

Hybrid systems of differential-algebraic equations – Analysis and numerical solution*

Volker Mehrmann[†] Lena Wunderlich[‡]

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Abstract

We consider hybrid systems of differential-algebraic equations and present a general framework for general nonlinear over- and underdetermined hybrid systems that allows the analysis of existence and uniqueness and the application of index reduction methods for hybrid differential-algebraic systems. A particular difficulty in the numerical simulation of hybrid systems is (numerical) chattering, i.e., fast oscillations between modes of operations. A regularization technique using sliding modes allows to regularize the system behavior in the case of chattering. Further, we show how chattering behavior during the numerical solution can be prevented using sliding mode simulation. The advantage of the sliding mode simulation is illustrated by numerical examples.

Keywords: differential-algebraic equation, hybrid system, switched system, index reduction, existence and uniqueness of solutions, strangeness index, chattering, sliding mode simulation.

1 Introduction

In the numerical simulation and control of constrained dynamical systems differential-algebraic equations (DAEs) are widely used, since they naturally arise in the modeling process. A particular feature of many complex dynamical systems is that they are *switched systems* or *hybrid systems*, i.e., the mathematical model changes with time, depending on certain indicators. Switched systems arise in numerous applications such as: mechanical systems with dry friction [7, 21], impact phenomena, robot manipulators [15], automatic gear-boxes [14], or in electronic circuits where different device models are used for different frequency ranges or switching elements like diodes or electric switches are used [27]. Further, switched systems also naturally arise in chemical engineering [2] and in control systems [32] where the value of a control switches. The continuous dynamics of the system in the different operation modes are described by sets of differential-algebraic equations and the change between different operation modes is modeled by discrete transitions resulting in switching between sets of equations. As the discrete and continuous dynamics interact they must be analyzed simultaneously. One of the basic difficulties in switched differential-algebraic systems is that after a mode switch takes place, the model dimension and the structure of the system as well as its properties such as the index, the number of algebraic or differential equations or redundancies may change. Furthermore, special phenomena that can occur during the simulation of hybrid systems, as e.g. numerical chattering, have to be treated in an appropriate way to ensure an efficient numerical integration. In practical applications the model is usually simulated in one of the modes, the switching points are determined and then the system is simulated in the next mode. For the active control of hybrid systems, however, this mode-by-mode approach is

*Supported by DFG Research Center MATHEON, ‘Mathematics for key technologies’ in Berlin.

[†]Institut für Mathematik, TU Berlin, D-10623 Berlin. mehrmann@math.tu-berlin.de.

[‡]Institut für Mathematik, TU Berlin, D-10623 Berlin. wunder@math.tu-berlin.de

typically not applicable, since the control may influence the switching and, hence, the hybrid system has to be considered as a whole. It is therefore necessary to study hybrid systems based on a general formulation that enables a simulation and control of the system with all its possible modes together. The mathematical theory of switched differential-algebraic systems, the control theory for such systems as well as the development of efficient and accurate numerical methods is still in an early stage. For an overview of modeling, analysis, simulation and control of hybrid systems, see e.g. [4, 22]. In [13, 14, 31] it was shown how the theory for general over- and underdetermined DAEs can be applied to hybrid differential-algebraic systems.

The paper is organized as follows. In Section 2 we give a formal definition of hybrid differential-algebraic systems. In Section 3 we briefly discuss DAEs and index reduction for DAEs. In Section 4 we present conditions for existence and uniqueness of continuous solutions of hybrid DAE systems. Further, in Section 5 we present mathematical methods to detect (numerical) chattering and show how the chattering behavior can be regularized by so-called *sliding modes*. Finally, in Section 6 we give a number of numerical examples.

2 Formulation of Hybrid Differential-Algebraic Systems

Hybrid systems are generally described by a collection of discrete subsystems, a collection of continuous subsystems and the possible interaction between these subsystems. There exists different approaches of an abstract modeling via hybrid systems, e.g. in [1, 3, 11, 22, 23], but until now no formulation has established itself as a standard. Here, we base the abstract modeling concept on the general theory of DAEs as described in [16] and consider hybrid systems that are composed of several different constrained nonlinear dynamical systems described by DAEs and transition conditions between these DAEs following an approach given in [4, 14]. We assume that the discrete and continuous subsystems only interact via instantaneous discrete transitions at distinct points in time called *events*.

Definition 1. Let $\mathbb{I} = [t_0, t_f] \subset \mathbb{R}$ be an interval that is decomposed into subintervals $\mathbb{I}_i = [\tau_i, \tau'_i]$ for $i = 1, \dots, N_{\mathbb{I}} - 1$ and $\mathbb{I}_{N_{\mathbb{I}}} = [\tau_{N_{\mathbb{I}}}, \tau'_{N_{\mathbb{I}}}]$, $N_{\mathbb{I}} \in \mathbb{N}$ such that $\mathbb{I} = \bigcup_{i=1}^{N_{\mathbb{I}}} \mathbb{I}_i$, with $\tau_1 = t_0$, $\tau'_{N_{\mathbb{I}}} = t_f$ and $\tau'_i = \tau_{i+1}$ for all $i = 1, \dots, N_{\mathbb{I}} - 1$ and $\tau_i < \tau'_i$ for all $i = 1, \dots, N_{\mathbb{I}}$. Further, let $\mathbb{M} := \{1, \dots, N_F\}$, $N_F \in \mathbb{N}$ be the set of modes and for each $\ell \in \mathbb{M}$ let D_ℓ be the union of certain intervals \mathbb{I}_i , such that $\bigcup_{\ell \in \mathbb{M}} D_\ell = \mathbb{I}$ and $D_\ell \cap D_k = \emptyset$ for $\ell, k \in \mathbb{M}$, $\ell \neq k$. Then, a hybrid system of differential-algebraic equations \mathcal{H} is defined as a collection of

- a set of N_F systems of nonlinear differential-algebraic equations

$$F^\ell(t, x^\ell, \dot{x}^\ell) = 0, \quad \ell \in \mathbb{M}, \quad (1)$$

with sufficiently smooth functions $F^\ell : D_\ell \times \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}^{m_\ell}$;

- an index set of autonomous transitions $J^\ell = \{1, 2, \dots, n_T^\ell\}$ for each mode $\ell \in \mathbb{M}$, where $n_T^\ell \in \mathbb{N}$ is the number of possible transitions of mode ℓ ;
- transition conditions $L_j^\ell(t, x^\ell, \dot{x}^\ell)$ for all transitions $j \in J^\ell$ and all modes $\ell \in \mathbb{M}$ with

$$L_j^\ell : D_\ell \times \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \rightarrow \{\text{TRUE}, \text{FALSE}\}; \quad (2)$$

- switching functions of the form

$$g_{j,i}^\ell : D_\ell \times \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}, \quad \text{for all } i = 1, \dots, n_j^\ell, \quad j \in J^\ell,$$

with $g_{j,i}^\ell(t, x^\ell, \dot{x}^\ell) > 0$ in mode ℓ ;

- mode allocation functions for all $\ell \in \mathbb{M}$ of the form

$$S^\ell : J^\ell \rightarrow \mathbb{M}, \quad \text{with } S^\ell(j) = k, \quad (3)$$

that determine the successor mode k after a mode change, and

- transition functions $T_\ell^k : \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}^{n_k \times 2}$ of the form

$$T_\ell^k(x^\ell(\tau'_i), \dot{x}^\ell(\tau'_i)) = [x^k(\tau_{i+1}), \dot{x}^k(\tau_{i+1})], \quad (4)$$

for all $\ell \in \mathbb{M}$ with successor mode $k \in \mathbb{M}$ that map the final values and derivatives of the variables in mode ℓ to the corresponding initial values in mode k at time $\tau'_i = \tau_{i+1} \in D_k$.

If in addition an initial value

$$x^{\ell_1}(t_0) = x_0^{\ell_1} \in \mathbb{R}^{n_{\ell_1}} \quad (5)$$

is given in some initial mode $\ell_1 \in \mathbb{M}$, then a hybrid system \mathcal{H} together with the initial condition (5) and initial mode ℓ_1 is called a hybrid initial value problem.

