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Models for Fare Planning in Public Transport

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Abstract

The optimization of fare systems in public transit allows to pursue objectives such as the maximization of demand, revenue, profit, or social welfare. We propose a non-linear optimization approach to fare planning that is based on a detailed discrete choice model of user behavior. The approach allows to analyze different fare structures, optimization objectives, and operational scenarios involving, e.g., subsidies. We use the resulting models to compute optimized fare systems for the city of Potsdam, Germany.

1 Introduction

Fares are a direct and flexible instrument to influence passenger behavior and cost recovery of a public transport system. Setting fares is therefore a fundamental problem for any mass transit company or authority. The importance of this task is further increased by technological progress such as the introduction of electronic ticketing systems, which offer opportunities to implement versatile fare structures in the future.

Public transport fares are well investigated in the economic literature. They are often studied from a macroscopic point of view in terms of elasticities, equilibrium conditions, and marginal cost analyses in order to derive qualitative insights, see, e.g., Samuelson [17], Oum, Waters, and Yong [15], Curtin [5], Goodwin [8], and Pedersen [16]. Glaister and Collings [7] and Nash [14] proposed to treat the setting of fares as an *optimization problem*, namely, to maximize objectives such as revenue, passenger miles, or social welfare subject to a budget constraint. They also considered a number of practical issues such as different public transport modes or peak and off-peak times, solve the first order conditions of their models numerically, and report on implementations of the results at London Transport. More details, in particular, aspects of the network structure, were added in the approaches of Kocur and Hendrickson [10] and De Borger, Mayeres, Proost, and Wouters [6], primarily to model costs in a more precise way.

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In the meantime, the theory of discrete choice had emerged as a viable approach to model the behavior of passengers of a public transport system, see Ben-Akiva and Lerman [3] and McFadden [13]. Empirical studies give evidence that travel choice is governed by a number of factors, most notably travel time, availability of a car and of discounted long term tickets, and fares, see Albers [1] and Vrtic and Axhausen [22]. Many of these factors depend on the network structure. It therefore makes sense to integrate detailed models of passenger behavior into a network fare optimization approach. Advanced bilevel logit models based on this idea have recently been formulated by Lam and Zhou [12]. In this paper we also integrate these aspects. Going one step further, we show how a number of objectives and constraints of practical relevance can be formulated and that the resulting models can be solved. We apply these methods to optimize fares for the city of Potsdam in Germany.

Our results show that the structure of the network does indeed influence the behavior of the passengers, i.e., we demonstrate a "network effect". For example, passengers with identical travel times make different choices according to the relative attractiveness of competition by the car.

Problems that are related to, but are different from, public transit fare optimization include toll optimization resp. road pricing, see, e.g., Labbé et al. [11], Verhoef [21], and van Dender [19], and tariff zone design, see Hamacher and Schöbel [9].

The article is organized as follows. Section 2 introduces the fare planning problem. We propose a discrete choice demand model in Section 2.1, and a family of five fare optimization models in Section 2.2. The models address the following objectives and constraints:

- MAX-R is a basic model; it assumes a fixed level of service and maximizes revenue.
- MAX-P includes costs that depend on line operation frequencies and subsidies; the objective is to maximize profit.
- MAX-D maximizes the demand, i.e., the number of public transport passengers, subject to budget and capacity constraints.
- MAX-B maximizes the user benefit subject to budget and capacity constraints.
- MAX-S maximizes social welfare subject to capacity constraints.

The models are calibrated in Section 2.3 and used to compute and analyze fare systems for the city of Potsdam in Section 3. Solving the models numerically, we show that different fare systems can be compared and evaluated in a quantitative way and that fare systems can be designed and optimized in order to achieve the goals specified above. As far as we know, an analysis at this level of detail has not been done before in the context of public transit fares.

2 Fare Planing

The fare planning problem involves a public transportation network, i.e., a directed graph $G = (V, E)$, where the nodes V represent stations and the arcs A connections that can be used for travel. There is a set $D \subseteq V \times V$ of *origin-destination pairs* (OD-pairs or traffic relations) between which passengers want to travel. We assume fixed passenger routes, i.e., for every OD-pair (s,t) there is a unique directed path P_{st} through the network that the passengers will take when using public transport. In our case, passengers use the time-minimal path.

Furthermore, we are given a finite set $\mathcal{C}$ of *travel choices*. The subset $\mathcal{C}' \subset \mathcal{C}$ represents the travel choices of public transport. Examples of travel choices that we have in mind are using single or monthly tickets, distance dependent tickets, the car, etc. Travel choices also include the number of trips during a time horizon, e.g., 30 trips during a month with a monthly ticket.

We consider n nonnegative fare variables $x_1, \dots, x_n$, which we call *fares*. Examples of fares are: a price per kilometer of travel, a price for crossing a zone, etc. A *fare vector* is a vector $\boldsymbol{x} \in \mathbb{R}^n_+$ of such fares.

We also consider price functions $p^i_{st} : \mathbb{R}^n_+ \to \mathbb{R}_+$ and demand functions $d^i_{st} : \mathbb{R}^n_+ \to \mathbb{R}_+$ for each OD-pair $(s,t) \in D$ and each travel choice $i \in \mathcal{C}$. The *price functions* $p^i_{st}(\boldsymbol{x})$ determine the price for traveling with travel choice i from s to t depending on the fare vector $\boldsymbol{x}$. All p^i_{st} appearing in this paper are affine functions and hence differentiable. The price functions for travel choices not using public transport do not depend on fares and are therefore constant. The *demand functions* $d^i_{st}(\boldsymbol{x})$ measure the number of passengers that travel from s to t with travel choice i, depending on the fare vector $\boldsymbol{x}$. The total demand for OD-pair $(s,t) \in D$ with public transport is

$$d^{\mathcal{C}'}_{st}(\boldsymbol{x}) := \sum_{i \in \mathcal{C}'} d^i_{st}(\boldsymbol{x}).$$

Usually, $d^{\mathcal{C}'}_{st}(\boldsymbol{x})$ is *non-increasing*. For one specific travel choice i, however, the demand function d^i_{st} does not necessarily have this property, because of substitution effects between different travel choices. In our application, the demand functions are differentiable and, in particular, continuous.

The remainder of this section is organized as follows. In Subsections 2.1 we specify a demand function. Fare planning models based on these definitions are proposed in Subsection 2.2. The models are calibrated with respect to data for the city of Potsdam in Subsection 2.3.

