AN OPTIMAL ADAPTIVE MIXED FINITE ELEMENT METHOD

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ABSTRACT. Various applications in fluid dynamics and computational continuum mechanics motivate the development of reliable and efficient adaptive algorithms for mixed finite element methods. In order to save degrees of freedom, not all but just some selected set of finite element domains are refined. Hence the fundamental question of convergence as well as the question of optimality require new mathematical arguments. The presented adaptive algorithm for Raviart-Thomas mixed finite element methods solves the Poisson model problem, with optimal convergence rate.

Chen, Holst, and Xu presented *convergence and optimality of adaptive mixed finite element methods* (2008) following arguments of Rob Stevenson for the conforming finite element method. Their algorithm reduces oscillations separately, before approximating the solution by some adaptive algorithm in the spirit of W. Dörfler (1996). The algorithm proposed here appears more natural in switching to either reduction of the edge-error estimator or of the oscillations.

1. INTRODUCTION

This paper suggests an optimal adaptive mixed finite element algorithm AMFEM for the Poisson model problem with unknown flux p and primal variable u with

$$p + \nabla u = 0 \text{ and } \operatorname{div} p = f \text{ in } \Omega, \text{ while } u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Given $f \in L^2(\Omega)$, the *mixed variational formulation* reads: Seek $(p, u) \in \mathrm{H}(\operatorname{div}, \Omega) \times L^2(\Omega)$ such that, for all $(q, v) \in \mathrm{H}(\operatorname{div}, \Omega) \times L^2(\Omega)$, it holds

$$(p, q)_{L^2(\Omega)} = (\operatorname{div} q, u)_{L^2(\Omega)} \text{ and } (\operatorname{div} p, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}.$$

Given finite-dimensional piecewise polynomial subspaces $RT_0(\mathcal{T}_\ell) \subseteq \mathrm{H}(\operatorname{div}, \Omega)$ and $P_0(\mathcal{T}_\ell) \subseteq L^2(\Omega)$, named after Raviart and Thomas and

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described in Section 2, the discrete problem reads: Seek $(p_\ell, u_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ such that, for all $(q_\ell, v_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$,

$$(p_\ell, q_\ell)_{L^2(\Omega)} - (\operatorname{div} q_\ell, u_\ell)_{L^2(\Omega)} + (\operatorname{div} p_\ell, v_\ell)_{L^2(\Omega)} = (f, v_\ell)_{L^2(\Omega)}\,. \tag{1.2}$$

The existence and uniqueness of a discrete solution follows from the inf-sup condition [11].

The a posteriori error control of mixed finite element methods dates back to the independent work of Alonso [2] and Carstensen [12]; the error reduction and convergence for adaptive mixed finite element methods was established by Carstensen and Hoppe [18, 17]. The main contribution of the new algorithm in this paper is the step MARK with either edge-oriented refinement or oscillation control. The oscillations on level ℓ are defined for the triangulation $\mathcal{T}_\ell$ of Ω into triangles by

$$\operatorname{osc}_\ell := \operatorname{osc}(f, \mathcal{T}_\ell) := \left(\sum_{T \in \mathcal{T}_\ell} \operatorname{osc}^2(f, T) \right)^{1/2} \tag{1.3}$$

with local oscillations on the domain $\omega \subseteq \Omega$ of area $|\omega|$ defined as

$$\operatorname{osc}(f, \omega) := |\omega|^{1/2} \, \|f - f_\omega\|_{L^2(\omega)} \text{ with}$$
$$f_\omega := \fint_\omega f \,\mathrm{dx} := |\omega|^{-1} \int_\omega f(x) \,\mathrm{dx}\,.$$

For the set of all interior edges $\mathcal{E}_\ell$ in the triangulation $\mathcal{T}_\ell$, the edge-error estimator reads

$$\eta_\ell := \eta_\ell(\mathcal{E}_\ell) \text{ with } \eta_\ell^2(\mathcal{M}) := \sum_{E \in \mathcal{M}} \eta_\ell^2(E) \text{ for } \mathcal{M} \subseteq \mathcal{E}_\ell \tag{1.4}$$

and local contributions

$$\eta_\ell(E) := |E|^{1/2} \, \|[p_\ell]_E\|_{L^2(E)} \text{ for all } E \in \mathcal{E}_\ell.$$

Here and in the sequel, $[q]_E := q|_{T_+} - q|_{T_-}$ denotes the jump of q across an edge $E = T_+ \cap T_-$ shared by the two elements $T_\pm \in \mathcal{T}$ and $\nu_E = \nu_{T_+}$ is the unit normal vector exterior to T_+ along E. Note that the normal component $[p_\ell] \cdot \nu_E$ vanishes because of $p_\ell \in \mathrm{H}(\operatorname{div}, \Omega)$ and so $\eta_\ell(E)$ solely measures the jump parallel to E.

The marking consists of the two alternatives (A) and (B) depending on the computable osc_ℓ and η_ℓ and some global parameter $\kappa > 0$.

In **Case (A)** $\operatorname{osc}_\ell^2 \le \kappa \eta_\ell^2$, compute $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ with minimal cardinality $|\mathcal{M}_\ell|$ such that

$$\theta \eta_\ell^2 \le \eta_\ell^2(\mathcal{M}_\ell)$$

and compute a shape-regular triangulation $\mathcal{T}_{\ell+1}$, where the edges in $\mathcal{M}_\ell$ are bisected plus a minimal number of other edges in some closure algorithm to avoid hanging nodes.

In **Case (B)** $\operatorname{osc}_\ell^2 > \kappa \eta_\ell^2$, run the *Thresholding Second Algorithm* plus completion from [6, 7] resulting in a triangulation $\mathcal{T}$ of nearly

minimal cardinality $|\mathcal{T}|$ such that

$$\operatorname{osc}^2(f,\mathcal{T}) \le \operatorname{Tol}^2 := \rho_B \operatorname{osc}_\ell^2, \text{ with } 0<\rho_B<1$$

and compute the overlay $\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \mathcal{T}$.

The *Thresholding Second Algorithm* of Section 4.4 of [6] is one possible example; the point here is to enforce an oscillation reduction with optimal complexity.

The algorithm is feasible in the sense, that the decision in MARK is solely based on computed quantities and realises the simultaneous reduction of η_ℓ and osc_ℓ. This avoids the computation of an initial triangulation $\mathcal{T}_0$, which approximates the data up to a given fixed tolerance as in [21].

The main theorem states optimal complexity (Theorem 5.7) for AMFEM as defined in detail in Section 2 for particular positive parameters α, β, κ, and $0<\rho_B$, $\theta<1$. Given (p,f) in some approximation class $\mathcal{A}_s$, the sequence of triangulations $(\mathcal{T}_\ell)_\ell$ from AMFEM with discrete fluxes $(p_\ell)_\ell$, Theorem 5.7 implies

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim (\varepsilon_\ell^2 + \operatorname{osc}_\ell^2)^{-1/(2s)} \approx \xi_\ell^{-1/s}$$

which is optimal with respect to $\mathcal{A}_s$ up to a multiplicative generic constant. The proof is based on overlay control (Theorem 3.3) and contraction (Lemma 5.2). In particular, for η_ℓ, and osc_ℓ as defined above and the exact error $\varepsilon_\ell := \|p-p_\ell\|_{L^2(\Omega)}$ of the flux there exists $0<\rho<1$ such that contraction holds for the weighted term

$$\eta_{\ell+1}^2 + \alpha\varepsilon_{\ell+1}^2 + \beta \operatorname{osc}_{\ell+1}^2 =: \xi_{\ell+1}^2 \le \rho\xi_\ell^2.$$

Here and in the sequel, $|\mathcal{T}_\ell| := \operatorname{card}(\mathcal{T}_\ell)$ denotes the number of elements in the finite set $\mathcal{T}_\ell$ and $A \lesssim B$ represents $A \le CB$ for some mesh-independent, positive generic constant C, whereas $A \approx B$ represents $A \lesssim B \lesssim A$. Moreover, standard notation of Lebesgue and Sobolev spaces is employed, e.g., the differential operators are defined for vector valued functions $v(x) \in \mathbb{R}^2$ for all $x \in \mathbb{R}^2$ as

$$\operatorname{div} v := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}, \qquad \operatorname{curl} v := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

and for scalar valued functions v for all $x \in \mathbb{R}^2$

$$\operatorname{Curl} v := \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right).$$

The remaining part of the paper is organised as follows. Section 2 introduces notation and describes further details of the proposed AMFEM. The focus is on the optimal oscillation reduction in MARK with the concept of the overlay $\mathcal{T}_\ell \oplus \mathcal{T}$, defined as the coarsest common regular refinement of both $\mathcal{T}_\ell$ and $\mathcal{T}$. Section 3 is based on the notion of trees and forests to represent refined meshes and overlays and to

combine the control of elements in Case (A) and (B). For the finite sequence $\mathcal{M}_\ell^{(0)}, \ldots, \mathcal{M}_\ell^{(K(\ell))}$ of sets of edges from Algorithm 3.2 with

$$\mathcal{T}_\ell \oplus \mathcal{T} = \textsc{Refine}\left(\mathcal{T}_\ell, \left(\mathcal{M}_\ell^{(k)}\right)_{k=0,\ldots,K(\ell)}\right)$$

the key estimate in Theorem 3.3 reads

$$\sum_{k=0}^{K(\ell)} \left|\mathcal{M}_\ell^{(k)}\right| \leq |\mathcal{T}| - |\mathcal{T}_0|\,.$$

Section 4 introduces the discrete stability and quasi-orthogonality for the proof of contraction and optimality in Section 5.

