.10 In the Itô process framework for the financial market S in this section the risky asset prices are modeled as continuous processes. This is like in the continuous time models in [12] or [10], and different from [7] where asset prices can also have jumps. On the other hand, differently from [12, 10, 7] our dynamics (4.1) do not impose any Markov structure on the evolution of S , neither alone nor jointly with additional factor processes.

Parameterization of strategies

Strategies φ in Φ_φ have as clear financial meaning in terms of wealth invested in the risky assets. Still, it will be technically convenient to re-parameterize strategies in terms of integrands with respect to $d\widehat{W} = \xi_t dt + dW_t$. To this end, let

$$C_t := \text{Im } \sigma_t^{\text{tr}}, \quad t \leq \bar{T},$$

denote the image (range) of $\sigma_t^{\text{tr}} \in \mathbb{R}^{n \times d}$. With any $\varphi \in \Phi$ one can associate the image of φ under σ^{tr} , i.e.

$$\phi_t = \sigma_t^{\text{tr}} \varphi_t \in C_t, \quad t \leq \bar{T},$$

and write the evolution of wealth (4.9) conveniently as

$$dV = \phi^{\text{tr}}(\xi dt + dW) = \phi^{\text{tr}} d\widehat{W}. \quad (4.10)$$

We let $\Phi := \Phi_\phi := \{\phi \mid \phi = \sigma_t^{\text{tr}} \varphi_t, \varphi \in \Phi_\varphi\}$. Then, by definition of Φ_φ ,

$$\Phi = \Phi_\phi = \left\{ \phi \mid \phi \text{ is predictable, } \phi_t \in C_t \forall t, E \left[\int_0^{\bar{T}} |\phi_t|^2 dt \right] < \infty \right\}.$$

By applying the pseudo-inverse $\sigma^{\text{tr}+} := (\sigma \sigma^{\text{tr}})^{-1} \sigma$ to $\phi = \sigma^{\text{tr}} \varphi$, one re-obtains $\varphi = \sigma^{\text{tr}+} \phi$. Hence, the relations $\varphi = \sigma^{\text{tr}+} \phi$ and $\phi = \sigma^{\text{tr}} \varphi$ provide a bijection between $\Phi = \Phi_\phi$ and Φ_φ . This allows to consider Φ as the set of permitted trading strategies.

Let $\Pi_t = \Pi_{C_t}$ and $\Pi_t^\perp = \Pi_{C_t^\perp}$ denote the orthogonal projections onto $C_t = \text{Im } \sigma_t^{\text{tr}} = (\text{Ker } \sigma_t)^\perp$ and $C_t^\perp = \text{Ker } \sigma_t = (\text{Im } \sigma_t^{\text{tr}})^\perp$, respectively, that are given by

$$\Pi_t : \mathbb{R}^n \rightarrow C_t, \quad z \mapsto \Pi_t(z) := \sigma^{\text{tr}} (\sigma_t \sigma_t^{\text{tr}})^{-1} \sigma_t z, \quad (4.11)$$

$$\Pi_t^\perp : \mathbb{R}^n \rightarrow C_t^\perp, \quad z \mapsto \Pi_t^\perp(z) := (\text{Id} - \Pi_t)(z). \quad (4.12)$$

Clearly, any z in $\mathbb{R}^n = C_t \oplus C_t^\perp = \text{Im } \sigma_t^{\text{tr}} \oplus \text{Ker } \sigma_t$ has a unique orthogonal decomposition

$$z = \Pi_t(z) + \Pi_t^\perp(z) = \Pi_{\text{Im } \sigma_t^{\text{tr}}}(z) + \Pi_{\text{Ker } \sigma_t}(z) \quad \text{in } C_t \oplus C_t^\perp. \quad (4.13)$$

Equivalent martingale measures

In the current setting of a Brownian filtration, the equivalent martingale measures $Q \in \mathcal{M}^e$ for S can be parameterized quite explicitly.

In order to construct a martingale measure, one can use the Girsanov transformation to eliminate the drift of S . For instance, since ξ is bounded,

$$\frac{d\widehat{Q}}{dP} := \mathcal{E} \left(- \int \xi dW \right)_{\bar{T}}$$

clearly defines a probability measure \widehat{Q} , which is known as the minimal martingale measure, see e.g. [25]. Under \widehat{Q} , the process

$$\widehat{W}_0 = 0, \quad d\widehat{W}_t = dW_t + \xi_t dt \quad \text{for } t \leq \bar{T}, \quad (4.14)$$

is Brownian motion.

More generally, the density process Z of any equivalent measure $Q \sim P$ must be a stochastic exponential

$$Z_t = \frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \mathcal{E}(L)_t = \mathcal{E} \left(\int \lambda dW \right)_t, \quad t \leq \bar{T},$$

with $dL := \frac{1}{Z} dZ$ being a local martingale of the form $L = \int \lambda dW$ for some predictable $\lambda = \lambda^Q$ with $\int_0^{\bar{T}} |\lambda|^2 dt < \infty$. By Girsanov's theorem and Lévy's characterization

$$W^Q = W - \int \lambda dt \quad \text{is a } Q\text{-Brownian Motion.}$$

Since

$$\sigma_t d\widehat{W}_t = \sigma_t ((\xi_t + \lambda_t) dt + dW_t^Q)$$

holds, Q is in \mathcal{M}^e if and only if $\sigma_t(\xi_t + \lambda_t) = 0$ holds ($P \times dt$ -a.e.), that is $\lambda_t^Q = -\xi_t + \eta_t$ with $\eta_t = \eta_t^Q \in \text{Ker } \sigma_t = C_t^\perp$. Hence, any equivalent martingale measure Q for S must have a density process of the form

