

Adaptive solution of elliptic PDE-eigenvalue problems. Part I: Eigenvalues.

V. Mehrmann * A. Miedlar †

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Abstract

We consider a new adaptive finite element (AFEM) algorithm for elliptic PDE-eigenvalue problems. In contrast to other approaches we incorporate the iterative solution of the resulting finite dimensional algebraic eigenvalue problems into the adaptation process. In this way we can balance the costs of the adaption process for the mesh with the costs for the iterative eigenvalue method. We present error estimates that incorporate the discretization errors, approximation errors in the eigenvalue solver and roundoff errors and use these for the adaptation process. We show that for the adaptation process it is possible to restrict to very few iterations of a Krylov subspace solver for the eigenvalue problem on coarse meshes. We present several examples and show that this new approach achieves much better complexity than previous AFEM approaches which assume that the algebraic eigenvalue problem is solved to full accuracy.

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1 Introduction

Many modern technological applications, e.g. computation of acoustic fields or energy levels in quantum mechanics, involve the solution of the eigenvalue problems for partial differential equations (PDEs). It is well understood that numerical methods for PDEs, such as the finite element method (FEM) with fine meshes, give good approximations but they typically lead to a very high computational effort. Therefore, it has been an important research topic in the last 30 years to design adaptively refined meshes to reduce the computational complexity, while retaining the overall accuracy. This approach is usually called the adaptive finite element method (AFEM). An adaptation of the mesh requires to determine the regions where the solution deviates from a regular behavior and concentrating grid points in these regions. To do this, a priori and a posteriori

*Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. mehrmann@math.tu-berlin.de. Supported by *Deutsche Forschungsgemeinschaft*, through the DFG Research Center MATHEON *Mathematics for Key Technologies* in Berlin.

†Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, D-10623 Berlin, FRG. miedlar@math.tu-berlin.de. Supported by *Berlin Mathematical School*, Berlin, Germany.

error estimates for the error between the exact solution and the computational solution are computed and used to control the mesh refinement. In most AFEM approaches it is assumed that the resulting finite dimensional algebraic problem (linear system or eigenvalue problem) is solved exactly and the computational costs for this part of the method as well as the fact that this problem is solved in finite precision arithmetic are typically ignored. This is acceptable if the costs for the algebraic problems are small and these problems are well-conditioned so that the solution of these problems to full accuracy is possible. However, in particular, in the context of eigenvalue problems, often the costs for the solution of the algebraic eigenvalue problem dominate the total costs, and (in particular for nonsymmetric problems) the desired accuracy may not be achievable due to ill-conditioning.

In this paper we therefore introduce a new approach for the adaptive finite element solution of elliptic PDE eigenvalue problems that incorporates the solution (in finite precision arithmetic) of the algebraic problem into the adaptation process. In order to stress this fact we call the new approach *AFEMLA*, where the letters 'LA' indicate that the adaptation also involves the numerical Linear Algebra part of the process. We will focus on the computation of the smallest real eigenvalues of selfadjoint elliptic PDE eigenvalue problems with associated real symmetric algebraic eigenvalue problems. The extension of these ideas to eigenvector computation and to the solution of non-selfadjoint problems is discussed in forthcoming papers.

We begin the discussion with a short historical overview of the development of adaptive finite element methods for elliptic selfadjoint eigenvalue problems.

First a priori error estimates for eigenvalues and eigenvectors were developed for elliptic operators by Strang and Fix [41]. Further improvements were established for selfadjoint operators by Chatelin [16], Raviart and Thomas [38], and Knyazev [27]. Babuška and Osborn [5], [6] proved estimates for compact operators. All these approaches, although optimal, contain mesh size restrictions, i.e. the mesh has to be sufficiently refined ($h \ll 1$) which cannot be verified or quantified, neither a priori nor a posteriori.

In 2006 Knyazev and Osborn [32] presented first truly a priori error estimates for symmetric eigenvalue problems. They introduced new a priori bounds for eigenvalues based on angles between subspaces and proved that the eigenvalue error depends on the approximability of the eigenfunctions in the corresponding invariant subspace both for single and multiple eigenvalues. Other works by Argentati et al. [3], Knyazev and Argentati [31] take advantage of this technique to obtain a priori Rayleigh-Ritz majorization error bounds and apply them in the context of the finite element method. Further results on a priori error estimates can be found in Raviart and Thomas [38] and Larsson and Tomée [34].

Since a priori error estimates yield information about theoretical properties such as asymptotic convergence rates or stability, one needs some fully computable lower and upper error bounds to control an adaptive mesh refinement procedure. On the other hand a posteriori error estimators, based on the numerical solution and initial data, control the whole adaptive process by indicating the error distribution.

A first approach on a posteriori error analysis for symmetric second order elliptic eigenvalue problems can be found in Verfürth [44]. These results, though of only suboptimal order, introduced a new way of analyzing eigenvalue problems

as parameter dependent nonlinear equations.

A combination of a posteriori and a priori analysis was used by Larson [33] to prove a posteriori estimates of optimal order. Under the assumption that the mesh is fine enough to guarantee that the computed eigenvalue is close to the exact one as well as appropriate regularity of the eigenfunction, it was proved that for smooth eigenvectors the error in eigenvalues and eigenvectors is bounded in terms of the mesh size, a stability factor, and the residual.

Durán, Padra, and Rodríguez [21] showed that the edge residual (i.e. the residual on the edges of the mesh) is an upper bound for the volumetric part of the residual. They constructed an explicit residual-based estimator equivalent to the error up to higher order terms. Mao, Shen, and Zhou [35] achieved similar results by applying a local averaging technique. Also Neymeyr [37], based on the analysis of the residual equations, presented an a posteriori error estimator that works on a subspace of eigenvector approximations obtained by the preconditioned inverse iteration.

Recently Carstensen and Gedicke [15] improved the results by Durán et al. [21] and Mao et al. [35] by showing that for all eigenvalues refinement is possible without the volume contribution in the estimator and that the higher order terms can in fact be neglected.

An approach for nonsymmetric elliptic eigenvalue problems was presented by Heuveline and Rannacher [25]. Using the general optimal control framework of Galerkin approximations of nonlinear variational equations by Becker and Rannacher [9] they developed a residual-based estimator with explicitly given remainder terms. Unfortunately also this result, as it needs the knowledge of the exact solution of the adjoint problem and provides only upper bounds of the error, is not a true a posteriori error estimate. A survey about a posteriori error estimation can be found in the books of Verfürth [44] and Ainsworth and Oden [1].

Recently some work has been carried out towards proving optimality of the adaptive finite element method for second order elliptic eigenvalue problems. Giani and Graham [23] proved convergence of an adaptive linear finite element method for computing eigenpairs with a refinement procedure that considers both a standard a posteriori error estimator and eigenfunction oscillations. The global convergence result by Carstensen and Gedicke [15] requires no additional mesh size assumptions and inner node properties. Additionally their AFEM with an averaging scheme has optimal empirical convergence rate. Nearly at the same time Garau, Morin and Zuppa [22] proved convergence of AFEM for any reasonable marking strategy and any initial mesh.

Also some complexity estimates for adaptive eigenvalue computations were obtained by Dahmen et al. [18].

As we have already noted, none of these discussed approaches is complete. In particular, in all these approaches the error and complexity of the algebraic eigenvalue problems is ignored. This may be partially justified for those elliptic boundary value problems, where the solution of small dense symmetric positive definite linear systems can be easily achieved with direct methods like the Cholesky decomposition, or iterative methods like the conjugate gradient methods. Even in the case of linear boundary value problems, however, this may not be justified if the algebraic problem is ill-conditioned.

In the context of eigenvalue problems, however, the cost for the solution of the algebraic eigenvalue problem often dominates the overall costs and the error

estimates for the solution of the algebraic eigenvalue problem with an iterative method have to be included.

The long term goal of our work is to develop fully adaptive algorithms for linear and nonlinear, selfadjoint and non-selfadjoint eigenvalue problems for PDEs. To achieve this objective, we first investigate elliptic eigenvalue problems for selfadjoint operators to find an alternative way of working when fundamental theoretical results, eg. min-max principles [19], are not available. In this paper we focus on the computation of a few smallest eigenvalues of elliptic selfadjoint PDE eigenvalue problems. This paper is organized as follows: In Section 2 we introduce the notation and basic facts about elliptic eigenvalue problems and the Ritz-Galerkin method. The construction of the AFEMLA algorithm is presented in Section 3. Section 4 contains error bounds for the discrete and continuous eigenvalues based on backward error analysis and a saturation assumption. Finally Section 5 contains some numerical examples.

2 Notation and preliminaries

Let Ω be a bounded, polygonal domain in \mathbb{R}^d , $d = 1, 2, 3$. We consider the *variational formulation* of an elliptic eigenvalue problem of the form

$$\begin{aligned} \text{Find an eigenpair } (\lambda, u) \in \mathbb{R} \times U \text{ s.t.} \\ a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V, \end{aligned} \tag{2.1}$$

where $(V, \|\cdot\|_V)$ and $(U, \|\cdot\|_U)$ are *Hilbert spaces*.

The inner products and their induced norms will be denoted by $(\cdot, \cdot)_V$, $\|\cdot\|_V := \sqrt{(\cdot, \cdot)_V}$ and $b(\cdot, \cdot)$, $\|\cdot\|_U := \sqrt{b(\cdot, \cdot)}$, respectively.

We assume that the bilinear form $a : V \times V \rightarrow \mathbb{R}$ is *bounded*, i.e.

$$|a(u, v)| \leq C \|u\|_V \|v\|_V \quad \text{for all } u, v \in V, \quad \text{with some constant } C > 0,$$

and *V-elliptic*, i.e.

