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The Line Connectivity Problem

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Abstract

This paper introduces the *line connectivity problem*, a generalization of the Steiner tree problem and a special case of the line planning problem. We study its complexity and give an IP formulation in terms of an exponential number of constraints associated with "line cut constraints". These inequalities can be separated in polynomial time. We also generalize the Steiner partition inequalities.

1 Introduction

The *line connectivity problem* (LCP) can be described as follows. We are given an undirected graph $G = (V, E)$, a set of *terminal nodes* $T \subseteq V$, and a set of *lines* L (simple paths) defined on the graph G, see the left of Figure 1 for an example. The lines have nonnegative costs $C \in \mathbb{R}^L_+$ and cover all edges, i.e., for every $e \in E$ there is an $\ell \in L$ such that $e \in \ell$. The problem is to find a set of lines $L' \subseteq L$ of minimal cost such that for each pair of distinct terminal nodes $t_1, t_2 \in T$ there exists a path from t_1 to t_2, which is completely covered by lines of L'.

LCP is a generalization of the Steiner tree problem (STP) since we get an STP if all lines have length one. In contrast to the STP with nonnegative costs, see [4, 5] for an overview, the optimal solution of the line connectivity problem does not have to be a tree. There can be two lines that form a cycle, but both are necessary to connect two terminal nodes, see the right of Figure 1. However, an optimal solution of LCP is minimally connected, i.e., if we remove a line from the solution, there exist at least two terminals which are not connected.

LCP is a special case of the *line planning problem* in which passenger routes are not fixed a priori, see [2] and the references therein for a detailed definition. Line planning deals with finding a set of lines and corresponding frequencies such that a given demand can be transported. Usually, the objective is to minimize cost and/or travel times. If we neglect travel time,

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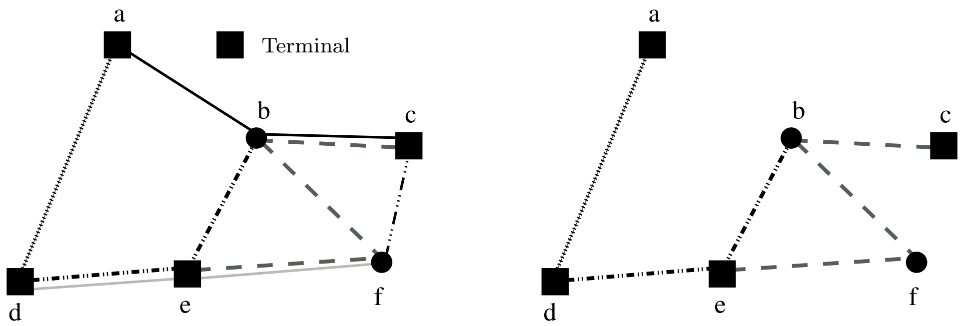

Figure 1: Example of a line connectivity problem.

capacity, and frequency constraints, the line planning problem reduces to LCP, namely, all stations that are departures or destinations of a passenger trip have to be connected by lines. Since line planning problems can not be solved to proven optimality for medium-sized and large instances, it is of interest to analyze LCP.

This article is structured as follows. In Section 2 we investigate the complexity of the LCP. An IP formulation and a polynomial time separation algorithm for a class of line cut inequalities associated with this formulation is proposed in Section 3. A polyhedral analysis is sketched in Section 4.

2 Complexity of LCP

Since the line connectivity problem is a generalization of the Steiner tree problem [5], it is strongly NP-hard in general. The complexity of two important special cases, for which the STP can be solved efficiently, is as follows:

Proposition 2.1. *1. LCP is polynomially solvable for $|T| = 2$.*

2. LCP is NP-hard for $T = V$.

Sketch of proof. 1. We can construct a directed graph D' similar to the one in Section 3 below. A shortest path in D' between two terminal nodes corresponds to a minimal cost connected line set in G.

