

# Weierstraß-Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint

ISSN 0946 – 8633

## Pricing CMS spreads in the Libor market model

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submitted: 12th December 2008

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No. 1386  
Berlin 2008



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2000 *Mathematics Subject Classification.* 60G44, 60G51, 91B28.

*Key words and phrases.* CMS spread option, Margrabe's formula, Libor market model.

Partial support by the Deutsche Forschungsgemeinschaft through the SFB 649 'Economic Risk' and the DFG Research Center MATHEON 'Mathematics for Key Technologies' in Berlin is gratefully acknowledged.

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## Abstract

We present two approximation methods for pricing of CMS spread options in Libor market models. Both approaches are based on approximating the underlying swap rates with lognormal processes under suitable measures. The first method is derived straightforwardly from the Libor market model. The second one uses a convexity adjustment technique under a linear swap model assumption. A numerical study demonstrates that both methods provide satisfactory approximations of spread option prices and can be used for calibration of a Libor market model to the CMS spread option market.

## 1 Introduction

Swap rate dependent derivatives have become increasingly popular in the interest rate market. Typical examples are the CMS spreads and CMS spread options. A constant maturity swap (CMS) spread note is a derivative with a payoff based on the difference of two swap rates of specific maturities. For example, a CMS spread note might pay quarterly coupons based on the difference between quarterly fixings of the 10-year and 5-year semi-annual swap rates. The coupons of such a structure depend on the slope of the yield curve; the Note would therefore be traded by parties who wish to take a view on future relative changes in different parts of the yield curve. The steeper the yield curve, the greater the coupon - giving rise to the term “steepener” for certain CMS spread instruments. Since the beginning of 2006, the yield curve in both the US and Europe has flattened dramatically and has thus led to a rapidly shrinking CMS spread. This has caused the classic steepener to underperform and investors who had positions in CMS spread instruments experienced large losses due to a significantly lower coupon. For these reasons, such products have become less attractive, and so their level of liquidity is thus also decreasing. However, mark-to-market valuations still need to be performed for the many outstanding products on investors’ books.

The price of CMS spread is usually computed under the forward measure (induced by next-after-maturity zero coupon bond). It is convenient to price CMS spread options in a suitable forward probability measure. If a lognormal approximation of the CMS rate and the correlation between swap rates is available under this measure, the price of the CMS spread option can be approximately computed via the well-known Margrabe’s formula. In this paper we present two lognormal approximations for CMS rates in a forward measure. The first one is derived straightforwardly from the Libor market model, where the Libor drift is approximated with it’s initial

value ('freezing'), or in a refined way with a suitable normal approximation. The second approach is based on a convexity adjustment technique (see, e.g., [2]). It is important to note that in contrast to the convexity adjustment approach, the lognormal approximation provides a natural way to approximate the correlation between logarithms of swap rates in the CMS spread pricing formula.

The structure of the paper is as follows. First, we recall some definitions of Libor market models and CMS spread options. Then we derive approximation formulas for swap dynamics under the terminal measure. Next the approach based on the CMS convexity adjustment will be presented. Finally we compare both methods numerically and study the calibration of the Libor model to market CMS spread prices.

## 2 Main setup

Consider a fixed sequence of tenor dates  $0 =: T_0 < T_1 < \dots < T_n$ , together with a sequence of so called day-count fractions  $\delta_i := T_{i+1} - T_i$ ,  $i = 1, \dots, n-1$ . With respect to this tenor structure we consider zero bond processes  $B_i$ ,  $i = 1, \dots, n$ , where each  $B_i$  lives on the interval  $[0, T_i]$  and ends up with its face value  $B_i(T_i) = 1$ . With respect to this bond system we deduce a system of forward rates, called Libor rates, which are defined by

$$L_i(t) := \frac{1}{\delta_i} \left( \frac{B_i(t)}{B_{i+1}(t)} - 1 \right), \quad 0 \leq t \leq T_i, \quad 1 \leq i \leq n-1.$$

Note that  $L_i$  is the annualized effective forward rate to be contracted at the date  $t$ , for a loan over a forward period  $[T_i, T_{i+1}]$ . Based on this rate one has to pay at  $T_{i+1}$  an interest amount of  $\$ \delta_i L_i(T_i)$  on a \$1 notional.

For a pre-specified volatility process  $\gamma_i \in \mathbb{R}^m$ , adapted to the filtration generated by some standard Brownian motion  $W \in \mathbb{R}^m$ , the dynamics of the corresponding Libor model have the form,

$$\frac{dL_i}{L_i} = [\dots]dt + \gamma_i^\top dW \tag{1}$$

$i = 1, \dots, n-1$ . The drift term, adumbrated by the dots, is known under different numeraire measures, such as the risk-neutral, spot, terminal and all measures induced by individual bonds taken as numeraire. If the processes  $t \rightarrow \gamma_i(t)$  in (1) are deterministic, one speaks of a Libor *market* model (for details see [1], [4] for example).