2.1 Demand Functions

A key feature of our fare planning models are the demand functions d^i_{st}. Based on a discrete choice logit approach, see Ben-Akiva and Lerman [3], we model them as follows. Consider a time horizon T, and assume that a passenger traveling from s to t performs a random number of trips $X_{st} \in \mathbb{Z}_+$ during T, i.e., X_{st} is a discrete random variable. We assume an upper bound N on X_{st}. In our approach, we define travel choices as follows. Passengers choose among a set A of *travel alternatives* (e.g., monthly ticket, single ticket, and car); we assume passengers do not mix alternatives, i.e., the same travel alternative is chosen for all trips. Let $A' \subset A$ be the set of travel alternatives involving public transport. The travel choices are then $\mathcal{C} = A \times \{1, \ldots, N\}$, and the travel choices for public transport are $\mathcal{C}' = A' \times \{1, \ldots, N\}$. Associated with each choice $(a, k) \in \mathcal{C}$ is a utility

$$U_{st}^{a,k}(\boldsymbol{x}) = V_{st}^{a,k}(\boldsymbol{x}) + \nu_{st}^{a},$$

which depends on the fare vector $\boldsymbol{x}$. Here, $V_{st}^{a,k}$ is a deterministic or observable utility, and ν_{st}^{a} is a random utility or disturbance term, which is $G(\eta, \mu)$ Gumbel distributed with $\eta = 0$. In our models, the deterministic utility is measured in monetary units and always includes the price function for public transport, i.e.,

$$V_{st}^{a,k}(\boldsymbol{x}) = W_{st}^{a,k} - p_{st}^{a,k}(\boldsymbol{x}). \tag{1}$$

Here, $W_{st}^{a,k}$ is a constant that subsumes all deterministic utilities that do not depend on fares such as travel time.

To simplify notation, we write $d_{st}^{a,k}(\boldsymbol{x})$ for the number of passengers traveling k times during T with alternative a from s to t and similarly $p_{st}^{a,k}(\boldsymbol{x})$ for the price of these trips. In logit models one assumes that each passenger takes the alternative of maximal utility. Using standard logit techniques, see Ben-Akiva and Lerman [3], it follows that the *expected* demand can be computed via an explicit formula as

$$d_{st}^{a,k}(\boldsymbol{x}) \;=\; \rho_{st} \cdot \frac{e^{\mu V_{st}^{a,k}(\boldsymbol{x})}}{\sum\limits_{b \in A} e^{\mu V_{st}^{b,k}(\boldsymbol{x})}} \cdot \mathbb{P}[X_{st} = k],$$

where ρ_{st} is the total number of passengers traveling from s to t. The last term computes the probability that passengers from s to t make k trips, while the middle term corresponds to the probability that they use alternative a. The formula expresses the expected demand over the probability spaces for X_{st} and the disturbance terms ν_{st}^{a}.

Note that $d_{st}^{a,k}(\boldsymbol{x})$ is continuous and even differentiable if the deterministic utilities $V_{st}^{a,k}(\boldsymbol{x})$ have this property. This is, for instance, the case for affine functions, see Subsection 2.3 and Section 3.

2.2 Fare Planning Models

We now propose five fare planning models that capture different aspects and objectives, whose appropriateness depends on the respective planning goals.

2.2.1 Maximizing Revenue

The first and most simple model maximizes revenue:

$$\begin{aligned}
(\textsc{Max-R}) \qquad \max \quad & \sum_{(s,t)\in D} \sum_{i\in\mathcal{C}'} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) \\
s.t. \quad & \boldsymbol{x} \in P.
\end{aligned}$$

Here, we assume that the fare vector $\boldsymbol{x}$ lies within a polyhedron $P \subseteq \mathbb{R}^n_+$ in the nonnegative orthant. P can be used to specify certain passenger interests or political goals, e.g., by stipulating upper bounds on the fare-variables. Note that the model also assumes constant costs, that is, a fixed level of service.

Max-R is similar to the revenue maximization model of Nash [14]; the model does, however, also consider different ticket types, and it includes substitution effects between public transport and car travel, see the definition of the demand function in Section 2.1. Therefore the "monopolistic exploitation" (as mentioned by Nash in case of revenue maximization) is limited.

2.2.2 Maximizing Profit

The second model, Max-P includes operating costs of lines. In principle, one would like to include a complete line planning model, see, e.g., Borndörfer, Grötschel, and Pfetsch [4]. Due to the complexity of line planning, however, we can at present only deal with a simplified version that plans frequencies of fixed lines. More precisely, we consider a pool $\mathcal{L}$ of lines, i.e., paths in the network, and associated continuous *frequencies* $f_\ell \geq 0$ for each line $\ell \in \mathcal{L}$. We assume that the lines are symmetric, i.e., f_ℓ is the frequency for the back and forth direction. We denote by $\boldsymbol{f}$ the vector of all frequencies. The operating costs for a line $\ell \in \mathcal{L}$ are $c_\ell \cdot f_\ell$, where $c_\ell \geq 0$ is a constant that depends on the length of the line. The transport capacity of line $\ell \in \mathcal{L}$ is $\kappa_\ell \cdot f_\ell$, where $\kappa_\ell > 0$ is a given *vehicle capacity.*

We can now express the maximization of profit, i.e., revenue minus costs. It is not unusual that the costs for transporting passengers in public transport are higher than the revenue arising from selling tickets. We therefore include a subsidy $\mathcal{S}$ in the model.

$$\begin{aligned}
(\textsc{Max-P}) \qquad \max \quad & \sum_{(s,t)\in D} \sum_{i\in\mathcal{C}'} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) - z \\
s.t. \quad & \sum_{\ell\in\mathcal{L}} c_\ell \cdot f_\ell - \mathcal{S} \leq z \\
& \sum_{\substack{(s,t)\in D \\ e\in P_{st}}} d^{\mathcal{C}'}_{st}(\boldsymbol{x}) \leq \sum_{\ell: e\in\ell} f_\ell \cdot \kappa_\ell \qquad \forall e \in E \\
& \boldsymbol{x} \geq 0 \\
& z \geq 0 \\
& \boldsymbol{f} \geq 0.
\end{aligned}$$

Since the objective maximizes $-z$, the first constraint, together with the inequality $z \geq 0$, guarantees that z is the maximum of cost minus subsidy and zero. Therefore, the subsidy can only be used for compensating costs. If the subsidy is zero, then z is equal to the cost. Hence, the model maximizes profit. The second set of constraints guarantees that sufficient transportation capacity on each arc is provided.

Glaister and Collings [7] consider a similar model for maximizing profit with a constraint that imposes a lower bound on the passenger-miles. They mention that this is dual to a model that maximizes passenger-miles subject to a budget constraint, which we will consider next. They calculate the costs in a coarser way, namely, depending on passenger miles instead of line frequencies.

