

# On cut-based inequalities for capacitated network design polyhedra \*

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## Abstract

In this article we study capacitated network design problems. We unify and extend polyhedral results for directed, bidirected and undirected link capacity models. Based on valid inequalities for a network cut we show that regardless of the link capacity model, facets of the polyhedra associated with such a cut translate to facets of the original network design polyhedra if the two subgraphs defined by the network cut are (strongly) connected. Our investigation of the facial structure of the cutset polyhedra allows to complement existing polyhedral results for the three variants by presenting facet-defining flow-cutset inequalities in a unifying way. In addition, we present a new class of facet-defining inequalities, showing as well that flow-cutset inequalities alone do not suffice to give a complete description for single-commodity, single-module cutset polyhedra in the bidirected and undirected case – in contrast to a known result for the directed case. The practical importance of the theoretical investigations is highlighted in an extensive computational study on 27 instances from the Survivable Network Design Library (SNDlib).

**Keywords:** capacitated network design, cutset polyhedra, flow-cutset inequalities, cut residual capacity inequalities, integer programming, mixed-integer rounding, SNDlib

## 1 Introduction

Capacitated network design problems arise in the strategic and tactical planning of telecommunication and public transport networks. Solving these problems is difficult as well as important for the profitable business of the network operators. We address in this paper the following variants of these problems: given point-to-point demands between locations and potential links of a network connecting these locations, a minimum cost assignment of capacity to the links has to be identified such that all demands can be realized by a network flow. A single capacity module can be installed on the network links in integer multiples. Each demand can be routed from its source to its destination along several paths (bifurcated routing). We distinguish three different types of capacity usage. The capacity of a link may be consumed by the flow in one direction only (DIRECTED link capacity model), of both directions independently (BIDIRECTED link capacity model), or it is shared between them (UNDIRECTED link capacity model).

In this paper, we analyze the facial structure of the underlying polyhedra and perform a computational study to investigate the practical importance of different classes of valid inequalities. Our main contributions are:

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- We prove a general lifting theorem, showing that facets of cutset polyhedra defined by the restriction of the problems to a cut of the network translate to facets of the original network design polyhedra if the two (directed) subgraphs defining the cut are (strongly) connected.
- For the DIRECTED link capacity model, flow-cutset inequalities with outflow and inflow-variables are known to be facet-defining (Atamtürk [4], Chopra et al. [12]). We transfer this class to the BIDIRECTED and UNDIRECTED case and provide necessary and sufficient conditions under which they define facets. This unifies the results for flow-cutset inequalities.
- We present the new class of facet-defining *cut residual capacity inequalities* for the BIDIRECTED and UNDIRECTED case. This class has no counterpart in the DIRECTED link capacity model. This is noticable because in the DIRECTED case, flow-cutset inequalities together with the trivial inequalities completely describe the single-commodity, single-module cutset polyhedron [4].
- We present an extensive computational study using 27 network design instances from SNDlib [27]. In all 27 cases, separating flow-cutset inequalities and cut residual capacity inequalities substantially improves the performance of CPLEX. The computation times are drastically reduced, and 14 instances can be solved to optimality within one hour, compared to 9 with the default settings of CPLEX. To our knowledge, this is the first study showing the practical strength of these cutting planes for capacitated network design problems on a large set of realistic instances.

This paper is structured as follows. After defining network design polyhedra and cutset polyhedra in Section 2 and Section 3, respectively, we show how facets of these polyhedra are related to each other, which is subsumed by the central Theorem 3.4. A detailed analysis of the facial structure of cutset polyhedra for the three link capacity models is provided in Section 4. We study a general class of flow-cutset inequalities in Section 4.1 and Section 4.2. The class of facet-defining cut residual capacity inequalities is introduced in Section 4.3. In Section 5 we present a generic separation procedure that can be used for all considered models and report on the effect of the generated inequalities in reducing computation times and gaps for a large set of test instances taken from SNDlib. Section 6 contains some concluding remarks.

## 2 Network design polyhedra

A network design instance is given by a directed graph  $G = (V, A)$  (DIRECTED link model) or an undirected graph  $H = (V, E)$  (BIDIRECTED and UNDIRECTED link model), a capacity module with capacity  $c \in \mathbb{Z}_+ \setminus \{0\}$  installable on the network links, and a set  $K$  of commodities. We assume  $G$  to be strongly connected and  $H$  to be connected, and explicitly allow for parallel arcs and edges. For notational convenience, we will nevertheless sometimes write  $e = \{i, j\}$  to denote the fact that edge  $e$  is defined between nodes  $i$  and  $j$ ; the precise meaning should be clear from the context.

For undirected graphs, in order to handle flow on edges, we define  $G = (V, A)$  to be the digraph that is obtained by introducing a pair of antiparallel arcs  $e^+$  and  $e^-$  for every edge  $e$  in  $E$ . Hence  $A := \{e^+ = (i, j), e^- = (j, i) : e = \{i, j\} \in E\}$  for the BIDIRECTED and UNDIRECTED link model. For all models,  $\delta_G^+(v)$  and  $\delta_G^-(v)$  denote the arcs in  $A$  that have  $v$  as their source and target node, respectively.

With every commodity  $k \in K$  we associate a vector  $d^k \in \mathbb{Z}^V$  of demands such that  $\sum_{v \in V} d_v^k = 0$ . There are mainly two approaches related to the definition of commodities and demands. The first is to consider an individual commodity for every point-to-point demand, resulting in  $|K| \in O(|V|^2)$  commodities. In our formulations, point-to-point traffic demands are assumed to be aggregated at their source nodes. Thus, every commodity  $k \in K$  can be identified with a unique source node  $s$  such that  $d_s^k > 0$  and  $d_v^k \leq 0$  for all  $v \in V \setminus \{s\}$ . There might be several target nodes with negative demand value. This approach leads to a significantly reduced problem size with  $|K| \in O(|V|)$ . The flow of a commodity can be split among several paths and is allowed to be fractional. Let  $f^k \in \mathbb{R}_+^A$  be the vector of flow variables corresponding to commodity  $k \in K$ . We define variables  $x_a, x_e$  to be the number of installed capacity modules on arc  $a \in A$  or edge  $e \in E$ , respectively.

Consider the following inequalities:

$$\sum_{a \in \delta_G^+(v)} f_a^k - \sum_{a \in \delta_G^-(v)} f_a^k = d_v^k \quad \forall v \in V, k \in K \quad (1)$$

$$\sum_{k \in K} f_a^k \leq c x_a \quad \forall a \in A \quad (\text{DIRECTED}) \quad (2a)$$

$$\max \left\{ \sum_{k \in K} f_{e^+}^k, \sum_{k \in K} f_{e^-}^k \right\} \leq c x_e \quad \forall e \in E \quad (\text{BIDIRECTED}) \quad (2b)$$

$$\sum_{k \in K} (f_{e^+}^k + f_{e^-}^k) \leq c x_e \quad \forall e \in E \quad (\text{UNDIRECTED}) \quad (2c)$$

The flow conservation constraints (1) ensure a feasible flow. Inequalities (2a), (2b), and (2c) are the capacity constraints for the three model types. The network design polyhedra for the link capacity models DIRECTED, BIDIRECTED, and UNDIRECTED are given by

$$\begin{aligned} \mathcal{ND}^{di} &= \text{conv} \left\{ (f, x) \in \mathbb{R}_+^{A \times K} \times \mathbb{Z}_+^A : (f, x) \text{ satisfies (1) and (2a)} \right\}, \\ \mathcal{ND}^{bi} &= \text{conv} \left\{ (f, x) \in \mathbb{R}_+^{A \times K} \times \mathbb{Z}_+^E : (f, x) \text{ satisfies (1) and (2b)} \right\}, \\ \mathcal{ND}^{un} &= \text{conv} \left\{ (f, x) \in \mathbb{R}_+^{A \times K} \times \mathbb{Z}_+^E : (f, x) \text{ satisfies (1) and (2c)} \right\}. \end{aligned}$$

The problem of optimizing a linear cost-function over one of these polyhedra is called a capacitated network design problem. This problem is known to be NP-hard already for special cases, see for instance Bienstock et al. [10] and Chopra et al. [12]. A valid inequality is called trivial if it is equivalent to one of the capacity or non-negativity constraints defining  $\mathcal{ND}^{di}$ ,  $\mathcal{ND}^{bi}$  or  $\mathcal{ND}^{un}$ . Notice that the capacity constraints (2b) for  $\mathcal{ND}^{bi}$  can be expressed by two linear inequalities for each edge, that  $\mathcal{ND}^{bi}$  is a relaxation of  $\mathcal{ND}^{un}$ , and that the constraint matrices and right-hand side vectors are integral. Since the capacity variables are not bounded and the underlying graphs are (strongly) connected we can construct a feasible flow for every demand vector (e.g., by applying a (min-cost) flow algorithm). Hence the given polyhedra are not empty. For special cases of these network design polyhedra (e.g., specific link capacity models, limited number of capacity modules), the following dimension result can be found in [4, 9, 23, 30]. There are no more implied equations than those given by the  $|K|(|V| - 1)$  linearly independent constraints among (1).

**Proposition 2.1.** *The dimension of  $\mathcal{ND}^{di}$  is  $|A|(|K| + 1) - |K|(|V| - 1)$ . The dimension of  $\mathcal{ND}^{bi}$  and  $\mathcal{ND}^{un}$  is  $|E|(2|K| + 1) - |K|(|V| - 1)$ .*

**Literature overview** The network design polyhedra considered in this paper have already been studied for special cases by several authors. Magnanti and Mirchandani [21], Magnanti et al. [22, 23] consider UNDIRECTED link models and a modular capacity structure with up to three different modules. The installable capacities are *divisible*, i.e., they are integer multiples of each other. Magnanti et al. initiate the study of network design polyhedra and introduce cutset inequalities, three-partition inequalities and arc residual capacity inequalities. Bienstock and Günlük [9] study polyhedra based on BIDIRECTED problems with two divisible base capacities. In addition to cutset and partition inequalities they consider a generalization of cutset inequalities to simple flow-cutset inequalities containing outflow variables. Chopra et al. [12] introduce general flow-cutset inequalities for DIRECTED models with a single module and show their validity. Atamtürk [4] gives a detailed analysis for DIRECTED cutset polyhedra. He studies the flow-cutset inequalities introduced in [12] and proves that they induce a complete description in the single-commodity, single-module case. Further, Atamtürk [4] generalizes DIRECTED flow-cutset inequalities to an arbitrary number of modules via lifting and states conditions under which these inequalities define facets.

Cutset polyhedra as introduced by Atamtürk [4] are based on network design problems for networks with two nodes (allowing parallel links). Hence every cut of a larger network together with the demands across the cut defines a cutset polyhedron. Most of the strong valid inequalities for network design polyhedra given in the literature are based on simple substructures of the network such as single arcs, cuts, three-partitions or, more general,  $k$ -partitions. These inequalities have

been derived as facets of simple structured relaxations, such as single-arcset polyhedra or network design polyhedra corresponding to simple  $k$ -node networks. Brockmüller et al. [11], van Hoesel et al. [17], Magnanti et al. [22] and Rajan and Atamtürk [31] consider single-arc sets. Magnanti et al. [22] and Bienstock et al. [10] study the capacity formulation that is obtained by projecting out all flow-variables in the classical multi-commodity link-flow formulation, see [7, 19]. They present a complete description of a three-node network design polyhedron. Agarwal [2] identifies facet-defining inequalities for the capacity formulation of the four-node UNDIRECTED problem.

It is of interest to know under which conditions facets for polyhedra based on these substructures translate to facets for the original network design polyhedra. For facets based on  $k$ -partitions of the node-set this question was partially answered by Agarwal [2]. He considers the UNDIRECTED capacity formulation with a single module and shows that every facet-defining inequality for the  $k$ -node problem based on capacity variables is a facet of the original problem if the subgraph of each component of the considered  $k$ -partition is connected. This result applies to inequalities in the space of the capacity variables only.

Cutset polyhedra can be seen as single node flow sets with unbounded integer capacity variables. The latter have been studied extensively in the literature initiated by Padberg et al. [28], Van Roy and Wolsey [33]. They introduce a special case of so-called flow-cover inequalities. Important work on strengthening flow-cover inequalities by superadditive lifting has been carried out by Gu et al. [15]. Reverse flow-cover inequalities were studied by Stallaert [32] and Atamtürk [3]. Louveaux and Wolsey [20] recently showed how strong valid flow-cover inequalities can be obtained by a MIR procedure applied to single node flow sets that has been introduced by Marchand and Wolsey [24]. The same procedure applied to cutset polyhedra, without complementing capacity variables, leads to the flow-cutset inequalities studied in the following sections.

### 3 Cutset polyhedra

Cutset polyhedra arise from the aggregation of flow conservation constraints for all nodes in a non-empty node-set  $S \subset V$ . The network design problems are restricted to the two artificial nodes  $S$  and  $V \setminus S$  and the corresponding network cut. Since a significant part of the characteristics of cutset polyhedra carries over to the related network design polyhedra, their polyhedral structure is important. After introducing the cutset polyhedra  $\mathcal{CS}_S^{di}$ ,  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ , we show that facet-defining inequalities for these cutset polyhedra also define facets of  $\mathcal{ND}^{di}$ ,  $\mathcal{ND}^{bi}$  or  $\mathcal{ND}^{un}$ , provided that rather mild conditions on the structure of the underlying graphs are satisfied. The results of this section, in particular Theorem 3.4, can be generalized to multi-module versions of network design polyhedra, see Appendix A and [30].

Let the network cut  $A_S \subseteq A$  in  $G$  be the set of arcs with one end-node in  $S$  and one end-node in  $V \setminus S$ , where  $A_S^+ \subseteq A_S$  is the set of arcs with source in  $S$  and  $A_S^- := A_S \setminus A_S^+$ . Similarly,  $E_S$  denotes the undirected cut in  $H$ , containing all edges with one end-node in  $S$  and one end-node in  $V \setminus S$ . For the BIDIRECTED and UNDIRECTED link model, every edge  $e$  in the cut  $E_S$  is represented by the two arcs  $e^+$  and  $e^-$  in  $A_S$  such that  $e^+ \in A_S^+$  and  $e^- \in A_S^-$ . Notice that the sets  $A_S^+$ ,  $A_S^-$  and  $E_S$  are all non-empty since  $G$  is strongly connected and  $H$  is connected.

Let  $d_S^k := \sum_{v \in S} d_v^k$  be the total demand over the cut given by  $S$  with respect to commodity  $k \in K$  and define

$$K_S^+ := \{k \in K : d_S^k > 0\}, \quad K_S^- := \{k \in K : d_S^k < 0\}, \quad K_S^0 := \{k \in K : d_S^k = 0\}$$

With the UNDIRECTED link capacity model, we may reverse the direction of every demand and exchange the corresponding flow variables without changing the problem. Hence, without loss of generality, we assume that  $K_S^- = \emptyset$  for UNDIRECTED models.

**Definition 3.1.** Given  $S \subset V$ , the cutset polyhedron  $\mathcal{CS}_S^{di}$  ( $\mathcal{CS}_S^{bi}$ ,  $\mathcal{CS}_S^{un}$ ) is the network design polyhedron  $\mathcal{ND}^{di}$  ( $\mathcal{ND}^{bi}$ ,  $\mathcal{ND}^{un}$ ) with respect to the graph  $G_S := (\{S, V \setminus S\}, A_S)$  ( $H_S := (\{S, V \setminus S\}, E_S)$ ) defined by the two artificial nodes  $S$  and  $V \setminus S$ , the cut arcs  $A_S$  (cut edges  $E_S$ ), and the aggregated demand vector  $(d_S^k, d_{V \setminus S}^k)$  for every  $k \in K$ .

Throughout we assume that there is demand across the considered cut, i. e.,  $K_S^+ \neq \emptyset$  w. l. o. g.. Note that the cutset polyhedra of  $S$  and  $V \setminus S$  are identical with  $d_S^k = -d_{V \setminus S}^k$  for all  $k \in K$ .