2. Adaptive mixed finite element method algorithm (Amfem)

This section is devoted to the design of an adaptive algorithm for the lowest-order mixed finite element method for solving the Poisson model problem (1.2).

2.1. **Outline of the adaptive algorithm.** Given a regular, initial coarse triangulation $\mathcal{T}_0$ of Ω into closed triangles, where two distinct elements are either disjoint or share exactly one node or one common edge. Moreover, each element of $T \in \mathcal{T}_0$ has at least one node in the interior of Ω. For any $T \in \mathcal{T}_0$, one edge from the set of its interior edges $\mathcal{E}(T)$ is selected and called its reference edge $E(T)$.

In successive loops of the basic steps Solve, Estimate, Mark and Refine, discrete solutions $(p_\ell, u_\ell) \in V_\ell$ are computed on each level $\ell \geq 0$ based on the current shape-regular triangulation $\mathcal{T}_\ell$ of Ω with the sets of its nodes $\mathcal{N}_\ell$, free nodes $\mathcal{K}_\ell := \Omega \cap \mathcal{N}_\ell$ and edges $\mathcal{E}_\ell$.

The adaptive algorithm is based on a combination of an edge-based error estimator and oscillation control in step Mark described in the sequel.

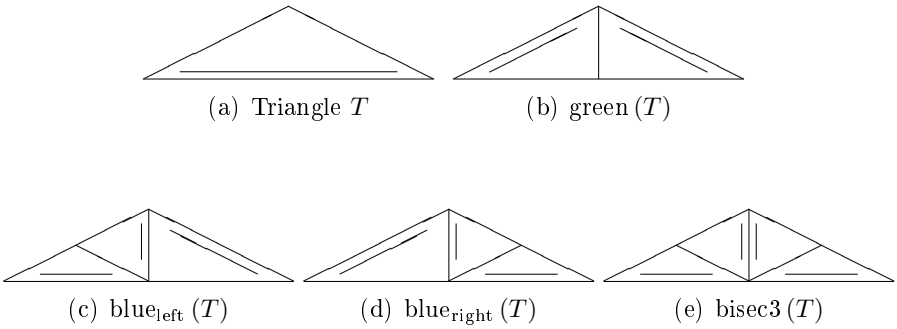
2.2. **Solve.** The Poisson model problem (1.2) is solved on the current triangulation $\mathcal{T}_\ell$ with the space of Raviart-Thomas finite elements of lowest order for the triangulation $\mathcal{T}_\ell$ of level ℓ [8, 11], namely

$$RT_0(\mathcal{T}_\ell) := \{q \in \mathrm{H}(\mathrm{div}, \Omega) \mid \forall T \in \mathcal{T}_\ell \\ \exists a \in \mathbb{R}^2\ \exists b \in \mathbb{R}\ \forall x \in T\ q(x) = a + bx\}.$$

Matlab implementations and documentations of Solve are provided in [5]. Notice that, in particular,

$$\mathrm{div}\, p_\ell + f_\ell = 0 \text{ a.e. in } \Omega$$

for the piecewise integral mean $f_\ell \in P_0(\mathcal{T}_\ell)$ defined by $f_\ell|_T = f_T = \fint_T f(x)\,\mathrm{dx}$ for all $T \in \mathcal{T}_\ell$.



(a) Triangle T (b) green (T)

(c) $\text{blue}_{\text{left}}\,(T)$ (d) $\text{blue}_{\text{right}}\,(T)$ (e) bisec3 (T)

FIGURE 2.1. Possible refinements of a triangle T. The reference edge of each (sub-)triangle is identified through an additional parallel line inside it.

2.3. **ESTIMATE.** The error estimator η_ℓ of (1.4) and the oscillations osc_ℓ of (1.3) allow reliable and efficient error control on the given triangulation $\mathcal{T}_\ell$.

Theorem 2.1 ([2, 16, 12])**.** *The error estimator η_ℓ of* (1.4) *plus the oscillations* osc_ℓ *of* (1.3) *are reliable and efficient in the sense that there exist positive constants C_{eff}, C_{rel}, which depend on the shape but not on the size of the element domains, with*

$$C_{\text{eff}}\eta_\ell^2 \le \varepsilon_\ell^2 := \|p - p_\ell\|^2_{L^2(\Omega)} \le C_{\text{rel}}(\eta_\ell^2 + \operatorname{osc}_\ell^2). \tag{2.1}$$ □

2.4. **Adaptive mesh-refinement: MARK and REFINE.** Given bulk parameter $0 < \theta < 1$ positive κ, the algorithm distinguishes Cases (A) for $\operatorname{osc}_\ell^2 \le \kappa\eta_\ell^2$ and (B) for $\kappa\eta_\ell^2 < \operatorname{osc}_\ell^2$.

The nonempty set $\mathcal{M}_\ell \subseteq \mathcal{E}_\ell$ is specified by MARK and used in REFINE to compute $\mathcal{T}_{\ell+1}$ by the Newest Vertex Bisection (NVB) and completion from [7, 6, 9, 30] w.r.t. $\mathcal{M}_\ell$.

2.5. **Case (A) for** $\operatorname{osc}_\ell^2 \le \kappa\eta_\ell^2$**.** Case (A) performs an error estimator reduction: Sort the set of all edges $\mathcal{E}_\ell$ in $(E_1, \dots, E_N)$ with $\eta_{E_1} \le \cdots \le \eta_{E_N}$. Compute a set $\mathcal{M}_\ell := \{E_N, \dots, E_{N-k}\}$ of minimal cardinality $|\mathcal{M}_\ell| = k + 1$ with

$$\theta\eta_\ell^2 \le \eta_\ell^2(\mathcal{M}_\ell). \tag{2.2}$$

Possible refinement rules *green*, *blue*, and *bisec3* are depicted in Figure 2.1. The refined triangulation $\mathcal{T}_{\ell+1} := \text{REFINE}(\mathcal{T}_\ell, \mathcal{C}\ell(\mathcal{M}_\ell))$ from $\mathcal{T}_\ell$ is uniquely defined in the way, that exactly the edges in $\mathcal{C}\ell(\mathcal{M}_\ell)$ are bisected. $\mathcal{C}\ell(\mathcal{M}_\ell)$ is the minimal subset of $\mathcal{E}_\ell$ which includes $\mathcal{M}_\ell$ and is closed in the sense that

$$\{E(T) \in \mathcal{E}_\ell \mid T \in \mathcal{T}_\ell \text{ and } \mathcal{E}(T) \cap \mathcal{C}\ell(\mathcal{M}_\ell) \neq \emptyset\} \subseteq \mathcal{C}\ell(\mathcal{M}_\ell). \tag{2.3}$$

2.6. **Case (B) for $\operatorname{osc}_\ell^2 > \kappa\eta_\ell^2$.** Case (B) reduces the oscillations: Given f, $\mathcal{T}_0$, $0 < \rho_B < 1$, set $\operatorname{Tol} := \rho_B^{1/2} \operatorname{osc}_\ell$ and run *Thresholding Second Algorithm* plus completion [6, 7] to compute an optimal $\mathcal{T}$ with

$$\operatorname{osc}^2(f,\mathcal{T}) \leq \operatorname{Tol}^2 \text{ and } |\mathcal{T}| - |\mathcal{T}_0| \lesssim \operatorname{Tol}^{-1/s}.$$

Hence, the overlay triangulation $\mathcal{T}_{\ell+1} := \mathcal{T} \oplus \mathcal{T}_\ell$, computed by means of the corresponding forests in Section 3, satisfies

$$\operatorname{osc}_{\ell+1}^2 \leq \operatorname{Tol}^2 \text{ and } |\mathcal{T}_{\ell+1}| - |\mathcal{T}_0| \lesssim \operatorname{Tol}^{-1/s}.$$

By definition of Tol in each level of Case (B), an oscillation reduction with $0 < \rho_B < 1$ holds

$$\operatorname{osc}_{\ell+1}^2 \leq \rho_B \operatorname{osc}_\ell^2. \tag{2.4}$$

The refinement in (B) is *not* level-oriented in the sense that one element domain K of $\mathcal{T}_\ell$ might contain a seemingly uncontrolled number of refined element domains in $\{T \in \mathcal{T}_{\ell+1} \mid T \subseteq K\} \subseteq \mathcal{T}_{\ell+1}$. The control requires the investigations of the subsequent section.