In this setting, (1) is the DAE that describes the dynamics of the hybrid system in mode $\ell \in \mathbb{M}$ and in each subinterval the dynamics of the system are governed by only one DAE. The hybrid system is said to be *in mode* $\ell \in \mathbb{M}$ if $t \in D_\ell$. The piecewise continuous functions $x^\ell : D_\ell \rightarrow \mathbb{R}^{n_\ell}$ describe the continuous state of the hybrid system in mode ℓ and $x^\ell(\tau'_i)$ is the smooth extension of x^ℓ to the interval boundary $\tau'_i = \tau_{i+1}$ of an interval $\mathbb{I}_i \in D_\ell$. We further define the *hybrid time trajectory* $T_\tau = \{\mathbb{I}_i\}_{i=1, \dots, N_\mathbb{I}}$ as a sequence of intervals and the *hybrid mode trajectory* $T_m = \{\ell_i\}_{i=1, \dots, N_\mathbb{I}}$ as the corresponding sequence of modes, where $\ell_i \in \mathbb{M}$ is the mode in interval \mathbb{I}_i . The hybrid time trajectory and the hybrid mode trajectory depend on the initial mode and initial conditions as well as on the defined switching conditions. The *set of event times* corresponding to a hybrid time trajectory is given by $\mathcal{E}(T_\tau) = \{\tau_i \mid i = 1, \dots, N_\mathbb{I}\}$. The hybrid system \mathcal{H} changes between different modes on the basis of the transition conditions. If $L_j^\ell(\hat{t}, x^\ell(\hat{t}), \dot{x}^\ell(\hat{t})) = FALSE$ for all $j \in J^\ell$ at a time $\hat{t} \in D_\ell$, then the system stays in the current mode. On the other hand, if there exists an integer $j \in J^\ell$ such that $L_j^\ell(\hat{t}, x^\ell(\hat{t}), \dot{x}^\ell(\hat{t})) = TRUE$ at time \hat{t} , then the system switches to another mode. The switch points are defined as the roots of the switching functions $g_{j,i}^\ell(t, x^\ell, \dot{x}^\ell)$ that are given as threshold functions, i.e., if $g_{j,i}^\ell(t, x^\ell, \dot{x}^\ell) > 0$ for all $i = 1, \dots, n_j^\ell, j \in J^\ell$, then the system stays in the current mode ℓ , but if $g_{j,i}^\ell(t, x^\ell, \dot{x}^\ell) \leq 0$ for a $j \in J^\ell$ and some i then the system may switch to a new mode. Note, that a transition condition L_j^ℓ is described by n_j^ℓ separated switching functions $g_{j,i}^\ell, i = 1, \dots, n_j^\ell$, and this logical combination determines if the transition condition L_j^ℓ is satisfied. In this way, the switching functions $g_{j,i}^\ell$ can be chosen as simple as possible, allowing an efficient and reliable computation of the switch points. Thus, each time a switching function crosses zero the associated transition condition may switch its logical value.

Each switching function has an associated *switching surface* in the state space given by

$$\Gamma_{j,i}^\ell = \{(t, x^\ell, \dot{x}^\ell) \in D_\ell \times \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_\ell} \mid g_{j,i}^\ell(t, x^\ell, \dot{x}^\ell) = 0\}, \quad j \in J^\ell, \ell \in \mathbb{M}, \quad (6)$$

i.e., mode switching occurs at points on these switching surfaces. For convenience of expression, in the following we assume that each transition condition L_j^ℓ is described by exactly one switching function g_j^ℓ (i.e., $n_j^\ell = 1$ for all $j \in J^\ell, \ell \in \mathbb{M}$) and the transition condition L_j^ℓ is satisfied if and only if $g_j^\ell \leq 0$. Then, the satisfaction of a transition condition corresponds to the crossing of the switching surfaces Γ_j^ℓ .

Further, we assume deterministic models, i.e., only one transition condition is becoming true at a time. Finally, the transition function T_ℓ^k transfers the state at the mode change from mode ℓ to mode k . This transfer can result in jumps in the state vector of the hybrid system. In order to obtain a solution in the new mode, the initial values obtained by the transition function have to be consistent with the DAE in mode k . For each interval \mathbb{I}_i we assume that consistent initial values $x_{\tau_i}^\ell$ for the currently valid mode $\ell \in \mathbb{M}$ exist, such that the solution in \mathbb{I}_i exists.

Remark 2. Hybrid systems of DAEs defined in this way have no limitations with respect to the types of the DAE $F^\ell = 0$ in the different modes, as long as these are solvable in the interval set D_ℓ , but there are restrictions concerning the transition processes. The number of mode changes must

be finite, as otherwise no reasonable numerical integration is possible. Furthermore, we assume that the integration intervals have a nonzero measure, i.e., an instantaneous multiple mode change is not possible.

Remark 3. In order to include descriptor control systems and standard DAEs in the same framework we have only required that the solution in the subintervals D_ℓ exists but not that it is unique. In a numerical solution method and in the active control of a hybrid system, however, a procedure that assigns the free variables (controls) in a unique way has to be designed.

3 Index Reduction for Hybrid DAEs

To describe the hybrid system, we consider in each mode a nonlinear DAE of the form (1). The index reduction procedure for nonlinear DAEs of the form (1) described in [16] can be applied for each mode of a switched systems as has been shown in [14]. We introduce a nonlinear derivative array \mathcal{F}_i^ℓ of level i in mode $\ell \in \mathbb{M}$ of the form

$$\mathcal{F}_i^\ell(t, x^\ell, \dot{x}^\ell, \dots, \frac{d^{i+1}}{dt^{i+1}}x^\ell) = 0, \quad (7)$$

which stacks the original equations of the DAE in mode $\ell \in \mathbb{M}$ and all its derivatives up to level i into one large system

$$\mathcal{F}_i^\ell(t, x^\ell, \dot{x}^\ell, \dots, x^{\ell(i+1)}) = \begin{bmatrix} F^\ell(t, x^\ell, \dot{x}^\ell) \\ \frac{d}{dt}F^\ell(t, x^\ell, \dot{x}^\ell) \\ \vdots \\ \frac{d^i}{dt^i}F^\ell(t, x^\ell, \dot{x}^\ell) \end{bmatrix}.$$

In the following, partial derivatives of \mathcal{F}_i^ℓ with respect to selected variables p from $(t, x^\ell, \dots, x^{\ell(i+1)})$ are denoted by $\mathcal{F}_{i;p}^\ell$, e.g.,

$$\mathcal{F}_{i;x^\ell}^\ell = \frac{\partial}{\partial x^\ell} \mathcal{F}_i^\ell, \quad \mathcal{F}_{i;\dot{x}^\ell, \dots, x^{\ell(i+1)}}^\ell = \begin{bmatrix} \frac{\partial}{\partial \dot{x}^\ell} \mathcal{F}_i^\ell \dots \frac{\partial^{(i+1)}}{\partial x^{\ell(i+1)}} \mathcal{F}_i^\ell \end{bmatrix}.$$

To obtain an existence and uniqueness result for general DAEs, a hypothesis was introduced in [18]. Adapted to the case of hybrid DAE systems the Hypothesis can be stated as follows.

Hypothesis 1. Consider a nonlinear differential-algebraic equations (1) in mode $\ell \in \mathbb{M}$. There exist integers $\mu^\ell, r^\ell, a^\ell, d^\ell$ and v^ℓ such that the solution set of the derivative array $\mathcal{F}_{\mu^\ell}^\ell$

$$\mathbb{L}_{\mu^\ell}^\ell = \{(t, x^\ell, \dot{x}^\ell, \dots, x^{\ell(\mu^\ell+1)}) \in D_\ell \times \mathbb{R}^{(\mu^\ell+2)n_\ell} \mid \mathcal{F}_{\mu^\ell}^\ell(t, x^\ell, \dot{x}^\ell, \dots, x^{\ell(\mu^\ell+1)}) = 0\}$$

is not empty, and the following properties hold:

1. The set $\mathbb{L}_{\mu^\ell}^\ell \subset \mathbb{R}^{(\mu^\ell+2)n_\ell+1}$ forms a manifold of dimension $(\mu^\ell + 2)n_\ell + 1 - r^\ell$.
2. We have $\text{rank } \mathcal{F}_{\mu^\ell; x^\ell, \dot{x}^\ell, \dots, x^{\ell(\mu^\ell+1)}}^\ell = r^\ell$ on $\mathbb{L}_{\mu^\ell}^\ell$.
3. We have $\text{corank } \mathcal{F}_{\mu^\ell; x^\ell, \dot{x}^\ell, \dots, x^{\ell(\mu^\ell+1)}}^\ell - \text{corank } \mathcal{F}_{\mu^\ell-1; x^\ell, \dot{x}^\ell, \dots, x^{\ell(\mu^\ell)}}^\ell = v^\ell$ on $\mathbb{L}_{\mu^\ell}^\ell$. (The corank is the codimension of the range and we use the convention that $\text{corank } \mathcal{F}_{-1; x^\ell}^\ell = 0$.)
4. We have $\text{rank } \mathcal{F}_{\mu^\ell; \dot{x}^\ell, \dots, x^{\ell(\mu^\ell+1)}}^\ell = r^\ell - a^\ell$ on $\mathbb{L}_{\mu^\ell}^\ell$ such that there are smooth full rank matrix functions $Z_{\ell,2}$ and $T_{\ell,2}$ defined on $\mathbb{L}_{\mu^\ell}^\ell$ of size $((\mu^\ell + 1)m_\ell, a^\ell)$ and $(n_\ell, n_\ell - a^\ell)$, respectively, satisfying

$$Z_{\ell,2}^T \mathcal{F}_{\mu^\ell; \dot{x}^\ell, \dots, x^{\ell(\mu^\ell+1)}}^\ell = 0, \quad \text{rank } Z_{\ell,2}^T \mathcal{F}_{\mu^\ell; x^\ell}^\ell = a^\ell, \quad Z_{\ell,2}^T \mathcal{F}_{\mu^\ell; x^\ell}^\ell T_{\ell,2} = 0$$

on $\mathbb{L}_{\mu^\ell}^\ell$.