2. We reduce the set covering problem to the line connectivity problem. In a set covering problem we are given a finite set S, a set $\mathcal{M} \subseteq 2^S$, and a positive integer k. The problem is to find a subset $\mathcal{M}' \subseteq \mathcal{M}$, $|\mathcal{M}'| \leq k$, such that for all $s \in S$ there exists an $M \in \mathcal{M}'$ with $s \in M$.

Given a set covering instance, we define a line connectivity problem in a graph $G = (V, E)$ as follows: The nodes are $V = S \cup \{v\}$ with v being one extra node. We first assume a complete graph and remove all edges that are not covered by a line after the construction of the lines. Let $V = \{v := s_0, s_1, s_2, \ldots\}$. For each set $M \in \mathcal{M}$ order the elements in M and construct a line beginning in node v and passing all nodes of M in the given order. The cost of this line is 1.

It can be easily seen that a cover $\mathcal{M}'$ with less than k elements exists if and only if we find a line set connecting all nodes with cost smaller or equal to k. $\square$

3 An Integer Programming Formulation

An integer program for LCP can be formulated as

$$
\begin{array}{llll}
(\mathrm{LCP}_{cut}) & \min \sum_{\ell \in L} C_\ell \, x_\ell & & \\
& \text{s.t} \quad \sum_{\ell \in L_{\delta(W)}} x_\ell & \geq 1 & \emptyset \subsetneq W \cap T \subsetneq T \\
& \qquad\qquad x_\ell & \in \{0,1\}. &
\end{array}
$$

Here, $L_{\delta(W)} := \{\ell \in L \,|\, \exists\, e \in \delta(W) \cap \ell\}$ is the set of all lines that cross a cut $\delta(W)$ at least one time. If $\delta(W)$ with $\emptyset \subsetneq W \cap T \subsetneq T$ is an (s,t)-cut we call $L_{\delta(W)}$ an (s,t)-*line cut* or shortly *line cut*. We call L' a *minimal* (s,t)-*line cut* with respect to x if

$$\sum_{\ell \in L'} x_\ell = \min\{\sum_{\ell \in \tilde{L}} x_\ell \,|\, \tilde{L} \text{ is an } (s,t)\text{-line cut}\}.$$

We call the inequalities in (LCP_{cut}) *line cut constraints*. Their number can be exponential in the size of the input. We therefore propose an efficient separation algorithm that decides whether a given point x^* is valid for the LP-Relaxation of (LCP_{cut}) or finds a violated line cut constraint. It will turn out that this problem can be formulated as a max flow/min cut problem in a suitable auxiliary digraph. The construction is as follows: We are given a graph $G = (V, E)$, a set of lines L, and two distinct nodes $s, t \in T \subseteq V$. Each line $\ell \in L$ has a value $x_\ell \geq 0$. We construct a directed graph $D' = (V', A')$ with node set

$$V' = \{s\} \cup \{t\} \cup \{v_\ell, w_\ell \,|\, \ell \in L\}$$

and the following arcs $a \in A'$ and capacities c_a

$$
\begin{array}{lll}
(s, v_\ell) & c_{sv_\ell} = x_\ell & \text{if } s \in \ell,\ \forall\, \ell \in L \\
(v_\ell, w_\ell) & c_{v_\ell w_\ell} = x_\ell & \forall\, \ell \in L \\
(w_{\ell'}, v_\ell) & c_{w_{\ell'} v_\ell} = \min\{x_\ell, x_{\ell'}\} & \forall\, \ell, \ell' \in L,\ \ell \neq \ell',\ \ \ell \text{ and } \ell' \text{ have} \\
& & \text{a node } v \in V \backslash \{s,t\} \text{ in common} \\
(w_{\ell'}, t) & c_{w_{\ell'} t} = x_{\ell'} & \text{if } t \in \ell',\ \forall\, \ell' \in L.
\end{array}
$$

Figure 2 illustrates this construction.