An interest rate swap is a contract to exchange a series of floating interest payments in return for a series of fixed rate payments. Consider a series of payment dates between  $T_{p+1}$  and  $T_q$ ,  $q > p$ . The fixed leg of the swap pays  $\delta_j K$  at each time  $T_{j+1}$ ,  $j = p, \dots, q-1$  where  $\delta_j = T_{j+1} - T_j$ . In return, the floating leg pays  $\delta_j L_j(T_j)$  at time  $T_{j+1}$ , where  $L_j(T_j)$  is the rate fixed at time  $T_j$  for payment at  $T_{j+1}$ . Thus,

the time  $t$  value of the interest rate swap is

$$\sum_{j=p}^{q-1} \delta_j B_{j+1}(t) (L_j(t) - K).$$

The swap rate  $S_{p,q}(t)$  is the value of the fixed rate  $K$ , such that the present value of the contract is zero, hence after some rearranging

$$S_{p,q}(t) = \frac{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t) L_j(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)} = \frac{B_p(t) - B_q(t)}{\sum_{j=p}^{q-1} \delta_j B_{j+1}(t)}. \quad (2)$$

So  $S_{p,q}$  is a martingale under the probability measure  $P_{p,q}$ , induced by the annuity numeraire  $B_{p,q} = \sum_{j=p}^{q-1} \delta_j B_{j+1}(t)$ . A *constant maturity swap (CMS)* is a contract to exchange floating swap rate in return to a fixed rate (or Libor) at a settlement dates. Pricing of *CMS spread options* essentially comes down to the valuation of a pay-off of the form

$$(S_{p,q'}(T_p) - S_{p,q}(T_p) - K)^+ \quad (3)$$

at a future date  $T_r$ ,  $T_r \geq T_p$ , where typically  $q' > q > p$ .

Assume that a close lognormal approximations for  $S_{p,q}$  and  $S_{p,q'}$  under the probability measure  $P_r$  (induced by the zero bond  $B_r$ ) are available,

$$\begin{aligned} S_{p,q}(T_p) &\approx S_{p,q}^*(0) \exp\left(-\frac{1}{2}(\sigma_{p,q}^*)^2 T_p + \sigma_{p,q}^* W_{T_p}^r\right), \\ S_{p,q'}(T_p) &\approx S_{p,q'}^*(0) \exp\left(-\frac{1}{2}(\sigma_{p,q'}^*)^2 T_p + \sigma_{p,q'}^* W_{T_p}^{r'}\right), \end{aligned} \quad (4)$$

where  $W^r$  and  $W^{r'}$  are standard Brownian motions under  $P_r$  with

$$\text{Cor}(W^r, W^{r'}) =: \rho.$$

Then, the value of CMS spread option with zero strike can be approximated via the well-known explicit Margrabe's formula,

$$E_r[S_{p,q'}(T_p) - S_{p,q}(T_p)]^+ \approx S_{p,q'}^*(0) \mathcal{N}(b_1) - S_{p,q}^*(0) \mathcal{N}(b_2) \quad (5)$$

with

$$\begin{aligned} b_1 &:= \frac{\ln \frac{S_{p,q'}^*(0)}{S_{p,q}^*(0)} + \frac{1}{2} \left[ (\sigma_{p,q}^*)^2 + (\sigma_{p,q'}^*)^2 - 2\sigma_{p,q}^* \sigma_{p,q'}^* \rho \right] T_p}{\sqrt{\left[ (\sigma_{p,q}^*)^2 + (\sigma_{p,q'}^*)^2 - 2\sigma_{p,q}^* \sigma_{p,q'}^* \rho \right] T_p}}, \\ b_2 &:= \frac{\ln \frac{S_{p,q'}^*(0)}{S_{p,q}^*(0)} - \frac{1}{2} \left[ (\sigma_{p,q}^*)^2 + (\sigma_{p,q'}^*)^2 - 2\sigma_{p,q}^* \sigma_{p,q'}^* \rho \right] T_p}{\sqrt{\left[ (\sigma_{p,q}^*)^2 + (\sigma_{p,q'}^*)^2 - 2\sigma_{p,q}^* \sigma_{p,q'}^* \rho \right] T_p}}. \end{aligned}$$

For non-zero strikes, the Margrabe's formula involves one-dimensional integration

$$E_r[\varepsilon(S_{p,q'}(T_p) - S_{p,q}(T_p) - K)]^+ \approx \int_{\mathbb{R}} \varepsilon[X_2 \mathcal{N}(\varepsilon d_1) - (X_1 + K) \mathcal{N}(\varepsilon d_2)] \varphi(x) dx \quad (6)$$

with  $\varepsilon = 1$  for a caplet,  $\varepsilon = -1$  for a floret,

$$\begin{aligned} X_1 &:= S_{p,q}^*(0) \exp \left[ \sigma_{p,q}^* x \sqrt{T_p} - \frac{1}{2} (\sigma_{p,q}^*)^2 T_p \right], \\ X_2 &:= S_{p,q'}^*(0) \exp \left[ \sigma_{p,q'}^* x \rho \sqrt{T_p} - \frac{1}{2} (\sigma_{p,q'}^*)^2 \rho^2 T_p \right], \\ d_{1,2} &:= \frac{\ln \frac{X_2}{X_1 + K} \pm \frac{1}{2} (\sigma_{p,q'}^*)^2 T_p (1 - \rho^2)}{\sqrt{(\sigma_{p,q'}^*)^2 T_p (1 - \rho^2)}}, \end{aligned}$$

and  $\varphi$  is the density of the standard normal distribution.

Thus, to price CMS spread via Margrabe's formula, we need (i) to find a close lognormal approximation for swap rates  $S_{p,q}$  and  $S_{p,q'}$  under the probability measure  $P_r$ , and (ii) to estimate the correlation between  $\ln S_{p,q}$  and  $\ln S_{p,q'}$ .