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \text{for all } v \in V, \quad \text{for some } \alpha > 0.$$

For the *symmetric* case, i.e. when

$$a(u, v) = a(v, u) \quad \text{for all } u, v \in V,$$

$a(\cdot, \cdot)$ defines an inner product for V and its induced norm $\|\cdot\|_a = \sqrt{a(\cdot, \cdot)}$, called *energy norm*, is equivalent to the $\|\cdot\|_V$ norm on V . In the following we will use a classical Galerkin approach with $U = V$.

Let \mathcal{T}_H be the partition of the domain $\bar{\Omega}$ into elements and let P_p denote the set of continuous piecewise polynomial functions of total degree $p \geq 1$ [10]. Then the *Ritz-Galerkin* discretization of (2.1) is given by

$$a(u_H, v_H) = \lambda_H b(u_H, v_H) \quad \text{for all } v_H \in V_H, \tag{2.2}$$

where $V_H \subset V$ is finite dimensional with dimension $\dim V_H = n_H$, i.e.

$$V_H(\Omega) := \{v \in V : v|_T \in P_p, \quad \text{for all } T \in \mathcal{T}_H\}.$$

Here, subscripts H, h correspond to diameters of the coarse and the fine space elements, respectively (i.e. $H > h$). Suppose that $\{\varphi_1^H, \dots, \varphi_{n_H}^H\}$, is a basis for V_H . Since globally the solution u_H is determined by its values at the n_H grid points it can be written as

$$u_H = \sum_{i=1}^{n_H} u_{H,i} \varphi_i^H.$$

Then (2.2) can be written as a *generalized eigenvalue problem* of the form

$$A_H \mathbf{u}_H = \lambda_H B_H \mathbf{u}_H,$$

where

$$A_H := [a(\varphi_i^H, \varphi_j^H)]_{1 \leq i, j \leq n_H}, \quad B_H := [b(\varphi_i^H, \varphi_j^H)]_{1 \leq i, j \leq n_H},$$

and

$$\mathbf{u}_H = [u_{H,i}]_{1 \leq i \leq n_H}.$$

The matrix A_H is usually called the *stiffness matrix* and B_H the *mass matrix*.

In the following we will concentrate on the solution of a **model problem** of the form

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

A corresponding weak formulation is of the form

$$\text{Determine } u \in V \text{ s.t. } a(u, v) = \lambda b(u, v) \quad \text{for all } v \in V, \tag{2.4}$$

where $V := H_0^1(\Omega)$ with usual norm $\|\cdot\|_V := \|\cdot\|_{H_0^1(\Omega)}$. The bilinear form $a(\cdot, \cdot)$ is bounded, V-elliptic and symmetric, $b(\cdot, \cdot)$ represents the standard $L_2(\Omega)$ inner product. It is known that (2.4) has a countable set of real eigenvalues [37]

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

and corresponding eigenfunctions

$$u_1, u_2, \dots$$

We discretize (2.4) using the standard finite element space of continuous piecewise linear elements to obtain $u_H \in V_H$ from the finite dimensional variational formulation

$$a(u_H, v_H) = \lambda_H b(u_H, v_H) \quad \text{for all } v_H \in V_H.$$

In every $T \in \mathcal{T}_H$, a function $v_H \in V_H$ has the form $v(x, y) = w_1 x + w_2 y + w_3$, and is uniquely defined by its values at the three vertices of the triangle.

The associated algebraic eigenvalue problem is given by

$$A_H \mathbf{u}_H = \lambda_H B_H \mathbf{u}_H, \tag{2.5}$$

where A_H and B_H are symmetric and positive definite matrices. The algebraic generalized eigenvalue problem (2.5) has a finite set of eigenvalues

$$0 < \lambda_{1,H} \leq \lambda_{2,H} \leq \dots \leq \lambda_{n_H,H}$$

and corresponding eigenvectors

$$\mathbf{u}_{1,H}, \mathbf{u}_{2,H}, \dots, \mathbf{u}_{n_H,H}.$$

It follows from the *Courant-Fischer minmax* theorem [19] that

$$\lambda_i \leq \lambda_{i,H} \quad \text{for all } i = 1, \dots, n_H.$$

In subsequent sections we will use different notation. To distinguish continuous, discrete and approximated eigenvalues with λ_i we denote an eigenvalue of the problem (2.3), $\lambda_{i,H}$ (or $\lambda_{i,h}$) will define the eigenvalues of the discretized algebraic eigenvalue problem associated with the space V_H (or V_h), while $\tilde{\lambda}_{i,H}$ (or $\tilde{\lambda}_{i,h}$) denote the approximation of $\lambda_{i,H}$ (or $\lambda_{i,h}$) computed by an iterative eigenvalue solver in finite precision arithmetic. In the following when no index is given in λ_i , $\lambda_{i,H}$ (or $\lambda_{i,h}$), $\tilde{\lambda}_{i,H}$ (or $\tilde{\lambda}_{i,h}$) then we mean λ_1 , $\lambda_{1,H}$ (or $\lambda_{1,h}$), $\tilde{\lambda}_{1,H}$ (or $\tilde{\lambda}_{1,h}$), respectively. The corresponding eigenfunctions, eigenvectors and computed eigenvectors are denoted in a similar fashion.

3 The AFEMLA Algorithm

The standard AFEM approach for eigenvalue problems is based on discretizing the variational formulation using the Ritz-Galerkin method on a given grid and solving the resulting generalized algebraic eigenvalue problem by an iterative solver. Based on this trial solution a posteriori error estimates are determined and used to refine the grid. This typically assumes that the algebraic eigenvalue problem is solved exactly. But often the computational costs for the algebraic eigenvalue problems dominate the total computational cost, since one may have to solve many algebraic eigenvalue problems related to finer and finer grids and information from the previous steps of the adaptive procedure like approximated eigenvalues is not used on the next level, which is a huge loss.

As an alternative one could think of the construction of iterative methods (like Krylov subspace methods) for the PDE formulation of the eigenvalue problem and then using some local discretization techniques, but this would require the solution of a PDE boundary value problem per iteration step.

We introduce a new adaptive finite element algorithm called *AFEMLA* which combines the two mentioned ideas and incorporates the information obtained during the iterative solution of the algebraic eigenvalues problems into the error estimation and refinement process. Since the accuracy of the computed eigenvalue cannot be better than the quality of the discretization, there is no need to solve the algebraic eigenvalue problem up to very high precision if the discretization scheme guarantees only small precision. The goal of adaptive methods is to achieve a desired accuracy with minimal computational effort. To achieve this goal, in order to determine the error estimates, we only solve the algebraic eigenvalue problem on the current coarse grid and use classical perturbation results from finite dimensional eigenvalue problems to determine the errors on the fine mesh.

As in the standard case of AFEM our adaptive finite element method consists typically of the loop

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}$$

After discretizing we *solve* the algebraic eigenvalue problem using a Krylov subspace method on the coarse mesh but we do not solve this problems exactly but stop the iteration early, when sufficient information is available. As stopping criteria in the iterative procedure we can either use a maximal number k of Arnoldi/Lanczos steps or a desired tolerance. This significantly reduces the cost in the algebraic eigenvalue solvers.

For a given matrix $M \in \mathbb{R}^{n_H \times n_H}$ (which in our case corresponds to $B_H^{-1}A_H$, and a nonzero starting vector $v_1 \in \mathbb{R}^{n_H}$ Krylov subspace methods generate the Krylov subspace $\mathcal{K}_m(M, v_1) = \text{span}\{v_1, Mv_1, M^2v_1, \dots, M^{m-1}v_1\}$ and determines an orthogonal basis for this subspace spanned by the columns of a matrix V_j . In general this is called the *Arnoldi method* or the *Lanczos method* if M is symmetric and an implementation via a three term recurrence is used.

The approximations to the eigenvalues of the matrix A are then determined via the eigenvalues (called *Ritz values*) of the Hessenberg matrix H_j which represents an orthogonal projection $H_j = V_j^T M V_j$ of the matrix M to $\mathcal{K}_m(M, v_1)$. Lifting the eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_j$ associated to the eigenvalues μ_1, \dots, μ_j of H_j by setting $\mathbf{u}_{k,H} = V_j \mathbf{u}_k$, $k = 1, 2, \dots, j$ then yields eigenvector approximations for the given matrix M , i.e. for the generalized eigenvalue problem (2.5).

The Arnoldi process is usually terminated at step j , when $\mathcal{K}_j(M, v_1)$ is approximately invariant under M or when a desired tolerance tol is reached. Then we have determined an approximation $\tilde{\lambda}_H$ to an eigenvalue λ_H of the generalized eigenvalue problem (2.5). With an approximation $\tilde{\mathbf{u}}_H$ to the corresponding eigenvector \mathbf{u}_H , it follows that the corresponding approximate eigenfunction is given by

$$\tilde{u}_H = \sum_{i=1}^{n_H} [\tilde{\mathbf{u}}_H]_i \varphi_i^H = \sum_{i=1}^{n_H} \tilde{u}_{H,i} \varphi_i^H.$$

We then want to check the quality of this solution and use to this information for adaptation. From a geometric point of view, it is our goal to enrich the space V_H corresponding to the coarse mesh \mathcal{T}_H by further functions. Since V_H corresponding to \mathcal{T}_H is a subspace of V_h corresponding to \mathcal{T}_h , which is obtained by a uniform refinement of \mathcal{T}_H , every function from the coarse space may be expressed as a linear combination of functions from the fine space. A uniform refinement of every single triangle, also called *red refinement*, is usually realized by joining the midpoints of the edges [44].