$$Z_t := \frac{dQ}{dP} \Big|_t = \mathcal{E} \left(\int \lambda dW \right)_t = \mathcal{E} \left(- \int \xi dW \right)_t \mathcal{E} \left(\int \eta dW \right)_t, \quad t \leq \bar{T}, \quad (4.15)$$

with $\lambda = -\xi + \eta$ satisfying $-\xi = \Pi_t(\lambda)$ and $\eta = \Pi_t^\perp(\lambda)$ ($P \times dt$ -a.e.). Since $\sigma\eta = 0$ implies $\eta^{\text{tr}}\xi = 0$, the second equality in (4.15) holds by Yor's formula.

If $d = n$ (as many risky assets as sources of noise), it thus holds that $\eta = \sigma^{-1}0 = 0$, hence \widehat{Q} is the unique equivalent local martingale measure for S . In that case, the market is complete by the strong predictable representation theorem.

The next result summarizes the convenient parameterization of \mathcal{Q} and \mathcal{Q}^{ngd} .

Proposition 4.0.11 *1. Any $Q \in \mathcal{Q}$ has a density process $Z = Z^Q$ of the form (4.15) with a predictable process $\lambda = -\xi + \eta$ with $\Pi_t(\lambda_t) = -\xi_t$ and $\Pi_t^\perp(\lambda_t) = \eta_t$, satisfying $\int_0^{\bar{T}} |\lambda_t|^2 dt < \infty$. In particular, $\xi^{\text{tr}}\eta = 0$. The processes λ , η , ξ are unique ($P \times dt$ -a.e.). For $Q \in \mathcal{Q}^{\text{ngd}} \subset \mathcal{Q}$, it holds in addition that $|\lambda|^2 = |\xi|^2 + |\eta|^2 \leq h^2$.*

2. In turn, any predictable λ with $|\lambda|^2 \leq h^2$ and $\Pi_t(\lambda_t) = -\xi_t$ ($P \times dt$ -a.e.) defines a density process Z of the form (4.15) for some $Q \in \mathcal{Q}^{\text{ngd}}$ with $\eta = \Pi_t^\perp(\lambda)$.

3. For $Q \in \mathcal{Q}$, the Doob-Meyer decomposition (2.7) for $-\log Z$ is given by

$$M_t = \int_0^t \xi - \eta dW, \quad (4.16)$$

$$A_t = \frac{1}{2} \int_0^t |\lambda_s|^2 ds, \quad t \leq \bar{T}. \quad (4.17)$$

Proof: Part 3 and the first claims of part 1 follows from the foregoing discussion, and $|\lambda| \leq h$ ($P \times dt$ -a.e.) then holds by Proposition 2.0.3, which implies that the

predictable set $B = \{|\lambda| > h\}$ is a nullset with respect to $P \times dt$ for $\lambda = \lambda^Q$ from $Q \in \mathcal{Q}^{\text{ngd}}$. Concerning part 2, assumption $|\lambda|^2 \leq h^2$ and boundedness of h imply that Z is a martingale and defines a measure $Q \sim P$, which is martingale measure by the foregoing discussion. The integrability condition (2.5) defining \mathcal{Q}^{ngd} is readily verified for Q from $|\lambda|^2 = |\xi|^2 + |\eta|^2 \leq h^2$. \square

Backward SDEs

This section introduces the notion and recalls some classical results on standard BSDEs whose generator satisfies a Lipschitz condition, as stated in [15].

For $p \in (1, \infty)$, we denote by $\mathcal{S}_T^p = \mathcal{S}_T^p(P)$ the space of real valued adapted RCLL processes Y with the norm $\|Y\|_{\mathcal{S}_T^p} := E[\sup_{t \leq T} |Y_t|^p]^{1/p} < \infty$. Let $\mathcal{H}_T^p = \mathcal{H}_T^p(P)$ denote the space of predictable \mathbb{R}^n -valued processes Z with the norm $\|Z\|_{\mathcal{H}_T^p} := E\left[\left(\int_0^T |Z_t|^2 dt\right)^{p/2}\right]^{1/p} < \infty$. Let $BMO(P)$ denote the subspace of those $Z \in \mathcal{H}_T^2(P)$ which satisfy that there is some $c \in \mathbb{R}_+$ such that $E_T\left[\int_T^{\bar{T}} |Z_t|^2 dt\right] < c$ for all stopping times T . For any Z in $BMO(P)$, the process $Z \cdot W$ is called a P -BMO martingale. The abbreviation BMO stands for ‘bounded mean oscillation’, see Chapter X in [17].

A (simplified) standard generator of a BSDE is a measurable function

$$f : (\Omega \times [0, \bar{T}] \times \mathbb{R}^n, \mathcal{P} \times \mathcal{B}(\mathbb{R}^n)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

which is such that $f_t(z) = f(\omega, t, z)$ satisfies $(f_t(0))_{t \leq \bar{T}} \in \mathcal{H}_T^2$, and $P \times dt$ -a.e.

$$|f_t(z) - f_t(z')| \leq L_f |z - z'| \quad \text{for all } z, z' \in \mathbb{R}^n,$$

with some Lipschitz constant $L_f < \infty$. For given BSDE standard parameters (f, X) , which we take to be given by a standard generator f and a terminal condition $X \in L^2(P)$, a solution to the BSDE

$$Y_{\bar{T}} = X \quad \text{and} \quad -dY_t = f_t(Z_t)dt - Z_t dW_t, \quad t \leq \bar{T}, \quad (4.18)$$

is a tuple (Y, Z) of processes in $\mathcal{S}_T^2 \times \mathcal{H}_T^2$, satisfying

$$Y_t = X + \int_t^{\bar{T}} f_u(Z_u) du - \int_t^{\bar{T}} Z dW \quad \text{for all } t \leq \bar{T}. \quad (4.19)$$

Since it is sufficient for our purposes, we consider simplified generators f , in comparison to [15], that do not depend on Y .