### 3 Lognormal swap rate approximations in the LMM

In general the dynamics of swap rates  $S_{p,q}$  under the swap measure  $P_{p,q}$  is given by

$$dS_{p,q} = S_{p,q} \sigma_{p,q}^\top dW^{p,q},$$

for some adaptive process  $\sigma_{p,q}$ . Our goal is to obtain the dynamics of  $S_{p,q}$  under some measure  $P_r$ ,  $p \leq r < q$ . With the Radon-Nykodym martingale defined by

$$M_t := E_{p,q}^{\mathcal{F}_t} \frac{dP_r}{dP_{p,q}}(T_r) = \frac{B_{p,q}(0)}{B_r(0)} \frac{B_r(t)}{B_{p,q}(t)}, \quad t \leq T_r,$$

we have by Girsanov's theorem that

$$S_{p,q} - \langle S_{p,q}, \ln M \rangle = S_{p,q} - \langle S_{p,q}, \ln \frac{B_r}{B_{p,q}} \rangle \quad (7)$$

is a  $P_r$ -martingale. Note that

$$\frac{B_r}{B_{p,q}} = \frac{B_r}{B_q} \frac{1}{\sum_{j=p}^{q-1} \delta_j B_{j+1}/B_q} = \frac{\prod_{k=r}^{q-1} (1 + \delta_k L_k)}{\sum_{j=p}^{q-1} \delta_j \prod_{k=j+1}^{q-1} (1 + \delta_k L_k)}$$

hence modulo finite variation  $dt$ -terms,

$$d \ln \frac{B_r}{B_{p,q}} = \sum_{j=r}^{q-1} \frac{\delta_j dL_j}{1 + \delta_j L_j} - \frac{\sum_{j=p}^{q-1} \delta_j d \prod_{k=j+1}^{q-1} (1 + \delta_k L_k)}{\sum_{j=p}^{q-1} \delta_j \prod_{k=j+1}^{q-1} (1 + \delta_k L_k)}.$$

After some transformations, we get (modulo  $dt$ -terms)

$$d \ln \frac{B_r}{B_{p,q}} = \sum_{k=p}^n \varsigma_k^{p,q,r} \frac{\delta_k dL_k}{1 + \delta_k L_k}$$

with

$$\varsigma_k^{p,q,r} := 1_{r=k=p} - 1_{r>p, p+1 \leq k < r} + \frac{B_{k,q}}{B_{p,q}} 1_{p < k < q}. \quad (8)$$

As the process (7) is  $P_r$ -martingale, we have (with  $\varsigma_k \equiv \varsigma_k^{p,q,r}$  for short)

$$\frac{dS_{p,q}}{S_{p,q}} = \sum_{k=p}^{q-1} \frac{\delta_k L_k}{1 + \delta_k L_k} \varsigma_k \sigma_{p,q}^\top \gamma_k dt + \sigma_{p,q}^\top dW^r, \quad (9)$$

or

$$\ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} = -\frac{1}{2} \int_0^{T_p} |\sigma_{p,q}|^2 dt + \sum_{k=p}^{q-1} \int_0^{T_p} \frac{\delta_k L_k}{1 + \delta_k L_k} \varsigma_k \sigma_{p,q}^\top \gamma_k dt + \int_0^{T_p} \sigma_{p,q}^\top dW^r. \quad (10)$$

Using formula (1.29) in [4] we may express  $\sigma_{p,q}$  via Libor volatility structure,

$$\sigma_{p,q} = \sum_{l=p}^{q-1} \frac{\delta_l L_l}{1 + \delta_l L_l} \left( \sum_{r=l}^{q-1} w_r^{p,q} + \frac{B_q}{B_p - B_q} \right) \gamma_l =: \frac{1}{S_{p,q}} \sum_{l=p}^{q-1} v_l^{p,q} L_l \gamma_l, \quad (11)$$

with

$$v_l^{p,q} := \frac{\delta_l B_{l,q} S_{p,q} - S_{l,q}}{B_{p,q} (1 + \delta_l L_l)} + w_l^{p,q}, \quad w_l^{p,q} := \frac{\delta_l B_l}{B_{p,q}}.$$

Note that all dynamics are 'exact' so far. Now we use a normal approximation from [3] of the form

$$\frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \doteq \delta_k L_k(0) \left( 1 + \int_0^t \gamma_k^\top dW^r \right), \quad (12)$$

We then obtain from (10),

$$\begin{aligned} \ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} &= -\frac{1}{2} \int_0^{T_p} |\sigma_{p,q}|^2 dt \\ &+ \sum_{k=p}^{q-1} \delta_k L_k(0) \int_0^{T_p} \left( 1 + \int_0^t \gamma_k^\top dW^{p+1} \right) \varsigma_k \sigma_{p,q}^\top \gamma_k dt + \int_0^{T_p} \sigma_{p,q}^\top dW^r \\ &= \int_0^{T_p} \left( -\frac{1}{2} |\sigma_{p,q}|^2 + \sum_{k=p}^{q-1} \delta_k L_k(0) \varsigma_k \sigma_{p,q}^\top \gamma_k \right) dt \\ &+ \int_0^{T_p} \left( \sum_{k=p}^{q-1} \delta_k L_k(0) \int_s^{T_p} \varsigma_k \sigma_{p,q}^\top \gamma_k du \gamma_k^\top + \sigma_{p,q}^\top \right) dW^r. \end{aligned} \quad (13)$$

In order to obtain a lognormal approximation for (13) we use the deterministic approximations

$$\begin{aligned}\sigma_{p,q}(s) &= \sum_{l=p}^{q-1} \left( \frac{v_l^{p,q} L_l}{S_{p,q}} \gamma_l \right) (s) \approx \sum_{l=p}^{q-1} \left( \frac{v_l^{p,q} L_l}{S_{p,q}} \right) (0) \gamma_l(s) =: \widehat{\sigma}_{p,q}, \\ \varsigma_k(s) &\approx \varsigma_k(0),\end{aligned}\quad (14)$$

by freezing weight related coefficients at  $t = 0$ . This yields the lognormal approximation

$$\begin{aligned}\ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} &\approx \int_0^{T_p} \left( -\frac{1}{2} |\widehat{\sigma}_{p,q}|^2 + \sum_{k=p}^{q-1} \delta_k L_k(0) \varsigma_k(0) \widehat{\sigma}_{p,q}^\top \gamma_k \right) dt \\ &+ \int_0^{T_p} \left( \sum_{k=p}^{q-1} \delta_k L_k(0) \varsigma_k(0) \int_s^{T_p} \widehat{\sigma}_{p,q}^\top \gamma_k du \gamma_k^\top + \widehat{\sigma}_{p,q}^\top \right) dW^r \\ &=: A_{p,q} + Z_{p,q}\end{aligned}\quad (15)$$

with  $Z_{p,q}$  being a zero-mean Gaussian random variable.

