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Fare Planning for Public Transport

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Abstract

In this paper we investigate the *fare planning model* for public transport, which consists in designing a system of fares maximizing the revenue. We discuss a discrete choice model in which passengers choose between different travel alternatives to express the demand as a function of fares. Furthermore, we give a computational example for the city of Potsdam and discuss some theoretical aspects.

1 Introduction

The design and the level of fares influence the passenger volume and consequently the revenue of a public transport system. Therefore, they are an important instrument to improve the profitability of the public transport system or to achieve other goals, e.g., to provide access to public transport for the general public.

Some articles in the literature deal with different approaches to find optimal fares for public transport. Hamacher and Schöbel [6] develop a model for designing fares and zones maximizing the similarity to a given fare system, e.g., a distance dependent one. Kocur and Hendrickson [7] and De Borger, Mayeres, Proost, and Wouters [5] introduce models for maximizing the revenue and the social welfare, respectively, subject to several budget constraints. The majority of the literature on public transport fares, however, discusses only theoretical concepts, e.g. marginal cost pricing (Pedersen [8]) and price elasticities (Curtin [4]).

In this article, we want to investigate a model to compute the fares that optimize the revenue for the public transport. This model is called the *fare planning model*. The main advantage of our approach is the inclusion of the public transport network. This allow us to distinguish different travel routes, e.g. between means of transportation like bus or subway, between slow and fast, short and long routes. Therefore it is possible to design complex and optimal fare systems.

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This work is a summary of the master thesis “Mathematische Preisplanung im ÖPNV”, that I wrote at the Zuse Institute Berlin. The advisor was Prof. Dr. Grötschel. Some parts of the master thesis were published in the Operations Research Proceedings 2005 [3], another summary can be found in [2]. The theoretical results in Section 3.2 include some new aspects.

2 General Fare Planning Model

We consider a traffic network whereas the nodes V represent the stations and $D \subseteq V \times V$ is a set of *origin-destination pairs* (OD-pairs or traffic relation). Furthermore, we are given a finite set $\mathcal{C}$ of *travel choices*. A travel choice can be a certain ticket type, e.g., single ticket or monthly ticket. Moreover, it can be a combination of a ticket type and a number of trips, which are performed within a certain time horizon, e.g., 40 trips with a monthly ticket in one month.

Let $p^i_{st} : \mathbb{R}^n \to \mathbb{R}_+$ be the *price function* for travel choice $i \in \mathcal{C}$ and OD-pair $(s,t) \in D$, i.e., it determines the price for the given travel choice and the given OD-pair. The price function depends on a *fare vector* $\vec{x} \in \mathbb{R}^n_+$ of $n \in \mathbb{N}$ fare variables $x_1, \ldots, x_n$, which we call *fares* in the following. The *demand functions* $d^i_{st}(\vec{x})$ measure the amount of passengers that travel from s to t with travel choice i, depending on the fare system $\vec{x}$; they are assumed to be *nonincreasing*.

We denote by $\vec{d}_{st}(\vec{x})$ the vector of all demand functions associated with OD-pair (s,t) and by $\vec{d}(\vec{x}) = (\vec{d}_{st}(\vec{x}))$ the vector of all demand functions. Analogous notation is used for $(p^i_{st}(\vec{x}))$. The *revenue* $r(\vec{x})$ can then be expressed as

$$r(\vec{x}) := \vec{p}(\vec{x})^{\mathrm{T}}\vec{d}(\vec{x}) = \sum_{i\in\mathcal{C}} \sum_{(s,t)\in D} p^i_{st}(\vec{x}) \cdot d^i_{st}(\vec{x}) \, .$$

With this notation our general model for the fare planning problem is:

$$\text{(FPP)} \qquad \begin{array}{rl} \max & \vec{p}(\vec{x})^{\mathrm{T}}\vec{d}(\vec{x}) \\ s.t. & \vec{x} \in P \, . \end{array} \tag{1}$$

All restrictions on the fare variables are included in the set $P \subseteq \mathbb{R}^n$. Here, one can also include social and political aspects like a minimum level of demand or a maximum level of fares.