3. Combining the two Cases in one optimal algorithm

This section is devoted to the overall control of the number of finite element domains treated in the two separate ways (A) and (B).

3.1. **Forests representing triangulations.** This subsection briefly recalls the concepts of *trees* and *forests* from [6] to clarify the notion of an overlay and to embed oscillation reduction into the successive loops of the estimator reduction in Case (B) and (A).

A *rooted tree* is a graph, where one *vertex* is designated to be the *root* and any two vertices are connected by exactly one path. If two vertices are connected by an edge, the vertex closer to the root is called *parent* the other its *child*. A vertex with at least one child is called *interior vertex* and otherwise *leaf*. A pairwise disjoint set of trees, is called a *forest*.

This paper focuses on triangulations generated from some coarse triangulation $\mathcal{T}_0$ by *NVB* with refinements of Figure 2.1. Any possible refined triangulation $\mathcal{T}_\ell$ from $\mathcal{T}_0$ is represented by one forest $\mathcal{F}_\ell$ and the refinement of each triangle T of the initial triangulation $\mathcal{T}_0$ corresponds to one tree with root T in $\mathcal{F}_\ell$. The leaves of all trees of $\mathcal{F}_\ell$ represent the elements of $\mathcal{T}_\ell$. Each vertex in $\mathcal{F}_\ell$ has either two children or none and all but the root in $\mathcal{T}_0$ have an ancestor.

In steps Mark and Refine of Case (A) in Amfem, a refined triangulation is computed by marking a set of edges followed by NVB. In Case (B), however, the current triangulation $\mathcal{T}_\ell$ is overlaid with a triangulation $\mathcal{T}$ represented by a forest $\mathcal{F}$ with $\operatorname{osc}(f,\mathcal{T})^2 \leq \operatorname{Tol}^2$. The subsequent subsection explains the definition and the key estimate for the overlay of triangulations. Then, on each level ℓ a triangulation $\mathcal{T}_\ell$ and its forest $\mathcal{F}_\ell$ is available.

3.2. **Overlay of two refinements of $\mathcal{T}_0$.** The coarsest common refinement $\mathcal{T}_\ell \oplus \mathcal{T}$ of two triangulations $\mathcal{T}_\ell$ and $\mathcal{T}$ refined from $\mathcal{T}_0$, called overlay, is defined by the union of their forests $\mathcal{F}_\ell \cup \mathcal{F}$. Its number of elements is bounded as stated in the following Lemma.

Lemma 3.1 ([20]). *The overlay $\mathcal{T}_\ell \oplus \mathcal{T}$ of two triangulations $\mathcal{T}_\ell$ and $\mathcal{T}$ refined from $\mathcal{T}_0$ by NVB satisfies*

$$|\mathcal{T}_\ell \oplus \mathcal{T}| - |\mathcal{T}_\ell| \leq |\mathcal{T}| - |\mathcal{T}_0|\,. \tag{3.1}$$ □

The following algorithm is for theoretical purposes only and allows a common refinement control for Case (A) and (B).

Algorithm 3.2 (Embed Oscillation Control)**.** This algorithm provides a finite sequence of sets of successively marked reference edges $\big(\mathcal{M}_\ell^{(k)}\big)_k$ in order to embed Case (B) of Amfem in the standard level-oriented overall adaptive mesh-refinement. The output of the algorithm realises a finite number of successive refinements, written

$$\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \mathcal{T} = \textsc{Refine}\Big(\mathcal{T}_\ell, \big(\mathcal{M}_\ell^{(k)}\big)_{k=0,\dots,K(\ell)}\Big), \tag{3.2}$$

where in each step k each triangulation is refined as shown in Figure 2.1, with respect to the set of marked edges $\mathcal{M}_\ell^{(k)}$ as follows.
Input: Given $\mathcal{T}_\ell$ and $\mathcal{T}_\ell \oplus \mathcal{T}$, set $\mathcal{T}_\ell^{(0)} := \mathcal{T}_\ell$, $\mathcal{E}_\ell^{(0)} := \mathcal{E}_\ell$, $k := 0$.
Loop: For $k = 0, 1, \dots$ until termination for $k = K(\ell)$ set

$$\mathcal{M}_\ell^{(k)} := \big\{E(T) \in \mathcal{E}_\ell^{(k)} \mid T \in \mathcal{T}_\ell^{(k)} \setminus \mathcal{T}_{\ell+1}\big\}$$

and run NVB to refine $\mathcal{M}_\ell^{(k)}$ in $\mathcal{T}_\ell^{(k)}$ with output

$$\mathcal{T}_\ell^{(k+1)} := \textsc{Refine}\big(\mathcal{T}_\ell^{(k)}, \mathcal{M}_\ell^{(k)}\big). \tag{3.3}$$

If $\mathcal{T}_\ell^{(k+1)} \subsetneqq \mathcal{T}_\ell \oplus \mathcal{T}$ update k, else stop with $k = K(\ell)$.
Output: A sequence of reference edges $\big(\mathcal{M}_\ell^{(k)}\big)_{k=0,\dots,K(\ell)}$.

The benefit of the artificial marked edges $\mathcal{M}_\ell^{(0)}, \dots, \mathcal{M}_\ell^{(K(\ell))}$ in Case (B) is, that the refinement (3.3) is level-oriented such that each triangle in $\mathcal{T}_\ell^{(k)}$ is refined as shown in Figure 2.1 to obtain $\mathcal{T}_\ell^{(k+1)}$.

3.3. **Refinement control in Case (B).**

Theorem 3.3. *Given two triangulations $\mathcal{T}_\ell$ and $\mathcal{T}$ refined from $\mathcal{T}_0$ by NVB, Algorithm 3.2 stops after a finite number $K(\ell) \geq 0$ of steps with*

$$\mathcal{T}_\ell^{(K(\ell)+1)} = \mathcal{T}_\ell \oplus \mathcal{T}$$

and outputs a finite sequence of sets $\big(\mathcal{M}_\ell^{(k)}\big)_{k=0,\dots,K(\ell)}$ with (3.2)-(3.3) and

$$\sum_{k=0}^{K(\ell)} \big|\mathcal{M}_\ell^{(k)}\big| \leq |\mathcal{T}| - |\mathcal{T}_0|\,.$$

Proof. Let $\mathcal{F}_\ell$, $\mathcal{F}_\ell^{(k)}$, and $\mathcal{F}_{\ell+1} = \mathcal{F}_\ell \cup \mathcal{F}$ denote the forests associated to the triangulations $\mathcal{T}_\ell$, $\mathcal{T}_\ell^{(k)}$, and $\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \mathcal{T}$, respectively. By mathematical induction, one oberserves that NVB leads to a nested sequence

$$\mathcal{F}_\ell = \mathcal{F}_\ell^{(0)} \subsetneqq \mathcal{F}_\ell^{(1)} \subsetneqq \cdots \subsetneqq \mathcal{F}_\ell^{(K(\ell))} \subsetneqq \mathcal{F}_\ell^{(K(\ell)+1)} = \mathcal{F}_{\ell+1}.$$

In fact, $\mathcal{T}_\ell^{(k)} \backslash \mathcal{T}_{\ell+1}$ denotes the triangles and $\mathcal{M}_\ell^{(k)}$ the marked edges to be refined in step k. Since $\mathcal{F}_{\ell+1} \setminus \mathcal{F}_\ell$ is finite, Algorithm 3.2 terminates after $K(\ell)$ steps with $\mathcal{F}_\ell^{(K(\ell)+1)} = \mathcal{F}_{\ell+1}$.