**Corollary 3.2.** *Let  $S \subset V$ . The dimension of  $\mathcal{CS}_S^{di}$  is  $|A_S|(|K| + 1) - |K|$ . The dimension of  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$  is  $|E_S|(2|K| + 1) - |K|$ .*

In the following we will work out the relation of cutset polyhedra to network design polyhedra. First note that valid inequalities for cutset polyhedra are also valid for network design polyhedra. This is true for all models and follows from the fact that for every point  $\hat{p}$  in a network design polyhedron, the subvector  $\hat{p}_S$ , obtained by restricting  $\hat{p}$  to the cut defined by  $S$ , is feasible for the corresponding cutset polyhedron. The following lemma provides a sufficient condition for the reverse statement and is one of the key arguments in the proof of Theorem 3.4. Let  $G[S]$ ,  $H[S]$  and  $G[V \setminus S]$ ,  $H[V \setminus S]$  be the subgraphs of  $G$ ,  $H$  induced by  $S$  and  $V \setminus S$ , respectively.

**Lemma 3.3.** *Let  $S \subset V$ . If  $G[S]$  and  $G[V \setminus S]$  are both strongly connected, then for every point  $p$  in  $\mathcal{CS}_S^{di}$  there is a point  $\hat{p}$  in  $\mathcal{ND}^{di}$  such that  $\hat{p}_S = p$ . If  $H[S]$  and  $H[V \setminus S]$  are connected, then for every point  $p$  in  $\mathcal{CS}_S^{bi}$  ( $\mathcal{CS}_S^{un}$ ) there is a point  $\hat{p}$  in  $\mathcal{ND}^{bi}$  ( $\mathcal{ND}^{un}$ ) such that  $\hat{p}_S = p$ .*

The point  $\hat{p}$  can be constructed by assigning sufficient capacity to all arcs (edges) in  $G[S]$  and  $G[V \setminus S]$  ( $H[S]$  and  $H[V \setminus S]$ ) and by solving flow problems for  $S$  and  $V \setminus S$  individually. Flow and capacity on the cut need not be changed, see [30]. It follows that if a valid inequality for a network design polyhedron is cut-based, i. e., it has nonzero coefficients for variables on some cut only, it is also valid for the corresponding cutset polyhedron, provided that the subgraphs defined by the cut are (strongly) connected.

We are now ready to prove the central lifting result of this section, stating that valid inequalities describing facets of cutset polyhedra also describe facets of the corresponding network design polyhedra if the mentioned condition on the structure of  $S$  and  $V \setminus S$  holds. For variants of the network design polyhedra considered here and special sub-classes of flow-cutset inequalities, this result has been proven, for instance, in [9, 23] (using similar techniques); in contrast, in the form presented here, it applies to *all* cut-based facet-defining inequalities and thus motivates investigations on the facial structure of two-node problems.

**Theorem 3.4** (Cutset lifting theorem). *Let  $S \subset V$  and*

$$\sum_{a \in A_S} \left( \sum_{k \in K} \gamma_a^k f_a^k + \beta_a x_a \right) \geq \pi$$

*be a facet-defining inequality of  $\mathcal{CS}_S^{di}$ . Then it also defines a facet of  $\mathcal{ND}^{di}$  if both  $G[S]$  and  $G[V \setminus S]$  are strongly connected. Similarly, let  $S \subset V$  and*

$$\sum_{e \in E_S} \left( \sum_{k \in K} \gamma_{e^+}^k f_{e^+}^k + \sum_{k \in K} \gamma_{e^-}^k f_{e^-}^k + \beta_e x_e \right) \geq \pi \quad (3)$$

*be a facet-defining inequality of  $\mathcal{CS}_S^{bi}$  ( $\mathcal{CS}_S^{un}$ ). Then it also defines a facet of  $\mathcal{ND}^{bi}$  ( $\mathcal{ND}^{un}$ ) if both  $H[S]$  and  $H[V \setminus S]$  are connected.*

*Proof.* We provide a proof for the BIDIRECTED case. The proof for the UNDIRECTED model is identical and a similar proof for the DIRECTED case can be found in Raack et al. [30]. We will first show that the induced face

$$F := \left\{ (f, x) \in \mathcal{ND}^{bi} : (f, x) \text{ satisfies (3) at equality} \right\}$$

is non-trivial, i. e., it is not empty and it does not equal  $\mathcal{ND}^{bi}$ . Then we will show that it is inclusion-wise maximal. Let

$$F_S := \left\{ (f, x) \in \mathcal{CS}_S^{bi} : (f, x) \text{ satisfies (3) at equality} \right\}$$

be the facet of  $\mathcal{CS}_S^{bi}$  defined by (3). Choose a point  $p \in F_S$ . By Lemma 3.3, there exists a point  $\hat{p} \in \mathcal{ND}^{bi}$  with  $\hat{p}_S = p$ . It follows that  $\hat{p}$  fulfills (3) at equality and hence  $\hat{p} \in F$ . Since  $F_S$  is a

facet of  $\mathcal{CS}_S^{bi}$ , there is a point  $q \in \mathcal{CS}_S^{bi}$  with  $q \notin F_S$ . Again by Lemma 3.3, there exists  $\hat{q} \in \mathcal{ND}^{bi}$  with  $\hat{q}_S = q$ . This point is not on the face  $F$ . We conclude that  $F$  is a non-trivial face of  $\mathcal{ND}^{bi}$ . It remains to show that  $F$  is inclusion-wise maximal. Choose a facet  $\tilde{F}$  of  $\mathcal{ND}^{bi}$  with  $F \subseteq \tilde{F}$  and let  $\tilde{F}$  be defined by

$$\sum_{e \in E} \left( \sum_{k \in K} \tilde{\gamma}_{e^+}^k f_{e^+}^k + \sum_{k \in K} \tilde{\gamma}_{e^-}^k f_{e^-}^k + \tilde{\beta}_e x_e \right) \geq \tilde{\pi}. \quad (4)$$

Every point in  $F$  satisfies (4) at equality. We will show that (4) equals (3) up to a linear combination of flow conservation constraints. First, for all  $e \in E \setminus E_S$ , we can construct a new point in  $F$  from  $\hat{p}$  by increasing the capacity of  $e$ , which implies  $\tilde{\beta}_e = 0$  for all  $e \in E \setminus E_S$ . In the following we assume w.l.o.g. that the capacity on edges in  $E \setminus E_S$  is large enough. By sending an epsilon flow on  $e^+$  and back on  $e^-$  for every edge  $e$  in  $E \setminus E_S$ , we conclude that  $\tilde{\gamma}_{e^+}^k = -\tilde{\gamma}_{e^-}^k$  for all  $e \in E \setminus E_S$  and  $k \in K$ . Next we choose a spanning tree  $T$  in  $H[S]$  ( $H[V \setminus S]$ ) which exists because  $H[S]$  ( $H[V \setminus S]$ ) is connected. By adding a linear combination of the flow conservation constraints (1) to (4) we may assume that  $\tilde{\gamma}_{e^+}^k = 0$  for all edges  $e$  in the tree. From  $\tilde{\gamma}_{e^+}^k = -\tilde{\gamma}_{e^-}^k$  we conclude  $\tilde{\gamma}_{e^+}^k = \tilde{\gamma}_{e^-}^k = 0$  for all  $e \in T$  and  $k \in K$ . Now let  $e$  be an edge in  $H[S]$  ( $H[V \setminus S]$ ) that is not in the tree. There is a unique cycle formed by  $e$  and edges in  $T$ . Sending circulation flows on that cycle eventually gives  $\tilde{\gamma}_{e^+}^k = \tilde{\gamma}_{e^-}^k = 0$  for all  $e \in E \setminus E_S$  and  $k \in K$ .

The coefficients of inequality (4) are nonzero only for edges on the cut. By Lemma 3.3, inequality (4) is thus valid for  $\mathcal{CS}_S^{bi}$ . Let  $\tilde{F}_S$  be the corresponding face of  $\mathcal{CS}_S^{bi}$ . By construction every point in  $F_S$  also fulfills (4) at equality. Since  $F_S$  is a facet, it follows  $\tilde{F}_S = F_S$ . Hence, (4) is (3) up to a scalar multiple and a linear combination of flow conservation constraints. We conclude that also  $F = \tilde{F}$ , and hence  $F$  defines a facet of  $\mathcal{ND}^{bi}$ .  $\square$

## 4 Facets of cutset polyhedra

Theorem 3.4 from the previous section motivates the analysis of the facial structure of cutset polyhedra, which will be done in more detail in this section. We will start by reviewing results for the class of so-called flow-cutset inequalities for  $\mathcal{CS}_S^{di}$  in Section 4.1. In the following section Section 4.2 we unify the existing results for the three link models by deriving flow-cutset inequalities for the BIDIRECTED and UNDIRECTED case. The presented class generalizes the well known cutset inequalities and simple flow-cutset inequalities for these models. We further present results on the strength of these inequalities. Eventually, Section 4.3 presents the new class of facet-defining cut residual capacity inequalities for the BIDIRECTED and UNDIRECTED case, which has no DIRECTED counterpart. This new class reflects the special structure of the polyhedra  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ .

We start by introducing the necessary notation. Let  $d_S^Q := \sum_{k \in Q} d_S^k$  be the total demand over the cut defined by  $S \subset V$  with respect to a non-empty commodity subset  $Q \subseteq K$ . We define

$$r(a, c) := a - c \left( \left\lceil \frac{a}{c} \right\rceil - 1 \right) > 0 \quad (5)$$

to be the remainder of the division of  $a \in \mathbb{R}$  by  $c \in \mathbb{R}_+ \setminus \{0\}$  if  $\frac{a}{c} \notin \mathbb{Z}$ , and  $c$  otherwise. For fixed  $S \subset V$  and for every commodity subset  $Q$  let

$$r^Q := r(d_S^Q, c), \quad \eta^Q := \left\lceil \frac{d_S^Q}{c} \right\rceil, \quad r_-^Q := r(-d_S^Q, c) \quad \text{and} \quad \eta_-^Q := \left\lceil \frac{-d_S^Q}{c} \right\rceil.$$

If  $\frac{d_S^Q}{c} \notin \mathbb{Z}$  then the relations  $\eta_-^Q = 1 - \eta^Q$  and  $r_-^Q = c - r^Q$  hold.

### 4.1 Flow-cutset inequalities for $\mathcal{CS}_S^{di}$

Flow-cutset inequalities for DIRECTED problems have been introduced by Chopra et al. [12] and studied in detail by Atamtürk [4]. We consider two subsets  $A_1^+ \subseteq A_S^+$  and  $A_2^- \subseteq A_S^-$  of the cut arcs  $A_S$  (see Figure 1). Define  $\bar{A}_1^+ := A_S^+ \setminus A_1^+$ . For any subset  $A^*$  of arcs we denote by  $x(A^*) := \sum_{a \in A^*} x_a$  the total number of installed modules. The total flow with respect to  $Q \subseteq K$  on  $A^*$  is denoted by  $f^Q(A^*) := \sum_{k \in Q} \sum_{a \in A^*} f_a^k$ .

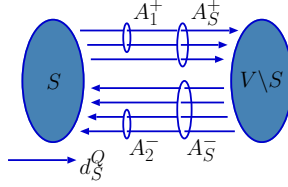


Figure 1: A directed cutset with selected cut arcs

**Lemma 4.1** (Atamtürk [4], Chopra et al. [12]). *The following flow-cutset inequality is valid for  $\mathcal{CS}_S^{di}$ :*

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + r^Q x(A_1^+) + (c - r^Q)x(A_2^-) \geq r^Q \eta^Q. \quad (6)$$

Inequality (6) is obtained by applying a  $\frac{1}{c}$ -MIR function (see [25, §II.1.7]) to the valid base inequality

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + c(x(A_1^+) - x(A_2^-)) \geq d_S^Q \quad (7)$$

with  $\bar{f}^Q(A_2^-) = cx(A_2^-) - f^Q(A_2^-)$ . We call inequality (6) a *simple flow-cutset inequality* if  $A_2^- = \emptyset$ . A simple flow-cutset inequality will be called a *cutset inequality* if additionally  $A_1^+ = A_S^+$ . Cutset inequalities are defined on the capacity variables only and have the form

$$x(A_S) \geq \eta^Q.$$

Notice that if  $d_S^Q$  is an integer multiple of  $c$ , then  $r^Q = c$  and (6) is a trivial aggregation of flow conservation, capacity and non-negativity constraints (see [4, 29, 30]). Unless otherwise stated, we assume  $r^Q < c$ , which means that  $c$  does not divide  $d_S^Q$ .

We will now review sufficient conditions for flow-cutset inequalities of type (6) to be facet-defining for  $\mathcal{CS}_S^{di}$ . We start with an important result for cutset inequalities. These inequalities are crucial for the performance of cutting-plane-based algorithms for network design problems, see Barahona [8], Bienstock et al. [10], Bienstock and Günlük [9]. In Section 5 we show that cutset inequalities outperform other types of flow-cutset inequalities within branch-and-cut algorithms.

**Theorem 4.2.** *The cutset inequality  $x(A_S^+) \geq \eta^{K_S^+}$  defines a facet of  $\mathcal{CS}_S^{di}$  if and only if  $r^{K_S^+} < c$ .*

For a simple proof of Theorem 4.2 we refer to Raack et al. [30]. Related results can be found for instance in [9, 10, 23]. The following two theorems by Atamtürk [4] introduce more classes of facet-defining flow-cutset inequalities for  $\mathcal{CS}_S^{di}$ . Theorem 4.4 is crucial for the theory of strong valid inequalities for DIRECTED network design polyhedra stating that in the single-commodity, single-module case, the trivial inequalities and all flow-cutset inequalities (6) completely describe  $\mathcal{CS}_S^{di}$ . In Section 4.3 we will show that this does not hold for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ .

**Theorem 4.3** (Atamtürk [4]). *Let  $\emptyset \neq Q \subseteq K_S^+$  and  $r^Q < c$ . The flow-cutset inequality (6) defines a facet of  $\mathcal{CS}_S^{di}$  if  $\emptyset \neq A_1^+ \subset A_S^+$  and  $\emptyset \neq A_2^- \subset A_S^-$ .*

**Theorem 4.4** (Atamtürk [4]). *Let  $|Q| = |K_S^+| = 1$ . The flow-cutset inequality (6) defines a facet of  $\mathcal{CS}_S^{di}$  if and only if  $r^{K_S^+} < c$  and  $A_1^+ \neq \emptyset$ . The inequalities (1), (2a), (6), and the non-negativity constraints completely describe  $\mathcal{CS}_S^{di}$ .*

In this paper we solely consider strongly connected digraphs, which implies  $A_S^- \neq \emptyset$ . If however  $A_S^- = \emptyset$ , then Theorem 4.2 and Theorem 4.4 do not hold. It can be shown that if  $x(A_S^+) \geq \eta^{K_S^+}$  defines a facet of  $\mathcal{CS}_S^{di}$  and  $|A_S^+| \geq 2$ , then either  $\eta^{K_S^+} \geq 2$  or  $A_S^- \neq \emptyset$ . In particular if  $A_S^- = \emptyset$ ,  $K_S^- = K_S^0 = \emptyset$ ,  $|A_S^+| \geq 2$ , and  $\eta^{K_S^+} = 1$  then the inequality  $x(A_S^+) \geq 1$  is not a facet of  $\mathcal{CS}_S^{di}$ . Taking  $A_1^+ \subset A_S^+$ , it is the sum of the flow-cutset inequalities

$$f^{K_S^+}(A_1^+) + r^{K_S^+} x(\bar{A}_1^+) \geq r^{K_S^+} \eta^{K_S^+} \quad \text{and} \quad f^{K_S^+}(\bar{A}_1^+) + r^{K_S^+} x(A_1^+) \geq r^{K_S^+} \eta^{K_S^+}$$

because  $r^{K_S^+} = d_S^{K_S^+}$  and  $f^{K_S^+}(A_S^+) = d_S^{K_S^+}$ . The condition  $\eta^{K_S^+} \geq 2$  is missing and implicitly assumed in Atamtürk [4, Theorem 1] and the corresponding proof.