5. We have $\text{rank } F_{;\dot{x}^\ell}^\ell T_{\ell,2} = d^\ell = m_\ell - a^\ell - v^\ell$ on $\mathbb{L}_{\mu^\ell}^\ell$ such that there exists a smooth matrix function $Z_{\ell,1}$ defined on $\mathbb{L}_{\mu^\ell}^\ell$ of size (m_ℓ, d^ℓ) with $Z_{\ell,1}^T F_{;\dot{x}^\ell}^\ell T_{\ell,2}$ having full rank.

The smallest possible μ^ℓ in Hypothesis 1 is called the *strangeness index* of the DAE (1) in mode ℓ and systems with vanishing strangeness-index are called *strangeness-free*. The *maximal strangeness index* μ_{max} of a hybrid system \mathcal{H} is given by

$$\mu_{max} = \max_{\ell \in \mathbb{M}} \{\mu^\ell\},$$

and a hybrid system \mathcal{H} is called *strangeness-free* if $\mu_{max} = 0$. Further, a DAE (1) that satisfies Hypothesis 1 with $n_\ell = m_\ell = d^\ell + a^\ell$ is called *regular*.

The corresponding numbers d^ℓ and a^ℓ are the numbers of differential and algebraic equations of the DAE in mode ℓ . It has been shown in [18] that Hypothesis 1 implies (locally) the existence of a reduced system (in the original variables) of the form

$$\begin{aligned} \text{(a)} \quad & \hat{F}^\ell(t, x_1^\ell, x_2^\ell, x_3^\ell, \dot{x}_1^\ell, \dot{x}_2^\ell, \dot{x}_3^\ell) = 0, \\ \text{(b)} \quad & x_3^\ell - \mathcal{R}^\ell(t, x_1^\ell, x_2^\ell) = 0, \end{aligned} \tag{8}$$

independently in each mode $\ell \in \mathbb{M}$, with $\hat{F}^\ell = Z_{\ell,1}^T F^\ell$.

An initial condition is consistent with the DAE in mode ℓ , if it satisfies the algebraic equation $x_3^\ell - \mathcal{R}^\ell(t, x_1^\ell, x_2^\ell) = 0$. Eliminating x_3^ℓ and \dot{x}_3^ℓ in (8a) with the help of (8b) and its derivative leads to a system

$$\begin{aligned} & \hat{F}^\ell(t, x_1^\ell, x_2^\ell, \mathcal{R}^\ell(t, x_1^\ell, x_2^\ell), \dot{x}_1^\ell, \dot{x}_2^\ell, \mathcal{R}_t^\ell(t, x_1^\ell, x_2^\ell) \\ & + \mathcal{R}_{x_1^\ell}^\ell(t, x_1^\ell, x_2^\ell) \dot{x}_1^\ell + \mathcal{R}_{x_2^\ell}^\ell(t, x_1^\ell, x_2^\ell) \dot{x}_2^\ell) = 0. \end{aligned}$$

By part 5 of Hypothesis 1 we may assume w.l.o.g. that this system can (locally via the implicit function theorem) be solved for \dot{x}_1^ℓ , leading to a system of the form

$$\begin{aligned} \dot{x}_1^\ell &= \mathcal{L}^\ell(t, x_1^\ell, x_2^\ell, \dot{x}_2^\ell), \\ x_3^\ell &= \mathcal{R}^\ell(t, x_1^\ell, x_2^\ell). \end{aligned} \tag{9}$$

Obviously, in this system $x_2^\ell \in C^1(D_\ell, \mathbb{R}^{u^\ell})$, with $u^\ell = n_\ell - d^\ell - a^\ell$, can be chosen arbitrarily (at least when staying in the domain of definition of \mathcal{R}^ℓ and \mathcal{L}^ℓ), i.e., it plays the role of a control. When x_2^ℓ has been chosen, then the resulting system has (locally) a unique solution for x_1^ℓ and x_3^ℓ , provided that a consistent initial condition is given. In this way, a strangeness-free formulation can be determined independently for each mode.

Remark 4. The system in the form (9) can be interpreted as a control system in behavior form [26], in which x^ℓ plays the role of an output and x_2^ℓ can be chosen freely, i.e. may be considered as input.

Theorem 5. [18] *Let F^ℓ in (1) be sufficiently smooth and satisfy Hypothesis 1 with μ^ℓ , r^ℓ , a^ℓ , d^ℓ , v^ℓ and $u^\ell = n_\ell - d^\ell - a^\ell$. Then every solution of (1) also solves the reduced problems (8) and (9) consisting of d^ℓ differential and a^ℓ algebraic equations.*

Remark 6. In the reduced systems (8) and (9) we have not used the quantity v^ℓ . This quantity measures the number of equations in the original system that give rise to trivial equations $0 = 0$, i.e., it counts the number of redundancies in the system. Together with a^ℓ and d^ℓ it gives a complete classification of the m_ℓ equations into d^ℓ differential equations, a^ℓ algebraic equations and v^ℓ trivial equations. Of course, trivial equations can be simply removed without altering the solution set.

In order to be able to derive a reduced system for each mode we introduce another hypothesis.

Hypothesis 2. *In each mode $\ell \in \mathbb{M}$ and each domain D_ℓ the strangeness index μ^ℓ is well-defined, i.e., the DAE (1) in mode ℓ satisfies Hypothesis 1 with constant characteristic values μ^ℓ , r^ℓ , a^ℓ , d^ℓ , and v^ℓ .*

If Hypothesis 2 is violated at a finite number of points, then we can introduce new modes and further switching points to satisfy Hypothesis 2.

A reduced system as in (8) or (9) with the same solution as the original DAE (1) in mode ℓ can be extracted independently in each mode and, therefore, also for the complete hybrid system. Connecting all the reduced systems together we locally obtain an equivalent reduced hybrid system denoted by $\hat{\mathcal{H}}$, which is strangeness-free, i.e., $\mu_{max} = 0$. By solving the corresponding transformed systems, the solution of the complete hybrid system can be computed for every time t .

We have the following theorem.

Theorem 7. *Let \mathcal{H} be a hybrid system as in Definition 1 with sufficiently smooth functions F^ℓ in each mode $\ell \in \mathbb{M}$ that satisfy Hypothesis 2. Then, every sufficiently smooth solution of \mathcal{H} is also a solution of $\hat{\mathcal{H}}$.*

Proof. Let $T_\tau = \{\mathbb{I}_i\}_{i=1, \dots, N_i}$ and $T_m = \{\ell_i\}_{i=1, \dots, N_i}$ be the hybrid time and mode trajectory corresponding to the hybrid system \mathcal{H} . If the sequence $\{x_{\ell_i}^*\}_{\ell_i \in T_m}$, corresponding to T_m is a sufficiently smooth solution of \mathcal{H} determined from the DAEs F^{ℓ_i} in each mode $\ell_i \in \mathbb{M}$, then the functions $x_{\ell_i}^*$ also solve the reduced strangeness-free systems of DAEs (8) and (9), since for every mode ℓ_i , and all $t \in D_{\ell_i}$ we have

$$(t, x_{\ell_i}^*(t), \dot{x}_{\ell_i}^*(t), \dots, (\frac{d}{dt})^{\mu_{\ell_i}+1} x_{\ell_i}^*(t)) \in \mathbb{L}_{\mu_{\ell_i}}^{\ell_i}.$$

Since the transition functions yield consistent initial values after each mode change as $\{x_{\ell_i}^*\}_{\ell_i \in T_m}$ is a solution, these values are also consistent for the reduced systems. Thus, the sequence $\{x_{\ell_i}^*\}_{\ell_i \in T_m}$ is also a solution of the reduced hybrid system $\hat{\mathcal{H}}$. \square

Note that the reduced hybrid system $\hat{\mathcal{H}}$ depends strongly on the choice and the consistency of the initial values in each mode. In each interval \mathbb{I}_i and for every mode ℓ , the initial values must be chosen in a consistent way, so that the solution in each mode exists and is unique if $u^\ell = 0$.