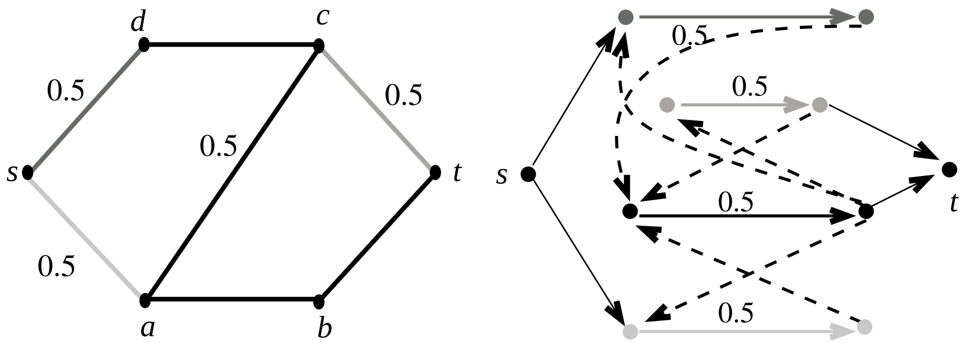

Figure 2: *Left:* Graph G with four lines ($\ell_1 = \{s, d\}$, $\ell_2 = \{s, a\}$, $\ell_3 = \{d, c, a, b, t\}$, $\ell_4 = \{c, t\}$) with value 0.5 and two terminal nodes s and t. *Right:* Corresponding directed graph D'. Here, each arc has capacity 0.5. The dashed arcs are of the form $(w_{\ell'}, v_\ell)$. The minimal (s,t)-cut has value 0.5.

Lemma 3.1.

1. *Each simple (s,t)-path has the form $(s, v_{\ell_1}, w_{\ell_1}, \dots, v_{\ell_k}, w_{\ell_k}, t)$, $k \geq 1$.*
2. *The only arc with target node w_ℓ is (v_ℓ, w_ℓ), $\forall\, \ell \in L$.*
3. *The only arc with source node v_ℓ is (v_ℓ, w_ℓ), $\forall\, \ell \in L$.*
4. *There is a directed (s,t)-cut with minimal capacity in D' such that all arcs over this cut are of the form (v_ℓ, w_ℓ), $\ell \in L$.* □

Proof. The first three parts can easily be seen. Consider part 4. Assume (s, v_ℓ) is in a minimal cut. Then we can replace this arc by (v_ℓ, w_ℓ) with the same value because this is the only arc with source node v_ℓ (Part 3). With a similar argument we can replace $(w_{\ell'}, t)$ by $(v_{\ell'}, w_{\ell'})$. Assume $(w_{\ell'}, v_\ell)$, $\ell \neq \ell'$, is in the cut and $x_\ell \leq x_{\ell'}$. Then we can replace this arc by (v_ℓ, w_ℓ) with same capacity because of Part 3 and $c_{w_{\ell'}, v_\ell} = \min\{x_\ell, x_{\ell'}\}$. If $x_{\ell'} \leq x_\ell$, we can replace it by $(v_{\ell'}, w_{\ell'})$ with same capacity because of Part 2 and the definition of the capacities. □

Proposition 3.2. *There is a one-to-one correspondence between minimal directed (s,t)-cuts in D' and minimal (s,t)-line cuts in G of the same capacity.*

Proof. We only show the forward direction. Let $\delta(W')$ be a minimal (s,t)-cut in D'. After applying part 4 of Lemma 3.1, let $L' = \{\ell \in L \,|\, (v_\ell, w_\ell) \in A', v_\ell \in W', w_\ell \in V' \backslash W'\}$. Assume L' is not an (s,t)-line cut. Then there exists a path from s to t in G that is covered by lines in $L \backslash L'$. Let $\ell_1, \dots, \ell_r$ be the lines that are used in this order when traversing the path. Then $(s, v_{\ell_1}, w_{\ell_1}, \dots, v_{\ell_r}, w_{\ell_r}, t)$ is a path from s to t in D'. This is a contradiction to the assumption that $\delta(W')$ is a cut in D'.