## 4.2 Flow-cutset inequalities for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$

For the BIDIRECTED and UNDIRECTED case, cutset inequalities and simple flow-cutset inequalities have been studied in the literature as well. In this section, we will generalize these inequalities to the class of flow-cutset inequalities analogously to the DIRECTED case, and extend the facet results of Magnanti and Mirchandani [21] and Bienstock and Günlük [9]. Additionally, we introduce a second class of facet-defining inequalities for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$  without a DIRECTED counterpart. For the sake of compactness the (rather technical) proofs of the main results can be found in the Appendix.

We consider two subsets  $E_1, E_2$  of the undirected cut edges  $E_S$  (see Figure 2). Edge  $e \in E_S$  of the cut corresponds to  $e^+ \in A_S^+$  and  $e^- \in A_S^-$ . Let  $A_1^+$  and  $A_1^-$  denote all forward and backward arcs with respect to  $E_1$  (similar  $A_2^+$  and  $A_2^-$  with respect to  $E_2$ ), while  $\bar{A}_1^+ := A_S^+ \setminus A_1^+$  and  $\bar{A}_1^- := A_S^- \setminus A_1^-$ .

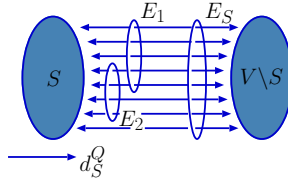


Figure 2: An undirected cutset with selected cut edges

**Lemma 4.5.** *The following flow-cutset inequality is valid for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ :*

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + r^Q x(E_1) + (c - r^Q)x(E_2) \geq r^Q \eta^Q. \quad (8)$$

*Proof.* The flow-cutset inequality (8) is the  $\frac{1}{c}$ -MIR inequality (see [25, §II.1.7]) for the base inequality

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + c(x(E_1) - x(E_2)) \geq d_S^Q. \quad (9)$$

with  $\bar{f}^Q(A_2^-) = cx(E_2) - f^Q(A_2^-)$ . Inequality (9) is the sum of all flow conservation constraints for node  $S$  and commodities in  $Q$ , the (BIDIRECTED) capacity constraints for flow on  $A_1^+$ , and the nonnegativity constraints for flow variables corresponding to  $A_S^- \setminus A_2^-$ .  $\square$

A simple flow-cutset inequality is a flow-cutset inequality (8) with  $E_2 = \emptyset$  and a cutset inequality is a simple flow-cutset inequality with  $E_1 = E_S$ , i. e., cutset inequalities reduce to

$$x(E_S) \geq \eta^Q.$$

Flow-cutset inequalities in the general form (8) have the nice property to be symmetric in  $S$  and  $V \setminus S$  in the sense that for every flow-cutset inequality of  $\mathcal{CS}_S^{bi}$  ( $\mathcal{CS}_S^{un}$ ) there exists a unique flow-cutset inequality for  $\mathcal{CS}_{V \setminus S}^{bi}$  ( $\mathcal{CS}_{V \setminus S}^{un}$ ). Adding the flow conservation constraint  $f^Q(A_S^-) - f^Q(A_S^+) = -d_S^Q$  to (8) gives

$$\begin{aligned} f^Q(\bar{A}_2^-) - f^Q(A_1^+) + (c - r_-^Q)x(E_1) + r_-^Q x(E_2) &\geq r^Q \eta^Q - d_S^Q = r^Q \eta^Q - c\eta^Q + c - r^Q \\ &= r^Q(\eta^Q - 1) - c(\eta^Q - 1) \\ &= r_-^Q \eta_-^Q. \end{aligned}$$

It turns out that if  $d_S^Q < 0$ , then (8) is equivalent to a flow-cutset inequality for  $\mathcal{CS}_{V \setminus S}^{bi}$  ( $\mathcal{CS}_{V \setminus S}^{un}$ ) with positive right-hand side. By interchanging  $S$  and  $V \setminus S$ , we can assume w. l. o. g. that  $d_S^Q > 0$ . Notice that the DIRECTED inequalities (6) are symmetric in the same way. Clearly not every combination of  $E_1$  and  $E_2$  can result in a facet. The following lemma provides necessary conditions for flow-cutset inequalities to define facets for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ .

**Lemma 4.6.** *If (8) is facet-defining for  $\mathcal{CS}_S^{bi}$  or  $\mathcal{CS}_S^{un}$  then*

- i)  $r^Q < c$  and  $E_1 \setminus E_2 \neq \emptyset$ .



ii) If (8) is a simple flow-cutset inequality with  $E_1 \neq E_S$  then either  $\eta^Q \geq 2$  or  $|Q| = 1$ .

iii) If (8) is a cutset inequality then  $\eta^Q = \eta^{K_S^+} \geq \eta_-^{K_S^-}$ . If additionally  $|E_S| > 1$  then  $\eta^{K_S^+} \geq 2$ .

If (8) is facet-defining for  $\mathcal{CS}_S^{un}$ , then  $E_1 \cap E_2 = \emptyset$ .

*Proof.*

- i) If  $r^Q = c$  then inequality (8) reduces to (9) which is a sum of model constraints (see Lemma 4.5). If  $r^Q < c$  but  $E_1 \setminus E_2 = \emptyset$  then inequality (8) can be written as

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + cx(E_1) + (c - r^Q)x(E_2 \setminus E_1) \geq r^Q \eta^Q = d_S^Q - (\eta^Q - 1)(c - r^Q),$$

which is dominated by  $f^Q(A_S^+) - f^Q(A_2^-) \geq d_S^Q$  since  $cx(E_1) \geq f^Q(A_1^+)$ ,  $\eta^Q \geq 1$ , and  $c > r^Q$ .

- ii) Suppose  $E_2 = \emptyset$ ,  $E_1 \subset E_S$ ,  $\eta^Q = 1$ , and  $Q = \{q_1, \dots, q_l\}$  with  $l \geq 2$ . It follows  $d_S^{q_i} \leq c$  for all  $i \in \{1, \dots, l\}$ ,  $d_S^Q = r^Q = \sum_{i=1}^l d_S^{q_i} = \sum_{i=1}^l r^{q_i}$  and  $\eta^Q = \eta^{q_i} = 1$ . Hence (8) is the sum of the following  $l$  valid simple flow-cutset inequalities:

$$f^{q_i}(\bar{A}_1^+) + r^{q_i} x(E_1) \geq r^{q_i} \eta^{q_i}.$$

- iii) Consider the cutset inequality  $x(E_S) \geq \eta^Q$ . The largest right-hand side is obtained if  $\eta^Q = \eta^{K_S^+}$ . Also  $x(E_S) \geq \eta_-^{K_S^-}$  is a valid cutset inequality for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$  and hence  $\eta^{K_S^+} \geq \eta_-^{K_S^-}$ . Suppose  $|E_S| > 1$  and  $\eta^{K_S^+} = 1$ . It follows that  $d_S^{K_S^+} = r^{K_S^+}$ . In this case, (8) is the sum of two non-trivial cut residual capacity inequalities of type (10) to be introduced in Section 4.3. Choose  $E_* \subset E_S$  such that  $E_*, \bar{E}_* \neq \emptyset$ . Anticipating Lemma 4.13,

$$\begin{aligned} cx(E_*) + (c - r^{K_S^+})x(\bar{E}_*) + f^Q(A_*^-) - f^Q(A_*^+) &\geq c - r^{K_S^+} \quad \text{and} \\ cx(\bar{E}_*) + (c - r^{K_S^+})x(E_*) + f^Q(\bar{A}_*^-) - f^Q(\bar{A}_*^+) &\geq c - r^{K_S^+} \end{aligned}$$

are both valid inequalities for  $\mathcal{CS}_S^{bi}$  (and  $\mathcal{CS}_S^{un}$ ) of the form (10) different from flow conservation constraints. Adding them up gives

$$\begin{aligned} (2c - r^{K_S^+})x(E_S) + f^Q(A_S^-) - f^Q(A_S^+) &\geq 2c - 2r^{K_S^+} \iff (2c - r^{K_S^+})x(E_S) - d_S^{K_S^+} \geq 2c - 2r^{K_S^+} \\ &\iff x(E_S) \geq 1 = \eta^{K_S^+}. \end{aligned}$$

It turns out that the cutset inequality  $x(E_S) \geq 1$  is the sum of non-trivial valid inequalities when  $|E_S| > 1$ .

To prove the last statement, we show that (8) is the sum of valid inequalities for  $\mathcal{CS}_S^{un}$  if  $E_1 \cap E_2 \neq \emptyset$ . Aggregating the UNDIRECTED capacity constraints for  $E_1 \cap E_2$  gives  $cx(E_1 \cap E_2) - f^Q(A_1^+ \cap A_2^+) - f^Q(A_1^- \cap A_2^-) \geq 0$ . Adding the flow-cutset inequality

$$f^Q(\bar{A}_1^+) + f^Q(A_1^+ \cap A_2^+) - f^Q(A_2^- \setminus A_1^-) + r^Q x(E_1 \setminus E_2) + (c - r^Q)x(E_2 \setminus E_1) \geq r^Q \eta^Q,$$

for the two edge-sets  $E_1 \setminus E_2$  and  $E_2 \setminus E_1$  results in

$$\begin{aligned} f^Q(\bar{A}_1^+) - f^Q(A_2^- \setminus A_1^-) - f^Q(A_1^- \cap A_2^-) + r^Q x(E_1 \setminus E_2) + cx(E_1 \cap E_2) + (c - r^Q)x(E_2 \setminus E_1) \\ \geq r^Q \eta^Q, \end{aligned}$$

which is (8). □

**Remark 4.7.** *The flow-cutset inequality (8) provides an analogon of (6) for BIDIRECTED and UNDIRECTED models and generalizes the cutset inequalities and simple flow-cutset inequalities studied in [8, 9, 21, 22, 23]. In contrast to the DIRECTED flow-cutset inequality (6), the two sets  $E_1$  and  $E_2$  are not necessarily disjoint. Lemma 4.6 implies that (8) cannot be strong if  $E_2$  is a (nonempty)*

subset of  $E_1$ ; in the UNDIRECTED case both subsets cannot even intersect in a facet-defining inequality. But there are still many facet-defining flow-cutset inequalities that are not simple, i. e.,  $E_2 \neq \emptyset$  (see Theorem 4.9 and 4.10). In the following we assume that  $Q$  is a subset of  $K_S^+$ . As mentioned, by reversing demands, we may assume that  $K_S^- = \emptyset$  in the UNDIRECTED case. When generating flow-cutset inequalities in practice this transformation can be carried out by implicitly exchanging forward and backward flow variables on a cut edge (a backward variable of a negative commodity can be handled as a forward variable for a positive commodity), which is not feasible for BIDIRECTED formulations.

After stating necessary conditions, the following results provide sufficient conditions for flow-cutset inequalities to be facet-defining for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ . We start with the well-known cutset inequalities [8, 9, 21, 22, 23] in Theorem 4.8. Theorem 4.9 can be seen as the analogon of Theorem 4.3 and generalizes the results of Bienstock and Günlük [9] and Magnanti et al. [23] for simple flow-cutset inequalities. Theorem 4.10 extends Theorem 4.9 further to the case  $E_1 \cup E_2 = E_S$  with  $E_1, E_2 \neq \emptyset$  and  $E_1 \neq E_2$ . Corollary 4.11 summarizes these results for the single-commodity case. Notice that for  $\mathcal{CS}_S^{un}$  the conditions  $\eta^{K_S^+} \geq \eta_-^{K_S^-}$  and  $d_S^{K_S^+} \geq |d_S^{K_S^-}|$  are trivially fulfilled.

**Theorem 4.8.** *The cutset inequality  $x(E_S) \geq \eta^{K_S^+}$  defines a facet of  $\mathcal{CS}_S^{bi}$  ( $\mathcal{CS}_S^{un}$ ) if and only if  $r^{K_S^+} < c$ ,  $\eta^{K_S^+} \geq \eta_-^{K_S^-}$  and if either  $\eta^{K_S^+} \geq 2$  or  $|E_S| = 1$ .*

*Proof.* Necessity was shown in Lemma 4.6 iii). Related sufficiency-results were proven by Bienstock and Günlük [9] for  $\mathcal{CS}_S^{bi}$  and Magnanti et al. [23] for  $\mathcal{CS}_S^{un}$ .  $\square$

**Theorem 4.9.** *Let  $\emptyset \neq Q \subseteq K_S^+$  and  $r^Q < c$ . The flow-cutset inequality (8) is facet-defining for  $\mathcal{CS}_S^{bi}$  if  $E_1 \setminus E_2 \neq \emptyset$ ,  $\bar{E}_1 \setminus E_2 \neq \emptyset$ , and one of the following conditions holds:*

- i)  $E_2 = \emptyset$  and either  $\eta^Q \geq 2$  or  $|Q| = 1$
- ii)  $E_2 \neq \emptyset$

The same holds for  $\mathcal{CS}_S^{un}$  if additionally  $E_1 \cap E_2 = \emptyset$ .

*Proof.* See Appendix B.1.  $\square$

**Theorem 4.10.** *Let  $\emptyset \neq Q = K_S^+$  with  $d_S^{K_S^+} \geq |d_S^{K_S^-}|$  and  $r^{K_S^+} < c$ .*

*The flow-cutset inequality (8) is facet-defining for  $\mathcal{CS}_S^{bi}$  if  $E_1 \setminus E_2 \neq \emptyset$ ,  $\bar{E}_1 \subseteq E_2 \neq \emptyset$ , and one of the following conditions holds*

- i)  $E_1 \cap E_2 = \emptyset$
- ii)  $E_1 \cap E_2 \neq \emptyset$ ,  $K_S^0 = \emptyset$  and either  $K_S^- = \emptyset$  or  $d_S^{K_S^+} > \max(|d_S^{K_S^-}|, c)$

The flow-cutset inequality (8) is facet-defining for  $\mathcal{CS}_S^{un}$  if  $K = K_S^+$ ,  $E_1, E_2 \neq \emptyset$  and  $\bar{E}_1 = E_2$ .

*Proof.* See Appendix B.2.  $\square$

**Corollary 4.11.** *Let  $|K| = |K_S^+| = 1$ . The flow-cutset inequality (8) is facet-defining for  $\mathcal{CS}_S^{bi}$  if and only if  $r^{K_S^+} < c$ ,  $E_1 \setminus E_2 \neq \emptyset$  and one of the following conditions holds:*

- i)  $E_2 = \emptyset$ ,  $\bar{E}_1 = \emptyset$  and either  $\eta^{K_S^+} \geq 2$  or  $|E_S| = 1$
- ii)  $E_2 \neq \emptyset$  or  $\bar{E}_1 \neq \emptyset$

The same holds for  $\mathcal{CS}_S^{un}$  if additionally  $E_1 \cap E_2 = \emptyset$ .

### 4.3 Cut residual capacity inequalities for $\mathcal{CS}_S^{bi}$ and $\mathcal{CS}_S^{un}$

For BIDIRECTED and UNDIRECTED models we consider a second class of strong valid inequalities that turns out to have no analogon in the DIRECTED case. Example 4.12 shows that in contrast to the directed case (see Theorem 4.4), flow-cutset inequalities of type (8) do not completely describe  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$  if  $|K| = 1$ .