If $\{\varphi_1^h, \dots, \varphi_{n_h}^h\}$ is a finite element basis for V_h then we have the identity

$$\tilde{u}_H = \sum_{i=1}^{n_H} \tilde{u}_{H,i} \varphi_i^H = \sum_{i=1}^{n_h} \hat{u}_{h,i} \varphi_i^h,$$

with an appropriate coefficient vector $\hat{\mathbf{u}}_h$.

Since every basis function $\{\varphi_i^H\}_{i=1}^{n_H}$ can be expressed as a linear combination of the basis functions $\{\varphi_i^h\}_{i=1}^{n_h}$, the relationship between the coefficient vectors $\hat{\mathbf{u}}_h$ and $\tilde{\mathbf{u}}_H$ can be described by multiplication with a prolongation matrix P_H^h , that is easily constructed [10, 12]. In the following, for simplicity, we leave off the dependency on the mesh sizes and write P instead of P_H^h when this is clear from the context.

Using this, for the corresponding prolonged coordinate vector associated with the computed eigenvector $\tilde{\mathbf{u}}_H$ in the fine space, we have

$$\hat{\mathbf{u}}_h = P \tilde{\mathbf{u}}_H. \quad (3.1)$$

We can then compute the corresponding residual

$$\hat{\mathbf{r}}_h = A_h \hat{\mathbf{u}}_h - \hat{\lambda}_h B_h \hat{\mathbf{u}}_h.$$

This gives us a natural way of *estimating* the error in the computed eigenfunction using the coarse grid solution combined with fine grid information, namely we can prolongate an already computed approximation $\tilde{\mathbf{u}}_H$ from the coarse grid to the fine grid. Then every entry in the residual vector $\hat{\mathbf{r}}_h$ corresponds to the appropriate basis function from the fine space. Furthermore, we know that if the i -th entry in the vector $\hat{\mathbf{r}}_h$ is large, then the i -th basis function has a huge influence on the solution, namely its support should be further investigated [26]. All those basis functions with large entries in the vector $\hat{\mathbf{r}}_h$ together with all basis functions from the coarse space form a basis for the new refined space. The decision on whether an entry in residual vector is small or large is based on different criteria, e.g. a prescribed tolerance or bulk strategy [20].

When we have identified the basis functions that should be added to enrich our trial space, we simply *mark* all edges that contain the corresponding nodes. In order to avoid hanging nodes or irregular triangulations, we mark some additional edges using a closure algorithm, i.e. if edge E is marked and it is not a reference edge (longest edge) of the element, we add the reference edge to the set of marked edges. If an element $T \in \mathcal{T}_H$ has one, two, or three edges marked, we *refine* it by green-, blue-, or red-refinement, respectively, [2, 11, 44]. After finishing the refinement procedure we have a new mesh, which will be an initial mesh for the next loop of our adaptive algorithm. Algorithm 1 presents a pseudo-code of this algorithm. In this algorithm we use a fixed number of steps k for the iterative method to solve the algebraic eigenvalue problem or we stop the iterative procedure based on a tolerance that is related to the discretization error. We will discuss this issue further in Section 4.

Having defined a basic refinement procedure, it is necessary to theoretically analyze this procedure.

4 A priori and a posteriori error bounds

In this section we analyze the refinement procedure theoretically. We discuss the question whether the residual vector provides sufficient information for the refinement procedure. In order to answer this question we will first analyze eigenvalue error bounds based on the backward error analysis for the algebraic eigenvalue problem arose after discretizing the PDE-eigenvalue problem.

We begin by recalling some classical perturbation and backward error results for eigenvalue problems.

Theorem 4.1 ([7, 45]). *Consider a real $n \times n$ matrix M .*

Let $(\tilde{\lambda}, \tilde{x})$ be a computed eigenpair of M with normalized \tilde{x} , i.e. $\|\tilde{x}\|_2 = 1$ and let $r = M\tilde{x} - \tilde{\lambda}\tilde{x}$. Then $(\tilde{\lambda}, \tilde{x})$ is an exact eigenpair of the matrix $M + E$, where the backward error matrix E satisfies $\|E\|_2 = \|r\|_2$.

This theorem shows that the backward error is of the size of the residual and thus it remains to analyze the conditioning of the eigenvalue which is characterized in the following theorem.

Algorithm 1 AFEMLA for elliptic Poisson eigenvalue problem

Input: An initial regular triangulation \mathcal{T}_H^i , a maximal number k of Arnoldi steps or a tolerance tol and a desired accuracy ϵ .

Output: Approximation $\tilde{\lambda}_1$ to the smallest eigenvalue λ_1 (2.1) together with the corresponding approximate eigenfunction u_1 .

Solve: Compute the smallest eigenvalue $\tilde{\lambda}_H$ and associated eigenvector $\tilde{\mathbf{u}}_H$ for the algebraic eigenvalue problem (2.5) associated with the coarse mesh \mathcal{T}_H^i , the Arnoldi method will be terminated after k steps or when desired a tolerance tol is reached.

Express $\tilde{\mathbf{u}}_H$ using basis functions from the mesh \mathcal{T}_h^i that is obtained by uniformly refining \mathcal{T}_H^i . With the prolongation matrix P from the coarse mesh \mathcal{T}_H^i to the fine mesh \mathcal{T}_h^i compute $\hat{\mathbf{u}}_h = P \cdot \tilde{\mathbf{u}}_H$.

Estimate: Determine the residual $\hat{\mathbf{r}}_h = A_h \hat{\mathbf{u}}_h - \tilde{\lambda}_h B_h \hat{\mathbf{u}}_h$ for the associated eigenvector $\hat{\mathbf{u}}_H$ and identify all large coefficients in $\hat{\mathbf{r}}_h$ and corresponding basis functions (nodes).

if $\|\hat{\mathbf{r}}_h\| < \epsilon$ **then**
 return $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$

else

Mark: Mark all edges that contain identified nodes and apply a closure algorithm.

Refine: Refine the coarse mesh \mathcal{T}_H^i using *RedGreenBlue* refinement to get \mathcal{T}_H^{i+1} .

 Start Algorithm 1 with \mathcal{T}_H^{i+1} .

end if

Remark 4.2. When we have the computed eigenvalue and eigenfunction then it is possible, at least in the interior of the elements, to compute the analytic residual for (2.3)

$$\tilde{r}_H = \Delta \tilde{u}_H - \tilde{\lambda}_H \tilde{u}_H.$$

Theorem 4.3 ([19], p. 149). Consider an $n \times n$ matrix M and let λ be a simple eigenvalue of M with right eigenvector x and left eigenvector y , normalized so that $\|x\|_2 = \|y\|_2 = 1$. Let $\tilde{\lambda} = \lambda + \delta\lambda$ be the corresponding eigenvalue of a perturbed matrix $M + E$. Then

$$\tilde{\lambda} - \lambda = \frac{y^* E x}{y^* x} + O(\|E\|^2),$$

and

$$|\tilde{\lambda} - \lambda| = |\delta\lambda| \leq \frac{\|E\|}{|y^* x|} + O(\|E\|^2),$$

where $y^* = \bar{y}^T$ is the conjugate transpose of y .

The quantity $\frac{1}{|y^* x|}$ is called the *condition number* of the simple eigenvalue λ of A .

From Theorems 4.1, 4.3 we see that if a simple eigenvalue λ is well-conditioned and if the backward error E has small norm, then the computed eigenpair $(\tilde{\lambda}, \tilde{x})$ is a good approximation of the exact eigenpair (λ, x) . In the symmetric case, where $\frac{1}{|y^* x|} = \frac{1}{|x^* x|} = 1$, it thus follows that a small residual $\|r\|_2$ implies a good eigenvalue approximation. A theorem of Weyl shows that in the symmetric case this is true for each eigenvalue individually.

Theorem 4.4 (Weyl, [19] p. 201.). Let M and E be $n \times n$ symmetric matrices. Let $\lambda_1 \leq \dots \leq \lambda_n$, be the eigenvalues of M and $\tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_n$ be the eigenvalues of $M + E$. Then

$$|\tilde{\lambda}_i - \lambda_i| \leq \|E\|_2.$$

These classical results are easily extended to generalized eigenvalue problems see [40]. We derive the necessary results for completeness.

Theorem 4.5. Let A, B be $n \times n$ matrices and let B be invertible. Let $\tilde{\lambda}$ be a computed eigenvalue for the matrix pair (A, B) and let \tilde{x} be an associated normalized eigenvector, i.e. $\|\tilde{x}\|_2 = 1$. Let $r = A\tilde{x} - \tilde{\lambda}B\tilde{x}$. Then $\tilde{\lambda}$ is an exact eigenvalue with associated eigenvector \tilde{x} of a pair $(A + E, B)$, where $\|E\|_2 = \|r\|_2$.

Proof. For the computed eigenpair $(\tilde{\lambda}, \tilde{x})$ of $Ax = \lambda Bx$ and the residual vector $r = A\tilde{x} - \tilde{\lambda}B\tilde{x}$ we have

$$r = r\tilde{x}^* \tilde{x},$$

since $\|\tilde{x}\|_2 = 1$. Thus, we obtain

$$A\tilde{x} - r = A\tilde{x} - r\tilde{x}^* \tilde{x} = (A - r\tilde{x}^*)\tilde{x} = \tilde{\lambda}B\tilde{x},$$

and hence $(\tilde{\lambda}, \tilde{x})$ is an exact eigenpair of $(A + E, B)$, where E is given by

$$E = -r\tilde{x}^*. \quad (4.1)$$

This completes the first part of the proof.