Proposition 4.0.12 *For given standard BSDE parameters (f, X) , there exists a unique solution $(Y, Z) \in \mathcal{S}_T^2 \times \mathcal{H}_T^2$ to the BSDE (4.18).*

This result holds by Theorems 2.1 and 5.1 in [15]. It ensures that unique solutions exists for the BSDEs in subsequent sections. Let us note that, more generally, there is even a unique (Y, Z) in $\mathcal{S}_T^p \times \mathcal{H}_T^p$ ($1 < p < \infty$) satisfying (4.19) if the BSDE data

satisfy $X \in L^p(P)$ and condition $(f_t(0))_{t \leq \bar{T}} \in \mathcal{H}_{\bar{T}}^2$ is replaced by $(f_t(0))_{t \leq \bar{T}} \in \mathcal{H}_{\bar{T}}^p$, see [15], Theorem 5.1.

The main argument for identifying solutions to optimal control problems in the sequel in terms of BSDE solutions is provided by the next result, which is a simplified version of Proposition 3.1 in [15].

Proposition 4.0.13 *For a family of standard parameters (f, X) and $\{(f^\alpha, X)\}$, with α from an arbitrary index set, let (Y, Z) and (Y^α, Z^α) denote the solution to the corresponding BSDEs. If there exists $\bar{\alpha}$ such that*

$$f_t(Z_t) = f_t^{\bar{\alpha}}(Z_t) = \operatorname{ess\,inf}_{\alpha} f_t^\alpha(Z_t), \quad P \times dt - a.e.,$$

then $Y_t = \operatorname{ess\,inf}_{\alpha} Y_t^\alpha = Y_t^{\bar{\alpha}}$ holds for all $t \leq \bar{T}$, P -a.s..

5 Good-deal valuation and hedging via BSDEs

In this section, we obtain a dynamic description for the good-deal valuation bounds, that arise from no-good-deal restrictions on optimal expected growth rates. The valuation bounds are given by the solutions to standard non-linear backward SDEs, whose generator satisfies a Lipschitz condition. Moreover, we develop a corresponding notion of hedging and show that also the hedging strategy is described by a BSDE.

Dynamic good-deal valuation bounds

In extension to Definition 2.0.8, let us define for $X \in L^2(P) \supset L^\infty$

$$\pi_t^u(X) := \operatorname{ess\,sup}_{Q \in \mathcal{Q}^{\text{ngd}}} E_t^Q[X], \quad t \leq \bar{T}. \quad (5.1)$$

Using an L^2 space for X fits conveniently with the present BSDE setting. As results in the present section show, (5.1) induces a mapping $L^2(P) \rightarrow L^2(P, \mathcal{F}_t)$. Using results from standard BSDE theory one can check that properties 1.-4. of Proposition 2.0.6 are maintained for $X \mapsto \pi_t^u(X)$ ($t \leq \bar{T}$) with $X \in L^2(P)$ and that properties 5.-6. follow from (5.1).

Lemma 5.0.14 *Let $Q \sim P$ with density process $\frac{dQ}{dP}|_{\mathbb{F}} =: D = \mathcal{E}(\int \lambda dW)$ for a predictable and bounded process λ . Then there exists a unique solution $(Y, Z) \in \mathcal{S}_{\bar{T}}^2 \times \mathcal{H}_{\bar{T}}^2$ to the linear BSDE*

$$Y_{\bar{T}} = X, \quad -dY_t = -\lambda_t^{\text{tr}} Z_t dt - Z_t dW_t, \quad t \leq \bar{T}, \quad (5.2)$$

for $X \in L^2(P)$. Moreover Y is a Q -martingale and $W^\lambda := W - \int \lambda dt$ is a Q Brownian motion, satisfying

$$Y_t = E_t^Q[X] = Y_0 + \int_0^t Z dW^\lambda, \quad t \leq \bar{T}.$$

If $X \in L^\infty$, then Y is bounded and Z is in $BMO(P)$, i.e. $Z \cdot W$ is a P -BMO martingale.

Proof: Since λ is bounded, the parameters of the linear BSDE (5.2) are standard and it has a unique solution (Y, Z) in $\mathcal{S}_{\bar{T}}^2 \times \mathcal{H}_{\bar{T}}^2$. By application of Itô's formula, the process DY is seen to be a local P -martingale. Because D is in $\mathcal{S}_{\bar{T}}^p(P)$ for any $p < \infty$, DY is in $\mathcal{S}_{\bar{T}}^{2-\varepsilon}(P)$ for any $\varepsilon > 0$ and thus a P -martingale. Hence, Y is a Q -martingale with $Y_t = E_t^Q[X]$. If $X \in L^\infty$, $Y - Y_0 = Z \cdot W^\lambda$ is bounded, so $Z \cdot W^\lambda$ is a Q -BMO martingale. This implies that $Z \cdot W$ is a P -BMO martingale by Theorem 3.6 from [19], noting that $dP/dQ|_{\mathbb{F}} = \mathcal{E}(-\lambda \cdot W^\lambda)$ holds for $\lambda \cdot W^\lambda$ being a Q -BMO martingale due to the boundedness of λ . \square