In the model (FPP) we assume constant costs and a constant level of service. In further investigations we included costs and maximized the profit, i.e., revenue minus costs. Other possible objectives were considered as well, e.g., maximization of the demand with respect to cost recovery. The goal is to make a first step with (FPP) towards a decision support tool for optimizing fare systems. We show the practicability of (FPP) on a prototype example in Section 3.1.

3 Fare Planing with a Discrete Choice Model

Our model expresses passenger behavior in response to fares by the demand function d_{st}^{i}. In this section, we use discrete choice models, especially the logit model, to obtain a realistic demand function. Therefore we assume that the passengers have full knowledge of the situation and act rationally with respect to the change of the fares. A thorough exposition of discrete choice analysis and logit models can be found in Ben-Akiva and Lerman [1].

In a discrete choice model for public transport, each passenger chooses among a finite set A of *alternatives* for the travel mode, e.g., single ticket, monthly ticket, bike, car travel, etc.

We consider a time horizon T and assume that a passenger which travels from s to t performs a random number of trips $X_{st} \in \mathbb{Z}_+$ during T, i.e., X_{st} is a discrete random variable. We assume that passengers do not mix alternatives, i.e., the same travel alternative is chosen for all trips. Furthermore, we assume an upper bound N on X_{st}. The travel choices are then $\mathcal{C} = A \times \{1, \ldots, N\}$.

Associated with each travel choice $(a, k) \in \mathcal{C}$ and each OD-pair $(s, t) \in D$ is a *utility* $U_{st}^{a,k}$, which may depend on the passenger. Each utility is the sum of an observable part, the deterministic utility $V_{st}^{a,k}$, and a random utility, or disturbance term ν_{st}^{a}. For $(a, k) \in \mathcal{C}$ we consider the utility $U_{st}^{a,k}(\vec{x}) = V_{st}^{a,k}(\vec{x}) + \nu_{st}^{a}$ that depends on the fare system $\vec{x}$.

Assuming that each passenger chooses the alternative with the highest utility, the probability of choosing alternative $a \in A$ in case of k trips is

$$P_{st}^{a,k}(\vec{x}) := \mathbb{P}\big[V_{st}^{a,k}(\vec{x}) + \nu_{st}^{a} = \max_{b \in A}(V_{st}^{b,k}(\vec{x}) + \nu_{st}^{b})\big]. \tag{2}$$

In case of the *logit model*, which introduces the Gumbel distribution for the disturbances ν_{st}^{a}, this probability can explicitly be computed by the formula

$$P_{st}^{a,k}(\vec{x}) = \frac{e^{\mu V_{st}^{a,k}(\vec{x})}}{\sum\limits_{b \in A} e^{\mu V_{st}^{b,k}(\vec{x})}} = \frac{1}{1 + \sum\limits_{b \in A \setminus \{a\}} e^{\mu (V_{st}^{b,k}(\vec{x}) - V_{st}^{a,k}(\vec{x}))}}. \tag{3}$$

Here $\mu > 0$ is a scale parameter for the disturbances ν_{st}^{a}.

We write $d_{st}^{a,k}(\vec{x})$ for the amount of passengers choosing $(a, k) \in \mathcal{C}$, i.e., traveling k times during T with alternative a from s to t and similarly $p_{st}^{a,k}(\vec{x})$ for the price of this travel. It follows that

$$d_{st}^{a,k}(\vec{x}) \;=\; d_{st} \cdot P_{st}^{a,k}(\vec{x}) \cdot \mathbb{P}[X_{st} = k] \;=\; d_{st} \cdot \frac{e^{\mu V_{st}^{a,k}(\vec{x})}}{\sum\limits_{b \in A} e^{\mu V_{st}^{b,k}(\vec{x})}} \cdot \mathbb{P}[X_{st} = k],$$

where d_{st} is the entry of the OD-matrix corresponding to $(s,t) \in D$. The revenue can then be written as:

$$r(\vec{x}) = \sum_{a \in A'} \sum_{k=1}^{N} \sum_{(s,t) \in D} p_{st}^{a,k}(\vec{x}) \cdot d_{st}^{a,k}(\vec{x}) = \vec{p}(\vec{x})^{\mathrm{T}} \vec{d}(\vec{x}) \,,$$

where A' is the set of public transport alternatives. This formula expresses the *expected* revenue over the probability spaces of X_{st} and disturbances ν_{st}^{a}.