It follows from (4.1) that

$$\|E\|_2 = \|-r\tilde{x}^*\|_2 \leq \|r\|_2\|\tilde{x}^*\|_2 = \|r\|_2, \quad (4.2)$$

but on the other hand we have that

$$\|r\|_2 = \|-r\|_2 = \|-r\tilde{x}^*\tilde{x}\| = \|E\tilde{x}\|_2 \leq \|E\|_2\|\tilde{x}\|_2 \leq \|E\|_2. \quad (4.3)$$

Inequalities (4.2) and (4.3) together give that $\|E\|_2 = \|r\|_2$. \square

The associated condition number is obtained as follows.

Theorem 4.6. *Consider a pair (A, B) of real $n \times n$ matrices and assume that B is invertible. Let λ be a simple eigenvalue of the pair (A, B) with right eigenvector x and left eigenvector y , normalized so that $\|x\|_2 = \|y\|_2 = 1$. Let $\tilde{\lambda} = \lambda + \delta\lambda$ be the corresponding eigenvalue of the pair $(A + E, B)$ with eigenvector $\tilde{x} = x + \delta x$. Then*

$$\tilde{\lambda} - \lambda = \frac{y^*Ex}{y^*Bx} + O(\|E\|^2),$$

and

$$|\tilde{\lambda} - \lambda| \leq \frac{\|E\|}{|y^*Bx|} + O(\|E\|^2).$$

Proof. Following the proof in [[19] p. 149], from Theorem 4.5 we see that $(\tilde{\lambda}, \tilde{x})$ is an exact eigenpair of a matrix pair $(A + E, B)$, i.e. we have

$$(A + E)\tilde{x} = \tilde{\lambda}B\tilde{x}$$

and

$$(A + E)(x + \delta x) = (\lambda + \delta\lambda)B(x + \delta x). \quad (4.4)$$

Subtracting $Ax = \lambda Bx$ from (4.4) we get

$$A\delta x + Ex + E\delta x = \lambda B\delta x + \delta\lambda Bx + \delta\lambda B\delta x.$$

Neglecting the second-order terms $E\delta x$ and $\delta\lambda B\delta x$, we get as first order approximation

$$A\delta x + Ex = \lambda B\delta x + \delta\lambda Bx. \quad (4.5)$$

Multiplying (4.5) with y^* from the left, we get (up to higher order terms)

$$y^*A\delta x + y^*Ex = \lambda y^*B\delta x + \delta\lambda y^*Bx. \quad (4.6)$$

Since $y^*A\delta x = \lambda y^*B\delta x$, it follows that (4.6) is equivalent to

$$y^*Ex = \delta\lambda y^*Bx.$$

Then, finally, from (4.7) we obtain

$$\begin{aligned} \tilde{\lambda} - \lambda &= \delta\lambda = \frac{y^*Ex}{y^*Bx} + O(\|E\|^2), \\ |\tilde{\lambda} - \lambda| &= |\delta\lambda| = \frac{\|y^*Ex\|_2}{\|y^*Bx\|_2} \leq \frac{\|E\|_2}{|y^*Bx|} + O(\|E\|^2). \end{aligned}$$

\square

The quantity $\frac{1}{|y^* Bx|}$ is again called the *condition number* of the simple eigenvalue λ . Since $\|E\|_2 = \|r\|_2$, for a small residual vector, good accuracy of the eigenvalue depends on this condition number.

In our special case of a symmetric generalized eigenvalue problem with positive definite B , we thus obtain the following result.

Corollary 4.7. *Let (λ, x) with $\|x\|_2 = 1$ be an exact eigenpair for a real symmetric matrix pair (A, B) with positive definite matrix B and let $(\tilde{\lambda}, \tilde{x})$ be a corresponding computed eigenpair. Then*

$$|\tilde{\lambda} - \lambda| \leq \|r\|_2 \|B^{-1}\|_2.$$

Proof. By Theorem 4.6 we have

$$|\tilde{\lambda} - \lambda| = \frac{\|y^* E x\|_2}{|y^* B x|}, \quad \|E\|_2 = \|r\|_2.$$

Since in our case the left eigenvector is real and satisfies $y = x$, we obtain

$$\frac{1}{|y^* B x|} \leq \frac{1}{\lambda_{\min}(B)} = \lambda_{\max}(B^{-1}) = \|B^{-1}\|_2,$$

where $\lambda_{\min}(B)$ is the minimal eigenvalue of B and $\lambda_{\max}(B^{-1})$ is the maximal eigenvalue of B^{-1} .

Hence with

$$|\tilde{\lambda} - \lambda| = \frac{\|x^T E x\|_2}{|x^T B x|} \leq \frac{\|r\|_2}{|x^T B x|} \leq \|r\|_2 \|B^{-1}\|_2$$

the proof is complete. \square

Using the previous theorems we will now prove some error bounds for the discrete and continuous eigenvalues. We will denote by (λ, u) the exact pair of eigenvalue and eigenfunction of (2.3), for $\text{diam} = h, H$, by $(\lambda_{\text{diam}}, \mathbf{u}_{\text{diam}})$ the exact and by $(\tilde{\lambda}_{\text{diam}}, \tilde{\mathbf{u}}_{\text{diam}})$ computed eigenpairs for the discrete formulation with respect to the finite space V_{diam} . We denote by $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ the eigenpair obtained from the prolongation of the eigenvector $\tilde{\mathbf{u}}_H$ to the finite space V_h . The corresponding residual vectors will be denoted by

$$\mathbf{r}_H = A_H \tilde{\mathbf{u}}_H - \tilde{\lambda}_H B_H \tilde{\mathbf{u}}_H, \quad (4.7)$$

$$\mathbf{r}_h = A_h \tilde{\mathbf{u}}_h - \tilde{\lambda}_h B_h \tilde{\mathbf{u}}_h, \quad (4.8)$$

$$\hat{\mathbf{r}}_h = A_h \hat{\mathbf{u}}_h - \hat{\lambda}_h B_h \hat{\mathbf{u}}_h. \quad (4.9)$$

We then have the following error bounds.

Theorem 4.8. *Let $(\lambda_H, \mathbf{u}_H)$, $(\lambda_h, \mathbf{u}_h)$ be the exact eigenvalues and associated eigenvectors of the matrix pairs (A_H, B_H) , (A_h, B_h) , respectively, and let $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$, $(\tilde{\lambda}_h, \tilde{\mathbf{u}}_h)$ be corresponding computed eigenpairs. Let the eigenvector $\mathbf{u}_H, \mathbf{u}_h$ be normalized, i.e. $\|\mathbf{u}_H\|_2 = \|\mathbf{u}_h\|_2 = 1$. Then the following estimates hold.*

$$|\tilde{\lambda}_H - \lambda_H| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2, \quad (4.10)$$

$$|\tilde{\lambda}_h - \lambda_h| \leq \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2, \quad (4.11)$$

$$|\hat{\lambda}_h - \lambda_h| \leq \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2. \quad (4.12)$$

Proof. With $\mathbf{r}_H = A_H \tilde{\mathbf{u}}_H - \tilde{\lambda}_H B_H \tilde{\mathbf{u}}_H$, by Theorem 4.7 we get

$$|\tilde{\lambda}_H - \lambda_H| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2.$$

Analogously, for $\mathbf{r}_h = A_h \tilde{\mathbf{u}}_h - \tilde{\lambda}_h B_h \tilde{\mathbf{u}}_h$ we get

$$|\tilde{\lambda}_h - \lambda_h| \leq \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2.$$

The last inequality then follows from (3.1) and the definition of $\hat{\mathbf{r}}_h = A_h \hat{\mathbf{u}}_h - \hat{\lambda}_h B_h \hat{\mathbf{u}}_h$. \square

It should be noted that in our algorithm we do not compute the fine grid eigenpair $(\tilde{\lambda}_h, \tilde{\mathbf{u}}_h)$. For this reason, the fine grid residual $\|\mathbf{r}_h\|_2$ is not available. Instead we use $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ as its approximation and have the following bounds.

Theorem 4.9. *Let $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$ be a computed eigenpair of the matrix pair (A_H, B_H) and let $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ be the computed eigenpair obtained by the prolongation (with the prolongation matrix P) of $\tilde{\mathbf{u}}_H$ to the fine space V_h as defined in (3.1). Assume that these vectors are normalized, i.e. $\|\tilde{\mathbf{u}}_H\|_2 = \|\hat{\mathbf{u}}_h\|_2 = 1$. Then*

$$|\tilde{\lambda}_H - \hat{\lambda}_h| \leq \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2}. \quad (4.13)$$

Proof. Following Theorem 4.5, the eigenpairs $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$, $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ are exact eigenpairs of the eigenvalue problems

$$(A_H + E_H) \tilde{\mathbf{u}}_H = \tilde{\lambda}_H B_H \tilde{\mathbf{u}}_H, \quad (4.14)$$

$$(A_h + \hat{E}_h) \hat{\mathbf{u}}_h = \hat{\lambda}_h B_h \hat{\mathbf{u}}_h, \quad (4.15)$$

respectively.