Recall that the density process of any $Q \in \mathcal{Q}^{\text{ngd}}$ is determined by the predictable process $\eta = \eta^Q = \Pi^\perp(\lambda^Q)$ from Proposition 4.0.11. For any $\eta = \eta^Q$ let $(Y^\eta, Z^\eta) \in \mathcal{S}_{\bar{T}}^2 \times \mathcal{H}_{\bar{T}}^2$ denote the solution to the BSDE

$$\begin{aligned} -dY_t &= (-\xi_t^{\text{tr}} \Pi_t(Z_t) + \eta_t^{\text{tr}} \Pi_t^\perp(Z_t)) dt - Z_t dW_t, \quad t \leq \bar{T}, \\ Y_{\bar{T}} &= X. \end{aligned} \quad (5.3)$$

We are going to demonstrate that the good-deal bound $\pi_t^u(X)$ is described by the solution to the BSDE

$$\begin{aligned} -dY_t &= \left(-\xi_t^{\text{tr}} \Pi_t(Z_t) + \sqrt{h_t^2 - |\xi_t|^2} |\Pi_t^\perp(Z_t)| \right) dt - Z_t dW_t, \quad t \leq \bar{T}, \\ Y_{\bar{T}} &= X. \end{aligned} \quad (5.4)$$

Theorem 5.0.15 *For $X \in L^2(P)$, let (Y, Z) and the family (Y^η, Z^η) (for $\eta = \eta^Q$ with $Q \in \mathcal{Q}^{\text{ngd}}$) be the solutions to the standard BSDEs (5.4) and (5.3), respectively. Then, there exists $\bar{Q} = Q^{\bar{\eta}} \in \mathcal{Q}^{\text{ngd}}$ corresponding (by (4.15)) to*

$$\bar{\eta} = \eta^{\bar{Q}} = \frac{\sqrt{h^2 - |\xi|^2}}{|\Pi_t^\perp(Z)|} \Pi_t^\perp(Z)$$

such that

$$Y_t = \text{ess sup}_\eta Y_t^\eta = Y_t^{\bar{\eta}}, \quad t \leq \bar{T}, \quad (5.5)$$

holds, with η ranging over all $\eta = \eta^Q$ for $Q \in \mathcal{Q}^{\text{ngd}}$. Moreover, the upper good-deal bound for X is given by

$$\pi_t^u(X) = \text{ess sup}_{Q \in \mathcal{Q}^{\text{ngd}}} E_t^Q[X] = E_t^{\bar{Q}}[X] = Y_t, \quad t \leq \bar{T}. \quad (5.6)$$

This result is not surprising. Noting the definition of the good-deal bound as a supremum of conditional expectation, the equality (5.6) is basically a special case of the dual representation for g -conditional risk measures (respectively, for non-linear g -expectations), which are defined by solutions to BSDEs with suitable generators g , see [4]. We give a short direct proof in our setting, to show how Proposition 4.0.13 provides the essential argument for the equality (5.6) and to identify $\bar{\eta}$ and \bar{Q} explicitly for π^u .

Proof: Comparing the BSDE generators in (5.4) and (5.3), evaluated at Z , one sees that for any η the inequality

$$-\xi_t^{\text{tr}} \Pi_t(Z_t) + \eta_t^{\text{tr}} \Pi_t^\perp(Z_t) \leq -\xi_t^{\text{tr}} \Pi_t(Z_t) + \sqrt{h^2 - |\xi_t|^2} |\Pi_t^\perp(Z_t)|$$

holds ($P \times dt$ -a.e.) by the definition of \mathcal{Q}^{ngd} , since $|-\xi + \eta|^2 = |\xi|^2 + |\eta|^2 \leq h^2$ with equality holding for $\bar{\eta} := \sqrt{h^2 - |\xi_t|^2} \Pi_t^\perp(Z) / |\Pi_t^\perp(Z)|$. By Proposition 4.0.13 thus follows (5.5). By Lemma 5.0.14, we have $Y_t^\eta = E_t^Q[X]$ for $\eta = \eta^Q$ and $Y_t^{\bar{\eta}} = E_t^{\bar{Q}}[X]$. Hence, (5.5) and the definition of $\pi^u(X)$ imply (5.6). \square

Dynamic good-deal hedging

We are going to show in this section, that a hedging strategy which minimizes a suitable dynamic coherent risk measure is naturally linked to the good-deal valuation bounds.

To this end, let \mathcal{P}^{ngd} denote the set of those equivalent probability measures Q whose density process is of the form $\mathcal{E}(\int \lambda dW)$ for a predictable bounded process λ with $|\lambda| \leq h$, that is

$$\mathcal{P}^{\text{ngd}} := \left\{ Q \sim P \mid \frac{dQ}{dP} \Big|_{\mathbb{F}} = \mathcal{E} \left(\int \lambda dW \right) \text{ for } \lambda \text{ predictable with } |\lambda| \leq h \right\}. \quad (5.7)$$