**Example 4.12.** We define a cutset polyhedron with  $|K| = 1$ ,  $|E_S| = 2$  for the BIDIRECTED link model:

$$P = \text{conv} \left\{ x \in \mathbb{Z}_+^2, f \in \mathbb{R}_+^4 : f_1 + f_2 - f_3 - f_4 = 7, \right. \\ \left. \begin{aligned} 0 \leq f_i \leq 3x_1 \quad \forall i \in \{1, 3\}, \\ 0 \leq f_i \leq 3x_2 \quad \forall i \in \{2, 4\} \end{aligned} \right\}$$

When adding all flow-cutset inequalities (8) to the LP-relaxation of  $P$ , the resulting polyhedron still has the two fractional vertices  $(\frac{1}{2}, \frac{15}{2}, 1, 0, \frac{1}{2}, \frac{5}{2})$  and  $(\frac{15}{2}, \frac{1}{2}, 0, 1, \frac{5}{2}, \frac{1}{2})$ , (computed with PORTA [13]). But we can formulate two valid inequalities cutting off these points, namely:

$$3x_1 + 2x_2 + f_3 - f_1 \geq 2 \quad \text{and} \quad 3x_2 + 2x_1 + f_4 - f_2 \geq 2.$$

The inequalities of the last example can be generalized to a large new class of valid inequalities for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ . In the following let  $E_*$  be a subset of the cut  $E_S$  and  $\bar{E}_* := E_S \setminus E_*$  its complement. We denote by  $A_*^+$ ,  $A_*^-$  and  $\bar{A}_*^+$ ,  $\bar{A}_*^-$  all forward, backward arcs with respect to  $E_*$  and  $\bar{E}_*$ , respectively and consider a commodity subset  $Q$ .

**Lemma 4.13.** *The following cut residual capacity inequality is valid for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ :*

$$cx(E_*) + (c - r^Q)x(\bar{E}_*) + f^Q(A_*^-) - f^Q(A_*^+) \geq c - r^Q. \quad (10)$$

*Proof.* If  $r^Q = c$  then inequality (10) reduces to  $cx(E_*) - f^Q(A_*^+) + f^Q(A_*^-) \geq 0$ , which is valid because of  $cx(E_*) \geq f^Q(A_*^+)$  and  $f^Q(A_*^-) \geq 0$ . Now consider  $r^Q < c$ . First assume that  $x(\bar{E}_*) = 0$ . All flow has to be routed through  $E_*$ . It follows that

$$f^Q(A_*^+) - f^Q(A_*^-) = d_S^Q \quad \text{and} \quad x(E_*) \geq \left\lceil \frac{|d_S^Q|}{c} \right\rceil \geq \eta^Q.$$

Hence

$$cx(E_*) - (f^Q(A_*^+) - f^Q(A_*^-)) \geq c\eta^Q - d_S^Q = c - r^Q.$$

If, on the other hand,  $x(\bar{E}_*) \geq 1$  then from  $cx(E_*) - f^Q(A_*^+) + f^Q(A_*^-) \geq 0$  we conclude that

$$cx(E_*) + (c - r^Q)x(\bar{E}_*) + f^Q(A_*^-) - f^Q(A_*^+) \geq c - r^Q. \quad \square$$

In contrast to flow-cutset inequalities, cut residual capacity inequalities are not symmetric in the sense described above, but for  $d_S^Q < 0$  they turn out to be weak. In this case inequality (10) is dominated by the sum of the capacity constraint  $cx(E_*) - f^Q(A_*^+) \geq 0$  and the simple flow-cutset inequality  $r_-^Q x(\bar{E}_*) + f^Q(A_*^-) \geq r_-^Q \eta_-^Q$ . Hence we may assume  $d_S^Q > 0$  also for (10).

**Theorem 4.14.** *Let  $\emptyset \neq Q = K_S^+$  with  $d_S^{K_S^+} \geq |d_S^{K_S^-}|$ .*

*The cut residual capacity inequality (10) defines a facet of  $\mathcal{CS}_S^{bi}$  if and only if  $r^{K_S^+} < c$  and one of the following conditions holds:*

- i)  $E_*, \bar{E}_* \neq \emptyset$
- ii)  $E_* = \emptyset$  and  $\eta^{K_S^+} = 1$  and  $|E_S| = 1$
- iii)  $\bar{E}_* = \emptyset$  and either  $\eta^{K_S^+} \geq 2$  or  $|E_S| = 1$

*Given that  $K = K_S^+$  the same holds for  $\mathcal{CS}_S^{un}$ .*

*Proof.* We prove necessity of the conditions i)–iii) here and refer to Appendix B.3 for sufficiency. If  $r^{K_S^+} = c$  then inequality (10) reduces to  $cx(E_*) - (f^{K_S^+}(A_*^+) - f^{K_S^+}(A_*^-)) \geq 0$ , which is the sum of capacity constraints and non-negativity constraints. Assume  $r^{K_S^+} < c$ . If  $E_* = \emptyset$ , then inequality (10) reduces to  $x(E_S) \geq 1$ , which is dominated by the cutset inequality  $x(E_S) \geq \eta^{K_S^+}$  if  $\eta^{K_S^+} \geq 2$ . If on the other hand  $\eta^{K_S^+} = 1$  and  $|E_S| > 1$ , then by Lemma 4.6 it cannot define a facet. Now suppose that  $\bar{E}_* = \emptyset$ . We can write (10) as

$$cx(E_S) + f^{K_S^+}(A_S^-) - f^{K_S^+}(A_S^+) \geq c - r^{K_S^+} \iff cx(E_S) \geq d_S^{K_S^+} + c - r^{K_S^+} = c\eta^{K_S^+}$$

Hence (10) reduces to the cutset inequality, which by Lemma 4.6 needs  $\eta^{K_S^+} \geq 2$  or  $|E_S| = 1$  to be facet-defining.  $\square$

For the BIDIRECTED cutset polyhedron of Example 4.12, all flow-cutset inequalities (8) and the two cut residual capacity inequalities give the convex hull. It remains an open question if the same holds for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$  in general if  $|K| = 1$ .

## 5 Computational results

In this section, we introduce a generic separation framework to find flow-cutset inequalities and cut residual capacity inequalities, and assess the added value of these classes of inequalities. With minor modifications, the separation procedure can be applied to all capacity models. We will first consider the complexity of separation and present our separation strategies in Section 5.1. The test settings and some implementational details are provided in Section 5.2. In an initial study presented in Section 5.3, we investigate the effect of the new facet-defining cut residual capacity inequalities for BIDIRECTED and UNDIRECTED models when generated in addition to flow-cutset inequalities. The main computational study in Section 5.4 reveals the joint strength of the cut-based inequalities within a branch-and-cut algorithm for all three link models with an emphasis on BIDIRECTED and UNDIRECTED models.

### 5.1 Separation

Given any of the network design polyhedra  $\mathcal{ND}^{di}$ ,  $\mathcal{ND}^{bi}$  and  $\mathcal{ND}^{un}$  and a (fractional) point  $(\hat{f}, \hat{x})$ , the separation problem for the presented cut-based inequalities reduces to the problem of simultaneously determining a node-set  $S \subset V$ , a commodity subset  $Q \subseteq K$  and arc or edge subsets of the cut  $A_S$  or  $E_S$  leading to a violated inequality.

Atamtürk [4] shows that in the special case of a single point-to-point commodity separating DIRECTED flow-cutset inequalities (6) is equivalent to the max-cut problem (which is NP-hard, [14]). The same holds for the BIDIRECTED and UNDIRECTED versions of the problem. Using the cost values

$$\kappa_e := \min\{r\hat{x}_e, (c-r)\hat{x}_e + \hat{f}_{e^+} - \hat{f}_{e^-}, c\hat{x}_e - \hat{f}_{e^-}, \hat{f}_{e^+}\} \quad (11)$$

for all  $e \in E$ , a most violated flow-cutset inequality (8) is found by solving a min-cut problem in  $H = (V, E)$  with respect to the endpoints of the commodity. Depending on where the minimum in (11) is attained, an edge  $e$  in the cut either belongs to  $E_1 \setminus E_2$ ,  $E_2 \setminus E_1$ ,  $E_1 \cap E_2$ , or  $E_S \setminus (E_1 \cup E_2)$ . Note that  $\kappa_e$  can be negative if the minimum is given by  $(c-r)\hat{x}_e + \hat{f}_{e^+} - \hat{f}_{e^-}$ . For general commodity sets, finding a most violated cutset inequality is known to be NP-hard (see Bienstock et al. [10]). For a fixed node-set  $S$ , the complexity of simultaneously determining  $Q$  and  $A_1^+$ ,  $A_2^-$  ( $E_1$ ,  $E_2$ ) to obtain most violated flow-cutset inequalities is not known. For fixed  $S$  and  $Q$ , however, suitable subsets of the cut edges can obviously be identified in linear time, see below. If a single point-to-point commodity is considered, the general separation problem for cut residual capacity inequalities can also be formulated as a min-cut problem with potentially negative costs. In this case a most violated inequality (10) is defined by a minimal cut with respect to the edge costs

$$\kappa_e := \min\{c\hat{x}_e + \hat{f}_{e^-} - \hat{f}_{e^+}, (c-r)\hat{x}_e\},$$

where  $e \in E_*$  (see (10)) if and only if  $\kappa_e = c\hat{x}_e + \hat{f}_{e^-} - \hat{f}_{e^+}$ . For fixed  $S$  and  $Q$ , a subset  $E_*$  giving a most violated cut residual capacity inequality can be identified in linear time, see below.

For our computational tests we compared three different algorithms, Algorithms A, B, and C, all of which decompose the separation problem. Algorithm A uses the following heuristical separation scheme. In a first step, cutsets are determined using a fast shrinking procedure proposed by Bienstock et al. [10] and Günlük [16] which is adapted to the three link capacity models. Given the slacks  $s_a(s_e)$  and the dual values  $\pi_a(\pi_e)$  of the capacity constraints in the current LP-solution, the arc  $a$  (edge  $e$ ) with the largest weight  $w_a(w_e)$  is shrunken, where

$$\begin{aligned} \text{DIRECTED link model:} & & w_a & := s_a + \pi_a & & a \in A, \\ \text{BIDIRECTED link model:} & & w_e & := \min(s_{e^+}, s_{e^-}) + \min(\pi_{e^+}, \pi_{e^-}) & & e \in E, \\ \text{UNDIRECTED link model:} & & w_e & := s_e + \pi_e & & e \in E. \end{aligned}$$

This procedure is applied until the shrunken graph has a predefined number of nodes. Then, all cuts in the remaining graph are enumerated. In this way, cuts are preferred that have many arcs (edges) with small slack. Since usually many capacity constraints have zero slack w. r. t. the current LP solution, the dual values are considered as a second sorting criterion. This shrinking procedure has the nice property that the shores of the resulting cuts are very often (strongly) connected, which is crucial for the strength of the separated inequalities, see Section 3. In addition to the cuts obtained by shrinking, all single node cuts are considered, i. e., all node-sets  $S = \{v\}$  with  $v \in V$ , in every separation round. For every cut, all singleton commodity subsets, some commodity subsets  $Q$  with  $|Q| = 2$ , and the whole set  $K_S^+$  ( $K_S^-$ ) are considered. Eventually, given a cut, a commodity subset, and a fractional point  $\hat{p} = (\hat{f}, \hat{x})$ , arc- and edge-sets leading to most violated flow-cutset inequalities and cut residual capacity inequalities are calculated in linear time using the following definitions:

(1) flow-cutset inequality, DIRECTED link model:

$$A_1^+ := \left\{ a \in A_S^+ : r^Q \hat{x}_a \leq \hat{f}_a^Q \right\}, \quad A_2^- := \left\{ a \in A_S^- : (c - r^Q) \hat{x}_a < \hat{f}_a^Q \right\}$$

(2) flow-cutset inequality, BIDIRECTED and UNDIRECTED link:

$$E_1 := \left\{ e \in E_S : r^Q \hat{x}_e \leq \hat{f}_{e^+}^Q \right\}, \quad E_2 := \left\{ e \in E_S : (c - r^Q) \hat{x}_e < \hat{f}_{e^-}^Q \right\}$$

(3) cut residual capacity inequality, BIDIRECTED and UNDIRECTED link model:

$$E_* := \left\{ e \in E_S : r^Q \geq \hat{f}_{e^+}^Q - \hat{f}_{e^-}^Q \right\}.$$

Algorithms B and C are variations of the general framework provided by Algorithm A. Algorithm B exclusively generates all cutset inequalities corresponding to the chosen cuts. General flow-cutset inequalities and cut residual capacity inequalities are not considered. Algorithm C follows a hierarchical approach favoring cutset inequalities. If no violated cutset inequalities can be found after several separation rounds also general flow-cutset inequalities and cut residual capacity inequalities are checked for violation using the commodity, arc, and edge-sets describes above. Recall that cutset inequalities form a subclass of both flow-cutset inequalities and cut residual capacity inequalities.

## 5.2 Setting

Our testbed consists of all DIRECTED, BIDIRECTED, and UNDIRECTED instances of the Survivable Network Design Library (SNDlib) [27] corresponding to the network design problems defined in Section 2. We consider both single-module and multi-module instances. These are 27 problem instances in total, 3 of which are DIRECTED, 12 BIDIRECTED, and 12 UNDIRECTED. The number of nodes of the underlying networks ranges from 10 to 65, the number of links from 18 to 172, and the number of demands from 22 to 1869. The objective is to minimize either capacity cost or capacity and flow cost, depending on the instance. We used the cost-values as provided in SNDlib and described in [27]. Only for BIDIRECTED models we calculated the total routing cost of a link as the given cost parameter times the sum of the flows in both directions, which differs from the definition in [27] where only the cost for the maximum of the two flow values is incurred.

Most of the MODULAR instances in SNDlib are multi-module versions of the network design problems we considered so far. The maximum number of different modules installable on the network links is 11. Since flow-cutset inequalities are based on MIR they can easily be generalized to the multi-module case, see Appendix A. For the cut residual capacity inequalities such a generalization is not known. We generate them only for single-module instances.

The separation algorithms are implemented as callbacks using the callable library of CPLEX 10.0 [18]. Notice that the default parameter settings of CPLEX have not been changed, hence all default separators are switched on. The number of violated inequalities identified by our separation scheme is enormous for most of the instances. Adding them all leads to unacceptable computation times since user-added cutting planes are never deleted in CPLEX. We used several techniques to select a small number of the most promising cuts following Achterberg [1], Raack et al. [29]. All computations were performed on a 3GHz x86 Linux machine with 2GB of memory and a time limit of one hour.

For the statistics we distinguish between *easy* and *hard* instances. An instance is *easy* if it can be solved to optimality within the time limit of one hour by any of the considered algorithms. All other instances are referred to as being *hard*. These can be solved within the time limit neither by default CPLEX nor by adding cutting planes using Algorithms A, B, or C. With respect to this definition, 14 instances turned out to be easy, compared to 13 hard instances.

### 5.3 Added value of cut residual capacity inequalities

In a first study we show, for single-module instances, how cut residual capacity inequalities (10) add to the BIDIRECTED and UNDIRECTED formulations compared to the flow-cutset inequalities (8). The only MODULAR single-module instances in SNDlib for the BIDIRECTED and UNDIRECTED case are *france-B* and *france-U*, respectively.

The results are reported in Table 1. The first two columns (*problem*) and (*sep*) state the problem and the applied separation algorithm. In the following three columns we report on the integrality gap at the end of the root node (*rootgap*), the total CPU time (*time*) elapsed in seconds and the final optimality gap (*endgap*). The *rootgap* is calculated as  $(best-root)/best$ , where *best* is the value of the best known SNDlib solution and *root* the value of the LP relaxation at the root node before branching. The *endgap* is given by  $(upper-lower)/upper$ , where *upper* and *lower* are the best primal and dual bounds at termination, respectively. The last three columns of Table 1 report the total number of nodes explored in the search tree, the number of flow-cutset inequalities (8) and cut residual capacity inequalities (10) added to the initial formulations. The two problems were

<i>problem</i>	<i>sep</i>	<i>rootgap</i>	<i>time</i>	<i>endgap</i>	<i>nodes</i>	#(8)	#(10)
france-B	<i>none</i>	8.6%	3600	4.1%	408573		
	(8)	7.6%	160	0.0%	3609	715	
	(8), (10)	6.8%	33	0.0%	817	578	53
france-U	<i>none</i>	4.2%	3600	1.9%	1102696		
	(8)	3.4%	4	0.0%	401	150	
	(8), (10)	3.4%	3	0.0%	319	140	14

Table 1: Performance of the cut-based inequalities (8) and (10) as cutting planes

solved with default CPLEX without additional separators (*none*), CPLEX augmented by Algorithm A only generating flow-cutset inequalities (8), and CPLEX calling Algorithm A to generate flow-cutset inequalities (8) as well as cut residual capacity inequalities (10).