2.2.3 Maximizing Demand

Models Max-R and Max-P aim at improving the profitability of a public transport system. We now consider three models that also cover social objectives. The goal of the first model is to maximize the number of passengers using public transport subject to a budget constraint.

$$
\begin{aligned}
(\textsc{Max-D}) \qquad \max \quad & \sum_{(s,t)\in D} d_{st}^{\mathcal{C}'}(\boldsymbol{x}) \\
s.t. \quad & \sum_{(s,t)\in D} \sum_{i\in\mathcal{C}'} p_{st}^{i}(\boldsymbol{x}) \cdot d_{st}^{i}(\boldsymbol{x}) + \mathcal{S} \geq \sum_{\ell\in\mathcal{L}} c_\ell \, f_\ell \\
& \sum_{\substack{(s,t)\in D \\ e\in P_{st}}} d_{st}^{\mathcal{C}'}(\boldsymbol{x}) \leq \sum_{\ell: e\in\ell} f_\ell \cdot \kappa_\ell \qquad \forall e \in E \\
& \boldsymbol{x} \geq 0 \\
& \boldsymbol{f} \geq 0.
\end{aligned}
$$

In case of zero subsidies $\mathcal{S}$, the objective is to maximize the number of transported passengers such that the costs are smaller than the revenue; in case of positive subsidies, the costs have to be smaller than revenue plus subsidy. The subsidies could also be negative. In that case, public transport has to yield a surplus.

The literature (e.g. Nash [14], Glaister and Collings[7]) usually maximizes demand in terms of passenger miles. In contrast to this objective, our approach maximizes the number of passenger that use public transport.

2.2.4 Maximizing Welfare

Fare planning models in the literature often consider fares that maximize a social welfare function. In general, the *social welfare* is the sum of a producer benefit and a user benefit. We consider the *producer benefit* as the profit, i.e., revenue minus cost. The *user benefit* is the difference between the generalized price the user is willing to pay and the actual generalized price. Note that the generalized price also includes parts that do not arise from fares, but that are also measured in (scaled) monetary units and hence change the willingness to pay. In case of a single fare variable and an invertible demand function, the user benefit can be easily derived by computing an integral. In the general case, we get the following, for the setting described in Section 2.1.

Lemma 1. *The total user benefit for given fares* $\boldsymbol{x}$ *is*

$$B(\boldsymbol{x}) := \sum_{(s,t)\in D} \sum_{k=1}^{N} \rho_{st} \cdot \mathbb{P}[X_{st} = k] \cdot \mathbb{E}\big[\max\{\max_{a\in A'} U_{st}^{a,k}(\boldsymbol{x}) - \max_{b\in A\setminus A'} U_{st}^{b,k}(\boldsymbol{x}),\ 0\}\big].$$

Proof. We sum up the benefit for one user multiplied by the number of users for all OD-pairs and the number of trips. The difference between the utilities of the best public transport alternative (with maximal utility) and the best non-public transport alternative gives the largest generalized price (utility) that a passenger is willing to pay for any public transport alternative, before switching to a non-public transport alternative. The maximum with 0 excludes passengers that choose a non-public transport alternative (i.e. the difference of the utilities is negative). Since the utilities are random variables, we take the expectation. $\square$

The next lemma shows how to compute the user benefit $B(\boldsymbol{x})$.

Lemma 2. *Let* I *be a finite nonempty set and* U_i*,* $i \in I$*, be Gumbel distributed random variables with parameters* (η_i, μ) *for* $i \in I$*. Let* $\emptyset \neq I' \subsetneq I$*. Then*

$$\begin{aligned} &\mathbb{E}\big[\max\{\max_{i\in I'} U_i - \max_{i\in I\setminus I'} U_i,\ 0\}\big] \\ &= \begin{cases} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cdot e^{n\cdot\frac{\alpha}{\beta}} & \text{if } -\frac{\alpha}{\beta} \geq 0 \\ \frac{\pi^2}{4} + \frac{(\frac{\alpha}{\beta})^2}{2} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cdot e^{-n\cdot\frac{\alpha}{\beta}} & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha = \frac{1}{\mu} \ln \sum_{i\in I'} e^{\mu\eta_i} - \frac{1}{\mu} \ln \sum_{i\in I\setminus I'} e^{\mu\eta_i}$ *and* $\beta = \frac{1}{\mu}$*.*

Proof. Let

$$U := \max_{i\in I'} U_i - \max_{i\in I\setminus I'} U_i.$$

Due to the properties of the Gumbel distribution, we know that U is logistically distributed with parameters (α, β). For $g(x) = \max\{x, 0\}$ and a random variable X with density function $f(x)$ it holds

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx = \int_{0}^{\infty} x\, f(x)\, dx.$$

Inserting the density function for the logistic distribution we obtain

$$\mathbb{E}[\max\{U,0\}] = \int_0^\infty x \cdot \frac{e^{-\frac{x-\alpha}{\beta}}}{\beta \cdot (1+e^{-\frac{x-\alpha}{\beta}})}\, dx.$$

The substitution $x = \alpha + \beta\xi$ and partial integration yields

$$\int_0^\infty x \cdot \frac{e^{-\frac{x-\alpha}{\beta}}}{\beta \cdot (1+e^{-\frac{x-\alpha}{\beta}})}\, dx = \beta \int_{-\frac{\alpha}{\beta}}^\infty \ln(1+e^{-\xi})\, d\xi.$$

For $-\frac{\alpha}{\beta} \geq 0$, the last integral can be expressed in terms of the Epstein zeta function

$$\int_{-\frac{\alpha}{\beta}}^\infty \ln(1+e^{-\xi})\, d\xi = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} \cdot e^{n \cdot \frac{\alpha}{\beta}}.$$

For $-\frac{\alpha}{\beta} < 0$, we get

$$\int_{-\frac{\alpha}{\beta}}^\infty \ln(1+e^{-\xi})\, d\xi = \frac{\pi^2}{4} + \frac{(\frac{\alpha}{\beta})^2}{2} - \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} \cdot e^{-n \cdot \frac{\alpha}{\beta}}.$$

$\square$

Note. The expressions in Lemma 2 can be used to evaluate the user benefit to any desired precision. The derivatives can be calculated directly by differentiating the integral.

We now consider two models that optimize welfare objectives. In model Max-B, which is similar to Max-D, we maximize the user benefit for public transport, subject to a budget constraint.

$$\begin{aligned}
(\textsc{Max-B}) \qquad \max \quad & B(\boldsymbol{x}) \\
s.t. \quad & \sum_{(s,t)\in D} \sum_{i \in \mathcal{C}'} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) + \mathcal{S} \geq \sum_{\ell \in \mathcal{L}} c_\ell\, f_\ell \\
& \sum_{\substack{(s,t)\in D \\ e \in P_{st}}} d^{\mathcal{C}'}_{st}(\boldsymbol{x}) \leq \sum_{\ell : e \in \ell} f_\ell \cdot \kappa_\ell \qquad \forall e \in E \\
& \boldsymbol{x} \geq 0 \\
& \boldsymbol{f} \geq 0.
\end{aligned}$$

Note that, without the budget constraint, the user benefit is maximal for zero fares, since the utilities for non-public transport alternatives are independent of the fare variables and the utilities for public transport alternatives are non-decreasing for increasing fares.