The notation \mathcal{P}^{ngd} is motivated by the following observation. In Section 3, it has been shown that the defining properties for $\mathcal{Q}^{\text{ngd}}(S)$ ensure two properties for any market $\bar{S} = (S, S')$ (3.2) that is enlarged by an additional price process S' which is obtained by some pricing measure Q from \mathcal{Q}^{ngd} . Firstly, the enlarged market is free of arbitrage since $Q \in \mathcal{M}^e(S)$. Secondly, it does not permit investment opportunities whose expected growth rates exceed the good-deal restrictions. How would the situation be, if we would take a step back and reduce instead of increase the number of risky assets? Suppose we start from a trivial initial market without risky assets S where only the riskless numeraire asset with price 1 is tradable. Since $\mathcal{M}^e(1) = \{Q \mid Q \sim P\}$, the integrability condition in the definition (2.5) of $\mathcal{Q}^{\text{ngd}}(1)$ yields $\mathcal{Q}^{\text{ngd}}(1) = \mathcal{P}^{\text{ngd}}$. That means, \mathcal{P}^{ngd} is such that any market $(1, S')$, whose price processes S' are given by (3.1) for some $Q \in \mathcal{P}^{\text{ngd}}$, does not permit trading opportunities that are ‘too good’.

In analogy to the definition of $\pi^u(X) = \pi^u(X; \mathcal{Q}^{\text{ngd}})$ in (2.16), we define

$$\rho_t(X) := \text{ess sup}_{Q \in \mathcal{P}^{\text{ngd}}} E_t^Q[X], \quad t \leq \bar{T}, \quad (5.8)$$

for $X \in L^2(P)$. That is, $\rho(X) = \pi^u(X; \mathcal{Q}^{\text{ngd}}(1))$ is of the same ‘good-deal’-type as $\pi^u(X) = \pi^u(X; \mathcal{Q}^{\text{ngd}}(S))$ but defined with respect to 1 instead of S .

Due to m -stability of \mathcal{P}^{ngd} , the mapping ρ satisfies by Proposition 2.0.6 the same good dynamic properties (on L^∞) as π^u and is thus a dynamic coherent risk measure. By Proposition 5.0.16, $X \mapsto \rho_t(X)$ induces a mapping $L^2(P) \rightarrow L^2(P, \mathcal{F}_t)$ and, in the same way as with π^u , one can show that for ρ the properties stated in Proposition 2.0.6 for $X \in L^\infty$ even hold on $L^2(P)$. By $\mathcal{P}^{\text{ngd}} \supset \mathcal{Q}^{\text{ngd}} = \mathcal{Q}^{\text{ngd}}(S)$, it is clear that $\rho(X) \geq \pi^u(X)$. We are going to show that $\pi_t^u(X)$ is obtained from $\rho_t \left(X - \int_t^{\bar{T}} \phi d\widehat{W} \right)$ by minimizing over all permitted trading strategies $\phi \in \Phi$.

To show firstly that $\rho_t(X)$ is described by the solution to the BSDE

$$\begin{aligned} -dY_t &= h|Z_t|dt - Z_t dW_t, \quad t \leq \bar{T}, \\ Y_{\bar{T}} &= X, \end{aligned} \quad (5.9)$$

we consider for any $Q \in \mathcal{P}^{\text{ngd}}$ the BSDE

$$\begin{aligned} -dY_t &= \lambda_t^{\text{tr}} Z_t dt - Z_t dW_t, \quad t \leq \bar{T}, \\ Y_{\bar{T}} &= X. \end{aligned} \quad (5.10)$$

where $\lambda = \lambda^Q$ denotes the process in (5.7) that determines the density of Q .

Proposition 5.0.16 *For $X \in L^2(P)$, let (Y, Z) and the family (Y^λ, Z^λ) (for $\lambda = \lambda^Q$ with $Q \in \mathcal{P}^{\text{ngd}}$) be the solutions to the standard BSDEs (5.9) and (5.10), respectively. Then, there exists $\hat{Q} = Q^{\hat{\lambda}} \in \mathcal{P}^{\text{ngd}}$ corresponding (by (5.7)) to $\hat{\lambda} = \lambda^{\hat{Q}} = hZ/|Z|$, such that*

$$Y_t = \text{ess sup}_{\lambda} Y_t^\lambda = Y_t^{\hat{\lambda}}, \quad t \leq \bar{T}, \quad (5.11)$$

holds, with λ ranging over all $\lambda = \lambda^Q$ for $Q \in \mathcal{P}^{\text{ngd}}$. Moreover,

$$\rho_t(X) = \text{ess sup}_{Q \in \mathcal{P}^{\text{ngd}}} E_t^Q[X] = E_t^{\hat{Q}}[X] = Y_t, \quad t \leq \bar{T}. \quad (5.12)$$

Since the proof for this result is very similar to the one of Theorem 5.0.15, we leave the details to the reader.

To motivate the next result on hedging, consider an investor who holds a contingent claim and is obliged to pay the liability X at maturity \bar{T} . If he measures his risk by the ‘a priori’ dynamic coherent risk measure ρ_t , he would assign at time t the monetary risk $\rho_t(X)$ to his liability if he had no access to the financial market S . By dynamic trading over the remaining period $(t, \bar{T}]$ according to some strategy $\phi \in \Phi$, he can transform his liability to $X - \int_t^{\bar{T}} \phi d\widehat{W}$. Accessing his risk in terms of the risk measure ρ_t , he should thus aim to trade according to some ϕ^* which minimizes $\rho_t \left(X - \int_t^{\bar{T}} \phi d\widehat{W} \right)$ at any time t . We are going to show that this is a well-posed optimal control problem whose value function turns out to be the good-deal bound $\pi_t^u(X)$ and whose optimal strategy ϕ^* can be obtained from the solution to the BSDE (5.4).