4 Existence, Uniqueness and Continuity of Solutions of Hybrid Systems

In this section we discuss necessary and sufficient conditions for the existence and uniqueness of continuous solutions of hybrid differential-algebraic systems. In general, solution concepts for hybrid systems have to deal with nonsmooth solutions and with changes in the number of equations or unknowns after mode changes. If we want overall continuous solutions, then the transition functions must guarantee this. However, if the number of equations or the number of free variables changes at a mode change, then this condition may be difficult to realize. In particular, we may face the situation that the solution is not unique after a mode change. In any case, we need consistency of the initial values with the DAE in the new mode for the existence of a solution. A major difficulty in defining a solution for the overall hybrid system \mathcal{H} is that at a switch point not only the index of the DAE, but also the number of unknowns of the system and the number of differential, algebraic and undetermined variables may change. Let $n := \max_{\ell \in \mathbb{M}} n_\ell$ be the maximal size of all solution vectors x^ℓ . If $n > n_\ell$ in some mode $\ell \in \mathbb{M}$, then it is not clear how some solution components can be continued in this mode after a mode change. In this case, we extend the system in mode ℓ by solution components \hat{x}^ℓ of size $n - n_\ell$, i.e., we consider the extended system

$$\tilde{F}^\ell(t, \tilde{x}^\ell, \dot{\tilde{x}}^\ell) = F^\ell(t, x^\ell, \hat{x}^\ell, \dot{x}^\ell, \dot{\hat{x}}^\ell) = 0, \quad (10)$$

with $\tilde{x}^\ell = [x^\ell, \hat{x}^\ell]^T$. The extended system (10) has the same strangeness index and, for the first n_ℓ components of the solution vector, the same solution as the original system in mode ℓ . But, now we have to deal with nonuniqueness of solutions, while the original system in mode ℓ may have been

uniquely solvable. With regard to the solution of the overall hybrid system this nonuniqueness was already present in the hybrid system before the extension, such that considering the extended systems (10) does not interfere with the solvability of the overall hybrid system. The nonuniqueness of solutions can be overcome by embedding the DAE (10) in a minimization problem in such a way that the undetermined state components are continued constantly with the last value of the previous mode. In the following, we assume without loss of generality that $n_\ell = n$ in all modes $\ell \in \mathbb{M}$. Locally in each mode $\ell \in \mathbb{M}$ and for every interval $\mathbb{I}_i \subseteq D_\ell$, we consider the classical solution concept.

Definition 8. Consider a nonlinear system (1) with sufficiently smooth function F^ℓ . A function $x^\ell : \mathbb{I}_i \rightarrow \mathbb{R}^{n_\ell}$, $\mathbb{I}_i \subseteq D_\ell$ is called a solution of (1) if $x^\ell \in C^1(\mathbb{I}_i, \mathbb{R}^n)$ and x^ℓ satisfies (1) pointwise. It is called a solution of the initial value problem (1)-(5) if x^ℓ is a solution of (1) and satisfies the initial condition (5).

The solution of a hybrid system \mathcal{H} depends on the initial mode, initial conditions, mode switching conditions and on the transition functions. For a given time trajectory T_τ , with corresponding mode trajectory T_m , the initial mode as well as the mode switching sequence due to the transition conditions are fixed and the solution of the overall hybrid system is a sequence of continuous functions $x^\ell : \mathbb{I}_i \rightarrow \mathbb{R}^{n_\ell}$, with $\mathbb{I}_i \subseteq D_\ell$.

Definition 9. A function $x \in C(\mathbb{I}, \mathbb{R}^n)$, with $\mathbb{I} = [t_0, t_f] = \bigcup_{i=1}^{N_i} \mathbb{I}_i$ is called a continuous solution of a hybrid system \mathcal{H} with hybrid time trajectory $T_\tau = \{\mathbb{I}_i\}_{i=1, \dots, N_i}$ and corresponding hybrid mode trajectory $T_m = \{\ell_i\}_{i=1, \dots, N_i}$ if

$$x|_{\mathbb{I}_i} \in C^1(\mathbb{I}_i, \mathbb{R}^n) \text{ for all } \mathbb{I}_i \in T_\tau, \ell_i \in T_m,$$

and $x|_{\mathbb{I}_i}$ is a solution of the DAE (1) in the corresponding mode ℓ_i . The function x is called a continuous solution of the hybrid initial value problem with initial condition $x_0 \in \mathbb{R}^n$ at t_0 if it is a continuous solution and satisfies the initial condition $x(t_0) = x_0$.

An initial condition $x(t_0) = x_0$ is called consistent with the hybrid system \mathcal{H} if the corresponding hybrid initial value problem has at least one solution.

To analyze nonlinear problems one usually uses the implicit function theorem to show that a solution is locally unique. To be able to apply the implicit function theorem we must require that for a given solution the derivative of F^ℓ has a continuous inverse. Therefore, we assume that the differential-algebraic system in each mode is sufficiently smooth in a small interval after the switch point such that the implicit function theorem can be applied. Furthermore, we have to add another assumption.

Assumption 1. Consider a hybrid system \mathcal{H} as in Definition 1 with nonlinear DAEs of the form (1) and $n_\ell = n$ in each mode $\ell \in \mathbb{M}$. Assume that F^ℓ is sufficiently smooth in $[\tau_i, \tau'_i + \epsilon]$ for small $\epsilon > 0$ for each interval $\mathbb{I}_i = [\tau_i, \tau'_i] \in D_\ell$. Further, let the strangeness-index μ^ℓ be well-defined for all modes $\ell \in \mathbb{M}$ and assume that the DAEs (1) in each mode are solvable, provided that consistent initial conditions are given.

Under Assumption 1 we can locally transform the nonlinear DAEs (1) to the reduced systems (8) or (9), and there exists a solution if at each mode change the transition function is such that the resulting initial condition is consistent. If moreover $u^\ell = 0$ for all $\ell \in \mathbb{M}$, then the solution is unique as well.

Theorem 10. Consider a hybrid system \mathcal{H} that satisfies Hypothesis 2 with hybrid time trajectory T_τ , corresponding hybrid mode trajectory T_m , and initial value $x_0 \in \mathbb{R}^n$. Let $\mathcal{E}(T_\tau)$ be the set of event times and assume that Assumption 1 holds. Then there exists a continuous solution x of the hybrid system \mathcal{H} in the sense of Definition 9 if and only if

1. the initial value x_0 is consistent for the DAE in the initial mode $\ell_1 \in T_m$,

2. the transition functions $T_{\ell_{i-1}}^{\ell_i}$ are the identity mappings, i.e.,

$$T_{\ell_{i-1}}^{\ell_i}(x(\tau'_{i-1}), \dot{x}(\tau'_{i-1})) = [x(\tau'_{i-1}), \dot{x}(\tau'_{i-1})] = [x(\tau_i), \dot{x}(\tau_i)],$$

and for every $\tau_i \in \mathcal{E}(T_\tau)$ the states $x(\tau_i)$ are consistent with the DAE in mode $\ell_i \in T_m$.

The continuous solution x is unique if and only if in addition $u^\ell = 0$ for all modes $\ell \in T_m$.

Proof. Due to Assumption 1, the DAE (1) in each mode is solvable provided that consistent initial values are given. A continuous solution of the hybrid system implies that no jumps in the state vector at the switch points occur and that at the switch points the state in the predecessor mode is consistent for the DAE in the successor mode. On the other hand, if the initial value x_0 is consistent for the DAE in the initial mode ℓ_1 there exists a continuous solution in the initial mode until the first mode change. If the consistency of the values given by the transition functions after each mode change is ensured, then a continuous solution exists for all subsequent modes. Further, if $T_{\ell_i}^{\ell_{i+1}}$ is the identity mapping we have $x^{\ell_{i+1}}(\tau_{i+1}) = x^{\ell_i}(\tau'_i)$ at all switch points $\tau'_i = \tau_{i+1}$ and the overall solution of the hybrid system is continuous. Obviously, if there are no free solution components, i.e., $u^\ell = 0$ for all modes $\ell \in \mathbb{M}$, and the initial conditions are consistent, then in each mode the solution exists and is unique. \square

Remark 11. In a hybrid system \mathcal{H} discontinuities or jumps in the solution can occur at the switch points $\tau_j \in \mathcal{E}(T_\tau)$. To deal with these discontinuities one could extend the space of potential solutions and consider distributional solutions, see e.g. [28, 31].