In model Max-S, we maximize the social welfare as a sum of user benefits and the profit of the public transport company.

$$
(\text{Max-S}) \qquad \max \quad B(\boldsymbol{x}) + \sum_{(s,t)\in D} \sum_{i\in\mathcal{C}'} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) - \sum_{\ell\in\mathcal{L}} c_\ell \cdot f_\ell
$$

$$
\begin{aligned}
s.t. \quad \sum_{\substack{(s,t)\in D \\ e\in P_{st}}} d^{\mathcal{C}'}_{st}(\boldsymbol{x}) &\leq \sum_{\ell:e\in\ell} f_\ell \cdot \kappa_\ell \qquad \forall e \in E \\
\boldsymbol{x} &\geq 0 \\
\boldsymbol{f} &\geq 0.
\end{aligned}
$$

2.2.5 Solving the Models

All of the above models are non-linear programs involving $|D| \cdot N \cdot |A'|$ demand functions, $|E|$ capacity constraints, and up to $n + |\mathcal{L}|$ variables. In our application, there will be at most $6830 \times 60 \times 3$ demand functions, 775 capacity constraints, and up to 42 variables.

We are able to solve all models for our data with the state-of-the-art non-linear programming package GAMS 2.50/Distribution 22.2 and the NLP-solver snopt. It turns out that Max-R is easily solved, while the others have computation times of up to three days. The most difficult instances arise from models Max-B and Max-S. These instances are numerically not well behaved, but we were able to compute locally optimal solutions. It is a challenge for future research to develop stable and fast solution methods for these models.

2.3 Data and Parameter Specification

That data that we use in our computations has been collected in 2005 for the city of Potsdam, Germany. It was provided to us in a joint project by the local public transport company ViP Verkehrsgesellschaft GmbH and the software company IVU Traffic Technologies AG. The data consists of the public transport network of Potsdam, which contains 36 lines and 775 edges, and a demand matrix for one day, which contains 6830 origin-destination pairs with positive demand.

Of the 209 315 trips in this matrix, 66 503 were done using public transport, and 142 812 using a car. Figure 1 illustrates the distribution of these trips according to travel time and distance, for public transport and car. On average, trip distances for public transport are similar to those for car travel, whereas the travel time for public transport is nearly three times as high as the average travel time for the car (the times for the cars do not include additional times for parking etc.). Another observation is that most passengers travel on connections with short distances, between 0.5 and 10 kilometers, for both public transport and car.

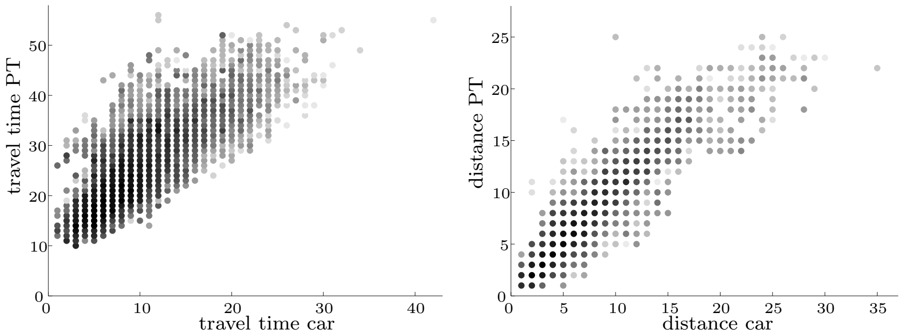

Figure 1: Passenger trips classified according to travel times (*left*) and travel distances (*right*) for car (x-axis) and public transport (y-axis). The darker the color of a point, the more passengers travel with the corresponding travel time and distance.

2.3.1 Defining Fare System and Utility Function

The public transport network of Potsdam is divided into three zones A, B, and C. Zones A, B and zones B, C, respectively, form tariff zone 1 (TZ 1). All three zones together constitute tariff zone 2 (TZ 2). We consider the following three travel alternatives for each tariff zone: "single ticket" (S), "monthly ticket" (M), and "car" (C), i.e., we have $A = \{S, M, C\}$ and $A' = \{S, M\}$. As we want to compare single and monthly tickets, we consider a time horizon of 30 days.

For each tariff zone $j \in \{1, 2\}$, the prices for public transport involve two fares: x_s^j is a single ticket fare, and x_m^j the monthly ticket fare (that has to be paid once a month). We write $\boldsymbol{x} = (x_s^1, x_m^1, x_s^2, x_m^2)$ and set the prices for alternatives single and monthly ticket to

$$p_{st}^{S,k}(\boldsymbol{x}) = x_s^j \cdot k \quad \text{and} \quad p_{st}^{M,k}(\boldsymbol{x}) = x_m^j, \qquad \forall\,(s,t) \in \text{TZ } j, \quad j = \{1, 2\},$$

respectively. In 2005, the prices for single ticket and monthly ticket were 1.45 € and 32.50 € for tariff zone 1 and 2.20 € and 49.50 € for tariff zone 2. For alternative "car", the price is the sum of a fixed cost Q and distance dependent operating costs q, i.e.,

$$p_{st}^{C,k}(\boldsymbol{x}) = Q + q \cdot \ell_{st}^c \cdot k;$$

here, ℓ_{st}^c denotes the shortest distance between s and t in kilometers for a car. We set $Q = 100$ € and $q = 0.1$ €. Note that $p_{st}^{C,k}(\boldsymbol{x}) \equiv p_{st}^{C,k}$ is independent of x.

The construction of the tariff zones allows the following: To travel in tariff zone 2, one can either buy a ticket for tariff zone 2 or two tickets for tariff zone 1 (one ticket for zones A, B and one ticket for zones B, C). To avoid unrealistic fares, we impose the following conditions on fares:

$$x_s^1 \leq x_s^2 \leq 2 \cdot x_s^1 \quad \text{and} \quad x_m^1 \leq x_m^2 \leq 2 \cdot x_m^1.$$

Table 1: Price and cross-price elasticity values for the Potsdam demand functions. Each entry in the table gives the demand elasticity for the travel alternative represented by the column if the fare of the alternative represented by the row is raised by 5%.

	single ticket	monthly ticket	car
single ticket	-0.72	0.21	0.12
monthly ticket	0.25	-0.93	0.15

The utilities for the travel alternatives are set up using affine functions for prices and travel times. Let t^c_{st} be the time for traveling from s to t with alternative car in minutes and t_{st} the time for traveling with public transport. The utilities depend on the number of trips k. More precisely, we set:

$$
\begin{aligned}
&U^{S,k}_{st}(x^1_s, x^1_m, x^2_s, x^2_m) = -x^j_s \cdot k - \delta \cdot t_{st} \cdot k + \nu^S_{st},\ (s,t) \in TZ\,j \\
&U^{M,k}_{st}(x^1_s, x^1_m, x^2_s, x^2_m) = -x^j_m - \delta \cdot t_{st} \cdot k + \nu^M_{st}, \quad (s,t) \in TZ\,j \\
&U^{C,k}_{st}(x^1_s, x^1_m, x^2_s, x^2_m) = -(Q + q \cdot \ell^c_{st} \cdot k) - \delta \cdot t^c_{st} \cdot k + y_{st} + \nu^C_{st} \\
&\text{for } j \in \{1,2\}.
\end{aligned}
$$

Here, δ is a parameter to express the travel time in monetary units; we use $\delta = 0.1$, i.e., 10 minutes of travel time are worth 1 €. In our first computations, we noticed that the behavior of the car users could not be explained solely in terms of travel time and costs. We therefore introduced an extra utility y_{st} for each OD-pair that is supposed to indicate a "convenience" of using a car. We computed y_{st} such that the 2005 fares, inserted in our demand function, resulted in the 2005 demand for public transport and car, respectively. This convenience utility y_{st} takes an average value of 83 € per month.