For each $E \in \mathcal{M}_\ell^{(k)}$ with $0 \le k \le K(\ell)$, at least one element in $\mathcal{T}_\ell^{(k)}$ is refined into at least two new elements in $\mathcal{T}_\ell^{(k+1)}$. Furthermore, if E is an interior edge, at least two elements in $\mathcal{T}_\ell^{(k)}$ are bisected to at least four new elements in $\mathcal{T}_\ell^{(k+1)}$. Therefore it holds

$$\left|\mathcal{M}_\ell^{(k)}\right| \le \left|\mathcal{T}_\ell^{(k+1)}\right| - \left|\mathcal{T}_\ell^{(k)}\right|. \tag{3.4}$$

Recall $\mathcal{T}_\ell^{(0)} = \mathcal{T}_\ell$, $\mathcal{T}_\ell^{(K(\ell)+1)} = \mathcal{T}_{\ell+1} = \mathcal{T}_\ell \oplus \mathcal{T}$ and apply (3.1), (3.4) to deduce

$$\sum_{k=0}^{K(\ell)} \left|\mathcal{M}_\ell^{(k)}\right| \le |\mathcal{T}_\ell \oplus \mathcal{T}| - |\mathcal{T}_\ell| \le |\mathcal{T}| - |\mathcal{T}_0|\,. \qquad \square$$

3.4. **Closure.** There are several strategies to realise the implementation of refining a mesh by NVB with respect to a set of marked edges $\mathcal{M}_\ell$. One way is first to run some Closure algorithm to compute the smallest superset $\mathcal{C}\ell(\mathcal{M}_\ell)$ of $\mathcal{M}_\ell$ with (2.3). Thereafter, second, refine each triangle according to Figure 2.1 and apply the indicated definition of reference edges.

The *overhead of* Closure is bounded in the following sense. A sequence of triangulations $(\mathcal{T}_\ell)_\ell$ and corresponding sets of marked edges $(\mathcal{M}_\ell)_\ell$ satisfies

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \le C_0 \sum_{j=0}^{\ell-1} |\mathcal{M}_j|$$

for some positive constant C_0 depending solely on $\mathcal{T}_0$ [6, 30]. This estimate is usually employed to Case (A), but holds in Case (B) in the sense that

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \le C_0 \sum_{j=0}^{\ell-1} \sum_{k=0}^{K(j)} \left|\mathcal{M}_j^{(k)}\right|, \tag{3.5}$$

where $\mathcal{M}_j^{(0)}, \ldots, \mathcal{M}_j^{K(j)}$ is the output of Algorithm 3.2 in Case (B), and where $\mathcal{M}_j^{(0)} := \mathcal{M}_j$ and $K(j) := 0$ in Case (A).

4. Further preliminaries

This section summarises some key arguments for the contraction property and optimal convergence. The following Lemmas apply to triangulations $\mathcal{T}_{\ell+k}$ refined from $\mathcal{T}_\ell$ in possibly more than one level of refinements from Figure 2.1.

In contrast to the new version of [21], the proof of Lemma 4.2 presents a direct verification of the discrete stability based on the following nonstandard Poincaré inequality.

Let $\mathrm{mid}(E)$ denote the midpoint of any edge E, and $\mathcal{E}_\ell^D$ the set of edges along the boundary $\partial\Omega$. Furthermore, let $\nabla_{\ell+k}$ denote the piecewise action of the gradient ∇ on $\mathcal{T}_{\ell+k}$.

The subsequent analysis of MFEM employs the nonconforming finite element spaces

$$P_1^{NC}(\mathcal{T}_\ell) := \{v \in P_1(\mathcal{T}_\ell) \mid \forall E \in \mathcal{E}_\ell,\, v \text{ continuous at } \mathrm{mid}(E)\},$$
$$P_{1,0}^{NC}(\mathcal{T}_\ell) := \{v \in P_1^{NC}(\mathcal{T}_\ell) \mid \forall E \in \mathcal{E}_\ell^D,\, v(\mathrm{mid}(E)) = 0\}.$$

Lemma 4.1 (Poincaré inequality)**.** *A nonstandard discrete Poincaré inequality for $a_{\ell+k} \in P_1^{NC}(\mathcal{T}_{\ell+k})$ and $T \in \mathcal{T}_\ell$ reads*

$$\|a_{\ell+k} - a_T\|_{L^2(T)} \lesssim h_T \, \|\nabla_{\ell+k} a_{\ell+k}\|_{L^2(T)}\,. \tag{4.1}$$

Proof. The estimate (4.1) is a consequence of the work of Brenner. For a proof one transforms T and its refined mesh $\mathcal{T}_{\ell+k}|_T$ onto the reference triangle T_{ref} with a refined mesh of right isosceles triangles. Furthermore $a_{\ell+k}$ and a_T are transformed onto T_{ref} to $\tilde{a}_{\ell+k}$ and $\tilde{a}_T$, respectively. Hence, the estimate from [10, Theorem 10.6.16] for arbitrary shape-regular meshes, cf. [10, p. 301f] simplifies to

$$\|\tilde{a}_{\ell+k} - \tilde{a}_T\|_{L^2(T_{\mathrm{ref}})} \lesssim \|\nabla_{\ell+k}\tilde{a}_{\ell+k}\|_{L^2(T_{\mathrm{ref}})}\,.$$

A careful transformation from T_{ref} to T yields the factor h_T in (4.1). □

Lemma 4.2 (Discrete Stability)**.** *Given a triangulation $\mathcal{T}_{\ell+k}$ refined from $\mathcal{T}_\ell$, $k \in \mathbb{N}$, let $p_{\ell+k}$ and $\hat{p}_{\ell+k}$ be the respective discrete MFEM solutions in $RT_0(\mathcal{T}_{\ell+k})$ with right-hand sides $f_{\ell+k}$ and f_ℓ, e.g.,*

$$\mathrm{div}\, p_{\ell+k} + f_{\ell+k} = 0 \text{ and } \mathrm{div}\, \hat{p}_{\ell+k} + f_\ell = 0.$$

Then, there exists some constant $C_1 > 0$ (depending solely on the shape-regularity of $\mathcal{T}_0$) such that

$$C_1^{-1}\, \|p_{\ell+k} - \hat{p}_{\ell+k}\|_{L^2(\Omega)} \le \Big(\sum_{T \in \mathcal{T}_\ell} h_T^2 \, \|f_{\ell+k} - f_\ell\|_{L^2(T)}^2 \Big)^{1/2} \le \mathrm{osc}_\ell\,.$$

Proof. Let $x_{\ell+k} \in P_0(\mathcal{T}_{\ell+k}; \mathbb{R}^2)$ denote the triangle midpoints, $x_{\ell+k}|_T := \mathrm{mid}(T)$ for $T \in \mathcal{T}_{\ell+k}$. It is well established [24, 5] that

$$p_{\ell+k}(x) = \nabla_{\ell+k} u_{\ell+k}^{NC} + (x - x_{\ell+k}) f_{\ell+k}/2 \text{ for } x \in T \in \mathcal{T}_{\ell+k},$$
$$\hat{p}_{\ell+k}(x) = \nabla_{\ell+k} \hat{u}_{\ell+k}^{NC} + (x - x_{\ell+k}) f_\ell/2 \text{ for } x \in T \in \mathcal{T}_{\ell+k},$$

while $u_{\ell+k}^{NC} \in P_1^{NC}(\mathcal{T}_{\ell+k})$ solves

$$\int_\Omega \nabla_{\ell+k} u_{\ell+k}^{NC} \cdot \nabla_{\ell+k} v_{\ell+k}^{NC} \,\mathrm{dx} = \int_\Omega f_{\ell+k} v_{\ell+k}^{NC} \,\mathrm{dx} \text{ for all } v_{\ell+k}^{NC} \in P_{1,0}^{NC}(\mathcal{T}_{\ell+k})$$

with piecewise constant right-hand side $f_{\ell+k} \in P_0(\mathcal{T}_{\ell+k})$.

When, $(\bullet - x_{\ell+k})$ represents the factor $(x - x_{\ell+k})$ of $x \in \Omega$, the definition $a_{\ell+k} := u_{\ell+k}^{NC} - \hat{u}_{\ell+k}^{NC}$ yields

$$\nabla_{\ell+k} a_{\ell+k} = \nabla_{\ell+k} u_{\ell+k}^{NC} - \nabla_{\ell+k} \hat{u}_{\ell+k}^{NC} = p_{\ell+k} - \hat{p}_{\ell+k} - (\bullet - x_{\ell+k})(f_{\ell+k} - f_\ell)/2.$$

Since $(\bullet - x_{\ell+k}) \perp_{L^2(\Omega)} P_0(\mathcal{T}_\ell)^2$ there holds

$$\begin{aligned} \|p_{\ell+k} - \hat{p}_{\ell+k}\|_{L^2(\Omega)}^2 &= \|(\bullet - x_{\ell+k})(f_{\ell+k} - f_\ell)/2\|_{L^2(\Omega)}^2 \\ &\quad + \|\nabla_{\ell+k} a_{\ell+k}\|_{L^2(\Omega)}^2 . \end{aligned} \tag{4.2}$$