The effect of the separation routine Algorithm A is remarkable. CPLEX in the default mode cannot solve the france instances within the time limit of one hour, whereas these problems are solved to optimality within only seconds when the considered cut-based inequalities are generated. The integrality gap at the root node and the total number of nodes evaluated is decreased significantly. It can also be observed that the new cut residual capacity inequalities (10) capture additional structure of UNDIRECTED and BIDIRECTED network design problems. When adding cut residual

capacity inequalities in addition to flow-cutset inequalities the number of nodes to solve the france problems is decreased from 3609 to 817 and from 401 to 319, respectively.

Based on these results, our conclusion is that flow-cutset inequalities compared to cut residual capacity inequalities are more important in strengthening the LP relaxations and improving the performance for UNDIRECTED and BIDIRECTED network design problems. However, cut residual capacity inequalities are still very useful in reducing computation times and gaps when generated in addition to flow-cutset inequalities.

## 5.4 Joint strength of cut-based inequalities

We proceed with a detailed presentation of the results for all considered cut-based inequalities and the different Algorithms A, B, and C. Figure 3 reports on the acceleration of the computations when adding cut-based inequalities compared to default CPLEX for easy instances. The bars represent the change in the absolute CPU time. A value of -80% means that the solution time of default CPLEX could be reduced by 80% using the respective algorithm. The black markers refer to the second y-axis, which displays the absolute solving times of default CPLEX in seconds (on a logarithmic scale), according to which the instances are sorted. Those instances reaching the time limit with default CPLEX are ordered according to the endgap. For those instances, the acceleration is computed with the time limit as an underestimation of the real computation time of default CPLEX. All instances are labeled with the name of the underlying network and the link model, where the letters U, B and D denote the UNDIRECTED, BIDIRECTED and DIRECTED model, respectively.

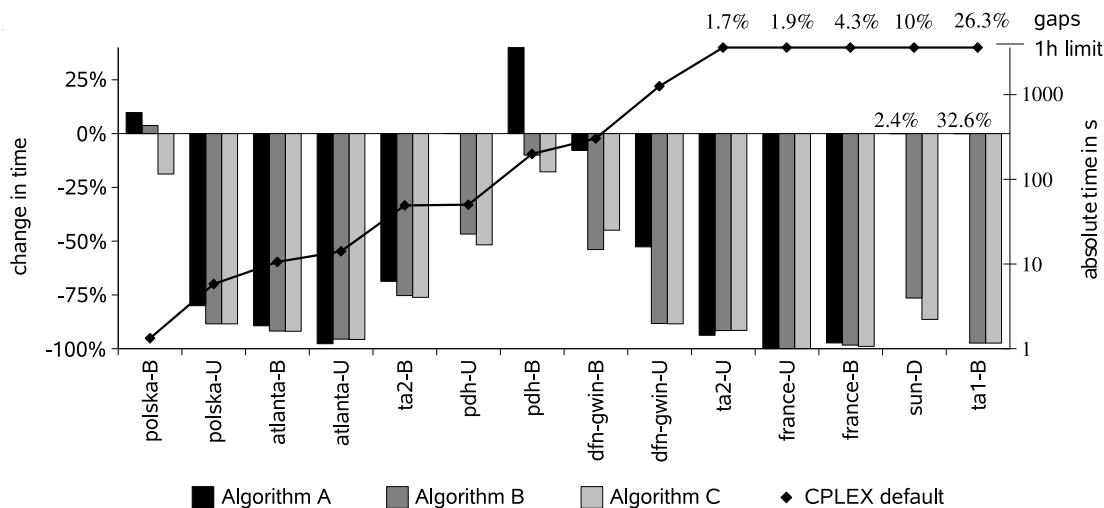


Figure 3: Easy instances: Relative change in solving time compared to default CPLEX [bars], as well as absolute solving time of default CPLEX [line with markers]. For instances unsolved within the time limit, the remaining endgap is reported.

Algorithms B and C solve all easy instances to optimality within the time limit. In contrast, CPLEX fails to solve five of these instances in the default settings, and two of these five are not solved by using Algorithm A. The remaining endgaps of CPLEX and Algorithm A are reported in the upper part of Figure 3. For nearly all of the easy instances the computation time is drastically reduced, in particular with Algorithms B and C. Even for the instances that cannot be solved by CPLEX in one hour, the solution time is less than 10 minutes. Comparing Algorithms B and C, it turns out that for most of the instances it suffices to add cutset inequalities to the initial formulation, but an additional speed-up can be obtained in particular cases by adding other types of cut-based inequalities using Algorithm C.

For those instances which are hard to solve, Figure 4 shows the relative change in the final lower and upper bounds. A value of 20% means that the corresponding final bound is 20% larger than

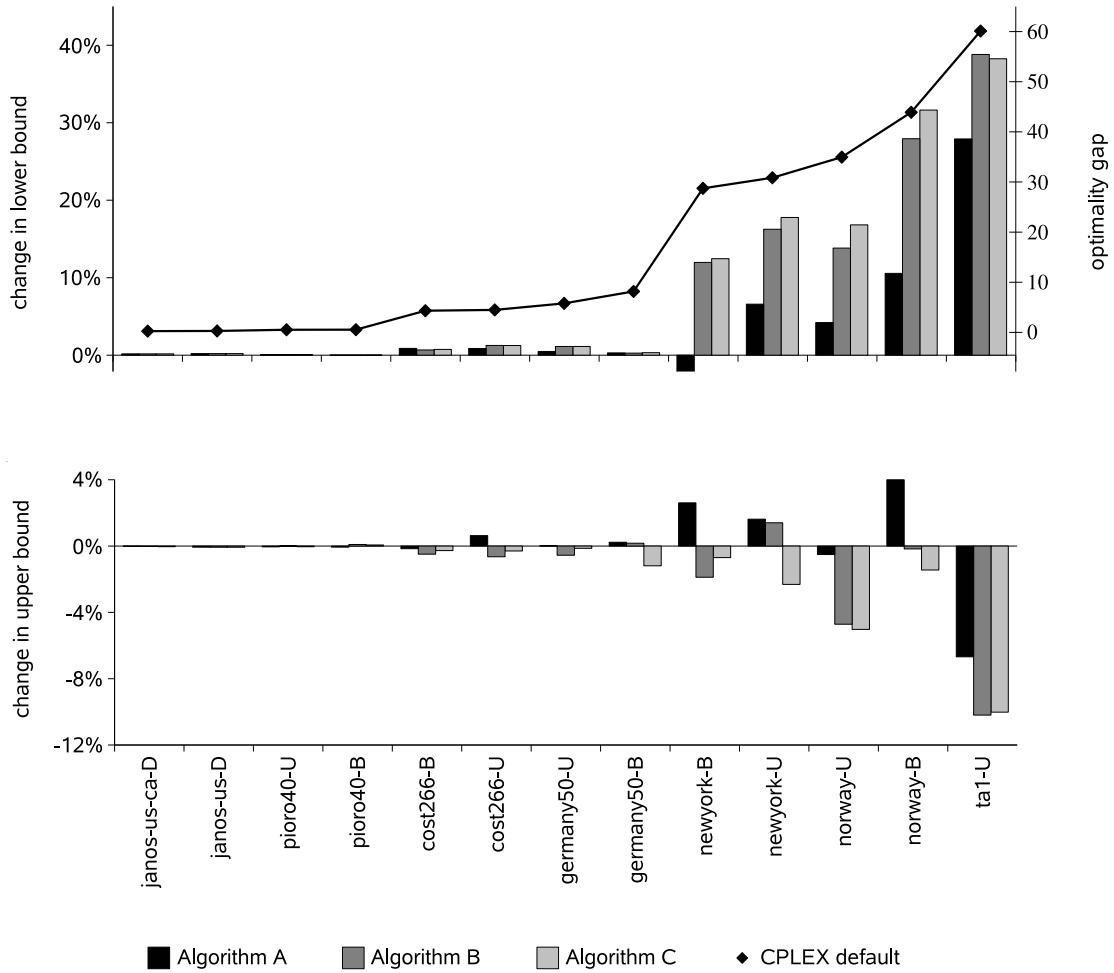


Figure 4: Hard instances: Relative change in lower and upper bound compared to default CPLEX [bars], remaining endgap of default CPLEX [line with markers]

the one obtained by default CPLEX after one hour of computation. The instances are sorted by the endgap of default CPLEX, displayed on the second y-axis. Both lower and upper bounds are improved by the generated cutting planes.

Again, Algorithms B and C outperform Algorithm A, and Algorithm C is slightly better than Algorithm B. It turns out that the improvement in the primal and dual bound is correlated to the endgap of default CPLEX. For the instance with the largest endgap, ta1-U, the lower bound can be increased by almost 40% and the upper bound is decreased by more than 10%. Whenever CPLEX has problems in finding cutting planes to increase the lower bound or in finding solutions to decrease the upper bound, the effect of adding flow-cutset inequalities and cut residual capacity inequalities is significant. On the other hand, the performance of CPLEX is not deteriorated by the cutting planes for those instances that have small gaps already in the default settings. The changes of the resulting endgaps are reported in Figure 5. Again the percentages are given with respect to the values obtained by CPLEX in the default settings. It can be seen that the cutting planes most often significantly reduce the gaps, sometimes by more than 90%.

Figure 6 summarizes the relative changes of several performance indicators when cut-based inequalities are added, compared to CPLEX in the default settings. For this statistic we calculated the ratio of the considered measure with and without the respective separation routine and averaged these ratios over all instances using the geometric mean. A percentage decrease in Figure 6



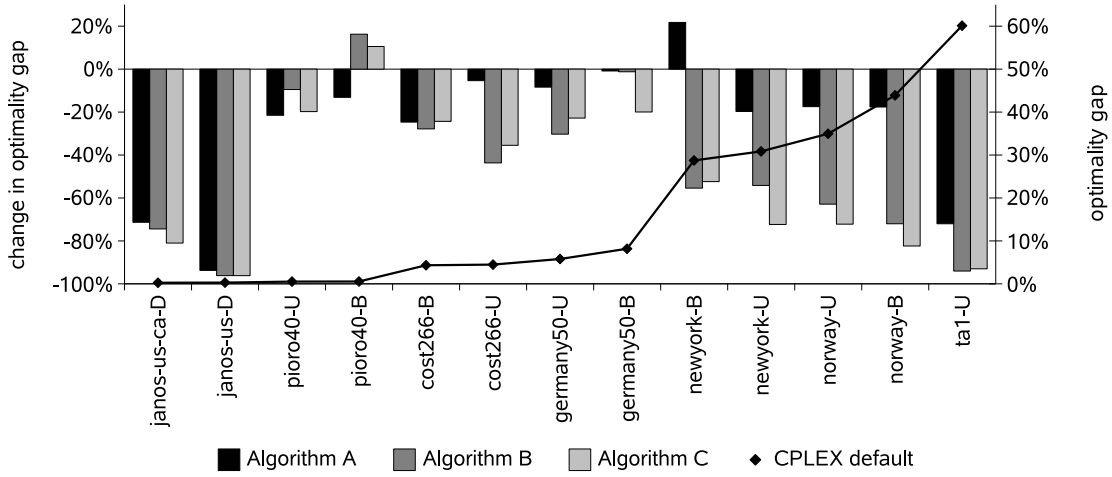


Figure 5: Hard instances: Relative change in the endgap compared to default CPLEX [bars], as well as the absolute endgap of default CPLEX [line with markers]

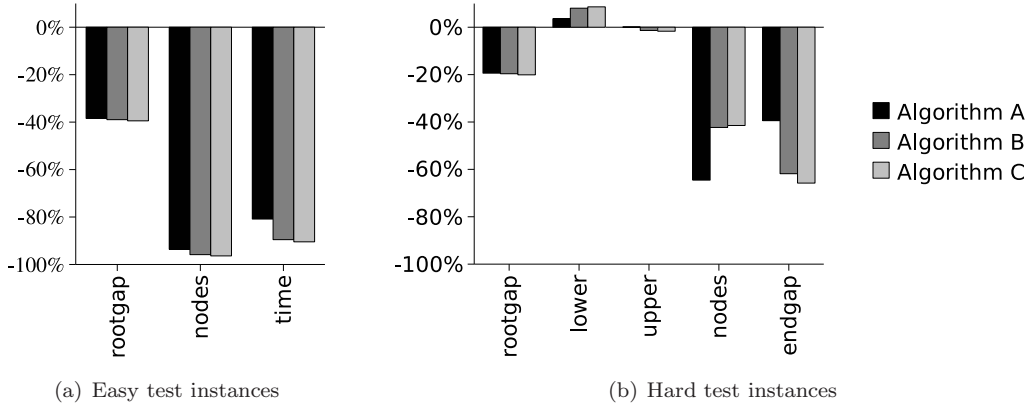


Figure 6: Relative changes of key measures compared to default CPLEX, geometric means

corresponds to a smaller mean ratio compared to default CPLEX. The figure shows the change in the remaining integrality gap at the root node before branching (*rootgap*), the solution time (*time*), the number of visited branch-and-bound nodes (*nodes*), the final lower and upper bounds (*lower*, *upper*), and the gap between *lower* and *upper* at termination (*endgap*).

For easy instances the saved computation time correlates with the number of visited nodes and the improved lower bound at the root node. The computation time is reduced by 90% on average, and the number of nodes is even reduced by almost 97%. For hard instances, adding cut-based inequalities leads to a decrease in the number of branch-and-bound nodes that can be visited within one hour of computation time. But since the cutting planes have a very positive effect on both the lower and upper bound, the endgap is reduced by 66% on average.

Among all tested separation algorithms, the hierarchical approach (Algorithm C) performed best. To understand the different behavior of the three algorithms we examined the distribution of the generated inequalities. Figure 7 shows the percentage of cutset inequalities (*cis*), of simple flow-cutset inequalities (*sfcis*) and of all other types of cut-based inequalities (*others*). With the integrated approach of Algorithm A, only 32.67% of all generated flow-cutset inequalities are cutset inequalities, whereas 100% and 55.21% are cutset inequalities with Algorithms B and C, respectively.

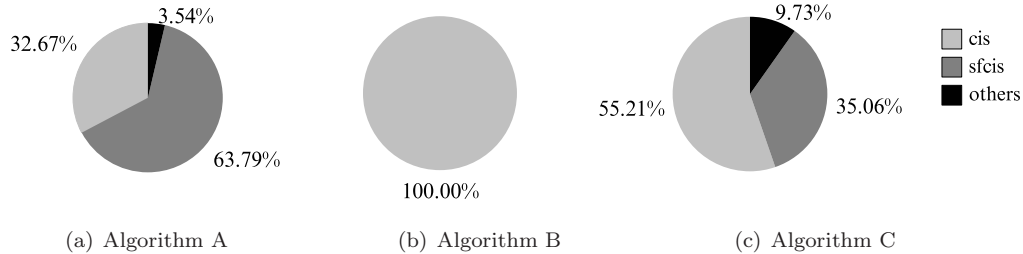


Figure 7: Average distribution of cut-based inequalities added by Algorithms A, B, and C

Given that Algorithm C only slightly increases the performance compared to Algorithm B, this means that cutset inequalities are responsible for most of the progress. All other types of flow-cutset inequalities should carefully be generated and only if no violated cutset inequalities are found. As already mentioned in Section 5.2, we restricted the total number of added flow-cutset inequalities to the number of rows in the initial formulation. With this restriction, the number of inequalities added amounted to 62%, 44% and 48% of the size of the initial formulation for Algorithms A, B, and C, respectively, averaged over all instances.