To this end, we consider for any permitted trading strategy $\phi \in \Phi$ the solution (Y^ϕ, Z^ϕ) to the BSDE

$$\begin{aligned} -dY_t &= (-\xi_t^{\text{tr}} \phi_t + h_t |\phi_t - Z_t|) dt - Z_t dW_t, \quad t \leq \bar{T}, \\ Y_{\bar{T}} &= X. \end{aligned} \quad (5.13)$$

This BSDE has a standard generator since h and ξ are bounded and $\phi \in \Phi = \mathcal{H}_{\bar{T}}^2$.

Theorem 5.0.17 *For $X \in L^2(P)$, let (Y, Z) and the family (Y^ϕ, Z^ϕ) (for $\phi \in \Phi$) be the solutions to the standard BSDEs (5.4) and (5.13), respectively. Then the strategy*

$$\phi^* = \frac{|\Pi^\perp(Z)|}{\sqrt{h^2 - |\xi|^2}} \xi + \Pi(Z) \quad (5.14)$$

is in Φ and satisfies

$$Y_t = \operatorname{ess\,inf}_{\phi \in \Phi} Y_t^\phi = Y_t^{\phi^*}, \quad t \leq \bar{T}. \quad (5.15)$$

Moreover, the upper good-deal bound for X is given by

$$\pi_t^u(X) = Y_t = \operatorname{ess\,inf}_{\phi \in \Phi} \rho_t \left(X - \int_t^{\bar{T}} \phi d\widehat{W} \right) = \rho_t \left(X - \int_t^{\bar{T}} \phi^* d\widehat{W} \right) \quad (5.16)$$

for $t \leq \bar{T}$, and ϕ^* is the unique $(P \times dt)$ minimizer from Φ for the infimum in (5.16).

Remark 5.0.18 1. Equation 5.16 shows, that the hedging strategy (5.14) minimizes the a priori risk measure ρ_t of the residual risk simultaneously for all $t \leq \bar{T}$. Being coherent, ρ is a monetary risk measure (see [2]) and the good deal bound can be interpreted as the minimal capital required to make the position acceptable, after optimal hedging according to ϕ^* .

2. By the relation $\pi^\ell(X) = -\pi^u(-X)$ between lower and upper good-deal bounds, the result also yields the lower good-deal bound and the corresponding hedging strategy. Since the BSDE (5.4) is non-linear, upper and lower bounds as well as the respective hedging strategies are different, in general.
3. By (5.16) and (5.14), not only the good deal bounds but also the hedging strategies are given explicitly in terms of solutions (Y, Z) to the BSDE (5.4).
4. If $X \in L^2(P)$ is replicable by some $\phi^X \in \Phi$, in the sense that $X = c + \phi^X \cdot \widehat{W}_{\bar{T}}$ with $c \in \mathbb{R}$, then the solution (Y, Z) to the linear BSDE $-dY_t = -\xi_t^{\operatorname{tr}} Z_t dt - Z_t dW_t$ with $Y_{\bar{T}} = X$ (see Lemma 5.0.14) satisfies $Z = \phi^X$ and $Y_t = E_t^Q[X] = c + \phi^X \cdot \widehat{W}_t$ for any $Q \in \mathcal{Q}^{\operatorname{ngd}}$. Since $\phi_t^X \in C_t = \operatorname{Ker} \Pi_t^\perp$ holds, uniqueness of the BSDE solution and (5.14) imply that $\phi^* = \phi^X$.
5. Since good-deal bounds (5.16) and the corresponding hedging strategies (5.14) are given in terms of solutions to standard BSDEs, they can be computed by available numerical methods for BSDEs. Monte Carlo simulation methods for BSDEs are of particular relevance for problems in higher dimensions. We refer to [8, 16, 6] and references therein for advances in this field.

Proof of Theorem 5.0.17: That ϕ^* from (5.14) is indeed in Φ follows since Z is in \mathcal{H}_T^2 , ξ is bounded (4.7), and $|h| - |\xi| > \varepsilon$ holds by (4.6). Comparing the BSDE generators evaluated at Z , we obtain by Lemma 6.0.20 that for any $\phi \in \Phi$ the inequality

$$-\xi_t^{\operatorname{tr}} \phi_t + h_t |\phi_t - Z_t| \geq -\xi_t^{\operatorname{tr}} \Pi_t(Z_t) + \sqrt{h_t^2 - |\xi_t|^2} |\Pi_t^\perp(Z_t)|$$

holds $(P \times dt)$ -a.e.) with equality holding for ϕ^* . Hence, equation (5.15) follows by Proposition 4.0.13. The first equality in (5.16) thus holds by Theorem 5.0.15 since the BSDE solution is unique. To show the remaining equalities in (5.16), let $\widetilde{Y}_t := Y_t - \phi^* \cdot \widehat{W}_t$ ($t \leq \bar{T}$). Noting that by definition of ϕ^* and Lemma 6.0.20

$$-\xi_t^{\operatorname{tr}} \Pi(Z_t) + \sqrt{h_t^2 - |\xi_t|^2} |\Pi_t(Z_t)| + \xi_t^{\operatorname{tr}} \phi_t^* = h_t |Z_t - \phi_t^*|$$

holds, it follows that

$$-d\tilde{Y}_t = -dY_t + \xi_t^{\text{tr}} \phi_t^* dt + \phi_t^* dW_t = h_t |Z_t - \phi_t^*| dt - (Z_t - \phi_t^*) dW_t.$$

Hence, equation $\tilde{Y}_t = \rho_t(\tilde{Y}_{\bar{T}})$ holds and adding $\phi^* \cdot \widehat{W}_t$ on both sides we obtain $Y_t = \rho_t \left(Y_{\bar{T}} - \int_t^{\bar{T}} \phi^* d\widehat{W} \right)$. By analogous arguments one obtains the inequality $Y_t \leq \rho_t \left(Y_{\bar{T}} - \int_t^{\bar{T}} \phi d\widehat{W} \right)$ for any $\phi \in \Phi$. This yields the last equality for (5.16). \square

Equation (5.16) justifies to define the unique minimizer ϕ^* as the good-deal hedging strategy for X corresponding to the good-deal valuation bounds π^u , with respect to the dynamic coherent risk measure ρ and the strategy set Φ .