Using the relation between the coarse and the fine mesh, i.e. that

$$P^T A_h P = A_H, \quad P^T B_h P = B_H,$$

it follows that (4.14) is equivalent to

$$(P^T A_h P + E_H) \tilde{\mathbf{u}}_H = \tilde{\lambda}_H (P^T B_h P) \tilde{\mathbf{u}}_H. \quad (4.16)$$

Multiplying (4.15) from the left by P^T gives

$$P^T A_h \hat{\mathbf{u}}_h + P^T \hat{E}_h \hat{\mathbf{u}}_h = \hat{\lambda}_h P^T B_h \hat{\mathbf{u}}_h. \quad (4.17)$$

Using the fact that $P \tilde{\mathbf{u}}_H = \hat{\mathbf{u}}_h$, we can rewrite (4.16) as

$$P^T A_h P \tilde{\mathbf{u}}_H + E_H \tilde{\mathbf{u}}_H = \tilde{\lambda}_H P^T B_h P \tilde{\mathbf{u}}_H. \quad (4.18)$$

Subtracting (4.17) from (4.18) we then get

$$(\tilde{\lambda}_H - \hat{\lambda}_h)(P^T B_h \hat{\mathbf{u}}_h) = P^T A_h \hat{\mathbf{u}}_h + E_H \tilde{\mathbf{u}}_H - P^T A_h \hat{\mathbf{u}}_h - P^T \hat{E}_h \hat{\mathbf{u}}_h$$

and, applying the triangle inequality, finally

$$\begin{aligned} |\tilde{\lambda}_H - \hat{\lambda}_h| &= \frac{\|E_H \tilde{\mathbf{u}}_H - P^T \hat{E}_h \hat{\mathbf{u}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} \leq \frac{\|E_H \tilde{\mathbf{u}}_H\|_2 + \|P^T \hat{E}_h \hat{\mathbf{u}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} \\ &\leq \frac{\|E_H\|_2 \|\tilde{\mathbf{u}}_H\|_2 + \|P^T \hat{E}_h\|_2 \|\hat{\mathbf{u}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} = \frac{\|E_H\|_2 + \|P^T \hat{E}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} \\ &\leq \frac{\|E_H\|_2 + \|P^T\|_2 \|\hat{E}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} = \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2}. \quad \square \end{aligned}$$

Having obtained an estimate between the computed eigenvalue on the coarse grid and the prolonged eigenvalue on the fine mesh we next obtain a comparison between the exact eigenvalue on the coarse grid and the prolonged eigenvalue.

Theorem 4.10. *Let $(\lambda_H, \mathbf{u}_H)$ be the exact eigenpair of the matrix pair (A_H, B_H) and let $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ be the eigenpair obtained by the prolongation (3.1) of $\tilde{\mathbf{u}}_H$ to the fine space V_h . Then the following bound holds.*

$$|\lambda_H - \hat{\lambda}_h| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2}.$$

Proof. Using the triangle inequality we get

$$|\lambda_H - \hat{\lambda}_h| = |\lambda_H - \tilde{\lambda}_H + \tilde{\lambda}_H - \hat{\lambda}_h| \leq |\lambda_H - \tilde{\lambda}_H| + |\tilde{\lambda}_H - \hat{\lambda}_h|.$$

Inserting the bounds (4.10) and (4.13) we get

$$|\lambda_H - \hat{\lambda}_h| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2}.$$

□

Applying again the triangle inequality

$$|\lambda_H - \lambda_h| = |\lambda_H - \hat{\lambda}_h + \hat{\lambda}_h - \lambda_h| \leq |\lambda_H - \hat{\lambda}_h| + |\hat{\lambda}_h - \lambda_h|$$

we get an error estimate between the exact fine and course grid eigenvalue

$$|\lambda_H - \lambda_h| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \quad (4.19)$$

and between the exact fine grid and computed coarse grid eigenvalues.

$$\begin{aligned} |\lambda_h - \tilde{\lambda}_H| &= |\lambda_h - \hat{\lambda}_h + \hat{\lambda}_h - \tilde{\lambda}_H| \leq |\lambda_h - \hat{\lambda}_h| + |\hat{\lambda}_h - \tilde{\lambda}_H|, \\ |\lambda_h - \tilde{\lambda}_H| &\leq \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2}. \end{aligned}$$

Combining these bounds and using the triangle inequality in different ways we obtain the following further bounds.

Theorem 4.11. *Let $(\lambda_H, \mathbf{u}_H)$, $(\lambda_h, \mathbf{u}_h)$ be the exact and let $(\tilde{\lambda}_H, \tilde{\mathbf{u}}_H)$, $(\tilde{\lambda}_h, \tilde{\mathbf{u}}_h)$ be the computed eigenpairs of the matrix pair (A_H, B_H) , (A_h, B_h) , respectively. Let furthermore $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ be the eigenpair obtained by the prolongation of $\tilde{\mathbf{u}}_H$ to the fine space V_h defined in (3.1). Then the following bound holds.*

$$\begin{aligned} |\tilde{\lambda}_h - \hat{\lambda}_h| &\leq \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2 + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2, \\ |\tilde{\lambda}_H - \tilde{\lambda}_h| &\leq \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2 + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2, \\ |\lambda_H - \tilde{\lambda}_h| &\leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2 \\ &\quad + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2. \end{aligned}$$

Proof. The proof follows by combing the previous bounds and using the triangle inequality. \square

Theorems 4.8 - 4.11 present error bounds with respect to the exact and computed eigenvalues for the discretized algebraic problems. Some of these bounds are computable but some have only theoretical meaning. Since in fact we are interested in errors between the eigenvalues of the continuous problem and those of the discrete problem, we would like to find computable bounds based on residual vectors. Since we were able to transform residual errors to backward errors it follows that for well-conditioned eigenvalues we can expect that small residuals will imply good accuracy of its approximations obtained by iterative solvers.

In order to relate the continuous and discrete eigenvalues we will use the so called *saturation assumption*, namely we will assume that the approximation of the eigenvalue computed on the fine space V_h is better than the approximation on the coarse space V_H . In practice this assumption is equivalent to the convergence of the AFEM procedure.

Theorem 4.12 ([37]). *Let λ be an exact eigenvalue of (2.3). Let λ_H, λ_h be the corresponding exact eigenvalues of the discretized problems on spaces V_H, V_h , respectively. Then the saturation assumption*

$$\lambda_h - \lambda \leq \beta(\lambda_H - \lambda), \quad (4.20)$$

with a positive $\beta < 1$, is equivalent to

$$\lambda_H - \lambda \leq \frac{1}{1-\beta}(\lambda_H - \lambda_h). \quad (4.21)$$

Remark 4.13. *Since for the symmetric eigenvalue problem the Courant-Fischer minmax theorem holds, the exact eigenvalue λ of the PDE eigenvalue problem and the eigenvalues λ_H, λ_h of the discretized problems satisfy the inequality*

$$\lambda \leq \lambda_h \leq \lambda_H.$$

Thus, the inequalities (4.20) and (4.21) are equivalent to $|\lambda_h - \lambda| \leq \beta(\lambda_H - \lambda)$ and $|\lambda_H - \lambda| \leq \frac{1}{1-\beta}(\lambda_H - \lambda_h)$, respectively.

Based on the saturation assumption and the estimates between the different and computed eigenvalues for the discretized eigenvalue problems we then obtain the following bounds.

Theorem 4.14. *Let λ be an exact eigenvalue of (2.3) and let u be the corresponding eigenfunction. Let λ_H be the corresponding exact eigenvalue of the discretized generalized eigenvalue problem (A_H, B_H) , and let $\hat{\mathbf{u}}_H$ be defined as in (3.1). Then with residuals $\mathbf{r}_H, \hat{\mathbf{r}}_h$ as defined in (4.7), (4.9) we have*

$$|\lambda_H - \lambda| \leq \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

Proof. Under the saturation assumption it follows from Theorem 4.12 that

$$|\lambda_H - \lambda| \leq \frac{1}{1-\beta}(\lambda_H - \lambda_h).$$

From (4.19) it follows that

$$|\lambda_H - \lambda_h| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2$$

and thus

$$|\lambda_H - \lambda| \leq \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

□

Theorem 4.15. *Let λ be an exact eigenvalue of (2.3), let λ_h be the corresponding exact fine grid eigenvalue and let $\hat{\mathbf{u}}_H$ be defined as in (3.1). Then with residuals \mathbf{r}_H , $\hat{\mathbf{r}}_h$ defined as in (4.7), (4.9) we have*

$$|\lambda_h - \lambda| \leq \left(1 - \frac{1}{1-\beta}\right) \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

Proof. Under the saturation assumption it follows from Theorem 4.12 that

$$|\lambda_h - \lambda| \leq \lambda_h - \lambda_H + \frac{1}{1-\beta} (\lambda_H - \lambda_h) = \left(1 - \frac{1}{1-\beta}\right) (\lambda_h - \lambda_H).$$

From (4.19) we have that

$$|\lambda_H - \lambda_h| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2$$

and hence

$$|\lambda_h - \lambda| \leq \left(1 - \frac{1}{1-\beta}\right) \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

□

Using the previous estimates we can also obtain a bound between the computed eigenvalue on the coarse mesh and the corresponding eigenvalues of the original PDE eigenvalue problem.