## 6 Concluding remarks

In this paper we have studied theoretical and computational aspects of capacitated network design polyhedra with DIRECTED, BIDIRECTED, and UNDIRECTED link capacity structure. We have shown that given a network cut, any facet of the corresponding cutset polyhedron translates to a facet of the network design polyhedron if both network components are (strongly) connected. We have amended the class of flow-cutset inequalities for the DIRECTED case to the BIDIRECTED and UNDIRECTED cases and introduced the new class of cut residual capacity inequalities for these cases. In contrast to the DIRECTED case, this class is necessary for a complete description of the single commodity single-module BIDIRECTED/UNDIRECTED cutset polyhedron. Its sufficiency could not be proven yet and is a direction for future research. Another direction for future research is the generalization of the cut residual capacity inequalities to the multi-module case. Whereas the flow-cutset inequalities can be generalized with mixed-integer rounding, it remains unclear how the cut residual capacity inequalities look like for multiple modules.

Our computational study shows that a state-of-the-art solver like CPLEX can be outperformed by adding flow-cutset inequalities. The best results were obtained by using a hierarchical approach that favors cutset inequalities over other types of flow-cutset inequalities. It turns out that this subclass of cutting planes, which contains only capacity variables but no flow variables, is responsible for most of the progress. The performance is only slightly better and varies over the instances when the larger class of flow-cutset inequalities is considered. The latter ones should be added carefully and only if no violated cutset inequalities can be found. We believe that a more elaborate selection of cuts and in particular of commodity subsets might even increase the impact of these inequalities. For almost all of the instances we had no difficulties in finding violated flow-cutset inequalities, but the challenge is to find the best ones. It turned out that the violation of flow-cutset inequalities is not a good measure to estimate their efficiency. Very often cutset inequalities are less violated (hence ignored by Algorithm A) but more effective in terms of reducing computation times and gaps.

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## A Flow-cutset inequalities in the multi-module case

In the sequel, we generalize flow-cutset inequalities (6) and (8) to the multi-module case. Consider a set of capacity modules  $M$  installable on every network arc  $a$  or edge  $e$ . A module  $m$  has a base capacity  $c^m \in \mathbb{Z}_+ \setminus \{0\}$  and is counted using the variables  $x_a^m$  or  $x_e^m$ . Two and three-module versions of cutset inequalities and flow-cutset inequalities have been studied already by Bienstock and Günlük [9], Magnanti and Mirchandani [21], Magnanti et al. [23] for the special case that the base capacities are divisible. Atamtürk [4] introduces DIRECTED flow-cutset inequalities for the general multi-module case by studying the lifting problem. We want to emphasize here that the lifting functions developed by Atamtürk are based on mixed-integer rounding [25, 26]. Valid lifting of the flow-cutset inequalities to the multi-module case can be done using the same subadditive MIR function for all models. Fixing node-, commodity- and arc-sets (resp. edge-sets) we obtain  $|M|$  potentially different multi-module flow-cutset inequalities, one for every available capacity module. For details we refer to Atamtürk [4], Raack et al. [29, 30], and references therein.

Let  $S \subset V$  and  $Q \subseteq K$  be fixed and let  $d_S^Q > 0$  be the corresponding cut demand as defined in Section 4.1. In a first step we generalize the base inequalities (7) and (9) to the multi-module case. By aggregating model inequalities and substituting  $\bar{f}^Q(A_2^-) := \sum_{m \in M} c^m x^m(A_2^-) - f^Q(A_2^-)$  or  $\bar{f}^Q(A_2^-) := \sum_{m \in M} c^m x^m(E_2) - f^Q(A_2^-)$  the base inequalities

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{m \in M} c^m (x^m(A_1^+) - x^m(A_2^-)) \geq d_S^Q, \quad (12)$$

$$f^Q(\bar{A}_1^+) + \bar{f}^Q(A_2^-) + \sum_{m \in M} c^m (x^m(E_1) - x^m(E_2)) \geq d_S^Q. \quad (13)$$

are valid for  $\mathcal{CS}_S^{di}$  and  $\mathcal{CS}_S^{bi}$ ,  $\mathcal{CS}_S^{un}$ , respectively. In a second step, a MIR-function is applied to the coefficients of (12) and (13). Given  $a, c, d \in \mathbb{R}$  with  $c > 0$  and  $\frac{d}{c} \notin \mathbb{Z}$ , define  $a^+ := \max(0, a)$  and consider the MIR function  $F_{d,c} : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$F_{d,c}(a) := \left\lceil \frac{a}{c} \right\rceil r(d, c) - (r(d, c) - r(a, c))^+.$$

$F_{d,c}$  is nondecreasing and subadditive, i. e.,  $F(a) + F(b) \geq F(a+b)$  for all  $a, b \in \mathbb{R}$  [25, 29]. It holds that  $F_{d,c}(0) = 0$  and  $\bar{F}_{d,c}(1) = \lim_{t \searrow 0} \frac{F(t)}{t} = 1$ .

The function  $F_{d,c}$  can be seen as the  $\frac{1}{c}$ -MIR function for  $\geq$ -base-inequalities with right-hand side  $d$ , scaled by the factor  $r(d, c)$ . Similar subadditive and superadditive functions based on MIR have been considered for instance by Atamtürk [5, 6] and Louveaux and Wolsey [20].

We set  $d := d_S^Q$  and  $c := c^s$  for some module  $s \in M$  and consider the function

$$F_s := F_{d_S^Q, c^s}.$$

Applying Theorem 7.4 of Chapter II.1 of Nemhauser and Wolsey [25] and resubstituting  $\bar{f}^Q(A_2^-)$ , gives the multi-module flow-cutset inequalities

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{m \in M} F_s(c^m) x^m(A_1^+) + \sum_{m \in M} (c^m + F_s(-c^m)) x^m(A_2^-) \geq F_s(d_S^Q), \quad (14)$$

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + \sum_{m \in M} F_s(c^m) x^m(E_1) + \sum_{m \in M} (c^m + F_s(-c^m)) x^m(E_2) \geq F_s(d_S^Q) \quad (15)$$

valid for  $\mathcal{CS}_S^{di}$  and  $\mathcal{CS}_S^{bi}$ ,  $\mathcal{CS}_S^{un}$ , respectively. These inequalities generalize (6) and (8) since  $F_s(c^s) = r(d_S^Q, c^s)$  and  $F_s(-c^s) = -r(d_S^Q, c^s)$ . If  $d_S^Q$  is an integer multiple of  $c^s$ , then (14) and (15) reduce to the base inequalities (12) and (13) because in this case  $F_s(a) = a$  for all  $a \in \mathbb{R}$ .

Notice that  $F_{d,c}(a), \bar{F}_{d,c}(a)$  are integral if  $a, c$ , and  $d$  are integral. Moreover,  $|F_{d,c}(a)| \leq |a|$  holds for all  $a \in \mathbb{R}$ , see Raack et al. [29]. This means that the considered inequalities have small integral coefficients as long as capacities and demands are small and integral. From a numerical point of view, this property is desirable in a cutting plane or in a branch-and-cut algorithm.

In case that there are no inflow-variables, i. e.,  $A_2^- = \emptyset$  or  $E_2 = \emptyset$ , the left-hand sides of (14) respectively (15) contain only non-negative coefficients which trivially can be strengthened to the

value of the right-hand side. Since  $F_s$  is non-decreasing, applying the strengthening and MIR can be exchanged, resulting in the same inequality. In particular, defining the function  $\mathcal{F}_s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by

$$\mathcal{F}_s(a) := F_s(\min(a, d_S^Q)) = \min(F_s(a), F_s(d_S^Q)),$$

the coefficient  $F_s(c^m)$  can be reduced to  $\mathcal{F}_s(c^m)$ . The resulting strengthened simple flow-cutset inequalities

$$f^Q(\bar{A}_1^+) + \sum_{m \in M} \mathcal{F}_s(c^m) x^m(A_1^+) \geq \mathcal{F}_s(d_S^Q), \quad (16)$$

$$f^Q(\bar{A}_1^+) + \sum_{m \in M} \mathcal{F}_s(c^m) x^m(E_1) \geq \mathcal{F}_s(d_S^Q) \quad (17)$$

are valid for  $\mathcal{CS}_S^{di}$  respectively  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$ . Notice that the function  $\mathcal{F}_s$  is nondecreasing and subadditive on  $\mathbb{R}_+$  with  $\mathcal{F}_s(0) = 0$  and  $\bar{\mathcal{F}}_s(1) = \lim_{t \searrow 0} \frac{\mathcal{F}_s(t)}{t} = 1$  [30]. In the single-module case the strengthening has no effect since  $F_s(c^s) = r(d_S^Q, c^s) \leq d_S^Q$ .

Inequalities (14) - (17) generalize all flow-cutset inequalities considered in the literature. We conclude this section with the observation that the lifting functions  $\phi_s^+$ ,  $\phi_s^-$ , and  $\tilde{\phi}_s^+$  obtained by Atamtürk [4] to lift DIRECTED single-module flow-cutset inequalities to the multi-module case are related to the MIR-functions above in the way that  $\phi_s^+(a) = F_s(a)$ ,  $\phi_s^-(a) = a + F_s(-a)$ , and  $\tilde{\phi}_s^+(a) = \mathcal{F}_s(a)$ , see Raack et al. [30]. It turns out that the multi-module flow-cutset inequalities (14) - (17) can be seen as being obtained by simultaneously lifting the flow-cutset inequalities (6) and (8) using the subadditive MIR-functions  $F_s$  and  $\mathcal{F}_s$ . In this context, flow-cutset inequalities are closely related to the concept of complemented-MIR inequalities introduced by Marchand and Wolsey [24] and to the MIR-lifting procedure for flow-cover inequalities presented by Louveaux and Wolsey [20]. The MIR procedure to obtain flow-cutset inequalities is identical to the one presented in [20] for flow-cover inequalities. The only difference is that capacity variables are not complemented since they are not bounded.

## B Omitted proofs

### B.1 Proof of Theorem 4.9

*Proof.* We will show that the related face

$$F =: \left\{ (f, x) \in \mathcal{CS}_S^{bi}(\mathcal{CS}_S^{un}) : (f, x) \text{ satisfies (8) with equality} \right\}$$

is non-trivial and then by contradiction, we will show that it defines a facet. This will be done for  $\mathcal{CS}_S^{bi}$  and  $\mathcal{CS}_S^{un}$  simultaneously. In the following we will construct points on the face  $F$ . Whenever we can ensure that  $E_1 \cap E_2 \neq \emptyset$  these points may only be valid for  $\mathcal{CS}_S^{bi}$  but not for  $\mathcal{CS}_S^{un}$ . Given  $e \in E_S$ , let  $b_e$  denote the unit vector in  $\mathbb{R}^{|E_S|+2|K||E_S|}$  for the integer design variable of  $e$  and let  $g_{e^+}^k, g_{e^-}^k$  be the unit vectors for the two continuous flow variables of  $e$  for commodity  $k \in K$ . We set  $d := d_S^Q$ ,  $\eta := \eta^Q$ ,  $r := r^Q < c$ ,  $\epsilon > 0$  small enough and  $\mathcal{M}$  a large integral number. Let  $\bar{Q} := K_S^+ \setminus Q$ . Choose  $l \in E_1 \setminus E_2$  and  $\bar{l} \in \bar{E}_1 \setminus E_2$ . We construct a point on the face  $F$  by sending all flow for  $Q$  on  $l$  and the flow for all other commodities on  $\bar{l}$ :

$$p := \eta b_l + \mathcal{M} b_{\bar{l}} + \sum_{k \in Q} d_S^k g_{l^+}^k + \sum_{k \in \bar{Q}} d_S^k g_{\bar{l}^+}^k + \sum_{k \in K_S^-} d_S^k g_{\bar{l}^-}^k.$$

The point  $p$  is on the face  $F$  by construction. Hence  $F$  is not empty.  $p + b_l$  is a point that is in  $\mathcal{CS}_S^{bi}(\mathcal{CS}_S^{un})$  but not on the face  $F$ .

It remains to show that  $F$  is inclusion-wise maximal. Choose a facet  $\tilde{F}$  of  $\mathcal{CS}_S^{bi}(\mathcal{CS}_S^{un})$  with  $F \subseteq \tilde{F}$  and let  $\tilde{F}$  be defined by

$$\sum_{e \in E_S} (\beta_e x_e + \sum_{k \in K} \gamma_{e^+}^k f_{e^+}^k + \sum_{k \in K} \gamma_{e^-}^k f_{e^-}^k) = \pi \quad (18)$$

where  $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k, \pi \in \mathbb{R}$ . We will show that (18) is (8) up to a scalar multiple and a linear combination of flow conservation constraints, proving that  $F = \tilde{F}$ .

Adding multiples of the  $|K|$  flow conservation constraints to (18) we can assume  $\gamma_{l^+}^k = 0$  for all  $k \in Q$ ,  $\gamma_{l^+}^k = 0$  for all  $k \in \bar{Q}$  and  $\gamma_{l^-}^k = 0$  for all  $k \in K_S^- \cup K_S^0$ . Set  $\beta := \beta_l$  and  $\bar{\beta} := \beta_{\bar{l}}$ . Since  $p$  lies on the hyperplane, we conclude that  $\beta\eta + \bar{\beta}\mathcal{M} = \pi$ . Now we modify  $p$  by installing a capacity of  $\mathcal{M} + 1$  on  $\bar{l}$ . This is another point on the face and thus  $\bar{\beta} = 0$ . It follows that

$$\beta\eta = \pi \quad (19)$$

The capacity on  $l$  is not saturated since  $d < c\eta$ . Modifying  $p$  by simultaneously increasing flow on  $l^+$  and  $l^-$  by  $\epsilon$  for  $k \in Q$  gives new points on the face and thus

$$\gamma_{l^+}^k, \gamma_{l^-}^k = 0 \quad \forall k \in Q$$

The same can be done on  $\bar{l}^+, \bar{l}^-$  for  $k \in K_S^- \cup K_S^0 \cup \bar{Q}$ , hence

$$\gamma_{l^+}^k, \gamma_{l^-}^k = 0 \quad \forall k \in K_S^- \cup K_S^0 \cup \bar{Q}.$$

Now consider the disjoint partition  $E_S = (E_1 \cap E_2) \cup (E_1 \setminus E_2) \cup (\bar{E}_1 \cap E_2) \cup (\bar{E}_1 \setminus E_2)$ . We will compute the coefficients  $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k$  for  $e$  in each of the four sets by constructing new points on the face  $F$ . They will obviously fulfill the flow conservation constraint and satisfy inequality (8) with equality. To see that they meet the (BIDIRECTED) capacity constraints just use that  $r < c$  and remember the relation  $c\eta = d + c - r$ . For edges in  $\bar{E}_1 \cap E_2$ ,  $E_1 \setminus E_2$  and  $\bar{E}_1 \setminus E_2$  we construct the points such that they additionally satisfy the UNDIRECTED capacity constraints given that  $K_S^- = \emptyset$ . Hence with  $E_1 \cap E_2 = \emptyset$  and  $K_S^- = \emptyset$  the theorem holds for  $\mathcal{CS}_S^{un}$ .

i)  $\bar{E}_1 \cap E_2$ : For  $e \in \bar{E}_1 \cap E_2$  and  $k \in Q$  define the following three points on  $F$ :

$$p + b_e + (c - r)g_{l^+}^k + (c - r)g_{e^-}^k \implies \beta\eta + \beta_e + (c - r)\gamma_{e^-}^k = \pi \quad (20)$$

$$p + b_l + cg_{l^+}^k + b_e + cg_{e^-}^k \implies \beta\eta + \beta + \beta_e + c\gamma_{e^-}^k = \pi \quad (21)$$

$$p + (c - r)g_{l^+}^k + b_e + \frac{r}{2}g_{e^+}^k + (c - \frac{r}{2})g_{e^-}^k \implies \beta\eta + \beta_e + \frac{r}{2}\gamma_{e^+}^k + (c - \frac{r}{2})\gamma_{e^-}^k = \pi \quad (22)$$

Comparison of (20) and (21) shows that  $-r\gamma_{e^-}^k = \beta$  for all  $e \in \bar{E}_1 \cap E_2$ , for all  $k \in Q$ . From (20) it follows then that  $\beta_e = \frac{\beta}{r}(c - r) \forall e \in \bar{E}_1 \cap E_2$ . From (22) we find that  $\frac{\beta}{r}(c - r) - (c - \frac{r}{2})\frac{\beta}{r} + \frac{r}{2}\gamma_{e^+}^k = 0$ , which implies that  $r\gamma_{e^+}^k = \beta$  for all  $e \in \bar{E}_1 \cap E_2$ , for all  $k \in Q$ .