Moreover, an elementwise integration by parts shows

$$\begin{aligned} \|\nabla_{\ell+k} a_{\ell+k}\|_{L^2(\Omega)}^2 &= (p_{\ell+k} - \hat{p}_{\ell+k}, \nabla_{\ell+k} a_{\ell+k})_{L^2(\Omega)} \\ &= -\left(\operatorname{div}\left(p_{\ell+k} - \hat{p}_{\ell+k}\right), a_{\ell+k}\right)_{L^2(\Omega)} \\ &\quad + \sum_{E \in \mathcal{E}_{\ell+k}} \int_E \left(p_{\ell+k} - \hat{p}_{\ell+k}\right) \cdot \nu_E [a_{\ell+k}]_E \,\mathrm{ds} \\ &= \sum_{T \in \mathcal{T}_\ell} \int_T (f_{\ell+k} - f_\ell)(a_{\ell+k} - a_T) \,\mathrm{dx} \end{aligned}$$

with the integral mean a_T of $a_{\ell+k}$ on a coarser triangle $T \in \mathcal{T}_\ell$. The last identity follows for $T \in \mathcal{T}_\ell$ and $E \in \mathcal{E}_{\ell+k}$ from

$$\int_T (f_{\ell+k} - f_\ell) \,\mathrm{dx} = 0 = \int_E [a_{\ell+k}]_E \,\mathrm{ds}\,.$$

Recall that h_ℓ is the piecewise constant function with $h_\ell|_T = h_T = \operatorname{diam}(T)$ for all $T \in \mathcal{T}_\ell$. The combination of (4.1) of Lemma 4.2 with the aforementioned arguments and orthogonality in (4.2) leads to

$$\|\nabla_{\ell+k} a_{\ell+k}\|_{L^2(\Omega)}^2 \lesssim \left(\sum_{T \in \mathcal{T}_\ell} h_T^2 \, \|f_{\ell+k} - f_\ell\|_{L^2(T)}^2 \right)^{1/2} \|\nabla_{\ell+k} a_{\ell+k}\|_{L^2(\Omega)} . \qquad \square$$

Lemma 4.3 (Quasiorthogonality)**.** *Let $\mathcal{T}_{\ell+k}$ be a triangulation refined from $\mathcal{T}_\ell$. Given $C_1 > 0$ from Lemma 4.2, for the exact solution (p, u) of (1.1), and the MFEM solutions $(p_{\ell+k}, u_{\ell+k}) \in RT_0(\mathcal{T}_{\ell+k}) \times P_0(\mathcal{T}_{\ell+k})$, and $(p_\ell, u_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ of (1.2), quasiorthogonality holds*

$$(p - p_{\ell+k}, p_{\ell+k} - p_\ell)_{L^2(\Omega)} \leq C_1 \varepsilon_{\ell+k} \operatorname{osc}_\ell; \tag{4.3}$$

$$\|p_{\ell+k} - p_\ell\|_{L^2(\Omega)}^2 \leq \varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 + 2C_1 \varepsilon_{\ell+k} \operatorname{osc}_\ell; \tag{4.4}$$

$$\varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 \leq \|p_{\ell+k} - p_\ell\|_{L^2(\Omega)}^2 + 2C_1 \varepsilon_{\ell+k} \operatorname{osc}_\ell\,. \tag{4.5}$$

Proof. Let $\hat{p}_{\ell+k} \in RT_0(\mathcal{T}_{\ell+k})$ be the intermediate solution of Lemma 4.2 with $\operatorname{div} \hat{p}_{\ell+k} + f_\ell = 0$. An integration by parts and $\operatorname{div}\big(\hat{p}_{\ell+k} - p_\ell\big) = 0$ show $(p - p_{\ell+k}, \hat{p}_{\ell+k} - p_\ell)_{L^2(\Omega)} = 0$.

Hence an application of Lemma 4.2 yields

$$\begin{aligned}\Big|(p - p_{\ell+k}, p_{\ell+k} - p_\ell)_{L^2(\Omega)}\Big| &\leq \varepsilon_{\ell+k} \, \|p_{\ell+k} - \hat{p}_{\ell+k}\|_{L^2(\Omega)} \\ &\leq C_1 \varepsilon_{\ell+k} \operatorname{osc}_\ell \,.\end{aligned}$$

This proves (4.3) and leads to (4.4) via

$$\begin{aligned}\|p_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} &= \varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 - 2\,(p - p_{\ell+k}, p_{\ell+k} - p_\ell)_{L^2(\Omega)} \\ &\leq \varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 + 2C_1 \varepsilon_{\ell+k} \operatorname{osc}_\ell \,.\end{aligned}$$

The same arguments yield (4.5), namely

$$\begin{aligned}\varepsilon_\ell^2 - \varepsilon_{\ell+k}^2 &= \|p_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} + 2\,(p - p_{\ell+k}, p_{\ell+k} - p_\ell)_{L^2(\Omega)} \\ &\leq \|p_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} + 2C_1 \varepsilon_{\ell+k} \operatorname{osc}_\ell \,.\end{aligned}$$

□

Lemma 4.4 ([21]). *Let $\mathcal{T}_{\ell+k}$ be a triangulation refined from $\mathcal{T}_\ell$. Given $C_1 > 0$ from Lemma 4.2, and the respective MFEM solutions of (1.2) $(p_{\ell+k}, u_{\ell+k}) \in RT_0(\mathcal{T}_{\ell+k}) \times P_0(\mathcal{T}_{\ell+k})$ and $(p_\ell, u_\ell) \in RT_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$, there exists a constant $C_2 > 0$, which solely depends on $\mathcal{T}_0$, such that*

$$\big|\mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}\big| \leq 3\big(\,|\mathcal{T}_{\ell+k}| - |\mathcal{T}_\ell|\,\big), \quad \text{and} \tag{4.6}$$

$$\|p_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} \leq C_2 \eta_\ell^2\big(\mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}\big) + C_1^2 \operatorname{osc}_\ell^2 \,. \tag{4.7}$$

Proof of (4.6) of Lemma 4.4. For each $E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}$ there is a refinement of the neighbourhood $\bar{\omega}_E := T_+ \cup T_-$ of E and each neighbouring $K \in \{T_+, T_-\} \subset \mathcal{T}_\ell$ with $E \subseteq \partial K$ is, at least, bisected in the refinements from $\mathcal{T}_\ell$ to $\mathcal{T}_{\ell+k}$. Let

$$\chi_j(K) := |\{T \in \mathcal{T}_j \mid T \subseteq \bar{\omega}_E\}| \text{ for } K \in \mathcal{T}_\ell \text{ and } j = \ell, \ell+1, \ldots;$$

$$m_E := \begin{cases} 1/2 & \text{if } E \text{ interior, } E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}, \\ 1 & \text{if } E \subseteq \partial\Omega, \; E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}$,

$$1 \leq m_E\big(\chi_{\ell+k}(\omega_E) - \chi_\ell(\omega_E)\big),$$

holds, and hence

$$\begin{aligned}|\mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}| &\leq \sum_{E \in \mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}} m_E \left(\chi_{\ell+k}(\omega_E) - \chi_\ell(\omega_E)\right) \\ &= \sum_{T \in \mathcal{T}_\ell} \sum_{E \in \mathcal{E}_\ell(T)} m_E \left(\chi_{\ell+k}(\omega_E) - \chi_\ell(\omega_E)\right) \\ &\leq 3 \sum_{T \in \mathcal{T}_\ell} \left(|\{K \in \mathcal{T}_{\ell+k} \mid K \subseteq T\}| - 1\right) = 3\left(|\mathcal{T}_{\ell+k}| - |\mathcal{T}_\ell|\right).\end{aligned}$$

This verifies (4.6) with factor 3 on the right-hand side. In fact, this factor could be 2 with a more detailled inspection and an assignment of one proper neighbour T_E of E with similar arguments. □

Proof of (4.7) *of Lemma 4.4.* Let $\hat{p}_{\ell+k} \in RT_0(\mathcal{T}_{\ell+k})$ be the intermediate solution of Lemma 4.2 with $\operatorname{div} \hat{p}_{\ell+k} + f_\ell = 0$.