As usual in logit models, we use disturbance terms ν^a_{st} that are Gumbel distributed. We set the parameters of the Gumbel distribution $G(\eta, \mu)$ to $\eta = 0$ and $\mu = 1/30$. The (discrete) probabilities for the number of trips X_{st} are defined by the function $1 - \frac{1}{1500} \cdot (k - 30)^2$ and then normalized. The resulting probabilities do not depend on a particular OD-pair $(s,t) \in D$ and are centered around 30 in an interval from 1 to $N := 60$; see the left of Figure 2.

2.3.2 Elasticities and Costs

As a consistency check, we approximated price elasticities of the resulting demand functions as follows. Taking the 2005 fares, we increased the fare for one ticket type by 5%, while keeping the fare of the other ticket type and the price for the car fixed. Table 1 shows the resulting price and cross-price

Table 2: Transportation capacities and costs (in €/km).

modes	bus	tram	city railroad	regional traffic	ferry
c_ℓ - capacity (in pass.)	57	114	536	600	39
κ_ℓ - costs (in €/km)	4.5	7.5	50	100	30

elasticities. A study of the Verkehrsverbund Berlin-Brandenburg (VBB) of December 2006 [20] reports price elasticities for single tickets between -0.43 and -0.76, which is similar to our computations. For monthly ticket and other long-term tickets, price elasticities between -0.03 and -0.34 are reported. In our model, the price elasticity for the single ticket is higher than the price elasticity for the monthly ticket. This means that the users of the monthly ticket are more price sensitive than the users of the single ticket. This is due to the fact that in our model we assume that every passenger has access to a car, resulting in a higher competition between car and monthly ticket than in reality. Actually, many passengers who buy long-term tickets do not own a car and are therefore less price sensitive. We do not consider this aspect, because no data on user groups was available.

We finally set line operation costs, depending on the modes bus, tram, city railroad, regional traffic, and ferry, and the transportation capacity of a line as listed in Table 2.

With the stated data, parameters, and assumptions we computed the revenue and the cost by fixing the fares in model Max-P to the 2005 fares. The revenue is around 2 086 317 € in total, the costs are 1 914 519 €. This would mean that no subsidies are needed. Note, however, that our model considers only operating costs, i.e., the overall costs can be (much) higher in reality.

3 Analyzing Fare Systems

In this section we discuss and analyze a number of fare planning scenarios for the public transportation system of Potsdam, using the proposed five fare planning models. We first investigate fare changes in the existing fare system, i.e., the influence of fares on different objectives such as demand, revenue, cost, and social welfare. Models Max-P and Max-D allow to analyze subsidies in Subsection 3.3. Going one step further, we address changes in the existing fare system and entirely different fare systems. Computing optimal fares, we investigate the advantages of different fare systems with respect to various objectives.

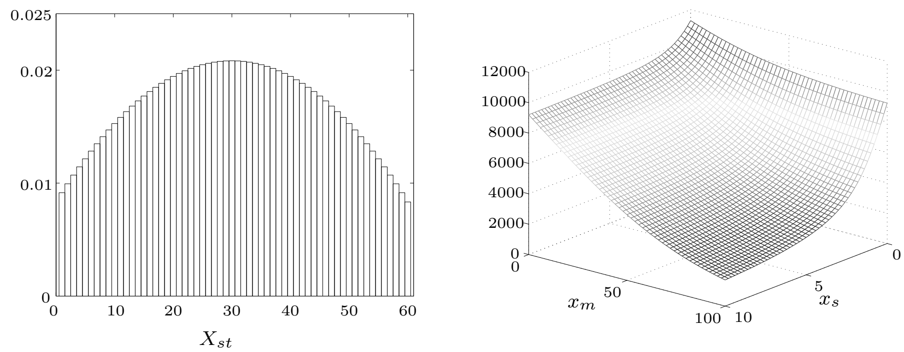

Figure 2: *Left:* Probability distribution for the number of trips X_{st}. *Right:* Demand function for single/monthly ticket fare system for Potsdam (tariff zone 2).

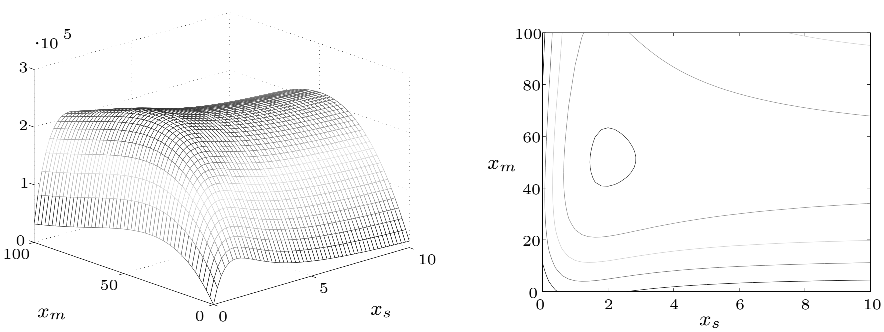

Figure 3: *Left:* Revenue function for the single/monthly ticket fare system for Potsdam (tariff zone 2). *Right:* Contour plot of the revenue function. The optimal fares for tariff zone 2 are $x_s = 1.98$ € and $x_m = 51.06$ € for model Max-R.

3.1 Basic Example – Maximizing Revenue

We start by illustrating our approach with a detailed discussion of our computations for the basic revenue maximization model Max-R, applied to the existing fare system of Potsdam. The computations for the following scenarios are similar, such that, hereafter, we will only state the results and omit a computational discussion.