To conclude that  $\gamma_{e^-}^k = 0$  for all  $k \in K_S^- \cup K_S^0 \cup \bar{Q}$  just modify the point in (20) by increasing flow on  $\bar{l}^+$  by some  $\epsilon$  and routing this  $\epsilon$ -flow back on  $e^-$ . Simultaneously increasing flow on  $e^+, e^-$  gives  $\gamma_{e^+}^k = 0$  for all  $k \in K_S^- \cup K_S^0 \cup \bar{Q}$ .

ii)  $E_1 \cap E_2$ : For  $e \in E_1 \cap E_2$  and  $k \in Q$  define:

$$v_e^k := p + b_e + cg_{e^+}^k + cg_{e^-}^k \implies \beta\eta + \beta_e + c\gamma_{e^+}^k + c\gamma_{e^-}^k = \pi \quad (23)$$

We can still increase flow on  $l^+$  by a small amount for commodity  $k$ . Decreasing flow on  $e^+$  at the same time gives another point on the face and thus  $\gamma_{e^+}^k = 0$  for all  $k \in Q$ . When having changed  $v_e^k$  this way, some flow for a commodity in  $K_S^- \cup K_S^0 \cup \bar{Q}$  can be routed on  $e^+$  while the same amount of flow increases on  $\bar{l}^-$ . Hence  $\gamma_{e^+}^k = 0$  for all  $k \in K_S^- \cup K_S^0 \cup \bar{Q}$ .

For  $k_1, k_2 \in Q, e \in E_1 \cap E_2$  consider the point

$$v_e^{k_1} - \epsilon g_{l^+}^{k_1} + \epsilon g_{l^+}^{k_2} - \epsilon g_{e^-}^{k_1} + \epsilon g_{e^-}^{k_2}$$

It is well defined and feasible because flow on  $l^+$  is positive for every  $k \in Q$  and flow on  $e^+$  is positive for  $k_1$ . It follows that  $\gamma_{e^-}^k := \gamma_{e^-}^{k_1} = \gamma_{e^-}^{k_2}$  for all  $k_1, k_2 \in Q$ .

To construct another point  $p_e$  on the face  $F$  we modify  $p$  by deleting one unit of capacity on  $l$  and installing one unit of capacity on  $e \in E_1 \cap E_2$ . A total flow of  $r$  has to be rerouted since  $c(\eta - 1) = d - r$ . We do so by decreasing flow of  $Q$  on  $l^+$  by a total of  $r$  and increasing it on

$e^+$ ,  $e^-$  by a total of  $c$  and  $(c-r)$  respectively. This can be done in such a way that flow is positive on  $e^+$ ,  $e^-$  for all  $k \in Q$ . If  $\varphi_{l^+}^k, \varphi_{e^+}^k, \varphi_{e^-}^k > 0$  denote the rerouted flows on  $l^+, e^+, e^-$ , then  $p_e$  can be written as

$$p_e := p - b_l + b_e - \sum_{k \in Q} \varphi_{l^+}^k g_{l^+}^k + \sum_{k \in Q} \varphi_{e^+}^k g_{e^+}^k + \sum_{k \in Q} \varphi_{e^-}^k g_{e^-}^k$$

with  $\sum_{k \in Q} \varphi_{l^+}^k = r$ ,  $\sum_{k \in Q} \varphi_{e^+}^k = c$  and  $\sum_{k \in Q} \varphi_{e^-}^k = c - r$ . From  $p_e \in F$  and the fact that  $\gamma_{l^+}^k = \gamma_{e^+}^k = 0$  and  $\gamma_{e^-}^k = \gamma_{e^-}$  for all  $k \in Q$  we conclude that

$$\beta\eta - \beta + \beta_e + (c-r)\gamma_{e^-} = \pi \quad (24)$$

Now comparing (23) and (24) gives

$$-r\gamma_{e^-} = -r\gamma_{e^-}^k = \beta \quad \forall k \in Q.$$

From (23) and (19) follows then

$$\beta_e = c \frac{\beta}{r} \quad \forall e \in E_1 \cap E_2.$$

Again considering the point  $p_e$ , the total flow on  $e^-$  is  $c - r$ , thus the capacity on  $e^-$  is not saturated. Increasing flow on  $l^+$  and  $e^-$  gives  $\gamma_{e^-}^k = 0$  for all  $k \in K_S^- \cup K_S^0 \cup Q$ .

iii)  $\bar{E}_1 \setminus E_2$  : For  $e \in \bar{E}_1 \setminus E_2$  consider the following point on  $F$ :

$$p + b_e \implies \beta\eta + \beta_e = \pi \quad (25)$$

The point can be modified by simultaneously increasing flow on  $l^+$  and  $e^-$ . This can be done for every commodity in  $Q$ , thus  $\gamma_{e^-}^k = 0$  for all  $k \in Q$ . Comparing (25) with (19) gives  $\beta_e = 0$  for all  $e \in \bar{E}_1 \setminus E_2$ .

To construct a new point  $q_e$  on the face  $F$  we modify  $p$  by deleting one unit of capacity on  $l$  and installing one unit of capacity on  $e \in \bar{E}_1 \setminus E_2$ . We decrease flow of  $Q$  on  $l^+$  by a total of  $r$  and increase it by the same amount on  $e^+$ . This can be done in such a way that flow is positive on  $e^+$  for all  $k \in Q$ . If  $\varphi^k > 0$  denotes the rerouted flow with respect to  $k \in Q$ , then  $q_e$  can be written as

$$q_e := p - b_l + b_e - \sum_{k \in Q} \varphi^k g_{l^+}^k + \sum_{k \in Q} \varphi^k g_{e^+}^k \implies \beta\eta - \beta + \sum_{k \in Q} \varphi^k \gamma_{e^+}^k = \pi \quad (26)$$

with  $\sum_{k \in Q} \varphi^k = r$ . Modifying  $q_e$  by simultaneously increasing flow on  $e^+$ ,  $\bar{l}^-$  and  $e^+$ ,  $e^-$  gives  $\gamma_{e^+}^k = \gamma_{e^-}^k = 0$  for all  $k \in K_S^- \cup K_S^0 \cup Q$ . It remains to show that  $\gamma_{e^+}^k = \frac{\beta}{r}$  for  $k$  in  $Q$ . We make use of the conditions i) and ii) of Theorem 4.9.

Assume first that  $E_2 = \emptyset$ . If  $|Q| = 1$ , it follows that  $\beta\eta - \beta + r\gamma_{e^+}^k = \pi$  and  $r\gamma_{e^+}^k = \beta$ . If  $|Q| > 1$  and  $\eta \geq 2$ , then  $d > c > r$  and  $q_e$  can be constructed such that flows are positive both on  $l^+$  and  $e^+$  for every commodity in  $Q$ . We choose  $k_1, k_2 \in Q$  and modify  $q_e$  by adding the flow  $\epsilon g_{l^+}^{k_2} - \epsilon g_{l^+}^{k_1} + \epsilon g_{e^+}^{k_1} - \epsilon g_{e^+}^{k_2}$ . This way we conclude that  $\gamma_{e^+}^{k_1} = \gamma_{e^+}^{k_2}$ . From (26) follows then  $r\gamma_{e^+}^k = \beta$  for all  $k \in Q$ . Now let us assume that there is an edge  $\bar{e}$  in  $E_1 \cap E_2$ . Modify  $q_e$  by installing one unit of capacity on  $\bar{e}$  and sending a flow of  $c$  on  $\bar{e}^+$  and  $\bar{e}^-$  for a commodity  $k_1 \in Q$ . Now adding  $\epsilon g_{\bar{e}^+}^{k_2} - \epsilon g_{\bar{e}^+}^{k_1} + \epsilon g_{e^+}^{k_1} - \epsilon g_{e^+}^{k_2}$  gives  $\gamma_{e^+}^{k_1} = \gamma_{e^+}^{k_2}$  and  $r\gamma_{e^+}^k = \beta$  for all  $k \in Q$  again since  $\gamma_{\bar{e}^+}^{k_1} = \gamma_{\bar{e}^+}^{k_2} = 0$ , as shown above. Finally assume that there is  $\bar{e}$  in  $\bar{E}_1 \cap E_2$ . For a commodity  $k \in Q$  consider the following vector:

$$\begin{aligned} p + (c-r)g_{l^+}^k + b_{\bar{e}} + b_e + c g_{\bar{e}^-}^k + r g_{e^+}^k &\implies \beta\eta + \beta_{\bar{e}} + \beta_e + c\gamma_{\bar{e}^-}^k + r\gamma_{e^+}^k = \pi \\ &\implies \beta\eta + (c-r)\frac{\beta}{r} - c\frac{\beta}{r} + r\gamma_{e^+}^k = \pi \\ &\implies \beta = r\gamma_{e^+}^k \quad \forall k \in Q \end{aligned}$$



iv)  $\mathbf{E}_1 \setminus \mathbf{E}_2$  : We construct the vector  $q_e$  again but for  $e \in E_1 \setminus E_2$ .

$$q_e := p - b_l + b_e - \sum_{k \in Q} \varphi^k g_{l^+}^k + \sum_{k \in Q} \varphi^k g_{e^+}^k \implies \beta\eta - \beta + \beta_e + \sum_{k \in Q} \varphi^k \gamma_{e^+}^k = \pi \quad (27)$$

with  $\sum_{k \in Q} \varphi^k = r$ . For  $k \in K$  add an  $\epsilon$ -flow to  $e^+$  and  $e^-$  to conclude that  $\gamma_{e^+}^k = -\gamma_{e^-}^k$  for all  $k \in K$ . Now we modify the point  $q_e$  by simultaneously increasing flow on  $e^+$  and  $l^-$  for a  $k$  in  $K$ . Hence  $\gamma_{e^+}^k = -\gamma_{l^-}^k = 0$ . It follows that  $\gamma_{e^-}^k = 0$  for all  $k \in K$  and  $\beta_e = \beta$  for all  $e \in E_1 \setminus E_2$ .

Plugging in all coefficients in (18) we arrive at:

$$\begin{aligned} \beta x(E_1 \setminus E_2) + \frac{\beta}{r} f^Q(\bar{A}_1^+ \setminus A_2^+) \\ + \frac{\beta}{r} (c - r)x(\bar{E}_1 \cap E_2) + \frac{\beta}{r} f^Q(\bar{A}_1^+ \cap A_2^+) - \frac{\beta}{r} f^Q(\bar{A}_1^- \cap A_2^-) \\ + c \frac{\beta}{r} x(E_1 \cap E_2) - \frac{\beta}{r} f^Q(A_1^- \cap A_2^-) = \beta\eta \end{aligned}$$

which is equivalent to

$$f^Q(\bar{A}_1^+) - f^Q(A_2^-) + rx(E_1) + (c - r)x(E_2) = r\eta$$

We have shown that the hyperplane (18) is a multiple of (8) plus a linear combination of flow conservation constraints. It follows that  $F = \tilde{F}$ . This concludes the proof.  $\square$

## B.2 Proof of Theorem 4.10

*Proof.* We proceed as in the proof of Theorem 4.9 and apply the definitions of the faces  $F, \tilde{F}$  with  $F \subseteq \tilde{F}$  and the vectors  $b_e, g_{e^+}^k, g_{e^-}^k$ . We set  $d := d_S^{K_S^+}, d^- := |d_S^{K_S^-}|, \eta := \eta^{K_S^+}, r := r^{K_S^+} < c, \epsilon > 0$  small enough and  $\mathcal{M}$  a large integral number.

In contrast to the proof of Theorem 4.9 the point  $p$  to start from is defined as follows. Choose  $l \in E_1 \setminus E_2$ . All demand is routed on  $l$  with capacity exactly  $c\eta$ , more precisely all flow for positive commodities is routed on  $l^+$  and all flow for negative commodities is routed on  $l^-$ . Notice that we assume  $d \geq d^-$  and  $r < c$ . The point  $p$  can be written as:

$$p := \eta b_l + \sum_{k \in K_S^+} d_S^k g_{l^+}^k + \sum_{k \in K_S^-} d_S^k g_{l^-}^k.$$

By considering  $p$  and  $p + b_l$  we conclude that  $\emptyset \neq F \neq \mathcal{CS}_S^{bi}$ . It is missing to prove that  $F = \tilde{F}$ . We will show that (18) is (8) up to a scalar multiple and a linear combination of flow conservation constraints. We can assume that  $\gamma_{l^+}^k = 0 \forall k \in K_S^+$  and  $\gamma_{l^-}^k = 0 \forall k \in K_S^- \cup K_S^0$  w.l.o.g. by adding multiples of the flow conservation constraints to (18).

Set  $\beta := \beta_l$ . Since  $p$  lies on the hyperplane, we conclude that

$$\beta\eta = \pi \quad (28)$$

Modifying  $p$  by simultaneously increasing flow on  $l^+$  and  $l^-$  by  $\epsilon$  for every commodity gives new points on the face and thus  $\gamma_{l^+}^k, \gamma_{l^-}^k = 0 \forall k \in K$ .

Now consider the disjoint partition  $E_S = (\bar{E}_1 \cap E_2) \cup (E_1 \cap E_2) \cup (E_1 \setminus E_2)$ . (Note that  $\bar{E}_1 \setminus E_2 = \emptyset$ ). We calculate the coefficients  $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k$  for  $e$  in each of the three sets by constructing new points on the face  $F$ . Note that all the points to be defined for edges in  $\bar{E}_1 \cap E_2$  and  $E_1 \setminus E_2$  additionally satisfy the UNDIRECTED capacity constraints when  $K = K_S^+$ . Hence with  $E_1 \cap E_2 = \emptyset$  and  $K_S^- \cup K_S^0 = \emptyset$  the theorem holds for  $\mathcal{CS}_S^{un}$ .

i)  $\bar{\mathbf{E}}_1 \cap \mathbf{E}_2$  : For  $e \in \bar{E}_1 \cap E_2$  and  $k \in K_S^+$  we define the points (20), (21) and (22) as in the proof of Theorem 4.9 and conclude that

$$-r\gamma_{e^-}^k = \beta, r\gamma_{e^+}^k = \beta \text{ and } \beta_e = \frac{\beta}{r}(c - r) \quad \forall e \in \bar{E}_1 \cap E_2, k \in K_S^+.$$

To see that  $\gamma_{e^+}^k = \gamma_{e^-}^k = 0 \forall k \in K_S^- \cup K_S^0$  modify the point in (20) by first increasing flows on  $l^-, e^+$  and then increasing flows on  $e^+, e^-$ . (This is not possible in the UNDIRECTED model.)