For pricing of spread options we assume (15) for  $q$  and  $q'$ , with  $q' > q > p$ . We then have

$$E_r Z_{p,q} Z_{p,q'} = (I) + (II) + (III),$$

where

$$\begin{aligned}E_r Z_{p,q} Z_{p,q'} &= \int_0^{T_p} \left( \sum_{k=p}^{q-1} \delta_k L_k(0) \varsigma_k(0) \int_s^{T_p} \widehat{\sigma}_{p,q}^\top \gamma_k du \gamma_k^\top + \widehat{\sigma}_{p,q}^\top \right) \times \\ &\left( \sum_{k'=p}^{q'-1} \delta_{k'} L_{k'}(0) \varsigma_{k'}(0) \int_s^{T_p} \widehat{\sigma}_{p,q'}^\top \gamma_{k'} du \gamma_{k'}^\top + \widehat{\sigma}_{p,q'}^\top \right) dt = (I) + (II) + (III)\end{aligned}$$

$$\begin{aligned}(I) &= \sum_{k=p}^{q-1} \sum_{k'=p}^{q'-1} \varsigma_k^{p,q}(0) \varsigma_{k'}(0) \delta_k \delta_{k'} L_k(0) L_{k'}(0) \times \\ &\int_0^{T_p} \gamma_k^\top \gamma_{k'} ds \int_s^{T_p} (\widehat{\sigma}_{p,q}^\top \gamma_k) dt \int_s^{T_p} (\widehat{\sigma}_{p,q'}^\top \gamma_{k'}) dt' \\ &= \frac{1}{S_{p,q} S_{p,q'}} \sum_{k=p}^{q-1} \sum_{k'=p}^{q'-1} \sum_{l=p}^{q-1} \sum_{l'=p}^{q'-1} \varsigma_k \varsigma_{k'} v_l^{p,q} v_{l'}^{p,q'} \delta_k \delta_{k'} L_k L_{k'} L_l L_{l'} \times \\ &\int_0^{T_p} \gamma_k^\top \gamma_{k'} ds \int_s^{T_p} \gamma_l^\top \gamma_k dt \int_s^{T_p} \gamma_{l'}^\top \gamma_{k'} dt',\end{aligned}$$

$$\begin{aligned}
(II) &= \sum_{k=p}^{q-1} \varsigma_k(0) \delta_k L_k(0) \int_0^{T_p} \sigma_{p,q}^\top \gamma_k ds \int_s^{T_p} (\sigma_{p,q}^\top \gamma_k) dt \\
&+ \sum_{k'=p+1}^{q'-1} \varsigma'_{k'}(0) \delta_{k'} L_{k'}(0) \int_0^{T_p} \sigma_{p,q}^\top \gamma_{k'} ds \int_s^{T_p} (\sigma_{p,q}^\top \gamma_{k'}) dt' \\
&= \frac{1}{S_{p,q} S_{p,q'}} \sum_{k=p}^{q-1} \sum_{l=p}^{q-1} \sum_{l'=p}^{q'-1} \varsigma_k v_l^{p,q} v_{l'}^{p,q'} \delta_k L_k L_l L_{l'} \int_0^{T_p} \gamma_{l'}^\top \gamma_k ds \int_s^{T_p} \gamma_l^\top \gamma_k dt \\
&+ \frac{1}{S_{p,q} S_{p,q'}} \sum_{k'=p}^{q'-1} \sum_{l=p}^{q-1} \sum_{l'=p}^{q'-1} \varsigma'_{k'} v_l^{p,q} v_{l'}^{p,q'} \delta_{k'} L_{k'} L_l L_{l'} \int_0^{T_p} \gamma_{l'}^\top \gamma_{k'} ds \int_s^{T_p} \gamma_{l'}^\top \gamma_{k'} dt', \\
(III) &\approx \frac{1}{S_{p,q} S_{p,q'}} \sum_{k=p}^{q-1} \sum_{k'=p}^{q'-1} v_k^{p,q} v_{k'}^{p,q'} L_k L_{k'} \int_0^{T_p} \gamma_k^\top \gamma_{k'} dt. \tag{16}
\end{aligned}$$

To apply the Margrabe's formula (5), or (6), straightforward, it is convenient to rewrite (15) in the form (4) with

$$\sigma_{p,q}^* := \sqrt{\frac{E_r Z_{p,q}^2}{T_p}}, \quad S_{p,q}^*(0) := S_{p,q}(0) \exp\left(A_{p,q} + \frac{\sigma_{p,q}^*}{2} T_p\right),$$

and

$$\rho := \frac{E_r Z_{p,q} Z_{p,q'}}{\sqrt{E_r Z_{p,q}^2} \sqrt{E_r Z_{p,q'}^2}} \tag{17}$$

Alternatively, we can approximate (10) with a lognormal process by simple freezing LL-ibor rates at  $t = 0$ ,

$$\frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} \doteq \delta_k L_k(0),$$

instead of using approximation (12). Analogously to (15), we than obtain the log-normal approximation

$$\ln \frac{S_{p,q}(T_p)}{S_{p,q}(0)} \approx A_{p,q} + Z_{p,q}^0, \tag{18}$$

where  $Z_{p,q}^0$  is a zero-mean Gaussian random variable and for  $q'$ , with  $q' > q > p$ , we have

$$E_r Z_{p,q}^0 Z_{p,q'}^0 = (III),$$

where (III) is given by (16). Then, we price CMS spread option via Margrabe's formula (5), or (6), with

$$\sigma_{p,q}^* := \sqrt{\frac{E_r (Z_{p,q}^0)^2}{T_p}}, \quad S_{p,q}^*(0) := S_{p,q}(0) \exp\left(A_{p,q} + \frac{\sigma_{p,q}^*}{2} T_p\right),$$

and

$$\rho^0 := \frac{E_r Z_{p,q}^0 Z_{p,q'}^0}{\sqrt{E_r (Z_{p,q}^0)^2} \sqrt{E_r (Z_{p,q'}^0)^2}}. \quad (19)$$

In Section 5 we compare the two approximations (15) and (18).