Inserting the demand function and the existing fare system, model Max-R takes the following explicit form:

$$\begin{aligned}
\max \quad & \sum_{k=1}^{N} \sum_{j=1}^{2} \sum_{s,t \in \mathrm{TZ}\, j} \rho_{st} \cdot \frac{x_s^j \cdot k \cdot e^{V_{st}^{S,k}(\boldsymbol{x})} + x_m^j \cdot e^{V_{st}^{M,k}(\boldsymbol{x})}}{\sum_{b \in \{S,M,C\}} e^{V_{st}^{b,k}(\boldsymbol{x})}} \cdot \mathbb{P}[X_{st} = k] \\
\text{s.t.} \quad & x_s^1 \le x_s^2 \le 2 \cdot x_s^1 \\
& x_m^1 \le x_m^2 \le 2 \cdot x_m^1 \\
& \boldsymbol{x} \ge 0.
\end{aligned}$$

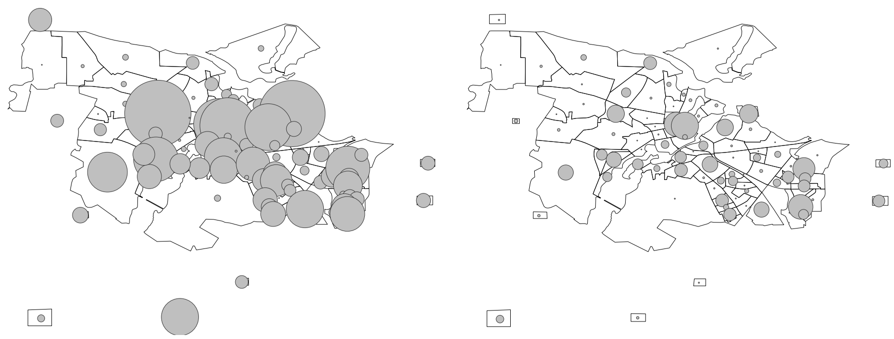

Figure 4: Results for optimizing the current fare system with single/monthly ticket using model Max-R. Potsdam is subdivided into 86 districts; the size of the circle in each district is proportional to the number of passengers which arrive at the district by car (*Left*) and public transport (*Right*).

Note that the objective function is differentiable.

The model produces the demand function shown on the right of Figure 2 and the revenue function shown in Figure 3. The optimal single ticket fare for tariff zone 1 is 1.75 € (up from 1.45 €) and 1.98 € for tariff zone 2 (down from 2.20 €). The optimal monthly ticket fare is 45.01 € (32.50 € currently) for tariff zone 1 and 51.06 € (49.50 € currently) for tariff zone 2. Comparing the revenue with the revenue for the current situation (which we also computed with our model, see end of Section 2.3), the revenue increases by around 4% to 2 165 282 €, and the demand decreases by around 14% to 57 021 passengers. Hence, the improvement in revenue is relatively small compared to the loss in the number of passengers.

Figure 4 shows Potsdam divided into 86 districts. The circles in the districts represent the number of passengers traveling to this district from the other districts by car (left) and by public transport (right of Figure 4), respectively. The figure illustrates the importance of the car as a travel alternative, which amounts to 73% of the total traffic.

3.2 Comparing Different Models

This subsection is devoted to an analysis of the current fare system. We compute optimal fares for all five proposed models. For this comparison, we consider the case of zero subsidies, i.e., we set $\mathcal{S} = 0$ in those models that include subsidies. The results are listed in Table 3. The first row of the table represents the 2005 solution. We can make the following observations:

- Compared to the current situation, the fares that maximize profit double – the increase in the single ticket fares is even higher – and the demand is halved. Doubled fares lead to a five-fold increase in profit. In fact, revenue decreases slightly while costs decrease dramatically. From an

Table 3: Results for Potsdam: Maximizing the profit for the current fare system providing a single ticket (x_s) and a monthly ticket (x_m). The computations are for the case of zero subsidies. (The costs for model Max-R are computed ex post. They are not part of the computation process.)

	x_s	x_m	revenue	demand	cost
2005	1.45 2.20	32.50 49.50	1 831 499 254 818	60 627.0 5876.0	1 914 519
Max-R	1.75 1.98	45.01 51.06	1 909 843 255 439	51 038.8 5 982.3	1 662 187
Max-P	3.96 7.93	64.66 87.59	1 613 537 170 892	29 819.2 2 310.8	912 876
Max-D	1.09 2.09	32.42 53.03	1 771 871 255 154	64 988.3 5 783.7	2 027 026
Max-B	1.11 1.92	32.14 52.62	1 775 660 255 271	64 788.5 5 970.8	2 030 931
Max-S	0.00 0.00	0.00 0.00	0 0	100 625.6 11 286.3	3 266 290

economical point of view, this is an appealing result; it might, however, not be possible nor desired to implement it in practice.

- The fares that maximize the demand are on a similar level as the 2005 fares. The demand increases by around 6% only. With respect to this objective, the current fares of Potsdam are quite well chosen.
- The fares of model Max-B are similar to the fares of model Max-D. There is only a slight difference between the two tariff zones. Model Max-B attracts more passengers for tariff zone 2.
- The revenue that results from demand maximization (Max-D) is higher than the revenue from profit maximization (Max-P), i.e., the effect of model Max-P is achieved by minimizing costs.
- For model Max-S we get zero fares. Obviously, the user benefit for zero fares is higher than the costs that are needed to establish a public transport system of zero fares with the current weighting of the parts in the objective of Max-S. We discuss zero fares in more detail in the next subsection.

Models Max-R and Max-P cover the efficiency of public transport from a purely economical point of view. Model Max-S adds interests of passengers in terms of user benefit. Models Max-D and Max-B focus solely on the passenger's point of view by optimizing the modal split and the user benefit, respectively. Which model is adequate for a particular application depends on the political, social, and theoretical side-constraints. To help in the

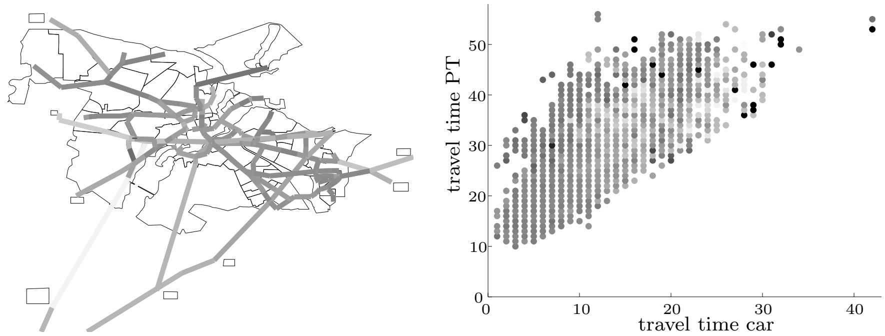

Figure 5: Comparing models Max-R and Max-P: The lighter the edge/point is colored the smaller the quotient of demand for model Max-P and demand for model Max-R is, i.e., the less passengers travel on the arc or OD-pairs with the related travel time for car and public transport with fares of model Max-P compared with fares of model Max-R.

planning process, optimization can bring quantitative arguments into the discussion and help to make a well-founded decision.

We finally take a more detailed look at our results by not only considering aggregate objectives, but by investigating changes in travel behavior. For this purpose, we compare a low fare and a high fare scenario, namely, the solutions of models Max-R and Max-P. Their only difference is that Max-P includes costs. The left of Figure 5 illustrates the resulting passenger flows. An edge is colored dark if the demands for model Max-P and model Max-R are similar. The bigger the relative difference between the flows, the lighter the edge is colored. We can see that the changes in the demand are quite similar for all edges. The right of Figure 5, however, shows that there is no simple pattern explaining the relative changes in demand for different OD-pairs. To predict how the passengers behave if the fares are doubled, all aspects that pertain to the utilities of car and public transport have to be considered, that is, in our case, travel time, lengths, and the extra utility for the car. This is an example of the network effect that was mentioned in the introduction.