- ii)  $\mathbf{E}_1 \cap \mathbf{E}_2$  : We can assume that  $K_S^0 = \emptyset$ . Let  $e \in E_1 \cap E_2$  and  $k \in K_S^+$ . By defining  $v_e^k$  as in (23) and with the same arguments it can be shown that

$$\gamma_{e^+}^k = 0 \quad \forall k \in K_S^+, K_S^- \quad \text{and} \quad \gamma_{e^-} := \gamma_{e^-}^{k_1} = \gamma_{e^-}^{k_2} \quad \forall k_1, k_2 \in K_S^+.$$

Modifying  $p$  by deleting one unit of capacity on  $l$  and installing one unit of capacity on  $e \in E_1 \cap E_2$  gives a point  $p_e$  on the face  $F$  as in the proof of Theorem 4.9. If  $K_S^- = \emptyset$  we can conclude  $-r\gamma_{e^-}^k = \beta \quad \forall k \in K_S^+$  and  $\beta_e = c\frac{\beta}{r} \quad \forall e \in E_1 \cap E_2$  in a similar way. Else if  $K_S^- \neq \emptyset$  we also have to reroute flow on  $l^-$  and  $e^-$  for commodities in  $K_S^-$ . We can assume  $d > d^-$  and  $d > c$  by assumption. It follows that a rerouting can be done in such a way that flow for  $k \in K_S^-$  is still positive on  $l^-$  and that the capacity for  $e^-$  is not saturated. If in this case  $\varphi_{l^+}^k, \varphi_{l^-}^k, \varphi_{e^+}^k, \varphi_{e^-}^k > 0$  denote the rerouted flows on  $l^+, l^-, e^+, e^-$  for  $k \in K$ , then  $p_e$  can be written as

$$p_e := p - b_l + b_e - \sum_{k \in K_S^+} \varphi_{l^+}^k g_{l^+}^k - \sum_{k \in K_S^-} \varphi_{l^-}^k g_{l^-}^k + \sum_{k \in K_S^+} \varphi_{e^+}^k g_{e^+}^k + \sum_{k \in K_S^+} \varphi_{e^-}^k g_{e^-}^k + \sum_{k \in K_S^-} \varphi_{e^-}^k g_{e^-}^k$$

with

$$\sum_{k \in K_S^+} \varphi_{l^+}^k = r, \quad \sum_{k \in K_S^+} \varphi_{e^+}^k = c, \quad \sum_{k \in K_S^+} \varphi_{e^-}^k = c - r \quad \text{and} \quad \sum_{k \in K_S^-} \varphi_{e^-}^k = \sum_{k \in K_S^-} \varphi_{l^-}^k < r$$

implying

$$\beta\eta - \beta + \beta_e + (c - r)\gamma_{e^-} + \sum_{k \in K_S^-} \varphi_{e^-}^k \gamma_{e^-}^k = \pi$$

We can decrease flow on  $l^-$  and increase it on  $e^-$  which gives  $\gamma_{e^-}^k = 0 \quad \forall k \in K_S^-$  and thus  $-r\gamma_{e^-}^k = \beta \quad \forall k \in K_S^+$  and  $\beta_e = c\frac{\beta}{r} \quad \forall e \in E_1 \cap E_2$  as above.

- iii)  $\mathbf{E}_1 \setminus \mathbf{E}_2$  : For  $e \in E_1 \setminus E_2$  we construct the point  $q_e$  as in the proof Theorem 4.9 but we also have to reroute flow for  $K_S^-$ :

$$q_e := p - b_l + b_e - \sum_{k \in K_S^+} \varphi^k g_{l^+}^k - \sum_{k \in K_S^-} \varphi^k g_{l^-}^k + \sum_{k \in K_S^+} \varphi^k g_{e^+}^k + \sum_{k \in K_S^-} \varphi^k g_{e^-}^k$$

with  $\sum_{k \in K_S^+} \varphi^k = r$  and  $\sum_{k \in K_S^-} \varphi^k \leq r$ . We conclude

$$\beta\eta - \beta + \beta_e + \sum_{k \in K_S^+} \varphi^k \gamma_{e^+}^k + \sum_{k \in K_S^-} \varphi^k \gamma_{e^-}^k = \pi \quad (29)$$

For  $k \in K$  add an  $\epsilon$ -flow to  $e^+$  and  $e^-$  to conclude that  $\gamma_{e^+}^k = -\gamma_{e^-}^k \quad \forall k \in K$ . If for all  $k \in K$  we can either show  $\gamma_{e^+}^k = 0$  or  $\gamma_{e^-}^k = 0$  we conclude  $\beta_e = \beta \quad \forall e \in E_1 \setminus E_2$  by using (29).

By assumption  $E_2 \neq \emptyset$ . First suppose that there is  $\bar{e}$  in  $E_1 \cap E_2$ . Modify  $q_e$  by installing one unit of capacity on  $\bar{e}$  and sending a flow of  $c$  on  $\bar{e}^+$  and  $\bar{e}^-$  for a commodity  $k \in K_S^+$ , which again gives a point on  $F$ . Now decrease flow on  $\bar{e}^+$  and increase it on  $e^+$  by  $\epsilon$ . Hence  $\gamma_{e^+}^k = 0 \quad \forall k \in K_S^+$ . Having done so simultaneously increasing flow on  $\bar{e}^+$  and  $e^-$  gives  $\gamma_{e^-}^k = 0 \quad \forall k \in K_S^- \cup K_S^0$ . Finally suppose that there is  $\bar{e}$  in  $\bar{E}_1 \cap E_2$ . For  $k \in K_S^+$  consider the vector

$$p + (c - r)g_{l^+}^k + b_{\bar{e}} + b_e + cg_{\bar{e}^-}^k + rg_{e^+}^k$$

Simultaneously increasing flow on  $l^-$  and on  $e^+$  for any commodity gives  $\gamma_{e^+}^k = 0 \quad \forall k \in K$ .

Plugging in all coefficients in (18) gives a multiple of (8) as in the proof of Theorem 4.9.  $\square$

### B.3 Proof of Theorem 4.14

*Proof.* We show sufficiency here. If  $E_* = \emptyset$  and  $d_S^{K_S^+} < c$ , then (10) reduces to the cutset inequality  $x(E_S) \geq 1$  which is facet-defining for  $\mathcal{CS}_S^{bi}$  ( $\mathcal{CS}_S^{un}$ ) if  $|E_S| = 1$  (see Theorem 4.8). If  $\bar{E}_* = \emptyset$ , then (10) reduces to  $x(E_S) \geq \eta^{K_S^+}$ , which is facet-defining if  $|E_S| = 1$  or  $d_S^{K_S^+} > c$  (see Theorem 4.8). For the rest of the proof we can assume that  $E_*, \bar{E}_* \neq \emptyset$ . For  $Q = K_S^+$  we define the face

$$F := \left\{ (f, x) \in \mathcal{CS}_S^{bi} (\mathcal{CS}_S^{un}) : (f, x) \text{ satisfies (10) with equality} \right\}$$

Given  $e \in E_S$  let  $b_e$  denote the incidence vector of the design variable of  $e$  and let  $g_{e^+}^k, g_{e^-}^k$  be the unit vectors for the flow variables for commodity  $k \in K$  of  $e$  in both directions. Suppose  $r^{K_S^+} < c$  and set  $d := d_S^{K_S^+}$ ,  $\eta := \eta^{K_S^+}$ ,  $r := r^{K_S^+}$  and  $\epsilon > 0$  small enough. Choose  $l \in E_*$  and  $\bar{l} \in \bar{E}_*$ . We construct a point  $p$  on the face  $F$  by installing  $\eta$  capacity units on the link  $l$  and by using this link to satisfy all demands. The point  $p$  is given by

$$p := \eta b_l + \sum_{k \in K_S^+} d_S^k g_{l^+}^k + \sum_{k \in K_S^-} d_S^k g_{l^-}^k.$$

Use  $c\eta - d = c - r$  to verify that  $p$  is on the face. Considering  $p + b_l$  we conclude  $\emptyset \neq F \neq \mathcal{CS}_S^{bi}$ . It remains to show that  $F$  is inclusion-wise maximal. Choose a facet  $\bar{F}$  of  $\mathcal{CS}_S^{bi}$  with  $F \subseteq \bar{F}$  and let  $\bar{F}$  be defined by (18). We may add multiples of the  $|K|$  flow conservation constraint to (18). Therefore we assume that  $\gamma_{\bar{l}^-}^k = 0$  for all  $k \in K$  w. l. o. g.. Set  $\beta := \beta_l$  and  $\bar{\beta} := \beta_{\bar{l}}$ . The point  $p$  lies on  $F \subseteq \bar{F}$ , hence

$$\beta\eta + \sum_{k \in K_S^+} d_S^k \gamma_{l^+}^k + \sum_{k \in K_S^-} d_S^k \gamma_{l^-}^k = \pi \quad (30)$$

Now we define a point  $p_e$  for all  $e \in E_S$  the following way. We modify  $p$  by deleting one unit of capacity on  $l$  and installing one unit of capacity on  $e \in E_S$ . We decrease flow for  $K_S^+$  on  $l^+$  by a total of  $r$  and increase it by the same amount on  $e^+$ . Some flow for  $K_S^-$  is also rerouted now using  $e^-$ . This can be done in such a way that flow is positive on  $e^+$  for all  $k \in K_S^+$ , that flow is positive on  $e^-$  for all  $k \in K_S^-$ , and that the capacity on  $e$  is not saturated. Note that  $p_e \in F$  for  $e \in E_*$  and also for  $e \in \bar{E}_*$ . If  $\varphi^k > 0$  denotes the rerouted flow for commodity  $k$ , then  $p_e$  can be written as

$$p_e := p - b_l + b_e - \sum_{k \in K_S^+} \varphi^k g_{l^+}^k - \sum_{k \in K_S^-} \varphi^k g_{l^-}^k + \sum_{k \in K_S^+} \varphi^k g_{e^+}^k + \sum_{k \in K_S^-} \varphi^k g_{e^-}^k,$$

with  $\sum_{k \in K_S^+} \varphi^k g_{l^+}^k = \sum_{k \in K_S^+} \varphi^k g_{e^+}^k = r$  and  $\sum_{k \in K_S^-} \varphi^k g_{l^-}^k = \sum_{k \in K_S^-} \varphi^k g_{e^-}^k \leq r$ . From  $p_e \in F \subseteq \bar{F}$  follows

$$\beta\eta - \beta + \beta_e + \sum_{k \in K_S^+} (d_S^k - \varphi^k) \gamma_{l^+}^k + \sum_{k \in K_S^-} (d_S^k - \varphi^k) \gamma_{l^-}^k + \sum_{k \in K_S^+} \varphi^k \gamma_{e^+}^k + \sum_{k \in K_S^-} \varphi^k \gamma_{e^-}^k = \pi. \quad (31)$$

Modifying  $p_e$  by simultaneously increasing flow on  $e^+$  and  $e^-$  by  $\epsilon$  for every commodity gives

$$\gamma_{e^+}^k = -\gamma_{e^-}^k \quad \forall e \in E_S, k \in K.$$

Now consider the disjoint partition  $E_S := E_* \cup \bar{E}_*$ . We calculate the coefficients  $\beta_e, \gamma_{e^+}^k, \gamma_{e^-}^k$  for  $e$  in each of the two sets by constructing new points from  $p$ . All these points are on the face  $F$  for  $\mathcal{CS}_S^{bi}$ . If  $K = K_S^+$ , then all the points additionally satisfy UNDIRECTED capacity constraints. Thus with  $K_S^- \cup K_S^0 = \emptyset$  the theorem holds for  $\mathcal{CS}_S^{un}$ .

i)  $E_*$  : For  $e \in E_*$  and  $k \in K_S^+$  consider the point

$$v_e^k := p_e + b_{\bar{l}} + (c - r)g_{e^+}^k + (c - r)g_{\bar{l}^-}^k.$$

Since  $v_e^k$  as well as  $p_e$  satisfy (18) and because  $\gamma_{\bar{l}^-}^k = 0$  we conclude  $\bar{\beta} + (c - r)\gamma_{e^+}^k = 0$ , and thus

$$\gamma_{e^-}^k = -\gamma_{e^+}^k = \frac{\bar{\beta}}{c - r} \quad \forall e \in E_*, k \in K_S^+. \quad (32)$$

We modify  $v_e^k$  by increasing flow on  $e^-$  and  $\bar{l}^+$  for commodities in  $K_S^- \cup K_S^0$ . (This is not possible in the UNDIRECTED model.)

$\gamma_{e^-}^k = -\gamma_{e^+}^k = 0 \ \forall e \in E_*, k \in K_S^- \cup K_S^0$ . The equations (30) and (31) (with  $e \in E_*$ ) now reduce to

$$\beta\eta - d\frac{\bar{\beta}}{c-r} = \pi \quad \text{and} \quad \beta\eta - \beta - d\frac{\bar{\beta}}{c-r} + \beta_e = \pi$$

which implies  $\beta_e = \beta \ \forall e \in E_*$ .

ii)  $\bar{E}_*$  : For  $e \in \bar{E}_*$  and  $k \in K_S^+$  define the point

$$w_e^k := p + b_e + (c-r)g_{l^+}^k + (c-r)g_{e^-}^k$$

on the face  $F$ . Since  $w_e^k$  satisfies (18) and because of (32), we get

$$\beta\eta + \beta_e - (d+c-r)\frac{\bar{\beta}}{c-r} - (c-r)\gamma_{e^+}^k = \pi \quad (33)$$

For commodities in  $K_S^- \cup K_S^0$  increasing flow on  $l^-$  and  $e^+$  gives

$$\gamma_{e^-}^k = -\gamma_{e^+}^k = \gamma_{l^-}^k = 0 \ \forall e \in \bar{E}_*, k \in K_S^- \cup K_S^0.$$

For a fixed commodity  $k \in K_S^+$  modify  $w_e^k$  by decreasing flow for  $k$  on  $l^+$ ,  $e^-$  and simultaneously increasing flow on  $l^+$ ,  $e^-$  for an arbitrary commodity  $k^* \in K_S^+$ . Hence

$$\gamma_e := \gamma_{e^-}^{k^*} = \gamma_{e^-}^k = -\gamma_{e^+}^{k^*} = -\gamma_{e^+}^k \ \forall e \in \bar{E}_*, k \in K.$$

The equation (31) with  $e \in \bar{E}_*$  now reduces to

$$\beta\eta - \beta - (d-r)\frac{\bar{\beta}}{c-r} + \beta_e - r\gamma_e = \pi \quad (34)$$

Evaluating (34) for  $e = \bar{l}$  and comparing with (30) gives

$$\beta = \frac{c\bar{\beta}}{c-r}$$

since  $\beta_{\bar{l}} = \bar{\beta}$  and  $-\gamma_{\bar{l}^+} = \gamma_{\bar{l}^+}^k = 0$  for all  $k \in K$ . Then from (33) and (34) follows that  $r\gamma_e = (r-c)\gamma_e$ . But  $c > r > 0$  and thus

$$\gamma_e = \gamma_{e^-}^k = -\gamma_{e^+}^k = 0 \ \forall e \in \bar{E}_*, k \in K.$$

Now comparing (33) with (30) results in

$$\beta_e = \bar{\beta} \ \forall e \in \bar{E}_*.$$

Plugging in all coefficients in (18) we arrive at:

$$\frac{c\bar{\beta}}{c-r}x(E_*) + \bar{\beta}x(\bar{E}_*) + \frac{\bar{\beta}}{c-r}f^{K_S^+}(A_*^-) - \frac{\bar{\beta}}{c-r}f^{K_S^+}(A_*^+) = \bar{\beta}$$

which by multiplying with  $\frac{c-r}{\bar{\beta}}$  reduces to (10) (with  $Q = K_S^+$ ). We have shown that the hyperplane (18) is a multiple of (10) plus a linear combination of flow conservation constraints. It follows that  $F = \bar{F}$ . This concludes the proof.  $\square$