## 4 Convexity adjustment approach

Now we show an alternative way to construct a lognormal approximation of a swap rate process (2) under measure  $P_r$ ,  $p \leq r < q$ . Let us consider the valuation of a cash payment  $S_{p,q}(T_p)$  (in \$ for instance) at date  $T_r$  seen at time  $T_0 = 0$ . We thus consider

$$E_r S_{p,q}(T_p) = \frac{B_{p,q}(0)}{B_r(0)} E_{p,q} S_{p,q}(T_p) \frac{B_r(T_p)}{B_{p,q}(T_p)}. \quad (20)$$

For evaluating (20) we here recall a convexity approach in [2] based on the assumption of a *linear swap market model*: For fixed  $p$  and  $q$  it is assumed that

$$\frac{B_r(T_p)}{B_{p,q}(T_p)} = \alpha + \beta_r S_{p,q}(T_p), \quad p \leq r, \quad (21)$$

where  $\alpha$  and  $\beta_r$  are constants which are determined as follows. By taking the expectation of (21) under the annuity measure we get

$$\frac{B_r(0)}{B_{p,q}(0)} = \alpha + \beta_r S_{p,q}(0),$$

hence

$$\beta_r = \frac{1}{S_{p,q}(0)} \left( \frac{B_r(0)}{B_{p,q}(0)} - \alpha \right)$$

It holds,

$$\begin{aligned} 1 &= \sum_{r=p}^{q-1} \frac{\delta_r B_{r+1}(T_p)}{B_{p,q}(T_p)} = \sum_{r=p}^{q-1} \delta_r (\alpha + \beta_{r+1} S_{p,q}(T_p)) \\ &= \alpha \left( 1 - \frac{S_{p,q}(T_p)}{S_{p,q}(0)} \right) \sum_{r=p}^{q-1} \delta_r + \frac{S_{p,q}(T_p)}{S_{p,q}(0)}, \end{aligned}$$

so we obtain

$$\alpha = \frac{1}{\sum_{r=p}^{q-1} \delta_r}, \quad \beta_r = \frac{\frac{B_r(0)}{B_{p,q}(0)} - \frac{1}{\sum_{r=p}^{q-1} \delta_r}}{S_{p,q}(0)} = \frac{B_r(0) - \frac{B_{p,q}(0)}{\sum_{r=p}^{q-1} \delta_r}}{B_p(0) - B_q(0)}.$$

Under assumption (21), we now derive from (20) straightforwardly,

$$E_r S_{p,q}(T_p) = S_{p,q}(0) \left( 1 + \frac{\beta_r \text{Var}_{p,q} [S_{p,q}(T_p)]}{S_{p,q}(0) (\alpha + \beta_r S_{p,q}(0))} \right) \quad (22)$$

(see [2]).

In a similar way we may derive for the second moment,

$$\begin{aligned} E_r S_{p,q}^2(T_p) &= \frac{B_{p,q}(0)}{B_r(0)} E_{p,q} \left( S_{p,q}^2 \frac{B_r}{B_{p,q}} \right) (T_p) \\ &= \frac{1}{\alpha + \beta_{p+1} S_{p,q}(0)} (\alpha E_{p,q} S_{p,q}^2(T_p) + \beta_{p+1} E_{p,q} S_{p,q}^3(T_p)). \end{aligned} \quad (23)$$

Under the market model for swaptions,  $S_{p,q}$  is a log-normal martingale under the annuity measure  $P_{p,q}$  induced by the annuity numeraire  $B_{p,q}$ . We so have

$$S_{p,q}(T_p) = S_{p,q}(0) \exp(\sigma_{p,q}^B W_{T_p}^{p,q} - \frac{1}{2} (\sigma_{p,q}^B)^2 T_p) \quad (24)$$

where  $\sigma_{p,q}^B$  is the so called Black volatility, and thus

$$E_{p,q} S_{p,q}^2(T_p) = S_{p,q}^2(0) \exp \left[ (\sigma_{p,q}^B)^2 T_p \right].$$

Hence, under the lognormal swap rate assumption (24), (22) yields

$$E_r S_{p,q}(T_p) = S_{p,q}(0) \left( 1 + \frac{\beta_r S_{p,q}(0) \left( \exp \left[ (\sigma_{p,q}^B)^2 T_p \right] - 1 \right)}{(\alpha + \beta_r S_{p,q}(0))} \right). \quad (25)$$

and further, since  $E_{p,q} S_{p,q}^3(T_p) = S_{p,q}^3(0) \exp \left[ 3 (\sigma_{p,q}^B)^2 T_p \right]$ , (23) yields

$$\begin{aligned} E_r S_{p,q}^2(T_p) &= \frac{1}{\alpha + \beta_r S_{p,q}(0)} S_{p,q}^2(0) \left( \alpha \exp \left[ (\sigma_{p,q}^B)^2 T_p \right] \right. \\ &\quad \left. + \beta_r S_{p,q}(0) \exp \left[ 3 (\sigma_{p,q}^B)^2 T_p \right] \right). \end{aligned} \quad (26)$$