Then, $\operatorname{div}\left(\hat{p}_{\ell+k} - p_\ell\right) = 0$ shows $(\hat{p}_{\ell+k} - p_{\ell+k}, \hat{p}_{\ell+k} - p_\ell)_{L^2(\Omega)} = 0$. This and Lemma 4.2 yield

$$\begin{aligned} \|p_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} &= \|\hat{p}_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} + \|p_{\ell+k} - \hat{p}_{\ell+k}\|^2_{L^2(\Omega)} \\ &\leq \|\hat{p}_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} + C_1 \operatorname{osc}_\ell^2 . \end{aligned} \tag{4.8}$$

It remains to bound $\|\hat{p}_{\ell+k} - p_\ell\|^2_{L^2(\Omega)}$. The discrete orthogonal Helmholtz decomposition [3] yields $a_{\ell+k} \in P^{NC}_{1,0}(\mathcal{T}_{\ell+k})$ and $b_{\ell+k} \in \hat{P}_1(\mathcal{T}_{\ell+k}) := \left\{v \in P_1(\mathcal{T}_{\ell+k}) \cap C\left(\bar{\Omega}\right) \mid \fint_\Omega v \,\mathrm{dx} = 0\right\}$ with

$$\hat{p}_{\ell+k} - p_\ell = \nabla_{\ell+k} a_{\ell+k} + \operatorname{Curl} b_{\ell+k}.$$

Let $b_\ell := \mathcal{I}_\ell b_{\ell+k}$ some (e.g., Scott-Zhang) quasi-interpolation with

$$\|b_{\ell+k} - \mathcal{I}_\ell b_{\ell+k}\|_{L^2(E)} \leq C h_E^{1/2} \, |b_{\ell+k}|_{H^1(\omega_E)}$$

for $E \in \mathcal{E}_\ell$ and its neighbourhood ω_E. Notice that

$$\|b_{\ell+k} - \mathcal{I}_\ell b_{\ell+k}\|_{L^2(E)} = 0 \text{ if } E \in \mathcal{E}_{\ell+k} \cap \mathcal{E}_\ell.$$

The L^2 orthogonality $(\hat{p}_{\ell+k} - p_\ell) \perp_{L^2(\Omega)} \nabla_{\ell+k} a_{\ell+k}$ verifies

$$\|\hat{p}_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} = (\hat{p}_{\ell+k} - p_\ell, \operatorname{Curl} b_{\ell+k})_{L^2(\Omega)} ,$$

while the discrete conditions (1.2) with test function $\operatorname{Curl} b_\ell \in RT_0(\mathcal{T}_\ell) \subseteq RT_0(\mathcal{T}_{\ell+k})$ lead to

$$(\hat{p}_{\ell+k}, \operatorname{Curl} b_{\ell+k})_{L^2(\Omega)} = 0 = (p_\ell, \operatorname{Curl} b_\ell)_{L^2(\Omega)} .$$

Hence,

$$\begin{aligned} \|\hat{p}_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} &= (p_\ell, \operatorname{Curl}(b_\ell - b_{\ell+k}))_{L^2(\Omega)} \\ &= \sum_{E \in \mathcal{E}_\ell} \int_E [p_\ell]_E \cdot \tau_E (b_\ell - b_{\ell+k}) \,\mathrm{ds} \\ &\quad - \sum_{T \in \mathcal{T}_\ell} \int_T \operatorname{curl}(p_\ell)(b_\ell - b_{\ell+k}) \,\mathrm{ds} \\ &= \sum_{E \in \mathcal{E}_{\ell+k} \setminus \mathcal{E}_\ell} \int_E [p_\ell]_E \cdot \tau_E (b_\ell - b_{\ell+k}) \,\mathrm{ds} . \end{aligned}$$

In 2D it holds $|b_{\ell+k}|_{H^1(\Omega)} \leq \|\hat{p}_{\ell+k} - p_\ell\|_{L^2(\Omega)}$. Thus,

$$\begin{aligned} \|\hat{p}_{\ell+k} - p_\ell\|^2_{L^2(\Omega)} &\leq \sum_{E \in \mathcal{E}_{\ell+k} \setminus \mathcal{E}_\ell} \|[p_\ell]_E\|_{L^2(E)} \, \|b_\ell - b_{\ell+k}\|_{L^2(E)} \\ &\leq C_2^{1/2} \eta_\ell \left(\mathcal{E}_\ell \setminus \mathcal{E}_{\ell+k}\right) \|\hat{p}_{\ell+k} - p_\ell\|_{L^2(\Omega)} . \end{aligned}$$

A division by $\|\hat{p}_{\ell+k} - p_\ell\|_{L^2(\Omega)}$ plus (4.8) lead to the assertion (4.7). □

5. Convergence and optimality of Amfem

This section is devoted to the convergence analysis of the adaptive mixed finite element method (Amfem).

5.1. **Contraction Property.**

Lemma 5.1. *There exists some constant $C_3 > 0$, which depends only on the triangulation $\mathcal{T}_0$, such that for any $\delta > 0$ and $C_\delta := C_3(1 + 1/\delta)$ and on any level ℓ with Case (A) and $0 < \theta < 1$ or with Case (B) and $k \geq 1$ it holds*

$$\eta_{\ell+1}^2 \leq (1+\delta)(1-\theta/2)\eta_\ell^2 + C_\delta \|p_{\ell+1} - p_\ell\|_{L^2(\Omega)}^2 \quad \text{in Case (A)}, \tag{5.1}$$

$$\eta_{\ell+k}^2 \leq (1+\delta)\eta_\ell^2 + C_\delta \|p_{\ell+k} - p_\ell\|_{L^2(\Omega)}^2 \quad \text{in Case (B)}. \tag{5.2}$$

Proof. The estimates are proven by applying Young's inequality and by exploiting that $p_{\ell+k} - p_\ell$ is piecewise constant on $\mathcal{T}_{\ell+k}$, $k \geq 1$ (cf. e.g., [14, 20]). □

Lemma 5.2 (Contraction)**.** *For any choice of $0 < \theta$, $\rho_B < 1$, there exists positive parameters α, β, κ_0 and $0 < \rho < 1$, such that for any $0 < \kappa < \kappa_0$ and on any level $\ell \geq 0$ in Case (A) or (B) of algorithm* Amfem *the weighted term ξ_ℓ^2 of exact error, estimated error and oscillations satisfies contraction*

$$\xi_{\ell+1}^2 \leq \rho \xi_\ell^2, \quad \xi_\ell^2 := \eta_\ell^2 + \alpha\varepsilon_\ell^2 + \beta \operatorname{osc}_\ell^2. \tag{5.3}$$

Proof. Given positive constants C_1, C_δ, C_{rel} from Lemmas 4.2, 5.1, and Theorem 2.1, respectively. Let $0 < \theta$, $\rho_B < 1$. Contraction is proven for the following choice of positive parameters A, B, D, α, β, δ, γ, κ_0, and $0 < \rho_A < 1$ and all $0 < \kappa < \kappa_0$

$$\begin{aligned}
\delta &:= \theta/(4-2\theta), & \rho_A &:= (1+\delta)(1-\theta/2),\\
\gamma &:= \frac{1}{2}\min\left\{1, \frac{1-\rho_A}{C_\delta C_{\mathrm{rel}}}\right\}, & B &:= \frac{3}{4}\min\left\{1, \frac{1-\rho_A}{C_\delta C_{\mathrm{rel}}}\right\},\\
A &:= 2(C_1 C_\delta/\gamma + C_\delta C_{\mathrm{rel}} B), & \alpha &:= C_\delta(1-\gamma),\\
\kappa_0 &:= \frac{1-\rho_A - C_\delta C_{\mathrm{rel}} B}{A}, & D &:= \frac{1/2 + \delta + C_\delta C_{\mathrm{rel}} B}{\kappa},\\
\beta &:= 2\max\left\{\frac{C_1 C_\delta/\gamma + C_\delta C_{\mathrm{rel}} B + D}{1-\rho_B}, A - C_1 C_\delta/\gamma - C_\delta C_{\mathrm{rel}} B\right\}.
\end{aligned}$$

The application of Young's inequality and reliability (2.1) to quasiorthogonality (4.4) leads to

$$\begin{aligned}
\|p_{\ell+1} - p_\ell\|_{L^2(\Omega)}^2 \leq{}& (1-B)\varepsilon_\ell^2 - (1-\gamma)\varepsilon_{\ell+1}^2\\
&+ (C_1/\gamma + C_{\mathrm{rel}} B)\operatorname{osc}_\ell^2 + C_{\mathrm{rel}} B \eta_\ell^2.
\end{aligned} \tag{5.4}$$

On each level with **Case (A)**, substituting (5.4) in (5.1) reveals

$$\begin{aligned}\eta_{\ell+1}^2 \leq (\rho_A + C_\delta C_{\mathrm{rel}} B)\, \eta_\ell^2 + C_\delta(1-B)\varepsilon_\ell^2 \\ - C_\delta(1-\gamma)\varepsilon_{\ell+k}^2 + C_\delta\,(C_1/\gamma + C_{\mathrm{rel}}B)\,\mathrm{osc}_\ell^2\,.\end{aligned}$$