5 Sliding Motion

A special phenomena that can occur during the simulation of hybrid systems is a fast changing between different modes of continuous operation, called *chattering*, for example if nearly equal thresholds for the transition conditions of different modes are given and the system starts to oscillate around these. These oscillations may be real in the physical model since hysteresis, delays and other dynamic nonidealities lead to fast oscillations. On the other hand, numerical errors may lead to numerical chattering as switching conditions may be satisfied due to local errors. The numerical solution of a hybrid system exhibiting chattering behavior requires high computational costs as small stepsizes are required to restart the integration after each mode change. In the worst case, the numerical integration breaks down, as it does not proceed in time, but chatters between modes. To prevent chattering and to reduce the computational costs, we can regularize the system dynamics along the switching surface in the regions where chattering can occur by adding an additional mode, the so-called *sliding mode*, that represents the dynamics during sliding. Sliding motion is well understood for ordinary differential equations, see e.g. [8, 9, 29, 30], we extend these results to DAEs.

In this section, for simplification we consider a hybrid DAE system \mathcal{H} that switches from mode ℓ to mode k . Let the two modes be separated by the j -th switching surface $\Gamma_j^\ell = \{(t, x) \in D_\ell \times \mathbb{R}^{n_\ell} \mid g_j^\ell(t, x) = 0\}$, $j \in J^\ell$, and assume that there exists a mode transition $\tilde{j} \in J^k$ such that $\Gamma_j^\ell = \Gamma_{\tilde{j}}^k = \{(t, x) \in D_k \times \mathbb{R}^{n_k} \mid g_{\tilde{j}}^k(t, x) = 0\}$, i.e., $g_j^\ell(t, x) = -g_{\tilde{j}}^k(t, x)$. Here, we restrict to switching functions independent of the state derivative \dot{x} . Under the assumption of regularity and well-definedness of the strangeness index in each mode, the differential-algebraic systems in the adjacent modes ℓ and k can be transformed to the corresponding reduced system (9). The hybrid differential-algebraic system exhibits chattering behavior if the dynamical part of the reduced system (9) fulfills a sliding condition. In particular, chattering occurs if all solutions near the surface Γ_j^ℓ approach it from both sides, see Figure 1, i.e., if the projections of the velocity vectors onto the surface gradient are of opposite signs and are directed towards the surface from both

sides in a neighborhood of the switching surface. We assume that $g_{j;x}^\ell(t, x) \neq 0$ in a neighborhood of the switching surface Γ_j^ℓ . Then, sliding occurs at a point $(t, x) \in \Gamma_j^\ell$ if the *sliding condition*

$$g_{j;x_1^\ell}^\ell(t, x_1^\ell, x_2^\ell)\mathcal{L}^\ell(t, x_1^\ell) < 0 \quad \text{and} \quad g_{j;x_1^k}^k(t, x_1^k, x_2^k)\mathcal{L}^k(t, x_1^k) > 0, \quad (11)$$

is satisfied. Here, we consider the directional derivatives of $g_j^\ell = -g_j^k$ along \mathcal{L}^ℓ , \mathcal{L}^k , respectively, which correspond to the projections of the vectors \mathcal{L}^ℓ and \mathcal{L}^k onto the gradient of the switching surface Γ_j^ℓ . The *sliding surface* $\Gamma_S^\ell \subseteq \Gamma$ can then be defined by

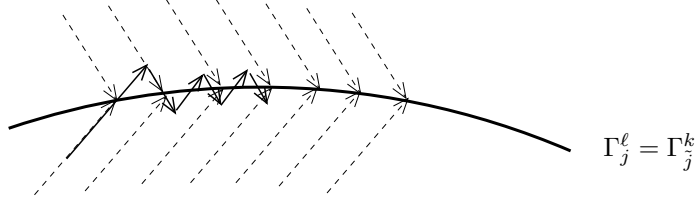


Figure 1: Chattering behavior along a switching surface

$$\Gamma_S^\ell := \{(t, x) \in \Gamma_j^\ell \mid g_{j;x_1^\ell}^\ell(t, x_1^\ell, x_2^\ell)\mathcal{L}^\ell(t, x_1^\ell) < 0, g_{j;x_1^k}^k(t, x_1^k, x_2^k)\mathcal{L}^k(t, x_1^k) > 0\},$$

as the part of the switching surface, where sliding occurs.

In the numerical simulation of hybrid systems an immediate switch back to mode ℓ after one or a few integration steps in mode k would result if the sliding condition is satisfied. To avoid this, we add an additional mode defining the dynamics of the DAE during sliding and switch to the sliding mode instead. The system should stay in the sliding mode as long as the solution trajectory stays on the sliding surface, and resume in mode ℓ or k , depending on the sign of the directional derivatives if the solution leaves Γ_S^ℓ . To define the system behavior during sliding the system dynamics is approximated in such a way that the state trajectory moves along the switching surface. We generalize an approach called *equivalence in dynamics*, or *Filippov regularization*, see e.g. [8, 9, 29]. In the following, w.l.o.g. we can restrict to the reduced strangeness-free systems (9). Then, the basic idea of the Filippov regularization is to approximate the velocity vector of sliding motion as a convex combination of the velocity vectors on both side of the switching surface in such a way that it lies on a plane tangential to the switching surface, see Figure 2.

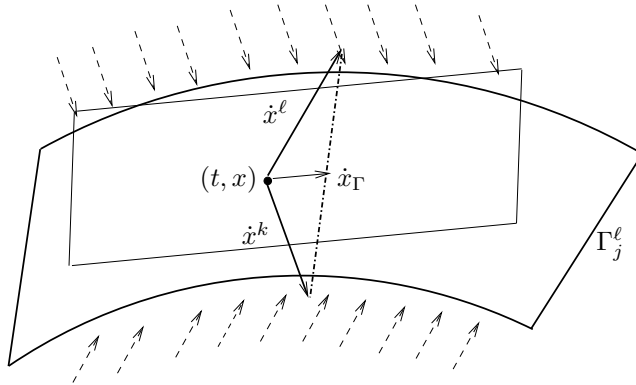


Figure 2: Filippov construction of equivalent dynamics

Let d^ℓ, d^k and a^ℓ, a^k denote the number of differential and algebraic equations in mode ℓ and mode k , i.e., the dimension of x_1^ℓ, x_1^k and x_2^ℓ, x_2^k in the corresponding reduced systems (9). Further,

let $d^\ell + a^\ell = d^k + a^k = n$ and without loss of generality assume that $d^\ell \geq d^k$ and $a^\ell \leq a^k$. Then, x_1^ℓ and x_2^k can be further partitioned into

$$x_1^\ell = \begin{bmatrix} x_{1,1}^\ell \\ x_{1,2}^\ell \end{bmatrix}, \quad x_2^k = \begin{bmatrix} x_{2,1}^k \\ x_{2,2}^k \end{bmatrix},$$

with $x_{1,1}^\ell \in \mathbb{R}^{d^k}$, $x_{1,2}^\ell \in \mathbb{R}^{d^\ell - d^k}$ and $x_{2,1}^k \in \mathbb{R}^{a^\ell}$, $x_{2,2}^k \in \mathbb{R}^{a^k - a^\ell}$. The corresponding reduced systems as in (9) can be partitioned accordingly into

$$\begin{aligned} \begin{bmatrix} \dot{x}_{1,1}^\ell \\ \dot{x}_{1,2}^\ell \end{bmatrix} &= \begin{bmatrix} \mathcal{L}_1^\ell(t, x_{1,1}^\ell, x_{1,2}^\ell) \\ \mathcal{L}_2^\ell(t, x_{1,1}^\ell, x_{1,2}^\ell) \end{bmatrix}, \\ x_2^\ell &= \mathcal{R}^\ell(t, x_{1,1}^\ell, x_{1,2}^\ell), \end{aligned} \quad (12)$$

and

$$\begin{aligned} \dot{x}_1^k &= \mathcal{L}^k(t, x_1^k), \\ \begin{bmatrix} x_{2,1}^k \\ x_{2,2}^k \end{bmatrix} &= \begin{bmatrix} \mathcal{R}_1^k(t, x_1^k) \\ \mathcal{R}_2^k(t, x_1^k) \end{bmatrix}. \end{aligned} \quad (13)$$

Using the ideas of the Filippov regularization we define the *differential-algebraic system in sliding motion* by

$$\dot{x}_1 = \alpha \begin{bmatrix} \mathcal{L}_1^\ell(t, x_1) \\ \mathcal{L}_2^\ell(t, x_1) \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{L}^k(t, x_1) \\ 0 \end{bmatrix}, \quad (14a)$$

$$x_2 = \alpha \begin{bmatrix} \mathcal{R}^\ell(t, x_1) \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{R}_1^k(t, x_1) \\ \mathcal{R}_2^k(t, x_1) \end{bmatrix}, \quad (14b)$$

$$0 = g_j^\ell(t, x_1, x_2). \quad (14c)$$

The extra algebraic variable α ensures that the solution stays on the manifold Γ_j^ℓ described by the algebraic constraint $g_j^\ell(t, x_1, x_2) = 0$. The differential equation (14a) describes the equivalent dynamics of the system during sliding motion and the algebraic equations (14b) is defined as a transformation of the constraint manifolds in the two modes such that $\mathcal{R}^\ell(t, x_1)$ is turned into $\mathcal{R}^k(t, x_1)$ across the discontinuity or vice versa depending on the direction of the discontinuity crossing. By the construction we can analyze the the DAE in sliding motion (14).