If we now also assume (though inconsistently) that  $S_{p,q}(T_p)$  is (approximately) log-normal distributed under  $P_r$ , i.e.,

$$S_{p,q}(T_p) = S_{p,q}^*(0) \exp \left[ -\frac{1}{2} (\sigma_{p,q}^*)^2 T_p + \sigma_{p,q}^* W_{T_p}^r \right], \quad (27)$$

for some  $S_{p,q}^*(0) > 0$  and  $\sigma_{p,q}^* > 0$ , we obtain by matching the first and second moment of (27) to (25) and (26) respectively,

$$S_{p,q}^*(0) = S_{p,q}(0) \left( 1 + \frac{\beta_r S_{p,q}(0) \left( \exp \left[ (\sigma_{p,q}^B)^2 T_p \right] - 1 \right)}{\alpha + \beta_r S_{p,q}(0)} \right), \quad (28)$$

$$(\sigma_{p,q}^*)^2 = (\sigma_{p,q}^B)^2 + \frac{1}{T_p} \ln \left[ \frac{\alpha + \beta_r S_{p,q}(0) \left( \alpha + \beta_r S_{p,q}(0) \exp \left[ 2 (\sigma_{p,q}^B)^2 T_p \right] \right)}{\left( \alpha + \beta_r S_{p,q}(0) \exp \left[ (\sigma_{p,q}^B)^2 T_p \right] \right)^2} \right].$$

Here, the implied Black's volatility can be expressed via Libor volatility structure (see, e.g., [4]),

$$\sigma_{p,q}^B = \frac{1}{T_p} \int_0^{T_p} \sigma_{p,q}^2 dt \approx \frac{1}{T_p} \int_0^{T_p} \widehat{\sigma}_{p,q}^2 dt = \frac{1}{T_p} (III),$$

with integral (III) given by (16).

Finally we note that the convexity adjustment technique does not give the correlation between  $\ln S_{p,q}$  and  $\ln S_{p,q'}$ . To price CMS spread via Margrabe's formula, we assume that the correlation is given by (17), or (19).

## 5 Numerical comparison of methods

Now we illustrate the algorithms by pricing the spread of 10-year and 2-year CMS for different maturities  $T_p$  under probability measure  $P_{p+1}$ . Further we show a calibration of the Libor model to market CMS spread prices.

In the market Libor model the dynamics of the forward Libors  $L_i(t)$  defined in the interval  $[0, T_p]$  for  $p \leq i < n$ , in the measure  $P_{p+1}$  are governed by the following system of SDE's,

$$dL_i = \sum_{j=p+1}^i \frac{\delta_j L_i L_j \gamma_i^\top \gamma_j}{1 + \delta_j L_j} dt + L_i \gamma_i^\top dW^{(p+1)}, \quad (29)$$

where  $t \rightarrow \gamma_i(t) = (\gamma_{i,1}(t), \dots, \gamma_{i,n-1}(t))$ ,  $0 \leq t \leq T_i$ , are deterministic volatility vector functions. For experiments we take  $W^{(p+1)}$  to be a standard  $n-1$ -dimensional Wiener process under  $P_{p+1}$  in (29), and use the volatility structure

$$\gamma_i(t) = cg(T_i - t)e_i, \quad \text{where } g(s) = g_\infty + (1 - g_\infty + as)e^{-bs},$$

where  $e_i$  are  $n$ -dimensional unit vectors, decomposing an input correlation matrix

$$\rho_{ij} = \exp \left[ -\frac{|j-i|}{m-1} \left( -\ln \rho_\infty + \eta \frac{i^2 + j^2 + ij - 3mi - 3mj + 3i + 3j + 2m^2 - m - 4}{(m-2)(m-3)} \right) \right],$$

$$1 \leq i, j \leq m, \quad \eta > 0, \quad 0 < \eta < -\ln \rho_\infty.$$

Further we take  $n = 30$ ,  $\delta_i \equiv 1\text{yr}$ , and the parameter values in Table 1. The parameters are chosen to be more or less typical with Libor volatilities ranging between 15% and 25%.

**Table 1.**

$a$	$b$	$g_\infty$	$\eta$	$\rho_\infty$	$c$
1.190	1.550	0.587	0.086	0.449	0.264

In Tables 2-4, we compare the values of the CMS spreads

$$E_{p+1} [\varepsilon(S_{p,p+10} - S_{p,p+2} - \text{strike})]^+, \quad \varepsilon = 1 \text{ for caplet, } \quad \varepsilon = -1 \text{ for floret,}$$

by the convexity adjustment approach (CA-method) and the lognormal swap rate approximation (LN-method) with (“true”) Monte Carlo simulated prices (MC), using small enough time steps. We denote by  $\text{LN}^0$  and  $\text{CA}^0$  the prices via the approximations (18) and (27)-(28), respectively, with the correlation given by (19). The prices via refined approximations (15) and (27)-(28) with the correlation (17) are denoted by LN and CA, respectively. From the tables we see that overall both CA- and LN-methods give a reasonable approximations of the spread option prices. The pricing error is at most 3 base points for 20yr maturity. Remarkably, for the floor examples the (LN) method performs slightly better, whereas for the caps the (CA) approach gives somewhat better results. The reason might be that the biases of both methods are in opposite directions. Note that the refined approximations CA and LN can give a significant improvement only for large maturity ( $T_p > 10$ ), and require higher computational time than approximations  $\text{CA}^0$  and  $\text{LN}^0$ . Therefore, the more crude approximations  $\text{CA}^0$  and  $\text{LN}^0$  are preferable for calibration procedure.