It is a natural idea to define a hedging strategy for a contingent claim as the optimal strategy which minimizes some risk measure. Contributions in this direction can be found in several papers, including [3, 4, 9, 18, 21], and the result of Theorem 5.0.17 with respect to ρ belongs to the same family. Mathematically, the result of Theorem 5.0.17 fits well within the general theory for inf-convolutions of (BSDE-induced) risk measures, see [3, 4]. Indeed, Theorem 5.0.17 proves via Lemma 6.0.20 that the generator in the BSDE (5.4) for π^u is equal to the inf-convolution of the generator in the BSDE (5.9) for ρ with the (formal) generator $-\xi_t Z_t$, restricted to $Z \in C_t$, for the risk measure that is induced by (super-)hedging opportunities in the market; see Section 3.8 in [4]. In this sense, we have worked out a concrete solution to a dynamic inf-convolution problem. In sections 3-4 of [21], a BSDE similar to (5.4) has been obtained for a prototypical model where the martingale component of the risky asset prices is given by independent Brownian motions. As with [3, 4], the focus of [21] is on the minimization of risk measures but proofs make less use of BSDE theory.

Despite these close relations, the perspective for our problem is in the following aspect opposite to that of the literature cited. For us, the starting point has been not a given a-priori risk measure from which a so-called market-consistent risk measure is to be found (see [4]) by optimal risk-sharing (hedging) with the market. Instead, we have started from the good-deal bounds π^u , which are already market consistent, and have constructed a suitable ‘a priori risk’ measure ρ in order to find a dynamic notion of good-deal hedging that can be associated to π^u . By this complementary perspective, we address the problem raised in the final conclusions of [7] about linking good-deal valuation to a suitable theory of hedging, what seems to have not been elaborated in the literature so far to our best knowledge.

Since $\pi_t^u(X)$ is the minimal risk (with respect to ρ_t) that is obtainable by optimal hedging when holding the (liability) position X , the position $X - Y_t$ ($Y_t \in L^2(P, \mathcal{F}_t)$) just becomes acceptable at t , in the sense that $\pi_t^u(X - Y_t) \leq 0$, for $Y_t = \pi_t^u(X)$. Considering $Y_t = \pi_t^u(X)$ as the minimal capital required at t (with respect to ρ_t) to hold position X , the process $\pi_0^u - \pi_t^u + \phi^* \cdot \widehat{W}_t$ can be interpreted as the tracking error of the hedging strategy ϕ^* and it is of interest to study its properties, following Remark 4.1 in [21]. In our setting, Theorem 5.0.17 readily yields

Corollary 5.0.19 *Under the assumptions and notations of Theorem 5.0.17, the tracking error $\pi_0^u(X) - \pi_t^u(X) + \int_0^t \phi_s^* d\widehat{W}_s$ ($t \leq \bar{T}$) of the good-deal hedging strategy ϕ^* is*

a submartingale under any $Q \in \mathcal{P}^{\text{ngd}}$ and a martingale under the measure $Q^\lambda \in \mathcal{P}^{\text{ngd}}$ corresponding (by (5.7)) to the bounded process

$$\lambda_t = -h_t \frac{\frac{|\Pi_t^\perp(Z_t)|}{\sqrt{h_t^2 - |\xi_t|^2}} \xi_t - \Pi_t^\perp(Z_t)}{\left| \frac{|\Pi_t^\perp(Z_t)|}{\sqrt{h_t^2 - |\xi_t|^2}} \xi_t - \Pi_t^\perp(Z_t) \right|}, \quad t \leq \bar{T}. \quad (5.17)$$

Noting that positive signs of the tracking error correspond to gains and negative signs to losses, the result can be interpreted as a robustness property of the hedging strategy with respect to the family \mathcal{P}^{ngd} of probability measures as generalized scenarios (cf. [2]), with Q^λ being a worst case scenario in terms of the (conditional) expectation of additional funding needed to maintain the capital requirements when holding on to position X .

In the special case where the contingent claim X is replicable in the sense of part 4 of Remark 5.0.18, the tracking error vanishes (hence is a martingale under any measure) and equation (5.17) yields $\lambda = 0$ and thus $Q^\lambda = P$.