Note that $r(\vec{x})$ is continuous and even differentiable if the deterministic utilities $V_{st}^{a,k}(\vec{x})$ (and the price functions $p_{st}^{a,k}(\vec{x})$) have this property.

3.1 Example with three Alternatives

In this section, we illustrate our discrete choice approach with a real world example for the city of Potsdam. We want to optimize the the current fare system including a single ticket (S) and a monthly ticket (M) for two different tariff-zones. The third travel alternative is the car (C), i.e., $A = \{M, S, C\}$. We consider a time horizon T of one month. The prices for public transport involve two fares for each tariff-zone; x_s, the single ticket fare, and x_m, the monthly ticket fare. We write $\vec{x} = (x_s, x_m)$ and set the prices for alternatives single, monthly ticket, and car to

$$p_{st}^{S,k}(\vec{x}) = x_s \cdot k, \qquad p_{st}^{M,k}(\vec{x}) = x_m, \qquad \text{and} \qquad p_{st}^{C,k}(\vec{x}) = Q + q \cdot \ell_{st} \cdot k.$$

For alternative “car”, the price is the sum of a fixed cost Q and distance dependent operating costs q. The parameter ℓ_{st} denotes the shortest distance between s and t in kilometers for a car. We set $Q = 100$ € and $q = 0.1$ €.

We assume that the utilities are affine functions of prices and travel times t_{st}^{a} for traveling from s to t with alternative a. The utilities depend on the number of trips k. More precisely, we set:

$$\begin{aligned}
U_{st}^{M,k}(x_M, x_S) &= -\delta_1 \cdot x_M - \delta_2 \cdot t_{st}^{M} \cdot k + \nu_{st}^{M} && \text{“monthly ticket”} \\
U_{st}^{S,k}(x_M, x_S) &= -\delta_1 \, (x_S \cdot k) - \delta_2 \cdot t_{st}^{S} \cdot k + \nu_{st}^{S} && \text{“single ticket”} \\
U_{st}^{C,k}(x_M, x_S) &= -\delta_1 \, (Q + q \cdot \ell_{st} \cdot k) - \delta_2 \cdot t_{st}^{C} \cdot k - y + \nu_{st}^{C} && \text{“car”.}
\end{aligned}$$

Here, δ_1 and δ_2 are weight parameters; we use $\delta_1 = 1$ and $\delta_2 = 0.1$, i.e., 10 minutes of travel time are worth 1 €. In first computations we noticed that the behavior of the motorists could not be explained only with travel time and costs. Therefore we introduced an extra positive utility y for the car indicating the convenience of the car. We set $y \approx 93$ € for the first tariff-zone and $y \approx 73$ € for the second tariff-zone to match the current demand for the given prices in our model. The (discrete) probabilities for the number of trips are centered around 30 in an interval from 1 to $N := 60$ for all OD-pairs.

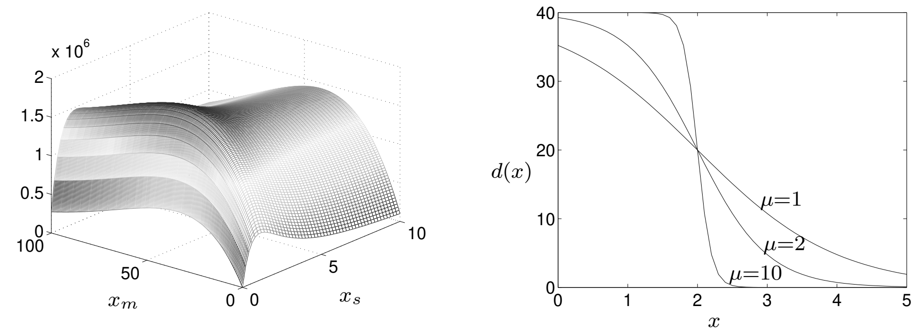

Figure 1: *Left:* Revenue function for example single monthly ticket. *Right:* Discrete choice demand functions for different μ.