**Table 2.** CMS spread caplet, strike 0.5%.

Maturity	CA	LN	MC(SD)
1	25.52	25.62	25.46(0.02)
2	28.00	28.20	27.90(0.03)
3	28.43	28.78	28.43(0.05)
4	27.27	27.75	27.35(0.05)
5	25.50	26.11	25.70(0.08)
6	23.46	24.20	23.73(0.08)
7	23.16	24.01	23.50(0.08)
8	23.47	24.49	23.69(0.14)
9	23.86	25.03	24.06(0.15)
10	24.58	25.91	24.79(0.16)
15	27.21	29.29	27.43(0.20)
20	32.45	35.23	32.50(0.25)

**Table 3.** CMS spread floret, strike 0.5%.

Maturity	CA <sup>0</sup>	CA	LN <sup>0</sup>	LN	MC(SD)
1	6.49	6.49	6.45	6.42	6.22(0.04)
2	10.82	10.84	10.74	10.69	10.36(0.05)
3	15.01	15.05	14.85	14.78	14.51(0.07)
4	20.61	20.68	20.36	20.25	20.01(0.08)
5	27.16	27.28	26.81	26.65	26.69(0.10)
6	34.73	34.88	34.27	34.02	34.44(0.12)
7	40.15	40.34	39.62	39.25	39.56(0.13)
8	44.26	44.48	43.67	43.18	43.71(0.14)
9	48.00	48.25	47.38	46.73	47.14(0.11)
10	50.88	51.16	50.25	49.42	50.19(0.16)
15	62.24	62.62	61.89	59.97	60.84(0.19)
20	63.81	64.24	64.03	61.02	61.91(0.20)

**Table 4.** CMS spread floret, strike -0.5%.

Maturity	CA <sup>0</sup>	CA	LN <sup>0</sup>	LN	MC(SD)
1	0.11	0.11	0.11	0.11	0.08(0.00)
2	0.64	0.64	0.64	0.63	0.49(0.01)
3	1.40	1.41	1.4	1.37	1.01(0.01)
4	2.65	2.68	2.65	2.58	2.17(0.02)
5	4.45	4.49	4.44	4.32	3.83(0.04)
6	6.93	7.01	6.93	6.71	6.03(0.05)
7	9.14	9.25	9.16	8.8	8.12(0.06)
8	11.03	11.17	11.08	10.66	9.80(0.12)
9	12.89	13.06	12.97	12.32	11.66(0.13)
10	14.45	14.65	14.59	13.78	13.00(0.14)
15	20.85	21.17	21.48	19.66	19.13(0.19)
20	22.24	22.61	23.53	20.76	20.00(0.20)

In Figures 1-2 we show the calibration of the Libor market model to the market CMS spread prices for a fixed strikes. To find the optimal parameters for Libors market model, we minimize the root-mean-square (RMS) error

$$\frac{1}{N} \sqrt{\sum_{i=1}^N (V^{mar}(\mathcal{T}_i) - V^{app}(\mathcal{T}_i; a, b, g_\infty, \eta, \rho_\infty))^2} \rightarrow_{a, b, g_\infty, \eta, \rho_\infty} \min.$$

Here,  $V^{mar}$  is a vector of market CMS spread prices (for fixed strike) for  $N$  maturities. In our case,

$$N = 12; \quad \mathcal{T}_1 = 1y, \mathcal{T}_2 = 2y, \dots, \mathcal{T}_{10} = 10y, \mathcal{T}_{11} = 15y, \mathcal{T}_{12} = 20y;$$

$V^{app}$  denotes an approximated CMS spread options,  $V^{app} = CA^0$ , or  $V^{app} = LN^0$ . Although both approximations provide a good fit for a fixed strike, the simultaneous

calibration for the whole CMS spread matrix is impossible. The problem can be solved by generalizing the lognormal approximation to the model with stochastic volatility.

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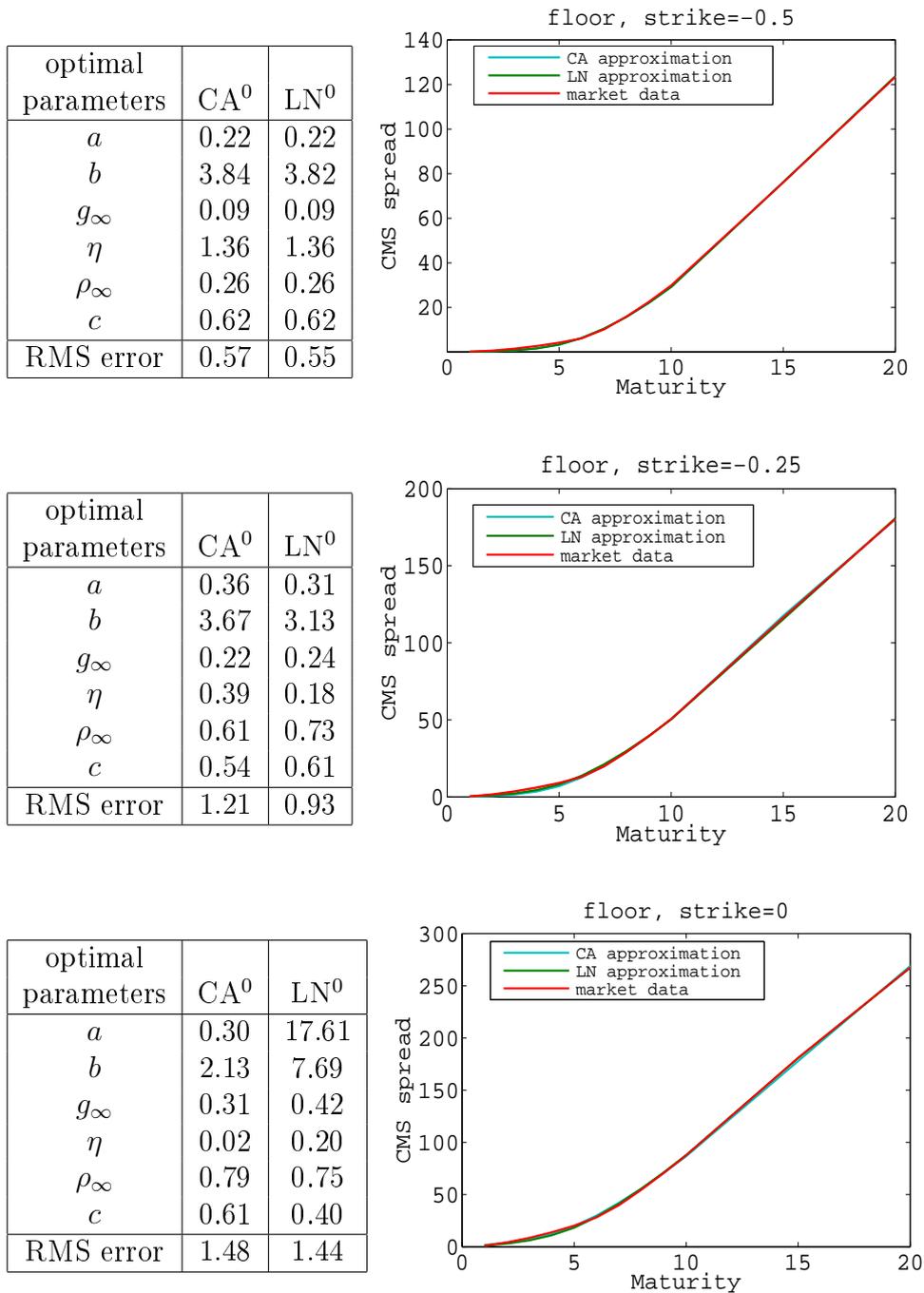