Theorem 12. *Consider a reduced hybrid system \mathcal{H} with regular DAEs of the form (9) in each mode that satisfies Hypothesis 2 and switches from mode ℓ to mode k along the smooth switching surface $\Gamma_j^\ell = \{(t, x) \in D_\ell \times \mathbb{R}^{n_\ell} \mid g_j^\ell(t, x) = 0\}$, $j \in J^\ell$. If*

$$g_{j;x_2}^\ell(t, x_1, x_2) (\mathcal{R}^k(t, x_1) - \mathcal{R}^\ell(t, x_1))$$

is nonsingular for all $(t, x_1, x_2) \in D_\ell \times \mathbb{R}^n$, then the system (14) is regular and of strangeness index $\mu = 1$.

Proof. As the DAEs in mode ℓ and k are regular also system (14) is regular due to construction. Differentiating the two algebraic constraints yields

$$\dot{x}_2 = \dot{\alpha} \begin{bmatrix} \mathcal{R}_1^\ell - \mathcal{R}_1^k \\ -\mathcal{R}_2^k \end{bmatrix} + \alpha \begin{bmatrix} \mathcal{R}_{;t}^\ell + \mathcal{R}_{;x_1}^\ell \dot{x}_1 \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{R}_{1;t}^k + \mathcal{R}_{1;x_1}^k \dot{x}_1 \\ \mathcal{R}_{2;t}^k + \mathcal{R}_{2;x_1}^k \dot{x}_1 \end{bmatrix}$$

and

$$0 = g_{j;t}^\ell + g_{j;x_1}^\ell \dot{x}_1 + g_{j;x_2}^\ell \dot{x}_2 \quad (15)$$

omitting the function arguments. Replacing the derivatives \dot{x}_1 and \dot{x}_2 in (15) yields

$$\begin{aligned} \dot{\alpha} g_{j;x_2}^\ell \begin{bmatrix} \mathcal{R}_1^k - \mathcal{R}^\ell \\ \mathcal{R}_2^k \end{bmatrix} &= g_{j;t}^\ell + g_{j;x_1}^\ell \left(\alpha \begin{bmatrix} \mathcal{L}_1^\ell \\ \mathcal{L}_2^\ell \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{L}^k \\ 0 \end{bmatrix} \right) \\ &+ g_{j;x_2}^\ell \left(\alpha \begin{bmatrix} \mathcal{R}_{;t}^\ell \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{R}_{1;t}^k \\ \mathcal{R}_{2;t}^k \end{bmatrix} \right) \\ &+ g_{j;x_2}^\ell \left(\alpha \begin{bmatrix} \mathcal{R}_{;x_1}^\ell \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{R}_{1;x_1}^k \\ \mathcal{R}_{2;x_1}^k \end{bmatrix} \right) \left(\alpha \begin{bmatrix} \mathcal{L}_1^\ell \\ \mathcal{L}_2^\ell \end{bmatrix} + (1 - \alpha) \begin{bmatrix} \mathcal{L}^k \\ 0 \end{bmatrix} \right) \end{aligned}$$

such that under the assumption that g_j^ℓ is differentiable and $g_{j;x_2}^\ell (\mathcal{R}^k - \mathcal{R}^\ell)$ is nonsingular we get an explicit ordinary differential equation for the variable α . \square

Remark 13. The construction of the DAE in sliding mode can be seen as a regularization since a potentially unsolvable problem is replaced by a solvable one. The idea of the Filippov regularization stem from the analysis of ODEs with right-hand side $f(t, x)$ that is discontinuous along a switching surface. While on both sides of the switching surface a unique solution exists, on the switching surface itself the classical solution concept is not applicable. Using the Filippov regularization a solution can be defined on the switching surface in terms of a differential inclusion, see [8].

Remark 14. Another possibility to prevent a hybrid system from chattering is to built in hysteresis that prevents the system from changing modes very quickly and thereby precluding the possibility of chattering. The hysteresis approach can be properly applied if numerical chattering occurs between two modes that have transition conditions that only differ in sign. In this case, a hysteresis can be realized by adding a term $\epsilon > 0$ to the transition conditions. For independent transition conditions between two modes ℓ and k or if numerical chattering between more than two modes occur, then the integration of a hysteresis is not so easy to realize. Furthermore, if the computation of the exact switch point is essential for the system behavior, then an artificial hysteresis cannot be inserted into the system.

The numerical integration of hybrid systems can be realized conceptually like the numerical integration of general DAEs, by generating, locally in each integration step, projectors $Z_{\ell,1}, Z_{\ell,2}$ that would lead to a strangeness-free formulation in the mode ℓ , if they were actually applied, see [18, 19, 20]. This integration is continued until one of the transition conditions (2) gives a zero crossing, i.e., a crossing of a threshold occurs. After determining this crossing point within a certain error tolerance as the root of a switching function, the system is transferred to the next mode via the transition function (4), and the numerical integration is continued in the new mode.

The time stepping procedure in the numerical integration in each mode ℓ can be achieved via any method for strangeness-free systems, e.g. by a BDF method [5] or by Runge-Kutta methods [10].

Thus, besides the pure integration process that may include an index reduction for simulating the hybrid system, we need a root finding procedure to determine the switch points, a process to compute consistent initial values in case of a mode change and, furthermore, an efficient organization of the different systems of DAEs F^ℓ . On the basis of these time stepping procedures a switch algorithm has been proposed in [14]. Further a hybrid mode controller, depicted in Figure 3, has been implemented that also allows sliding mode simulation, see [31].

Remark 15. One of the difficulties in the numerical integration of hybrid differential-algebraic equations is the computation of consistent initial values at the switch points. To determine a consistent initial value, we must solve the underdetermined system

$$\mathcal{F}_{\mu}^\ell(\tau_i, x_0, y_0) = 0 \tag{16}$$

for (x_0, y_0) . We use the Gauss-Newton method [25] started with a sufficiently good initial guess $(\tilde{x}_0, \tilde{y}_0)$ to solve this underdetermined systems in a least squares sense. Since the Jacobian of