It can be easily seen that L' and $\delta(W')$ have the same capacity. □

Theorem 3.3. *The separation problem for line cut constraints can be solved in polynomial time.*

Computing for every two terminals $s, t \in T$ the minimum (s,t)-cut in D' can be done in polynomial time. If and only if the value of this cut is smaller than 1, we can construct a violated line cut constraint. $\square$

4 Polyhedral Analysis

Let $P_{\text{LCP}} := \text{conv}\{\boldsymbol{x} \in \{0,1\}^L \mid \boldsymbol{x} \text{ satisfies the line cut constraints}\}$ be the *line connectivity polytope.* We assume that the line connectivity polytope is non-empty, i.e., the graph G is connected.

Using the results for the set covering polytope of Balas and Ng [1], we get the following information about P_{LCP}.

Corollary 4.1.

1. *The LCP-polytope P_{LCP} is full dimensional if and only if there exists no valid cut $\delta(W)$ with $|L_{\delta(W)}| = 1$.*

   In the following we assume P_{LCP} to be full dimensional.

2. *The inequality $x_\ell \geq 0$ defines a facet of P_{LCP} if and only if $|L_{\delta(W)}| \geq 3$ for all W with $\ell \in L_{\delta(W)}$ and $\emptyset \subsetneq W \cap T \subsetneq T$.*

3. *All inequalities $x_\ell \leq 1$ define facets of P_{LCP}.*

4. *All facet defining inequalities $\alpha x \geq \alpha_0$ for P_{LCP} have $\alpha \geq 0$ if $\alpha_0 > 0$.*

5. *A line cut inequality is facet defining if and only if the following two properties are satisfied:*

   (a) *There exists no W', $\emptyset \subsetneq W' \cap T \subsetneq T$, such that $L_{\delta(W')} \subsetneq L_{\delta(W)}$.*

   (b) *For each two W_1, W_2, $\emptyset \subsetneq W_i \cap T \subsetneq T$, with $|L_{\delta(W_i)} \backslash L_{\delta(W)}| = 1$, $i = 1, 2$ and $L_{\delta(W_1)} \backslash L_{\delta(W)} = L_{\delta(W_2)} \backslash L_{\delta(W)}$, we have*

   $$|L_{\delta(W_1)} \cap L_{\delta(W_2)} \cap L_{\delta(W)}| \geq 1.$$

6. *The only nontrivial facet defining inequalities for P_{LCP} with integer coefficients and righthand side equal to 1 are the line cut inequalities.*

Similar to the Steiner tree problem we can define *partition inequalities.* Let $P = (V_1, \ldots, V_k)$ be a partition of the node set V where $V_i \cap T \neq \emptyset$ for $i = 1, \ldots, k$ and $k \geq 3$, i.e., P is a Steiner partition. Let G_P be the graph that arises by contracting each node set V_i to a single node.

Lemma 4.2. *The* line partition inequality

$$\sum_{\ell \in L} a_\ell \cdot x_\ell \geq k-1, \quad a_\ell := (\textit{number of nodes in } G_P \textit{ visited by } \ell) - 1$$

is valid for the line connectivity problem. □

Note that if $k = 2$ we get a line cut constraint.

Analogous to the properties which are necessary for a Steiner partition inequality to be facet defining, c. f. Grötschel and Monma [3], we can formulate the following Proposition.

Proposition 4.3. *Let $\tilde{L} := \{\ell \in L \mid a_\ell = 0\}$. The line partition inequality is facet defining if the following properties are satisfied.*

1. *$G(V_i)$ is connected by $\tilde{L}$, $i = 1, \ldots, k$.*
2. *$G(V_i)$ contains no line cut $L' \subseteq \tilde{L}$ with $|L'| = 1$, $i = 1, \ldots, k$.*
3. *Each line visits at most two nodes in G_P, i.e., $a_\ell \in \{0, 1\}\, \forall \ell \in L$.*
4. *The shrunk graph G_P is 2-line-connected, i.e., if we remove any node with all adjacent lines, the resulting graph is connected.* □

Examples can be constructed in which a line partition inequality is facet defining, but does not satisfy all of the first three properties of Lemma 4.3. Indeed, only Property 4 is necessary.

Proposition 4.4. *If the shrunk graph G_P is not 2-line-connected, the partition inequality is not facet defining for P_{LCP}.* □

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