Figure 1: Calibration to CMS spread (floor) for fixed strike. The optimal values of LMM parameters are given in the tables (left).

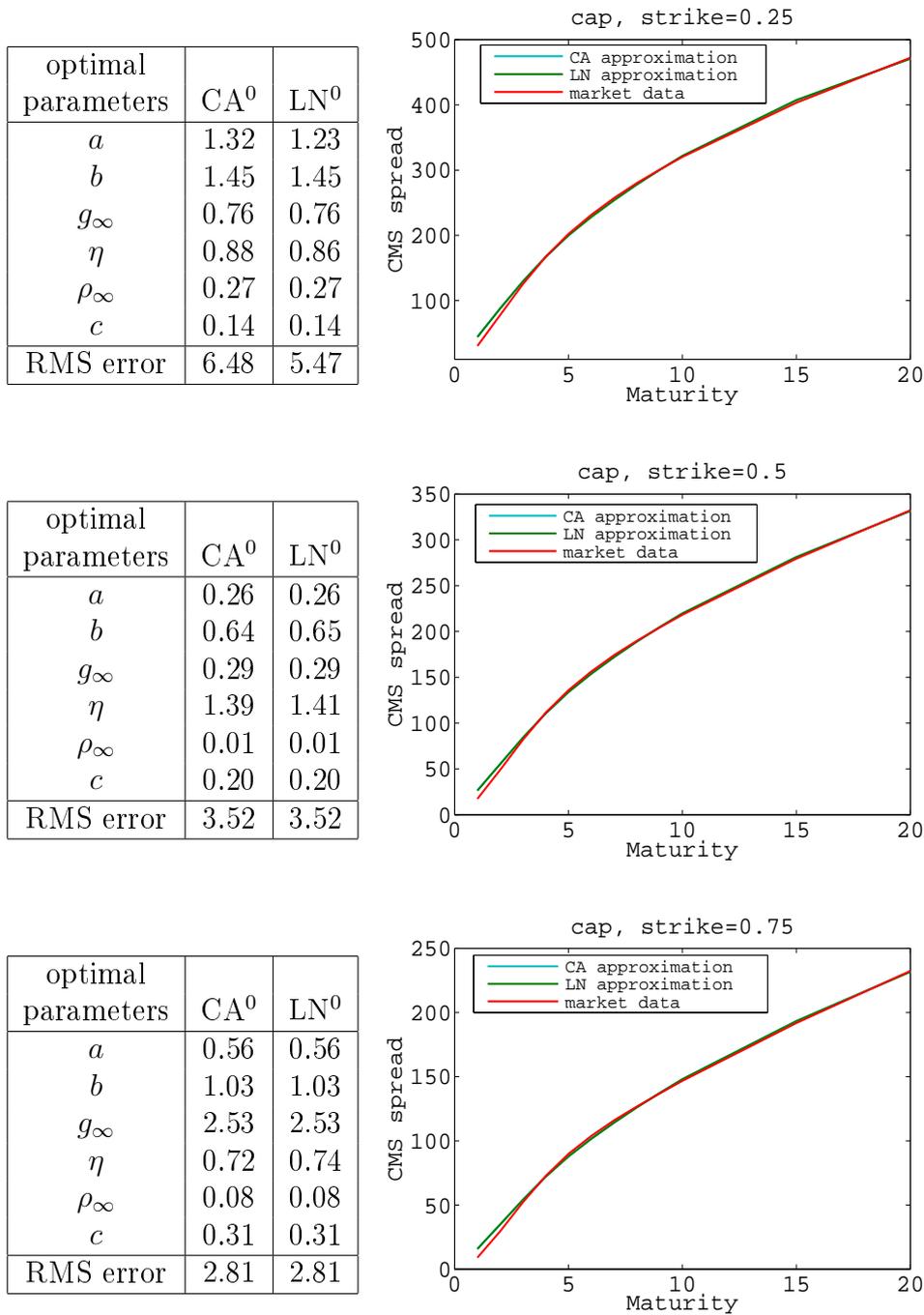


Figure 2: Calibration to CMS spread (cap) for fixed strike. The optimal values of LMM parameters are given in the tables (left).