3.3 Including Subsidies

We now study the effect of subsidies by computing optimal fares for models Max-P and Max-D, setting subsidies to $\mathcal{S} = 1\,000\,000$ €. The results are shown in Table 4.

For the profit maximization model Max-P, the subsidies are used to establish a level of service with a cost of exactly $1\,000\,000$ €, which is a bit more than the cost of Max-P in the 0-subsidy case. Actually, the subsidized variant of model Max-P amounts to a revenue maximization under the restriction that the costs are equal to the amount of subsidies. In the current investigation, where public transport can be operated without extra money,

Table 4: Effects of 1 000 000 € of subsidies: Maximizing the profit for the current fare system of single ticket (x_s) and monthly ticket (x_m).

	x_s	x_m	revenue	demand	cost
2005	1.45	32.50	1 831 499	60 627.0	1 914 519
	2.20	49.50	254 818	5876.0	
MAX-P	3.46	62.23	1 683 464	32 560.5	1 000 000
	6.93	83.25	183 202	2 597.7	
MAX-D	0.57	18.98	1 293 622	80 034.0	2 527 431
	1.13	37.95	233 809	7 651.9	

it does not seem to be reasonable to consider subsidies in combination with the aim of maximizing a profit.

In the demand maximization case, the fare for the single ticket for tariff zone 1 is more than halved, and all other fares are reduced by around 25% to 45%. The demand increases by around 21 183 passengers, which is 32% more than the 2005 demand and around 24% more than for the zero-subsidy case. This gives rise to the question whether there is a certain "best" amount of subsidies. We therefore investigate how different subsidies influence the modal split.

We computed the solutions of model MAX-D for 20 different values of subsidies between zero and 5 000 000 €. The results are plotted on the left of Figure 6. They show the dependency between subsidy and total demand. One can see that indeed the demand increases for rising subsidies. The marginal increase gets smaller and smaller and becomes zero for a subsidy higher than 3 200 000 €. In fact, this amount of subsidy is needed to establish a service with zero fares. This would result in around 112 000 passengers using public transport. In total, there are 209 315 passengers (by car or public transport). Around 97 000 passengers, however, do not change to public transport even in case of zero fares. For these passengers the convenience of the car or its shorter travel time are more important than costs.

The literature on zero fares in public transport, e.g., Baum [2] and Storchmann [18], does not give a clear picture. In some cases the demand for public transport increased by extremely large amounts when switching to zero fares. Often, this increase is due to additional traffic such as passengers that used bike or went by foot before. Because such substitution effects are not considered in our computations, the subsidies needed for zero fares could be to small. Our results agree with some outcomes of the mentioned studies in the fact that only 55% of all passengers would use public transport instead of the car in case of zero fares.

The right of Figure 6 shows the ratio of subsidies and passengers for different subsidies in an attempt to estimate "the value of an additional

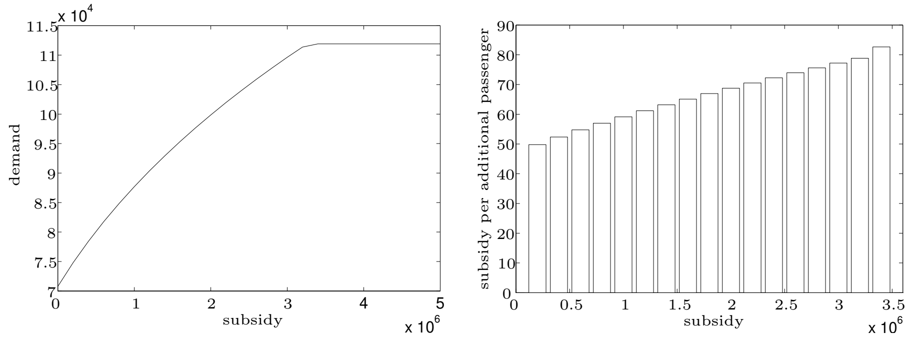

Figure 6: Single/monthly ticket fare system: *Left:* The dependency between subsidy and total demand for the model maximizing the revenue (MAX-D). *Right:* Dependency between global subsidy and subsidy for an additional passenger.

passenger". According to this criterion, there is no best amount of subsidies, because the ratio of rising subsidies and rising demand is nearly linear. In this case, one would therefore have to find a compromise between the number of passengers using public transport and the amount of subsidies needed to induce this demand. If one wants to have around 88 000 passengers using public transport, one needs subsidies of around 1 000 000 €. In this scenario, each additional passenger costs about 60 € per month.

3.4 Including a New Ticket Type

We are now going one step further and expand the current fare system by a third alternative for public transport. To this purpose, we introduce a new travel alternative, in which the passengers have the opportunity to buy single tickets at a 50% discount, if they pay a certain amount for one month. The resulting travel alternatives for each tariff zone are "single ticket" (S), "monthly ticket" (M), "reduced single ticket" (R), and "car" (C).

The prices for public transport involve three fares for each tariff zone $j \in \{1, 2\}$, the two fares x_s^j and x_m^j for single and monthly tickets as in the current fare system and a basic fare x_r^j that has to be paid once a month in order to buy reduced single tickets. We write $\boldsymbol{x} = (x_s^1, x_m^1, x_r^1, x_s^2, x_m^2, x_r^2)$ and set the prices for alternative "reduced single ticket" to

$$p_{st}^{R,k}(\boldsymbol{x}) = x_r^j + \tfrac{1}{2}\, x_s^j \cdot k \quad \text{ if } (s,t) \in TZ\, j.$$

The results for the corresponding revenue, profit, and demand maximizations are listed in Table 5. Compared with the previous computations without the new ticket type (see Table 3), demand and revenue increase: The revenues for the models MAX-R and MAX-P increase by around 30%, and the demand for model MAX-D increases by around 23% as well. The corresponding changes in passenger behavior can be classified according to

Table 5: Results for optimizing a fare system including a single ticket, a monthly ticket, and a reduced single ticket (x_s is the single ticket fare, x_m the monthly ticket fare, and x_r is a basic fare for one month in order to buy 50%-reduced single tickets). The computations are for zero subsidies.

	x_s	x_m	x_r	revenue	demand	cost
2005	1.45	32.50		1 831 499	60 627.0	1 914 519
	2.20	49,50		254 818	5876.0	
MAX-R	1.73	48.71	28.60	2 477 350	59 298.0	1 746 496
	1.98	55.43	31.12	321 249	6 751.4	
MAX-P	3.09	67.15	36.15	2 205 178	39 262.7	1 197 794
	5.86	89.15	42.34	225 743	3019.1	
MAX-D	0.92	31.30	19.88	2 175 001	79 646.2	2 494 532
	1.76	51.40	29.07	319 532	7222.0	

the number of trips. Single ticket, reduced single ticket, and monthly ticket are tickets for infrequent, frequent, and heavy users of the public transport, respectively.