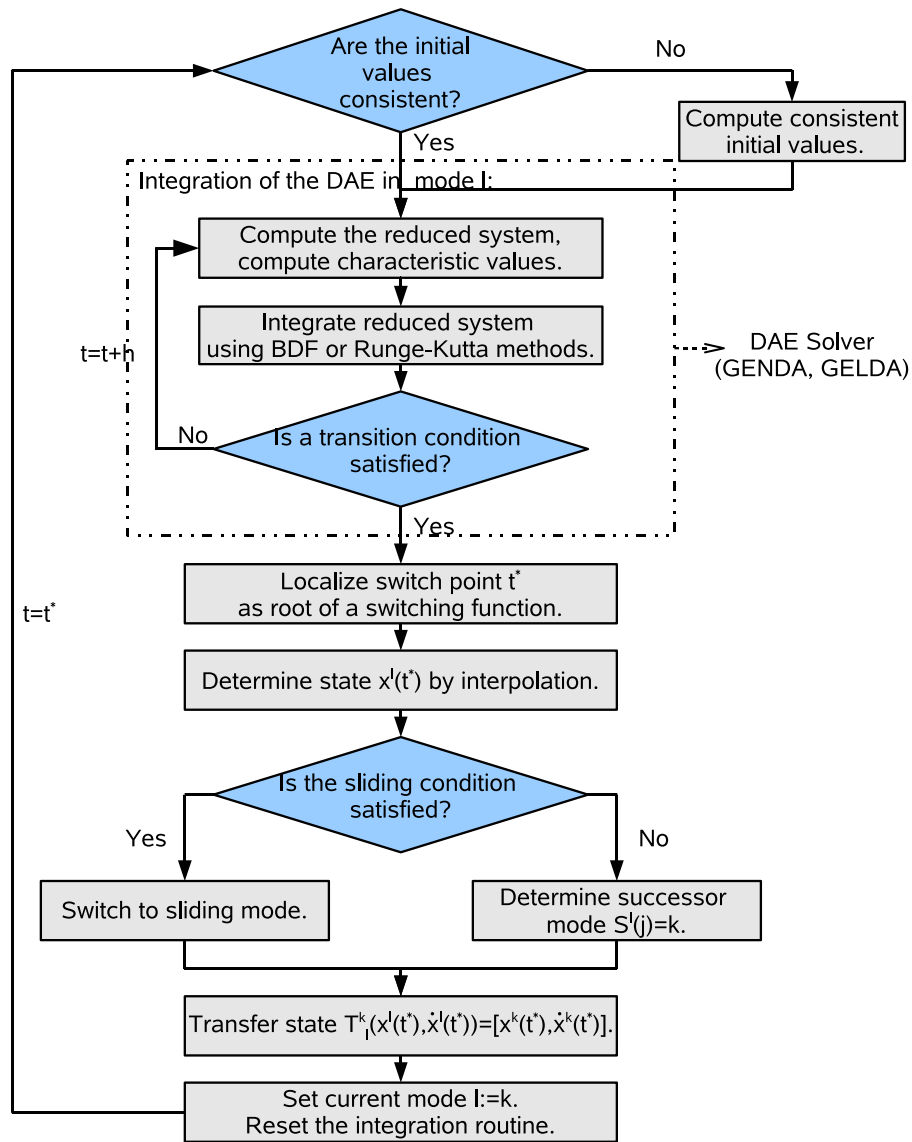


Figure 3: The hybrid mode controller

(16) has full row rank at a solution (x_0, y_0) by Hypothesis 1, we have local quadratic convergence of the Gauss-Newton method, provided that the initial guess $(\tilde{x}_0, \tilde{y}_0)$ is sufficiently close to the solution. To continue the integration of a hybrid system after a mode transition a consistent initial value has to be computed on the basis of a given but probably inconsistent initial state. While algebraic components need to be chosen consistently with the DAE in the current mode, differential and undetermined components should proceed continuously over a switch point, if the transition functions provides continuity. It is possible to prescribe initial values for certain components of an initial guess and only compute consistent values for the algebraic variables. Setting the columns of the Jacobian of (16) that corresponds to differential components to zero, then these components of the initial guess \tilde{x}_0 are kept fixed during the Gauss-Newton iteration. This approach will lead to a rank drop in the Jacobian if any of the algebraic variables are fixed. Note that even if the transition function T_ℓ^k provides continuity of the variables over a switch point, the consistent reinitialization can cause discontinuities in the solution.

6 Numerical Examples

As first example we consider the simple model of an evaporator vessel from [24]. There is a constant inflow of liquid f_{in} into a tank and an outflow that depends on the pressure in the tank and the Bernoulli resistance R_b . To keep the level of fluid in the evaporator vessel at or below a pre-specified maximum, an overflow mechanism is activated when the level of fluid L in the evaporator exceeds the threshold value L_{th} . This causes a flow through a narrow pipe with resistance R_p and inertia I that builds up flow momentum p . The system is modeled in two distinct operation modes: mode 1, where there is no overflow (the fluid level is below the overflow level), and mode 2, where the overflow mechanism is active. The ordinary differential equations describing the system in the two operation modes are given by

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{L} \end{bmatrix} &= \begin{bmatrix} -R_p & 0 \\ 0 & -\frac{1}{R_b} \end{bmatrix} \begin{bmatrix} p \\ L \end{bmatrix} + \begin{bmatrix} 0 \\ f_{in} \end{bmatrix} & \text{(mode 1),} \\ \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{L} \end{bmatrix} &= \begin{bmatrix} -R_p & 1 \\ -1 & -\frac{1}{R_b} \end{bmatrix} \begin{bmatrix} p \\ L \end{bmatrix} + \begin{bmatrix} 0 \\ f_{in} \end{bmatrix} & \text{(mode 2).} \end{aligned}$$

Supposing that initially the system is in mode 1, the inflow causes the tank to start filling, and this in turn causes an outflow through resistance R_b . In this mode the outflow through the narrow pipe is zero. If L exceeds the threshold level L_{th} , a switch from mode 1 to mode 2 occurs at the point in time when $L = L_{th}$. The hybrid system is solved with the hybrid system solver GESDA [31] (at first without the use of sliding mode simulation) using the parameters $R_b = 1, R_p = 0.5, I = 0.5, C = 15, f_{in} = 0.25, L_{th} = 0.08$ and a hysteresis parameter $\varepsilon = 10^{-4}$ defining the transition conditions

$$L^1 : L > L_{th} + \varepsilon, \quad L^2 : L < L_{th} - \varepsilon.$$

For the numerical integration of the differential systems in the two modes the DAE solver GELDA [20] is embedded in the solver GESDA using BDF methods with a combined stepsize and order control. The results of the computations, i.e., the numerical solutions for p and L and the phase portrait, are presented in Figure 4. The hybrid system exhibits chattering behavior around the threshold value $L = L_{th}$ yielding a large number of mode switches. Using the proposed sliding mode simulation this chattering behavior can be prevented. The DAE system in sliding mode as defined in (14) is given by

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{L} \end{bmatrix} &= \left(\alpha \begin{bmatrix} -R_p & 0 \\ 0 & -\frac{1}{R_b} \end{bmatrix} + (1 - \alpha) \begin{bmatrix} -R_p & 1 \\ -1 & -\frac{1}{R_b} \end{bmatrix} \right) \begin{bmatrix} p \\ L \end{bmatrix} + \begin{bmatrix} 0 \\ f_{in} \end{bmatrix}, \\ 0 &= L - L_{th}. \end{aligned}$$

Solving the hybrid system using sliding mode simulation yields the results given in Figure 5. Compared to the first results the oscillations around $L = L_{th}$ have vanished and the computational

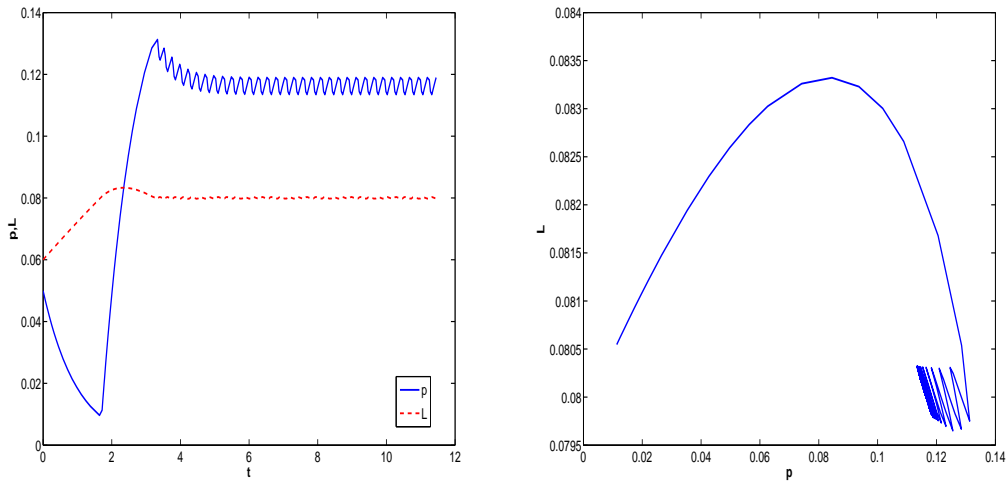


Figure 4: Numerical solution of the evaporator vessel and phase portrait

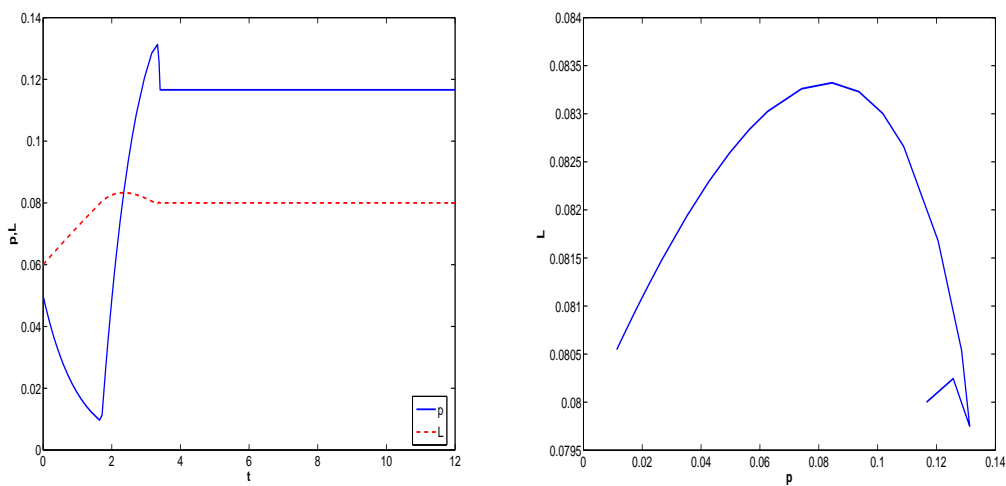


Figure 5: Numerical solution of the evaporator vessel using sliding mode simulation

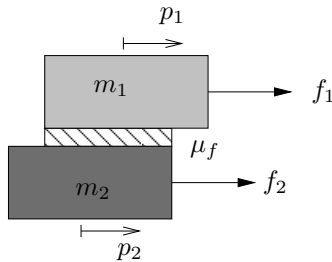


Figure 6: Stick-slip friction between rigid bodies

costs can be reduced drastically. In the first case 65 switch points are detected while in the second case only 3 mode changes occur.