Altogether, the fare planning problem we want to consider has the form:

$$\begin{aligned} \max \quad & \sum_{k=1}^{N} \sum_{(s,t)\in D} d_{st} \cdot \frac{x_M \cdot e^{\mu V_{st}^{M,k}(\vec{x})} + x_S \cdot k \cdot e^{\mu V_{st}^{S,k}(\vec{x})}}{\sum\limits_{b\in A} e^{\mu V_{st}^{b,k}(\vec{x})}} \cdot \mathbb{P}[X_{st} = k] \\ \text{s.t.} \quad & \vec{x} \geq 0 \end{aligned}$$

We set $\mu = \frac{1}{30}$. Note that the revenue function is differentiable.

The revenue function is shown on the left of Figure 1. The optimal fares for the two tariff zones are $x_s = 1.57$ (currently 1.45), $x_m = 43.72$ (32.50) for tariff-zone 1 and $x_s = 1.79$ (2.20) and $x_m = 48.21$ (49.50) for tariff-zone 2. The revenue increased by about 3% up to 2 129 971€.

3.2 Some Theoretical Results

In this section, we analyze the revenue function in case of a discrete choice demand function with a small random utility.

The second part of equation (3) emphasizes the importance of the difference of the deterministic utilities which is weighted by the parameter μ. The higher μ, the more important is the difference of the deterministic utilities for the decision, i.e., the influence of the random utility decreases. The right of Figure 1 shows a demand function for different values of μ. The choice is getting deterministic if μ tends to infinity, i.e., $P_{st}^{a,k}(x) = 1$ if a is the alternative with the maximum deterministic utility and $P_{st}^{a,k}(x) = 0$ otherwise. In this case the demand function becomes a step function.

For further analysis we omit the number of trips k and consider two travel alternatives with the following utility function $V_i^1(x) = -x,\ V_i^2(x) = -i$ for OD-pair i. The demand function for alternative 1 is

$$d_i(x) = d_i \cdot \frac{e^{-\mu \cdot x}}{e^{-\mu \cdot x} + e^{-\mu \cdot i}}.$$

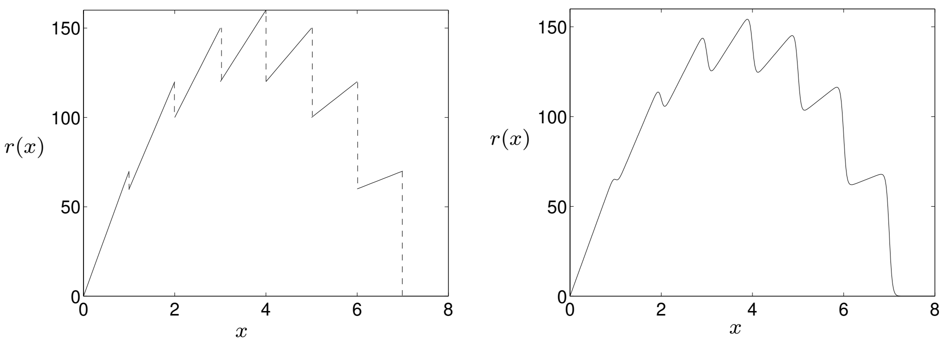

Figure 2: *Left:* Revenue function for a discrete choice demand function for $m = 7$ OD-pairs without random utility. *Right:* Revenue function for the same example with "small" random utility ($\mu = 30$).

If we set the price functions of alternative 1 to $p_i^1(x) = x$, we obtain for the revenue function for alternative 1

$$r(x) = \sum_{i=1}^{m} d_i \cdot \frac{e^{-\mu \cdot x}}{e^{-\mu \cdot x} + e^{-\mu \cdot i}} \cdot x.$$

For $\mu \to \infty$ $\qquad \tilde{r}(x) := \lim_{\mu \to \infty} r(x) = \sum_{i=1}^{m} d_i \cdot \begin{cases} x & \text{if } x \leq i \\ 0 & \text{otherwise} \end{cases}.$

For $x = i$, $i = \{1, \ldots, m\}$ the revenue function $\tilde{r}$ is not continuous and has m local maxima, see left of Figure 2. This means, that if the deterministic utility approximates the utility of the alternative quite well (the random utility is small), the revenue function has m local maxima, see right of Figure 2.

It is likely to construct examples with upto m^n local maxima in case of n fare variables. Therefore, the better the utilities are known, the closer the demand function is to reality. On the other hand, more local optima can appear and the problem may be hard to solve.

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