Together with reliability, $\mathrm{osc}_{\ell+1}^2 \leq \mathrm{osc}_\ell^2$ and $\kappa\eta_\ell^2 \leq \mathrm{osc}_\ell^2$, this verifies

$$\begin{aligned}\xi_{\ell+1}^2 \leq (\rho_A + A\kappa + C_\delta C_{\mathrm{rel}} B)\, \eta_\ell^2 + C_\delta\,(1-B)\,\varepsilon_\ell^2 \\ + (\beta + C_1 C_\delta/\gamma - A + C_\delta C_{\mathrm{rel}} B)\,\mathrm{osc}_\ell^2\,.\end{aligned}$$

Thus, with the proposed choice of parameters, on any level with Case (A) contraction is realised, i.e., $\xi_{\ell+1}^2 \leq \rho_1 \xi_\ell^2$ holds with ρ_1 defined by

$$\begin{aligned}0 < \rho_1 := \max\{(\rho_A + A\kappa + C_\delta C_{\mathrm{rel}} B)\,, C_\delta\,(1-B)\,/\alpha, \\ (\beta + C_1 C_\delta/\gamma - A + C_\delta C_{\mathrm{rel}} B)\,/\beta\} < 1.\end{aligned}$$

In **Case (B)**, similarly to Case (A), substituting (5.4) in (5.2) and applying $\mathrm{osc}_{\ell+1}^2 \leq \rho_B\,\mathrm{osc}_\ell^2$, as well as $\kappa\eta_\ell^2 \leq \mathrm{osc}_\ell^2$ proves

$$\begin{aligned}\xi_{\ell+1}^2 \leq (1 + \delta + C_\delta C_{\mathrm{rel}} B - D\kappa)\eta_\ell^2 + C_\delta(1-B)\varepsilon_\ell^2 \\ + (C_1 C_\delta/\gamma + C_\delta C_{\mathrm{rel}} B + \beta\rho_B + D)\,\mathrm{osc}_\ell^2\,.\end{aligned}$$

Thus, for the special choice of parameters, contraction of ξ_ℓ^2 on any level with Case (B), i.e., $\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2$ holds with ρ_2 defined as

$$\begin{aligned}0 < \rho_2 := \max\{1 + \delta + C_\delta C_{\mathrm{rel}} B - D\kappa, C_\delta(1-B)/\alpha, \\ +(C_1 C_\delta/\gamma + C_\delta C_{\mathrm{rel}} B + \beta\rho_B + D)/\beta\} < 1.\end{aligned}$$

Hence, there exist parameters α, β, $\kappa_0 > 0$ such that for all $0 < \kappa < \kappa_0$ on any level, irrespective of the relation between η_ℓ and osc_ℓ contraction (5.3) of the weighted term ξ_ℓ with $\rho := \max\{\rho_1, \rho_2\}$ is ensured. □

Remark 5.3 (Upper and lower bounds). Let $\mathcal{T}_{\ell+k}$ be some refinement of $\mathcal{T}_\ell$. Given C_{eff}, C_{rel}, C_1, α, β, κ from Theorem 2.1, and Lemmas 4.2, 5.2, respectively, and set

$$\begin{aligned}C_{\mathrm{a}} &:= 1 + \alpha C_{\mathrm{rel}} + (\alpha C_{\mathrm{rel}} + \beta)\kappa, \qquad C_{\mathrm{b}} := C_{\mathrm{a}}/\kappa, \\ C_{\mathrm{c}} &:= \max\left\{2(\alpha + C_{\mathrm{eff}}^{-1})/\alpha, \left(4C_1^2(\alpha + C_{\mathrm{eff}}^{-1}) + \beta\right)/\beta\right\}.\end{aligned}$$

Then, the weighted terms ξ_ℓ and $\xi_{\ell+k}$ of the MFEM errors ε_ℓ, $\varepsilon_{\ell+k}$, the estimated errors η_ℓ, $\eta_{\ell+k}$, and oscillations osc_ℓ, $\mathrm{osc}_{\ell+k}$ satisfy

$$\xi_\ell^2 \leq \begin{cases} C_{\mathrm{a}}\eta_\ell^2 & \text{if Case (A) applies on level } \ell, \\ C_{\mathrm{b}}\,\mathrm{osc}_\ell^2 & \text{if Case (B) applies on level } \ell, \end{cases} \tag{5.5}$$

$$\varepsilon_{\ell+k}^2 \leq 2\varepsilon_\ell^2 + 4C_1^2\,\mathrm{osc}_\ell^2, \qquad \alpha\varepsilon_{\ell+k}^2 \leq \xi_{\ell+k}^2 \leq C_{\mathrm{c}}\xi_\ell^2, \tag{5.6}$$

$$\xi_\ell^2 \approx \varepsilon_\ell^2 + \mathrm{osc}_\ell^2\,. \tag{5.7}$$

Proof. (5.5) is proven by applying reliability and the specific relation of the estimated error and oscillations in Case (A) and (B); while the estimates (5.6) are proven by applying quasiorthogonality and Young's inequality, which yields

$$\begin{aligned}\varepsilon_{\ell+k}^2 &\leq \varepsilon_\ell^2 + 2C_1\varepsilon_{\ell+k}\operatorname{osc}_\ell - \|p_{\ell+k}-p_\ell\|_{L^2(\Omega)}^2 \\ &\leq \varepsilon_\ell^2 + 2C_1^2\operatorname{osc}_\ell^2 + \varepsilon_{\ell+k}^2/2 - \|p_{\ell+k}-p_\ell\|_{L^2(\Omega)}^2,\end{aligned}$$

plus efficiency for the second estimate of (5.6). (5.7) follows directly from efficiency and reliability (2.1) of η_ℓ. □

5.2. **Optimal convergence.** This subsection is devoted to the optimal convergence rate [6, 29] of the adaptive algorithm AMFEM.

Definition 5.4 (Approximation Class)**.** Given an initial regular triangulation $\mathcal{T}_0$ of Ω and $s>0$, let

$$\begin{aligned}\mathcal{A}_s &:= \{(p,f)\in \mathrm{H}(\operatorname{div},\Omega)\times L^2(\Omega) \mid \|(p,f)\|_{\mathcal{A}_s}<\infty\} \text{ with}\\ \|(p,f)\|_{\mathcal{A}_s} &:= \sup_{N\in\mathbb{N}}\left(N^s \inf_{|\mathcal{T}|-|\mathcal{T}_0|\leq N}\left(\varepsilon^2(\mathcal{T})+\operatorname{osc}^2(f,\mathcal{T})\right)^{1/2}\right),\\ \varepsilon(\mathcal{T}) &:= \|p-p_{\mathcal{T}}\|_{L^2(\Omega)}.\end{aligned}$$

The infimum is with respect to all NVB-generated refinements $\mathcal{T}$ of $\mathcal{T}_0$, called admissible triangulations, with number of element domains $|\mathcal{T}|\leq N+|\mathcal{T}_0|$ and with the exact error $\varepsilon(\mathcal{T})$ of the flux-part $p_{\mathcal{T}}\in RT_0(\mathcal{T})$ of the MFEM solution.

Remark 5.5. The approximation class $\mathcal{A}_s$ can be characterised [29, p.255 Remark 5.1, p.263 l.17] by $(p,f)\in\mathcal{A}_s$ if and only if for all $\epsilon>0$ there exists an admissible triangulation $\mathcal{T}_\epsilon$ such that the associated MFEM solution (p_ϵ,u_ϵ) in $RT_0(\mathcal{T}_\epsilon)\times P_0(\mathcal{T}_\epsilon)$ satisfies

$$\varepsilon^2(\mathcal{T}_\epsilon)+\operatorname{osc}^2(f,\mathcal{T}_\epsilon)\leq\epsilon^2 \qquad\text{and}\qquad |\mathcal{T}_\epsilon|-|\mathcal{T}_0|\lesssim\epsilon^{-1/s}\|(p,f)\|_{\mathcal{A}_s}^{1/s}.$$

Remark 5.6. Our main result states optimal convergence of the AMFEM in the following sense: Given $(p,f)\in\mathcal{A}_s$ and $\ell\in\mathbb{N}$, the algorithm generates a triangulation $\mathcal{T}_\ell$ with discrete solutions (p_ℓ,u_ℓ) in $RT_0(\mathcal{T}_\ell)\times P_0(\mathcal{T}_\ell)$ such that

$$|\mathcal{T}_\ell|-|\mathcal{T}_0|\lesssim\xi_\ell^{-1/s}\approx\left(\varepsilon_\ell^2+\operatorname{osc}_\ell^2\right)^{-1/(2s)}\approx\left(\varepsilon_\ell+\operatorname{osc}_\ell\right)^{-1/s}.$$