Theorem 4.16. *Let λ be an exact eigenvalue of (2.3) and let $\tilde{\lambda}_H$ be the computed coarse grid eigenvalue and let $\hat{\mathbf{u}}_h$ be defined as in (3.1). Then with residuals \mathbf{r}_H , $\hat{\mathbf{r}}_h$ defined as in (4.7), (4.9), we have*

$$\begin{aligned} |\tilde{\lambda}_H - \lambda| &\leq \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right) \\ &\quad + \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2. \end{aligned}$$

Proof. From (4.7) we know that

$$|\tilde{\lambda}_H - \lambda_H| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2$$

and from Theorem 4.14 we have that

$$|\lambda_H - \lambda| \leq \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

Hence,

$$|\tilde{\lambda}_H - \lambda| \leq \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right) + \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2.$$

□

Theorem 4.17. *Let λ be an exact eigenvalue of (2.3) and let $\hat{\mathbf{u}}_H$ be the computed coarse grid eigenvector. Furthermore, let $(\hat{\lambda}_h, \hat{\mathbf{u}}_h)$ be the corresponding eigenpair obtained by the prolongation of $\hat{\mathbf{u}}_H$ to the fine space V_h defined as in (3.1). Then with residuals $\mathbf{r}_H, \hat{\mathbf{r}}_h$ as defined in (4.7), (4.9) we have*

$$|\hat{\lambda}_h - \lambda| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

Proof. From Theorem 4.10 it follows that

$$|\lambda_H - \hat{\lambda}_h| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2}$$

and from Theorem 4.14 we have that

$$|\lambda_H - \lambda| \leq \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right)$$

and hence,

$$|\hat{\lambda}_h - \lambda| \leq \|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \frac{1}{1-\beta} \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right).$$

□

Theorem 4.18. *Let λ be an exact eigenvalue of (2.3), $\tilde{\lambda}_h$ the computed fine grid eigenvalue. Then with residuals $\mathbf{r}_H, \mathbf{r}_h, \hat{\mathbf{r}}_h$ as defined in (4.7), (4.8), (4.9) we have*

$$|\tilde{\lambda}_h - \lambda| \leq \left(1 - \frac{1}{1-\beta}\right) \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right) + \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2$$

Proof. From Theorem 4.8 it follows that

$$|\tilde{\lambda}_h - \lambda_h| \leq \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2$$

and from Theorem 4.15 we have that

$$|\lambda_h - \lambda| \leq \left(1 - \frac{1}{1-\beta}\right) \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right)$$

and hence

$$\begin{aligned}
|\tilde{\lambda}_h - \lambda| &\leq \left(1 - \frac{1}{1 - \beta}\right) \left(\|\mathbf{r}_H\|_2 \|B_H^{-1}\|_2 + \frac{\|\mathbf{r}_H\|_2 + \|P^T\|_2 \|\hat{\mathbf{r}}_h\|_2}{\|P^T B_h \hat{\mathbf{u}}_h\|_2} \right. \\
&\quad \left. + \|\hat{\mathbf{r}}_h\|_2 \|B_h^{-1}\|_2 \right) + \|\mathbf{r}_h\|_2 \|B_h^{-1}\|_2.
\end{aligned}$$

□

In summary, we have obtained bounds between the exact eigenvalues and eigenvalue approximations. We can immediately notice, that all bounds except for the last bound in Theorem 4.18 are fully computable. As a matter of fact the last bound has only theoretical value because we do not want to compute the residual \mathbf{r}_h on the fine mesh. But it follows from all these bounds, that under the saturation assumption, and if we consider problems with well-conditioned eigenvalues, then the information about a small residual vector is equivalent to the good accuracy of the computed eigenvalues.

5 Numerical experiments

In this section, we present some numerical results that illustrate our algorithm. The numerical tests were partially realized with help of the finite element program openFFW [13] published under GNU General Public License v.3. We consider the eigenvalue problem

$$\begin{aligned}
\Delta u &= \lambda u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{5.1}$$

on different domains Ω .

5.1 L-shape domain

Let us consider the eigenvalue problem (5.1) with L-shape domain $\Omega = [-1, 1] \times [0, 1] \cup [-1, 0] \times [-1, 0]$. An approximation of the smallest eigenvalue was given in [43], where the authors obtained that

$$\lambda_1 \approx 9.639723844.$$

Figure 1 shows the adaptively refined mesh on the 8-th level of refinement. We note that the mesh constructed by our algorithm contains more elements around the singularity. Additionally the convergence history is given in Figure 2, namely the log-log plot of the approximation error versus the number of degrees of freedom. This error is $|\lambda - \tilde{\lambda}|$, where the discrete eigenvalue problem is solved using the Arnoldi/Lanczos method, namely the MATLAB function *eigs* [36]. The squares show the approximation error based on the computation on the uniformly refined grid, while the triangles illustrate the approximation for our residual based refined grid. This shows that with our algorithm we may reach the same accuracy of the computed eigenvalue with fewer degrees of freedom. Tables 1 and 2 present the convergence history data for both strategies. Comparing the last columns of both tables, where we present the MATLAB CPU times of running the iterative algebraic eigenvalue solver for both algorithms, we notice that to reach the accuracy 10^{-2} for the first algorithm we have to

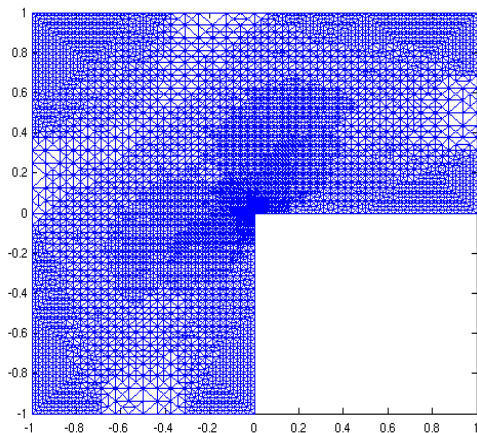


Figure 1: AFEMLA mesh on 8th refinement level of the L-shape domain.

ref. level	#DOF	$\tilde{\lambda}_1$	$ \lambda_1 - \tilde{\lambda}_1 $	CPU time (s)
1	5	13.1992	3.5595	0.01
2	33	10.5740	0.9342	0.03
3	161	9.9165	0.2768	0.04
4	705	9.7284	0.0886	0.16
5	2945	9.6698	0.0301	0.90
6	12033	9.6504	0.0107	7.60

Table 1: Approximation of the smallest eigenvalue of (5.1) on uniformly refined L-shape domain.

work with 12033 degrees of freedom and spend 7.6 s in solving the algebraic eigenvalue problem, while we need only 2745 degrees of freedom and 0.7 s in the case of our algorithm.

The second goal of our algorithm is to reduce the huge computational costs of solving the algebraic eigenvalue problem up to final accuracy, to get the best possible approximation of the eigenvalue. At this point one may try to reduce the size of the corresponding Krylov spaces generated in each run of the Arnoldi/Lanczos method. By setting the parameter p for the MATLAB function *eigs* [36], we restrict the number of Lanczos basis vectors used in the iterative solver. Figures 4 and 5 present numerical examples in this direction.

Table 3 presents the results for our algorithm for $p = 2$ Lanczos basis vectors. The number of degrees of freedom is slightly increased, moreover we observe slower decrease of the error, the convergence history for this case is presented on the Figure 5. However, if we run the standard algorithm with uniformly refined mesh we do not get any reasonable approximation of the eigenvalue, we get 0 in each level. Of course, also in our approach at some point, we have to perform some additional steps to achieve a good final accuracy.

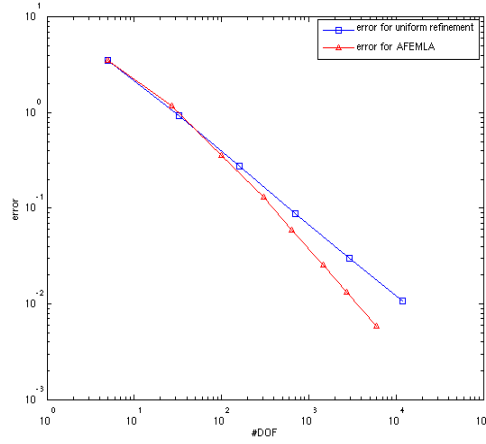


Figure 2: Convergence history for uniform refinement and AFEMLA for (5.1) on the L-shape domain.

ref. level	#DOF	$\tilde{\lambda}_1$	$ \lambda_1 - \tilde{\lambda}_1 $	CPU time (s)
1	5	13.1992	3.5595	0.02
2	27	10.8173	1.1775	0.02
3	99	9.9982	0.3584	0.03
4	306	9.7721	0.1323	0.07
5	641	9.6982	0.0585	0.14
6	1461	9.6652	0.0255	0.33
7	2745	9.6528	0.0131	0.70
8	5961	9.6455	0.0058	2.14

Table 2: Approximation of the smallest eigenvalue determined by AFEMLA for (5.1) on the L-shape domain.

ref. level	#DOF	$\tilde{\lambda}_1$	$ \lambda_1 - \tilde{\lambda}_1 $	CPU time (s)
1	5	13.2156	3.5758	0.01
2	27	10.8390	1.1993	0.01
3	98	10.0582	0.4184	0.02
4	305	9.8166	0.1769	0.06
5	712	9.7346	0.0949	0.14
6	1491	9.7221	0.0824	0.33
7	3186	9.7055	0.0657	0.82
8	6167	9.6991	0.0593	2.14

Table 3: Approximation of the smallest eigenvalue of (5.1) determined by AFEMLA on the L-shape domain with $p = 2$ Lanczos basis vectors.

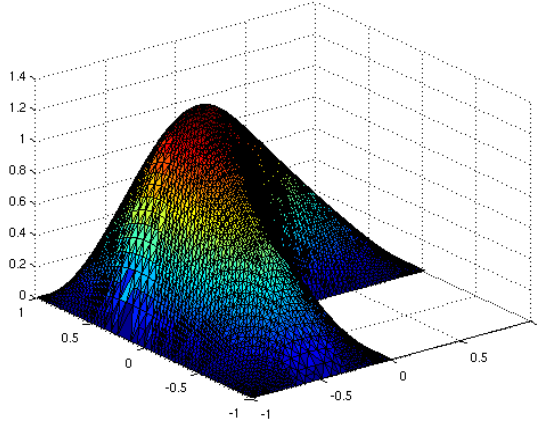


Figure 3: The eigenfunction corresponding to the eigenvalue $\tilde{\lambda} = 9.6455$ on the L-shape domain.

ref. level	1	2	3	4	5	6
#DOF	5	33	161	705	2945	12033
mesh \mathcal{T}_1	13.1992	10.5740	9.9165	9.7284	9.6698	9.6504
mesh \mathcal{T}_2	13.1992	10.5930	9.9261	9.7315	9.6707	9.6507

Table 4: Approximation of the smallest eigenvalue of (5.1) on the uniformly refined grids with initial mesh \mathcal{T}_1 and \mathcal{T}_2 .