The second example considers stick-slip friction between two rigid bodies as depicted in Figure 6, see also [7]. The equations of motion of the multibody system consisting of two masses with dry friction between them are given by

$$\begin{aligned}
 \dot{p}_1 &= v_1, \\
 \dot{p}_2 &= v_2, \\
 m_1 \dot{v}_1 &= f_1 - \mu_f \|F_N\| \operatorname{sign}(v_1 - v_2), \\
 m_2 \dot{v}_2 &= f_2 + \mu_f \|F_N\| \operatorname{sign}(v_1 - v_2),
 \end{aligned} \tag{17}$$

where p_1, p_2 describe the positions of the bodies and v_1, v_2 are the corresponding velocities. Further, f_1 and f_2 are the applied forces, μ_f is the coefficient of friction, and F_N is the normal force on the surface between the two bodies. Thus, the system (17) represents a hybrid system consisting of two modes depending on the direction of the relative velocity between the bodies, i.e., the system is in mode 1 if $\operatorname{sign}(v_1 - v_2) = 1$ and the system is in mode 2 if $\operatorname{sign}(v_1 - v_2) = -1$. Note that the model equations (17) cannot be applied if the relative velocity is $v_{rel} = v_1 - v_2 = 0$, since $\operatorname{sign}(0)$ is not defined. We apply a small hysteresis band $[-\varepsilon, \varepsilon]$ around zero relative velocity to define the transition conditions

$$\begin{aligned}
 L_1^1(v_1, v_2) &= v_1 - v_2 < -\varepsilon, \\
 L_1^2(v_1, v_2) &= v_1 - v_2 > \varepsilon.
 \end{aligned}$$

The hybrid system (17) is solved with the solver GESDA using the DAE solver GELDA in BDF mode. Since (17) is an ordinary differential equation, the characteristic values in both modes are given by $\mu = 0$, $d = 4$, $a = v = u = 0$. The solution is computed in the interval $[0, 10]$ using $m_1 = m_2 = 1$, $f_1 = \sin(t)$, $f_2 = 0$, $F_N = 1$, $\mu_f = 0.4$, and $\varepsilon = 0.003$ with relative and absolute error tolerance $ATOL = RTOL = 10^{-8}$ and initial values $p_1(0) = p_2(0) = 1$, $v_1(0) = v_2(0) = 0$ in initial mode 1. The computed solution of the system and the relative velocity v_{rel} between the two bodies are presented in Figure 7. We can see that in the beginning both bodies move together, since the applied force is less than the friction force. If the applied force exceeds the friction force, the velocity of the first body becomes greater than the velocity of the second body and the bodies slid over each other until the applied force is again lower than the friction force and the two bodies stick together once more. The relative velocity between the bodies can be seen to oscillate around zero during stiction, resulting in a large number of integration steps and a high computational effort. Altogether, the code requires 8209 integration steps, and 396 mode switches are detected. Physically, these oscillations do not occur, since in the case of stiction both bodies move together and the relative velocity is zero. Using sliding mode simulation we can define the system behavior

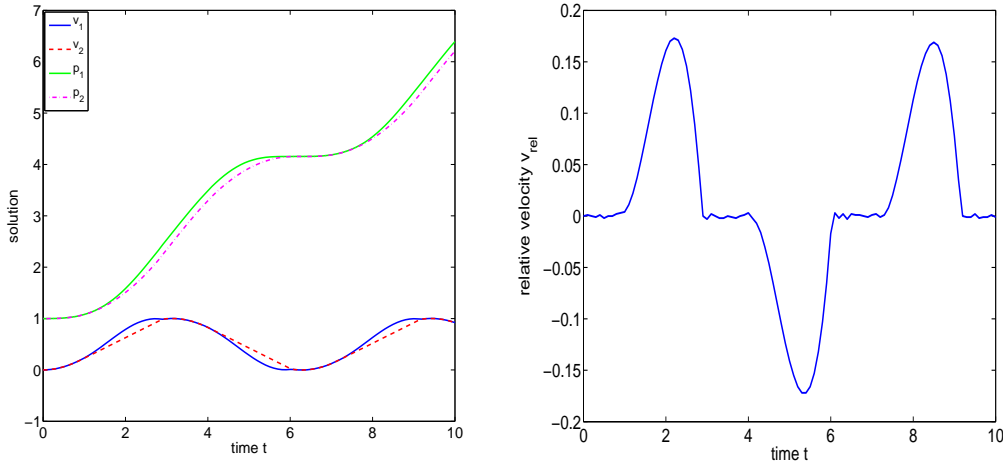


Figure 7: Solution of (17) and relative velocities between the two bodies

during stiction for $v_{rel} = 0$ by

$$\begin{aligned}
 \dot{p}_1 &= v_1, \\
 \dot{p}_2 &= v_2, \\
 m_1 \dot{v}_1 &= f_1 - \mu_f \|F_N\| + \lambda_S, \\
 m_2 \dot{v}_2 &= f_2 + \mu_f \|F_N\| - \lambda_S, \\
 0 &= v_1 - v_2,
 \end{aligned} \tag{18}$$

defining the DAE in sliding mode (mode 3). Now, system (18) represents a DAE with characteristic quantities $\mu = 1$, $d = 4$, and $a = 1$. The system stays in sliding mode as long as the applied force is less than the friction force. If the applied force exceeds the friction force, then the system leaves the sliding mode, i.e., if $\sin(t) - 2\mu_s > 0$, then the system switches back to mode 1, and if $\sin(t) + 2\mu_s < 0$, then the system switches back to mode 2. The system is solved with GESDA using sliding mode simulation. The solution of the system together with the relative velocity v_{rel} using sliding mode simulation whenever chattering is detected by the code is presented in Figure 8. We can see that the oscillations in the relative velocity during stiction disappear. Altogether, 12 switch points are detected during the numerical integration using sliding mode simulation and the number of integration step is drastically reduced to 4737. Note that in Figure 7 and in Figure 8 the solution is plotted only at predefined output points.

7 Conclusions

We have presented a general framework for the analysis, numerical simulation and control of hybrid systems of differential-algebraic equations. We have analyzed the solution behavior and constructed a hybrid mode controller. Together with the extension of Fillipov regularization and sliding mode computation we have substantially improved the performance of standard techniques. The properties of the new numerical approach are demonstrated with some numerical examples.

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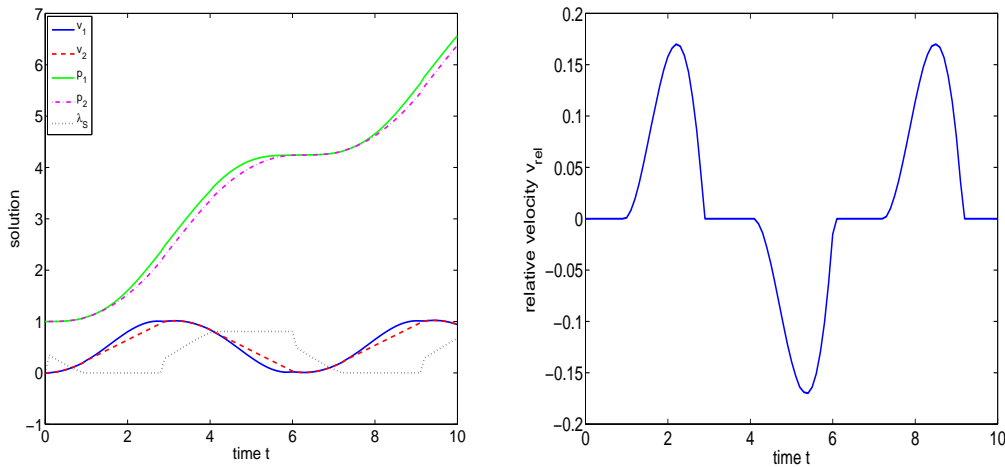


Figure 8: Solution and relative velocity using sliding mode simulation

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