Theorem 5.7 (Optimal Convergence Rates)**.** *Given positive constants C_{eff}, C_1, C_2 from Theorem 2.1, Lemmas 4.2, 4.4. Furthermore, let $\mathcal{T}_0$ be some intial triangulation, $s>0$, and $0<\theta<\min\{C_{\text{eff}}/C_2,1\}$. Then, for all $0<\rho_B<1$ there exists positive parameters α, β, κ_0, and $0<\rho_A$, $\rho<1$ from Lemma 5.2, such that for all $0<\kappa<\min\{\kappa_0, C_{\text{eff}}-C_2\theta/(2C_1^2)\}$ and $(p,f)\in\mathcal{A}_s$ the algorithm* AMFEM *generates triangulations which satisfy*

$$|\mathcal{T}_\ell|-|\mathcal{T}_0|\lesssim\xi_\ell^{-1/s},\ \textit{for all } \ell\geq 0.$$

Proof. Moreover, given positive constants $C_{\rm eff}$, $C_{\rm rel}$, C_0, C_1, C_2, $C_{\rm a}$ from Theorem 2.1, (3.5), Lemmas 4.2, 4.4, and Remark 5.3, as well as positive parameters α, β, κ_0, and $0<\rho_A$, ρ_B, $\rho<1$ from Lemma 5.2. The key of the proof is to verify for any level ℓ in either Case (A) or (B) there exist $K(\ell)\in\mathbb{N}$ sets of marked edges $\mathcal{M}_\ell^{(0)},\dots,\mathcal{M}_\ell^{K(\ell)}$ with

$$\sum_{k=0}^{K(\ell)}\big|\mathcal{M}_\ell^{(k)}\big|\lesssim\xi_\ell^{-1/s}. \tag{5.8}$$

In fact, once (5.8) is verified, (3.5) shows

$$|\mathcal{T}_\ell|-|\mathcal{T}_0|\le\sum_{j=0}^{\ell-1}\big(\,|\mathcal{T}_{j+1}|-|\mathcal{T}_j|\,\big)\le C_0\sum_{j=0}^{\ell-1}\sum_{k=0}^{K(j)}|\mathcal{M}_j^{(k)}|\lesssim\sum_{j=0}^{\ell-1}\xi_j^{-1/s}.$$

This and the contraction property (5.3) from Lemma 5.2 reveal optimal global convergence owing to the subsequent arguments

$$|\mathcal{T}_\ell|-|\mathcal{T}_0|\le\xi_\ell^{-1/s}\sum_{k=1}^{\ell}\rho^{-k/(2s)}=\frac{1-\rho^{-(\ell+1)/(2s)}}{1-\rho^{-1/(2s)}}\xi_\ell^{-1/s}\lesssim\xi_\ell^{-1/s}.$$

Hence it remains to prove (5.8). Given the triangulation $\mathcal{T}_\ell$ and discrete solution (p_ℓ,u_ℓ) of (1.2). Due to the choice of κ, in **Case (A)** τ can be chosen to satisfy

$$0<\tau^2\le\frac{\alpha\,(C_{\rm eff}-2C_1^2\kappa-C_2\theta)}{2C_{\rm a}C_{\rm c}},$$

with $C_{\rm a}$, $C_{\rm c}>0$ from Remark 5.3. Setting $\epsilon:=\tau\xi_\ell$, Remark 5.5 leads to the existence of some admissible triangulation $\mathcal{T}_\epsilon$ refined from $\mathcal{T}_0$ such that

$$\xi^2(\mathcal{T}_\epsilon):=\eta^2(\mathcal{T}_\epsilon)+\alpha\varepsilon^2(\mathcal{T}_\epsilon)+\beta\,\mathrm{osc}^2(f,\mathcal{T}_\epsilon)\le\epsilon^2,$$

with the exact error $\varepsilon(\mathcal{T}_\epsilon)$, the estimated error $\eta(\mathcal{T}_\epsilon)$ and oscillations $\mathrm{osc}(f,\mathcal{T}_\epsilon)$ on $\mathcal{T}_\epsilon$. Then, by applying Lemma 3.1 the overlay $\mathcal{T}:=\mathcal{T}_\epsilon\oplus\mathcal{T}_\ell$ and its associated quantities $\xi^2(\mathcal{T})=\eta^2(\mathcal{T})+\alpha\varepsilon^2(\mathcal{T})+\beta\,\mathrm{osc}^2(f,\mathcal{T})$, with estimated error $\eta(\mathcal{T})$, $\varepsilon(\mathcal{T})=\|p-p_{\mathcal{T}}\|_{L^2(\Omega)}$, and oscillations $\mathrm{osc}(f,\mathcal{T})$ satisfy

$$|\mathcal{T}|-|\mathcal{T}_\ell|=|\mathcal{T}_\epsilon\oplus\mathcal{T}_\ell|-|\mathcal{T}_\ell|\le|\mathcal{T}_\epsilon|-|\mathcal{T}_0|\lesssim\epsilon^{-1/s}\approx\xi_\ell^{-1/s}.$$

In the next step we prove that $\mathcal{E}_\ell\setminus\mathcal{E}_{\mathcal{T}}$ fulfils the bulk criterion (2.2), namely

$$\theta\eta_\ell^2\le\eta_\ell^2(\mathcal{E}_\ell\setminus\mathcal{E}_{\mathcal{T}}). \tag{5.9}$$

Efficiency, $\mathrm{osc}_\ell^2\le\kappa\eta_\ell^2$, and the combination of (5.5)-(5.6)

$$\alpha\varepsilon^2(\mathcal{T})\le\xi^2(\mathcal{T})\le C_{\rm c}\xi^2(\mathcal{T}_\epsilon)\le C_{\rm c}\tau^2\xi_\ell^2\le C_{\rm a}C_{\rm c}\tau^2\eta_\ell^2$$

lead to

$$\begin{aligned} C_2\theta\eta_\ell^2 &\le \left(C_{\rm eff} - 2\kappa C_1^2\right)\eta_\ell^2 - \left(2\tau^2 C_{\rm a}C_{\rm c}/\alpha\right)\eta_\ell^2 \\ &\le \varepsilon_\ell^2 - 2C_1^2\,{\rm osc}_\ell^2 - 2\varepsilon^2(\mathcal{T}), \end{aligned}$$

while (4.5) and Lemma 4.4 show

$$\begin{aligned} \varepsilon_\ell^2 - 2\varepsilon^2(\mathcal{T}) &\le \|p_{\mathcal{T}} - p_\ell\|_{L^2(\Omega)}^2 + C_1^2\,{\rm osc}_\ell^2 \\ &\le 2C_1^2\,{\rm osc}_\ell^2 + C_2\eta_\ell^2(\mathcal{E}_\ell \setminus \mathcal{E}_{\mathcal{T}}). \end{aligned}$$

The combination of the previous estimations results in (5.9). Since $\mathcal{M}_\ell$ was chosen with minimal cardinality and $\theta\eta_\ell^2 \le \eta_\ell^2(\mathcal{M}_\ell)$, Lemma 4.4 yields

$$|\mathcal{M}_\ell| \le |\mathcal{E}_\ell \setminus \mathcal{E}_{\mathcal{T}}| \lesssim |\mathcal{T}| - |\mathcal{T}_\ell| \lesssim \xi_\ell^{-1/s}.$$

For each level ℓ with Case (A), set $K(\ell) := 0$ and $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$.

In **Case (B)**, $\kappa\eta_\ell^2 < {\rm osc}_\ell^2$, let $\mathcal{T}$ be some refinement of $\mathcal{T}_\ell$ with

$${\rm osc}^2(f,\mathcal{T}) \le {\rm Tol}^2 \ \text{ and } \ |\mathcal{T}| - |\mathcal{T}_0| \lesssim {\rm Tol}^{-1/s},\ {\rm Tol}^2 := \rho_B\,{\rm osc}_\ell^2\,.$$

Algorithm 3.2 computes a finite sequence $\mathcal{M}_\ell^{(1)}, \dots, \mathcal{M}_\ell^{(K(\ell))}$ of marked reference edges such that

$$\mathcal{T}_{\ell+1} := \mathcal{T} \oplus \mathcal{T}_\ell = {\rm Refine}\left(\mathcal{T}_\ell, \left(\mathcal{M}_\ell^{(k)}\right)_{1\le k\le K(\ell)}\right).$$

Finally, Theorem 3.3 and (5.5) for Case (B) with $\rho_B\,{\rm osc}_\ell^2 = {\rm Tol}^2$ verifies

$$\sum_{k=0}^{K(\ell)} |\mathcal{M}_\ell^{(k)}| \le |\mathcal{T}| - |\mathcal{T}_0| \lesssim {\rm Tol}^{-1/s} \lesssim {\rm osc}_\ell^{-1/s} \lesssim \xi_\ell^{-1/s},$$

which proves (5.8) in Case (B). □

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