5.2 Dependence on the layout of the initial grid

In Figure 6 and 7 we present two very similar initial meshes, however running the standard algorithm with uniform refinement on these meshes at each step provides interesting results. Analyzing results summarized in Table 4 we observe a faster convergence of the algorithm with initial mesh \mathcal{T}_1 , with the same number of grid points generated in each point. After some steps we get the errors of the same order but we observe slightly better behavior for the initial mesh \mathcal{T}_1 . This result shows, that not only the number of degrees of freedom plays a role in the convergence of the AFEM, but also the distribution over the domain.

5.3 More complicated domains

Let us consider the eigenvalue problem (5.1) with domain Ω as in Figure 8. Since in this case we do not know a priori any good approximation of the smallest eigenvalue, for comparison we will use the values obtained by exact procedure run on the uniformly refined grid in each step.

Comparing the values listed in Table 5 and 6 we notice that to obtain the approximation $\tilde{\lambda} \approx 11.97$ using uniformly refined grids we need around 69825 degrees of freedom while in our algorithm we work with 9584 degrees of freedom. Moreover, when we look at the run time for the iterative procedure, for the uniform algorithm we need around 788 s, while in our case we only need 4.6 s.

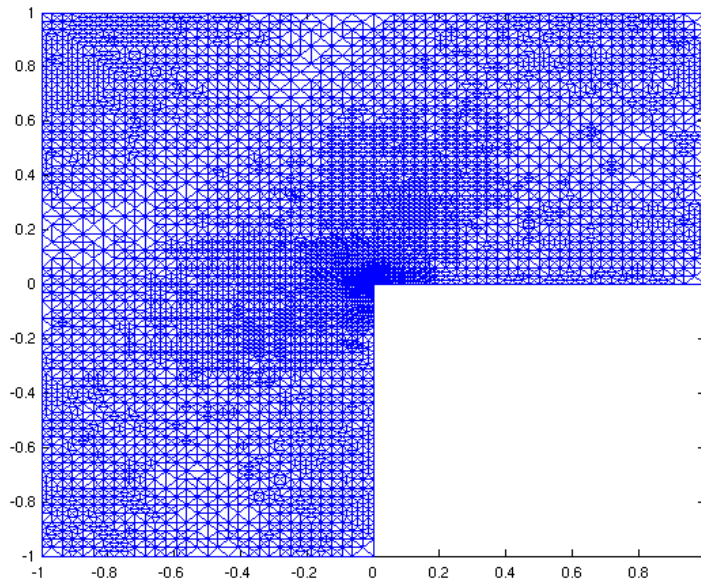


Figure 4: AFEMLA mesh on 8th refinement level of the L-shape domain with $p = 2$ Lanczos basis vectors.

The adaptively refined grid in Figure 8 constructed by our algorithm fully recognizes the critical regions of the domain, refining the mesh more around singularities.

5.4 Slit

Here we investigate the behavior of the AFEMLA on a slit domain (see Figure 9). The mesh generated on level 8 is shown in Figure 10. Because an exact value of the smallest eigenvalue of (5.1) for the slit domain is not known, as a reference value we use eigenvalue approximations from Table 7 obtained by solving the problem on a very fine grid. Table 8 contains approximations computed by AFEMLA. The first eigenfunction is shown in Figure 11. We notice again that AFEMLA saves CPU time and significantly reduces the number of degrees of

ref. level	#DOF	$\tilde{\lambda}_1$	CPU time (s)
1	44	13.6075	0.26
2	225	12.5102	0.07
3	1001	12.1579	0.23
4	4209	12.0352	1.43
5	17249	11.9905	15.45
6	69825	11.9737	788.02

Table 5: Approximation of the smallest eigenvalue of (5.1) obtained by the standard FEM on a uniformly refined grid.

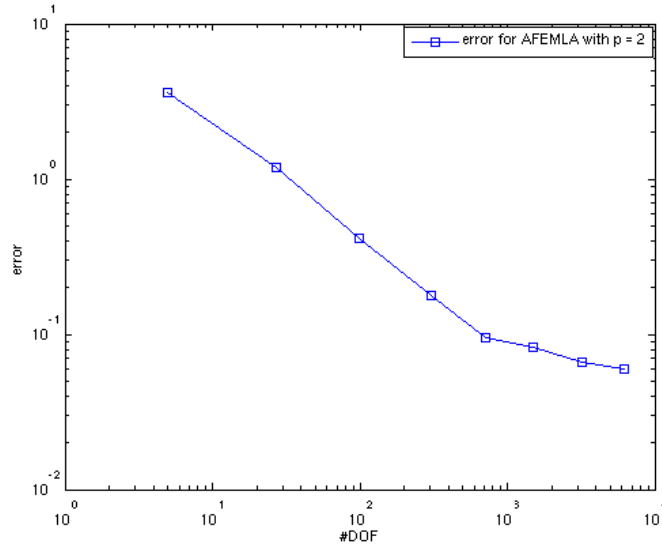


Figure 5: Convergence history of AFEMLA for (5.1) on the L-shape domain with $p = 2$ Lanczos vectors.

freedom.

5.5 More eigenvalues - refinement based on all residual vectors

Since often one is interested in determining all the eigenvalues in a certain area, it is important to compute more than one eigenvalue at once. The marking strategy used in AFEMLA based on the entries of a residual vector can be extended to use several residual vectors corresponding to the desired eigenvalues. The marking procedure will identify the large entries in all the residual vectors and take the union of the corresponding basis functions. Table 9 contains the

ref. level	#DOF	$\tilde{\lambda}_1$	CPU time (s)
1	44	13.6075	0.03
2	110	12.6818	0.04
3	262	12.2930	0.06
4	552	12.1340	0.12
5	1112	12.0474	0.25
6	2348	12.0037	0.59
7	4622	11.9834	1.49
8	9584	11.9728	4.60

Table 6: Approximation of the smallest eigenvalue of (5.1) obtained by the AFEMLA.

ref. level	#DOF	$\tilde{\lambda}_1$	CPU time (s)
1	2	5.1429	0.29
2	19	3.8704	0.04
3	101	3.5444	0.04
4	457	3.4538	0.11
5	1937	3.4253	0.52
6	7969	3.4150	3.99
7	32321	3.4109	120.53

Table 7: Approximation of the smallest eigenvalue of (5.1) obtained by the standard FEM on a uniformly refined slit domain.

ref. level	#DOF	$\tilde{\lambda}_1$	CPU time (s)
1	2	5.1429	0.02
2	12	3.9949	0.01
3	46	3.6020	0.02
4	148	3.4895	0.04
5	354	3.4462	0.09
6	723	3.4271	0.15
7	1496	3.4173	0.34
8	3030	3.4125	0.81

Table 8: Approximation of the smallest eigenvalue of (5.1) obtained by the AFEMLA on the slit domain.

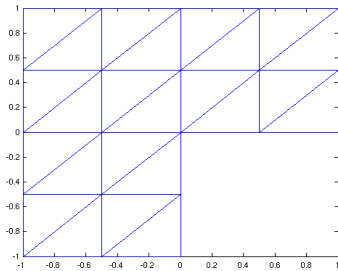


Figure 6: The initial mesh \mathcal{T}_1 .

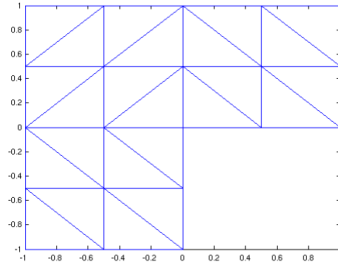


Figure 7: The initial mesh \mathcal{T}_2 .

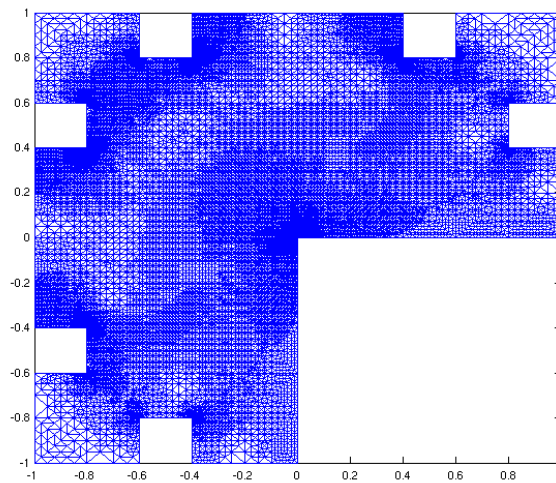


Figure 8: AFEMLA mesh on 8th refinement level.

approximations of the three smallest eigenvalues of the problem (5.1) obtained by AFEMLA, additionally in Table 10 the errors $|\lambda_i - \tilde{\lambda}_i|$, $i = 1, 2, 3$ are presented. The corresponding adaptively refined grid is depicted in Figure 12, while all three eigenfunctions are presented in Figure 13. An approximations of the three smallest eigenvalues of (5.1) was given in [43], where the authors obtained that

$$\lambda_1 \approx 9.639723844, \quad \lambda_2 \approx 15.197252, \quad \lambda_3 \approx 19.739209.$$

6 Conclusions

In this paper we have introduced a new adaptive algorithm (AFEMLA) for elliptic PDE-eigenvalue problems. We have concentrated on problems with symmetric and positive definite matrix pencils. The algorithm based on the residual values is able to indicate the important behavior of the solution and to construct a series of meshes. As we have shown, our algorithm reduces the computational

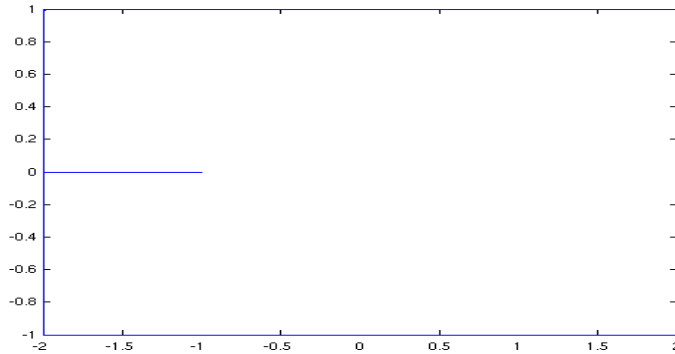


Figure 9: Slit domain.