Proof of Corollary 5.0.19: First, note that the tracking error process is in $\mathcal{S}_T^{2-\varepsilon}(Q)$ ($\varepsilon > 0$) for any $Q \in \mathcal{P}^{\text{ngd}}$ since it is in $\mathcal{S}_T^2(P)$ and the density dQ/dP is in L^q for any $q < \infty$. Using Theorem 5.0.17, it follows from (5.14) and (5.4) with $Y_t = \pi_t^u(X)$ that

$$-dY_t + \phi_t^* d\widehat{W}_t = \frac{h_t^2}{\sqrt{h_t^2 - |\xi_t|^2}} |\Pi_t^\perp(Z_t)| dt + \left(\frac{|\Pi_t^\perp(Z_t)|}{\sqrt{h_t^2 - |\xi_t|^2}} \xi_t - \Pi_t^\perp(Z_t) \right) dW_t \quad (5.18)$$

for $t \leq \bar{T}$. Since the minimum

$$\min_{\lambda_t} \lambda_t^{\text{tr}} \left(\frac{|\Pi_t^\perp(Z_t)|}{\sqrt{h_t^2 - |\xi_t|^2}} \xi_t - \Pi_t^\perp(Z_t) \right) = - \frac{h_t^2}{\sqrt{h_t^2 - |\xi_t|^2}} |\Pi_t^\perp(Z_t)|,$$

taken over all λ_t in \mathbb{R}^n with $|\lambda_t| \leq h_t$, is attained ($P \times dt$ -a.e.) by the predictable process λ from (5.17), the claim follows from (5.18) by a change of measure argument. \square

6 Appendix

This section states a result on a deterministic convex optimization problem, which is needed for the proof of Theorem 5.0.17. We recall our convention that $\frac{0}{0} = 0$.

Lemma 6.0.20 *Assume $h > 0$ and $\xi, z \in \mathbb{R}^n$. With $d \leq n$, let $\sigma \in \mathbb{R}^{d \times n}$ be a matrix with full rank d , and let Π and Π^\perp be the orthogonal projections on the linear subspaces $C := \text{Im } \sigma^{\text{tr}}$ respectively $C^\perp = (\text{Im } \sigma^{\text{tr}})^\perp = \text{Ker } \sigma$ in \mathbb{R}^n . Let $\xi \in C$ and assume $h > |\xi|$.*

1. For $f_0(\phi) := -\xi^{\text{tr}}\phi + h|\phi - z|$, the ordinary convex minimization program

$$\min_{\phi} f_0(\phi) \quad \text{with } \phi \in \mathbb{R}^n \text{ and linear constraint } \Pi^{\perp}(\phi) = 0 \quad (6.1)$$

attains its minimum value

$$f_0(\phi^*) = -\xi^{\text{tr}}\Pi(z) + \sqrt{h^2 - |\xi|^2} |\Pi^{\perp}(z)| \quad (6.2)$$

at the unique minimum

$$\phi^* = \frac{|\Pi^{\perp}(z)|}{\sqrt{h^2 - |\xi|^2}} \xi + \Pi(z) \quad \text{in } C. \quad (6.3)$$

2. The maximization problem $\max_{\lambda} \lambda^{\text{tr}}z$ over $\lambda \in \mathbb{R}^n$ with constraint $|\lambda| \leq h$ attains its maximum at $\lambda^* = h \frac{z}{|z|}$ with $(\lambda^*)^{\text{tr}}z = h|z|$.

Proof: 1. We prove that the Kuhn-Tucker conditions are satisfied. Since the convex function f_0 is differentiable for $\phi \neq z$, its subgradient $\partial f_0(\phi)$ at ϕ is simply the gradient

$$\partial f_0(\phi) = -\xi + \frac{h}{|\phi - z|}(\phi - z) \quad \text{for } \phi \neq z.$$

Noting that

$$|\phi^* - z| = \left| \frac{|\Pi^{\perp}(z)|}{\sqrt{h^2 - |\xi|^2}} \xi - \Pi^{\perp}(z) \right| = \frac{h}{\sqrt{h^2 - |\xi|^2}} |\Pi^{\perp}(z)|,$$

it follows that

$$\partial f_0(\phi^*) = -\frac{\sqrt{h^2 - |\xi|^2}}{|\Pi^{\perp}(z)|} \Pi^{\perp}(z) \quad \text{when } \phi^* \neq z. \quad (6.4)$$

At $\phi = z$, the subgradient of f_0 includes the closed ball in \mathbb{R}^n with radius $h - |\xi| > 0$ around the origin, hence

$$\{g \in \mathbb{R}^n \mid |g| \leq h - |\xi|\} \subset \partial f_0(\phi^*) \quad \text{when } \phi^* = z. \quad (6.5)$$

In either case, there exists by (6.4) and (6.5) a vector of Lagrange multipliers $\lambda^* \in \mathbb{R}^n$ such that

$$\Pi^{\perp}(\lambda^*) \in \partial f_0(\phi^*). \quad (6.6)$$

The constraint $\Pi^{\perp}(\phi^*) = 0$ is satisfied by ϕ^* since ξ and $\Pi(z)$ are in $C = \text{Ker } \Pi^{\perp}$. Since (ϕ^*, λ^*) satisfies the Kuhn-Tucker conditions, optimality of ϕ^* in (6.3) is ensured by the Kuhn-Tucker theorem, see [24]. Direct computation yields (6.2). If $\Pi^{\perp}(z) = 0$, then f_0 is strictly convex at $\phi^* = z$ since $h > |\xi|$. Otherwise, if $\Pi^{\perp}(z) \neq 0$, the restriction of f_0 onto $C = \text{Im } \Pi = \text{Ker } \Pi^{\perp}$ is strictly convex, in particular at ϕ^* from (6.3). Overall, this implies uniqueness of the minimum.

2. This is obvious from $|\lambda^{\text{tr}}z| \leq |\lambda||z| \leq h|z|$.



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Author information

Dirk Becherer, Institut für Mathematik
Humboldt Universität zu Berlin
Unter den Linden 6
D-10099 Berlin, Germany.
Email: becherer[at]math.hu-berlin.de