Complex fare systems with many ticket types may alienate passengers. On the other hand, the computations clearly show that demand is covered better. Therefore, more detailed fare systems can lead to an improvement in the profitability as well as the attractiveness of public transport.

3.5 Designing a New Fare System

We now design an alternative system with distance dependent fares and compare it with the current fare system. We consider the travel alternatives "standard ticket" (D), "reduced ticket" (B), and "car" (C).

In the new fare system, the prices for public transport involve two fares: x_d, a distance fare per kilometer for standard tickets, and x_b, a basic fare that has to be paid once a month in order to buy reduced tickets with a 50%

Table 6: Results for the distance dependent fare system for Potsdam, x_d is the distance fare per kilometer, x_b is the basic fare per month to buy a reduced ticket.

	x_b	x_k	revenue	demand	cost
MAX-R	27.34	0.26	1 901 102	59 673.9	1 456 154
MAX-P	33.30	0.65	1 568 256	33 989.0	728 189
MAX-D	20.68	0.18	1 822 608	71 253.3	1 822 608
MAX-B	15.44	0.21	1 797 885	70 949.2	1 797 885
MAX-S	0.00	0.00	0	111 911.9	3 266 291

Table 7: Comparison of two fares systems single/monthly and distance dependent. The row "2005" shows the results of the current situation.

		demand	revenue	costs
2005		66 503.0	2 072 106	3 597 604
Max-R	single/monthly	57 021.1	2 165 282	1 662 187
	distance dependent	59 673.9	1 901 102	1 456 154
Max-P	single/monthly	32 130.0	1 784 429	912 876
	distance dependent	33 989.0	1 568 256	728 189
Max-D	single/monthly	70 772.0	2 027 026	2 027 026
	distance dependent	71 253.3	1 822 608	1 822 608
Max-B	single/monthly	70 772.0	2 030 931	2 030 931
	distance dependent	70 949.2	1 797 885	1 797 885
Max-S	single/monthly	111 911.9	3 266 291	3 266 291
	distance dependent	111 911.9	3 266 291	3 266 291

discount on standard tickets. We write $\boldsymbol{x} = (x_b, x_d)$ and set the prices for alternatives standard and reduced ticket to

$$p_{st}^{D,k}(\boldsymbol{x}) = x_d \cdot \ell_{st} \cdot k \qquad \text{and} \qquad p_{st}^{B,k}(\boldsymbol{x}) = x_b + \tfrac{1}{2}\, x_d \cdot \ell_{st} \cdot k,$$

where ℓ_{st} denotes the shortest distance in the public transport network between s and t in kilometers. Table 6 shows the optimal fares for the five models. We leave a comparison of the results for the new system to the reader and focus on comparing the new fare system with the current one.

Table 7 compares revenue, demand, and costs. In all cases, the revenue produced by the fare system with single/monthly ticket is more than 10% higher than for distance dependent fares. For costs the opposite holds. In all models the number of passengers using public transport is higher for the distance dependent fare system than for the fare system with single/monthly ticket. It increases by around 5% for models Max-R and Max-P, but only slightly for model Max-D and Max-B.

Therefore, the single/monthly ticket fare system seems to be more operator friendly, whereas the distance dependent fare system seems to be more customer oriented. This interpretation is corroborated by considering the results for user benefit maximization. In fact, the user benefit for the single/monthly ticket fare system is 6 390 682 € and for the distance dependent fare system 6 784 373 €, which is an increase of 6.2%. The social welfare is equal for both fare system because it is optimal for zero fares.

The left of Figure 7 illustrates travel behavior for both fare systems according to model Max-D. An edge is colored gray if the number of passengers using this edge in the fare system with single/monthly tickets is at

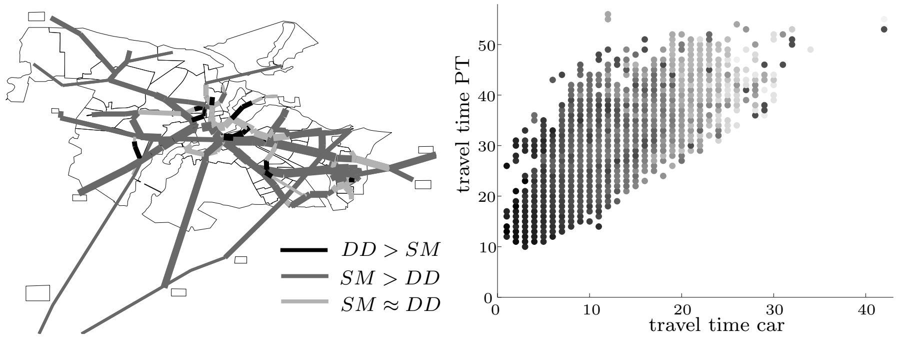

Figure 7: Comparing optimized single/monthly ticket fare system (SM) with the optimized distance dependent fare system (DD) for model MAX-D. *Left:* The thickness of the arcs corresponds to the number of passengers traveling on this arc with single/monthly ticket. The arc is colored gray if more than 5% passengers would use single/monthly ticket fare system compared to the distance dependent one; the arc is colored black if the relation is the other way round and light-gray if the difference is smaller than 5%. *Right:* Dark colored points imply that more passengers travel with the distance dependent fare system. Light colored points imply higher usage of the single/monthly ticket fare system.

Table 8: Number of passengers using single/monthly ticket and a distance dependent ticket, respectively depending on the distances of the OD-pairs.

distance	passengers for single/monthly ticket	distance dependent tickets
0 to 5 km	33 794.1	38 762.4
5 to 10 km	26 702.9	24 814.6
≥10 km	10 275.0	7 676.2

least 5% larger than the number of passengers in the system with distance dependent fares. It is colored black if more than 5% passengers travel on this arc with distance dependent tickets. The arc is colored light gray if the difference is smaller than 5%. The figure shows that, on most arcs, the current fare system with single and monthly tickets induces a higher load. However, the overall number of passengers traveling with the distance dependent fare system is slightly larger than the number of passengers traveling with the single/monthly ticket fare system. This, at first sight contradictory result, is due the fact that distance dependent fares are more attractive for passengers traveling on short distances, whereas the single/monthly ticket fare system is more attractive for passengers traveling long distances, compare with Table 8. The right of Figure 7 also shows that passengers with short travel times are more attracted by the distance dependent fare system. This is not surprising, because short travel times are often related with short distances.

4 Conclusion

Fare planning with its interdependencies between passenger behavior and costs is a complex optimization problem. The analyses conducted in this paper show that setting fares can have a significant impact on passenger behavior and, in particular, travel choice. Objectives ranging from cost recovery to welfare maximization can be handled in this way. We therefore believe that mathematical fare optimization can be a valuable decision support tool for planners.

It seems that further progress in fare optimization requires the inclusion of combinatorial aspects of network planning, e.g., to obtain a better model of the real cost structure, the choice of travel routes, transfer times, etc. This is, of course, computationally difficult. A more direct impact can be achieved by improving the quality and the breadth of the data basis and by improving the demand forecast.

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