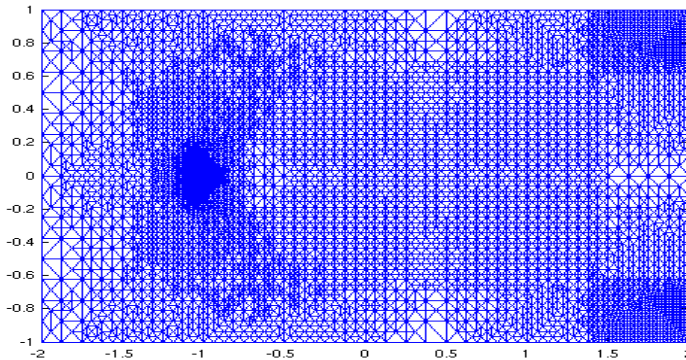


Figure 10: The AFEMLA mesh on 8th refinement level.

effort and the number of degrees of freedom considered in each step together with keeping the overall accuracy.

But it reduces not only the number of degrees of freedom which already simplifies the work load on the side of the iterative solver, but additionally saves much work also by reducing the sizes of Krylov subspaces generated in each step of the iterative procedure for the solution of the algebraic eigenvalue problem. Further work is geared towards restarting our algorithm with previously computed eigenvectors as a good initial guess for the iterative solver and creating its block version.

7 Acknowledgments

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We thank J. Gedicke and L. Grasedyck for fruitful discussions and useful comments.

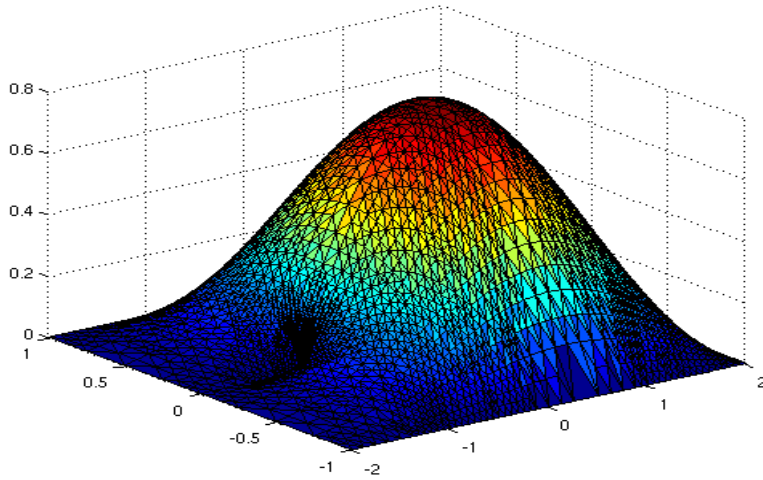


Figure 11: The eigenfunction corresponding to the eigenvalue $\tilde{\lambda} = 3.4125$ of (5.1) on the slit domain.

ref. level	1	2	3	4	5	6	7	8
#DOF	5	33	133	465	1306	2770	4997	11499
$\tilde{\lambda}_1$	13.1992	10.5542	9.9192	9.7376	9.6817	9.6591	9.6496	9.6440
$\tilde{\lambda}_2$	22.0215	16.9097	15.6315	15.3211	15.2421	15.2184	15.2085	15.2024
$\tilde{\lambda}_3$	32.0000	22.9075	20.5262	19.9515	19.8089	19.7760	19.7569	19.7482

Table 9: Approximation of the three smallest eigenvalues of (5.1) on the L-shape domain obtained by the AFEMLA.

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ref. level	1	2	3	4	5	6	7	8
$ \lambda_1 - \tilde{\lambda}_1 $	3.5595	0.9144	0.2795	0.0979	0.0420	0.0194	0.0099	0.0043
$ \lambda_2 - \tilde{\lambda}_2 $	6.8242	1.7125	0.4342	0.1239	0.0448	0.0211	0...0112	0.0051
$ \lambda_3 - \tilde{\lambda}_3 $	12.2608	3.1683	0.7870	0.2123	0.0697	0.0367	0.0177	0.0090

Table 10: Errors corresponding to the three smallest eigenvalues of (5.1) on the L-shape domain obtained by the AFEMLA.

ref. level	1	2	3	4	5	6	7	8
CPU time (s)	0.0074	0.0281	0.0465	0.1131	0.3498	0.8469	1.8888	7.0342

Table 11: CPU time (s) for running the iterative algebraic eigenvalue solver.

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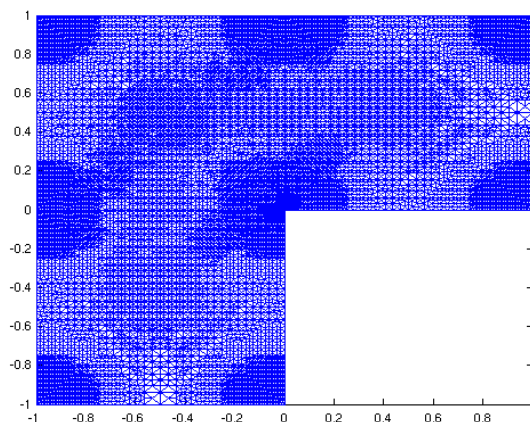


Figure 12: AFEMLA mesh for 8th level of refinement in computing three smallest eigenvalues on the L-shape domain.

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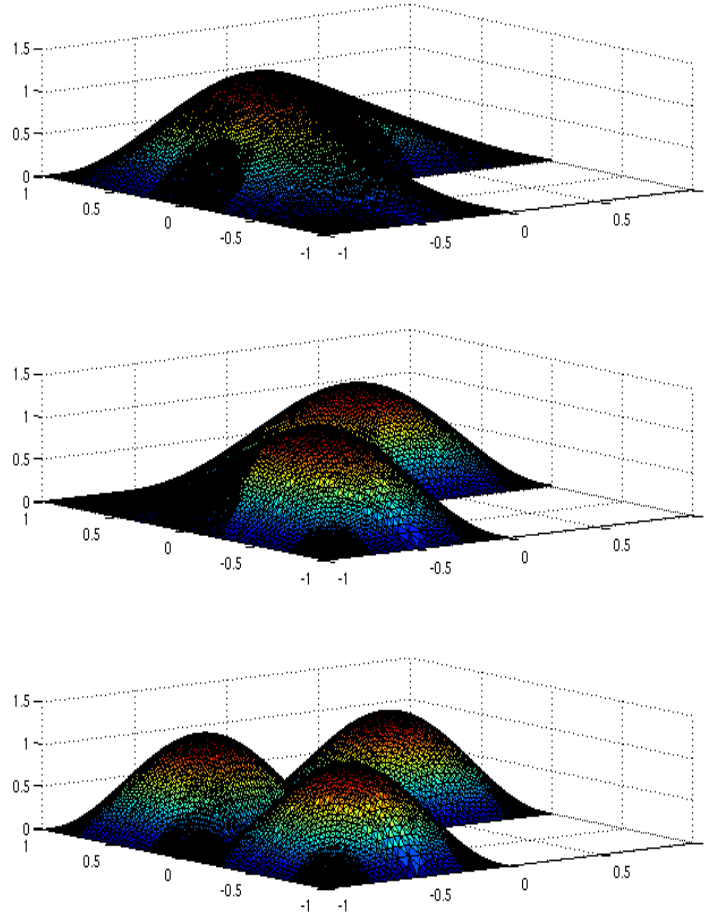


Figure 13: The computed eigenfunctions corresponding to the eigenvalues $\tilde{\lambda}_1 = 9.644$, $\tilde{\lambda}_2 = 15.2024$, $\tilde{\lambda}_3 = 19.7482$.

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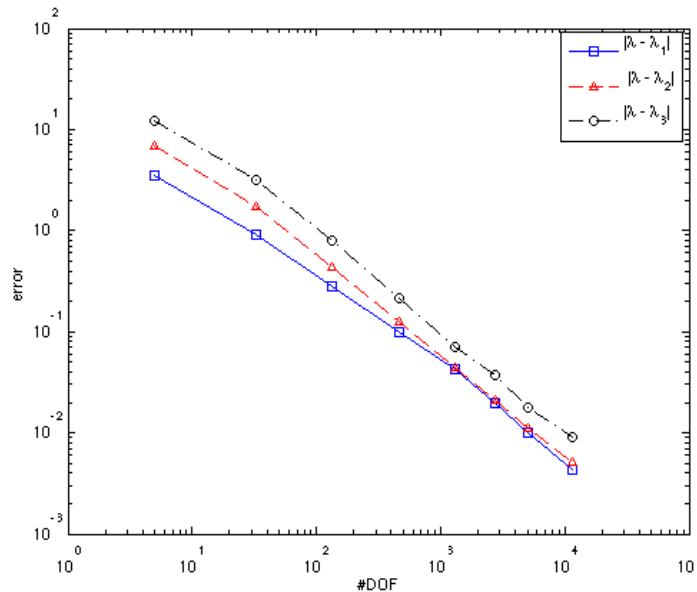


Figure 14: Convergence history for AFEMLA for computing the three smallest eigenvalues of (5.1) on the L-shape domain.

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